

Zeddouk, Fadoua; Devolder, Pierre

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Article

Pricing of Longevity Derivatives and Cost of Capital

Fadoua Zeddouk * and Pierre Devolder

Institute of Statistics, Biostatistics and Actuarial Sciences, Voie du Roman Pays 20,
1348 Louvain-La-Neuve, Belgium; Pierre.Devolder@uclouvain.be

* Correspondence: fadoua.zeddouk@uclouvain.be

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Abstract: Annuities providers become more and more exposed to longevity risk due to the increase in life expectancy. To hedge this risk, new longevity derivatives have been proposed (longevity bonds, q-forwards, S-swaps...). Although academic researchers, policy makers and practitioners have talked about it for years, longevity-linked securities are not widely traded in financial markets, due in particular to the pricing difficulty. In this paper, we compare different existing pricing methods and propose a Cost of Capital approach. Our method is designed to be more consistent with Solvency II requirement (longevity risk assessment is based on a one year time horizon). The price of longevity risk is determined for a S-forward and a S-swap but can be used to price other longevity-linked securities. We also compare this Cost of capital method with some classical pricing approaches. The Hull and White and CIR extended models are used to represent the evolution of mortality over time. We use data for Belgian population to derive prices for the proposed longevity linked securities based on the different methods.

Keywords: stochastic longevity risk; S-forward; S-swap; solvency capital requirement

1. Introduction

Significant improvements in longevity have been experienced in most developed countries. For annuity providers, longevity risk, i.e., the risk that future mortality trends differ from those anticipated constitutes an important risk factor.

Under the new European regulatory environment for the insurance industry, Solvency II, it becomes a requirement for insurers to measure and evaluate longevity risks (Levantesi and Menzietti (2006)). As a consequence, the level of the capital required for longevity is increasing (EIOPA (2011)), and this creates the need for some longevity risk management solutions.

In the past, annuity providers were only able to transfer the longevity risk to reinsurer who can offer a standardized, or a customized hedging with no residual basis risk, but this risk transfer arrangement has become increasingly expensive and present a large counterparty credit risk. In addition, the capacity of reinsurers is limited because longevity risk cannot be diversified, and, as a consequence, just few reinsurers accept taking longevity risk (Blake 2008). Hence, finding other alternatives to transfer the longevity risk, notably securitization, became a necessity (Cox et al. 2013; Li et al. 2019). Another de-risking approach for a pension provider are the pension buy-in and the pension buy-out (D'Amato et al. 2018).

In recent years, some longevity derivatives arose. These products are based on mortality/longevity rates, similar to those existing in the financial market (Blake et al. 2011).

The two main longevity linked securities that were proposed in the literature are longevity bonds (Blake and Burrows 2001), and survival swaps (Dowd et al. 2006).

Longevity swaps are considered better because their transaction costs are lower, are more flexible and could be customized to suit individual circumstances. Moreover, they do not require the existence

of a liquid market. Therefore, longevity swaps appear to be the most relevant derivative to consider for hedging longevity risk.

Like the interest rate swaps, the survival swaps can be broken down into a collection of more simple derivative: the Survival-forwards (Coughlan et al. 2007).

These derivatives have attracted many researchers and practitioners who have published many papers on this subject, but because of the pricing difficulties, and the fact that these products remove only the longevity risk, they are still not widely traded in the financial market (Lin et al. 2015).

Some authors have introduced a new methodology inspired by Solvency II: the pricing of longevity derivatives using the Cost of Capital approach. A version of this method has been proposed by (Levantesi and Menziatti 2006) in a discrete time model. In this paper, we generalize this approach in two directions: first, we use a risk measurement more in line with Solvency II and based on successive one-year time horizons; secondly, we model longevity risk with continuous time processes, as they offer analytical tractability and can be used for the pricing of financial instruments. We also study the consistency between this Cost of Capital approach (COC) and three important classical pricing methods used in finance: risk-neutral pricing, Wang transform and Sharpe ratio. This paper is organized as follows: in Section 2, we give the general pricing framework for the S-forward with the classical COC approach, then the same framework is provided in Section 3 for the S-swaps. In Section 4, we translate the theoretical framework into a numerical illustration, enabling the analysis of the consistency between the proposed and the existing classical pricing approaches. We show that none of these methods is completely consistent with the COC approach. Finally, in Section 5, we provide conclusions.

2. Survival-Forwards

As an interest-rate swap is essentially a portfolio of forward-rate agreement (Hull 2009), the survival swap can be broken down into a collection of a more simple derivative: the Survival-forward. Therefore, we will begin our study by focusing on them.

2.1. Definition and Structure

A Survival forward (or S-forward) is an agreement between two counterparties to exchange at a future date T (the maturity of the contract), an amount equal to the realized survival rate of a given population cohort (floating leg), in return for a fixed survival rate agreed at the inception of the contract (fixed rate payment), as shown in Figure 1.

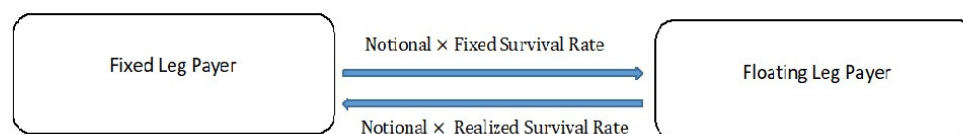


Figure 1. S-forward.

For easiness of representation, we consider the notional amount equal to one monetary unit. The payoff of the S-forward is then given by:

$$\text{Payoff}(T) = {}_T p_x - {}_T \hat{p}_x, \quad (1)$$

where ${}_T p_x$ is the realized survival rate (\mathcal{F}_T measurable), and ${}_T \hat{p}_x$ is a fixed probability of an individual aged x at time 0 to be alive at age $x + T$ (\mathcal{F}_0 measurable).

2.2. Assumptions

In order to price the S-forward and S-swap, we will need to make some assumptions. For the whole paper, we will consider the following:

- The longevity is stochastic through one explanatory variable, the mortality intensity denoted by $\mu_x(t, \omega)$, which represents the force of mortality for an individual aged $x + t$ at time t .
- For this mortality intensity, we consider two affine stochastic time-continuous models (Luciano and Vigna 2015):

1. Hull and White model (HW):

$$d\mu_x(t) = b(\xi(t) - \mu_x(t))dt + \sigma dw(t), \tag{2}$$

2. Cox-Ingersoll-Ross (CIR) extended model:

$$d\mu_x(t) = b(\xi(t) - \mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw(t). \tag{3}$$

With $\xi(t) = \frac{Ae^{bt}}{b}$ and A, B, b, σ all positive numbers, w a standard Brownian motion under the real world probability measure \mathbb{P} .

The CIR model guarantees the positivity of the force of mortality.

- The spot interest rate $r(t, \omega)$ is stochastic and is solution of an SDE (Stochastic Differential Equation) of the form:

$$dr(t, \omega) = h(t, r)dt + z(t, r)d\bar{w}(t, \omega), \tag{4}$$

where h and z are two given functions.

- w and \bar{w} are independent standard Brownian motions under the real probability measure \mathbb{P} .
- We denote by $V(t, T)$ the price at a given time t of an S-forward with maturity T .
- The survival index at time t of an individual initially aged x , alive at time t and surviving $T-t$ years more is given by the following process:

$$I(x + t, T - t) = e^{-\int_t^T \mu_x(u, \omega)du}. \tag{5}$$

- ${}_t p_x^{obs} = e^{-\int_0^t \mu_x(s)ds}$ is the observed survival function (\mathcal{F}_t measurable).
- The two models being affine, the expectation of their survival index is directly given by (Dowd et al. 2006):

$$E_{\mathbb{P}}(I(x + t, T - t) | \mathcal{F}_t) = e^{\alpha_{\mathbb{P}}(t, T) - \beta_{\mathbb{P}}(t, T)\mu_x(t)}. \tag{6}$$

1. In the HW model, we have:

$$\begin{cases} \alpha_{\mathbb{P}}^{HW}(t, T) = \frac{A}{b} [e^{-bT} \frac{e^{(B+b)T} - e^{(B+b)t}}{B+b} - \frac{e^{bT} - e^{bt}}{b}] - \frac{\sigma^2}{2b^2} [\frac{1}{b}(1 - e^{-b(T-t)}) - T + t] - \frac{\sigma^2}{4b^3}(1 - e^{-b(T-t)})^2, \\ \beta_{\mathbb{P}}^{HW}(t, T) = \frac{1}{b}(1 - e^{-b(T-t)}). \end{cases} \tag{7}$$

2. In the CIR extended model, we have:

$$\begin{cases} \alpha_{\mathbb{P}}^{CIR}(t, T) = 2A \int_t^T \frac{e^{Bs}(e^{\gamma(T-s)} - e^{-\gamma(T-s)})}{(2\gamma+b)e^{\gamma(T-s)} + (2\gamma-b)e^{-\gamma(T-s)}} ds, \\ \beta_{\mathbb{P}}^{CIR}(t, T) = \frac{\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2}b \sinh(\gamma(T-t))}. \end{cases} \tag{8}$$

$$\text{with } \gamma = \frac{(b^2 + 2\sigma^2)^{\frac{1}{2}}}{2}$$

- $P(t, T) = E_{\mathbb{Q}}(e^{-\int_t^T r(s)ds} | \mathcal{F}_t)$ is the price at time t of a zero coupon bond with maturity T under the risk-neutral probability measure \mathbb{Q} .
- There is no basis risk: the insurer's reference population is the same as in the longevity linked security.
- There is no counterparty default risk.

2.3. S-Forward Pricing by Cost of Capital

There is an ongoing debate and many authors have addressed the question of how to price longevity-linked securities, but the calibration of the market price of longevity risk remains an open question as “ in one hand, there is a quest for actuarial or economic methods of how to derive prices “conceptually” and, on the other hand, there is the problem of how to derive prices given a certain methodology, i.e., what data to use for calibration purposes ” (Bauer et al. 2010).

Some authors have suggested that, following the directive of Solvency II, the price of longevity linked securities could be associated with the Solvency Capital Requirement (SCR) to cover unexpected losses (Börger 2010).

Levantesi and Menzietti (2006) have proposed to price a S-forward using the risk margin implicit within the calculation of the technical provisions as defined by Solvency II.

The overall idea is based on linking the price of longevity-linked securities with the capital that the insurer should hold to cover unexpected losses: according to Solvency II, insurance liabilities that cannot be hedged are evaluated as the sum of a best estimate and a risk margin (RM), which is the market value of the uncertainty on insurance liabilities. The RM is determined by the cost of capital approach based on the SCR, the capital required to cover with 99.5% probability the unexpected losses on a one-year time horizon.

In other words, by entering an S-forward, the insurer can lower its exposition to the longevity risk and, in consequence, the SCR for longevity risk can be mitigated, or even reduced to zero if the longevity risk is completely covered, and so does the corresponding RM. Hence, the price of an S-forward shouldn't be higher than the cost of holding the SCR; otherwise, insurers won't be interested in buying it. In fact, this approach provides an estimation of the maximum price that the insurer would be willing to pay to transfer longevity risk.

Let's consider a T -years S-forward contract (Figure 2).

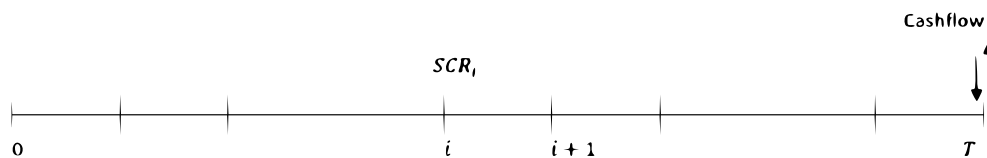


Figure 2. T -years S-forward contract.

Under this approach, the price of the S-forward at time t is given by:

$$V_{COC}(t, T) = BE_t^{\mathbb{P}} + RM_t, \tag{9}$$

where $BE_t^{\mathbb{P}}$ is the best estimate, equal to the discounted expectation of the payoff of the S-forward at time t under the real-world measure, and RM_t the risk margin.

At time 0, we obtain:

$$V_{COC}(0, T) = BE_0^{\mathbb{P}} + RM_0, \tag{10}$$

where $BE_0^{\mathbb{P}}$ is as follows:

$$BE_0^{\mathbb{P}} = P(0, T)(E_{\mathbb{P}}(I(x, T) - {}_T\hat{p}_x)). \tag{11}$$

The risk margin is defined as the present value of future returns on the successive SCRs:

$$RM_0 = C\% \sum_{i=0}^{T-1} SCR_i P(0, i + 1), \tag{12}$$

where SCR_i is the solvency capital Requirement to cover with 99.5% probability the unexpected losses for year i , and C is the Cost of Capital rate (6% in Solvency II).

These future SCR_i are random variables. In order to compute the initial risk margin RM_0 , we need to use their estimation at time 0 denoted by: $\hat{S}CR_i = SCR_i |_0$.

The risk margin at time 0 becomes then:

$$RM_0 = C\% \sum_{i=0}^{T-1} S\hat{C}R_i P(0, i + 1). \tag{13}$$

In order to determine the price of an S-forward, we will need to calculate the estimation the future solvency capitals. Solvency II provides different methods to calculate these capitals. We will consider the VaR (Value at Risk) method, which measures the potential loss in value of a risky asset or liability over a defined time horizon for a given confidence level, say 99.5% (Solvency II standard).

2.3.1. Levantesi and Menzietti’s Specification

A first approach developed by (Levantesi and Menzietti 2006) is based on the VaR of the payoff at 99.5% from the moment of calculation until maturity, as represented by Figure 3:

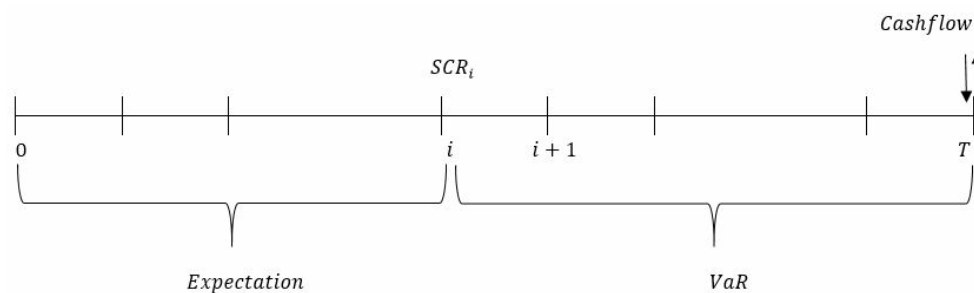


Figure 3. Computation of the successive SCR_i in the Levantesi and Menzietti’s specification.

Explicitly, the SCR_i are given by:

$$SCR_i = VaR_{99,5\%}(P(i, T) Payoff_T^{VaR_{99,5\%}} - BE_i^{\mathbb{P}}) \tag{14}$$

$$= P(i, T) VaR_{99,5\%}(I(x, T)_T \hat{p}_x) - P(i, T) (E_{\mathbb{P}}(I(x, T) - T \hat{p}_x)) \tag{15}$$

$$= VaR_{99,5\%}(P(i, T) (I(x, T)_T \hat{p}_x)) - P(i, T) (E_{\mathbb{P}}(I(x, T) - T \hat{p}_x)) \tag{16}$$

$$= P(i, T) (I(x, i) [VaR_{99,5\%}(I(x + i, T - i)) - E_{\mathbb{P}}(I(x, T))]), \tag{17}$$

$$SCR_i |_0 = P(i, T) [E_{\mathbb{P}}(I(x, i)) VaR_{99,5\%}(I(x + i, T - i)) - E_{\mathbb{P}}(I(x, T))]. \tag{18}$$

From expression (18), we can see that the Value at Risk is measured on the whole interval $[i, T]$.

The price of the S-forward is then equal to:

$$V_{COC}(0, T) = P(0, T) (E_{\mathbb{P}}(I(x, T)) - T \hat{p}_x) + 6\% \sum_{i=0}^{T-1} P(i, T) P(0, i + 1) [E_{\mathbb{P}}(I(x, i)) VaR_{99,5\%}(I(x + i, T - i)) - E_{\mathbb{P}}(I(x, T))]. \tag{19}$$

2.3.2. Our Specification

To be more consistent with the logic of Solvency II, we propose to adopt another formulation: the SCR at time i should cover with 99.5% probability the unexpected losses on a one-year time horizon (the interval $[i, i + 1]$), and we consider mortality evolution’s up to time i , and from $i + 1$ to T according to their best estimates (Figure 4) :

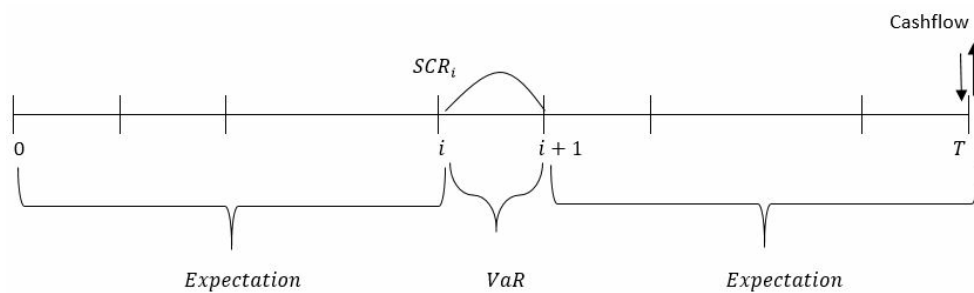


Figure 4. Computation of the successive SCR_i at time 0 in our specification.

Then, the expression of the SCR_i becomes:

$$SCR_i = VaR_{99,5\%}[BE_{i+1}^{\mathbb{P}}P(i, i + 1) - BE_i^{\mathbb{P}}], \tag{20}$$

where

$$BE_{i+1}^{\mathbb{P}} = (I(x, i + 1)E_{\mathbb{P}}(I(x + i + 1, T - i - 1) - {}_T\hat{p}_x))P(i + 1, T). \tag{21}$$

In addition,

$$BE_i^{\mathbb{P}} = (I(x, i)E_{\mathbb{P}}(I(x + i, T - i) - {}_T\hat{p}_x))P(i, T). \tag{22}$$

The SCR_i is then given by:

$$\begin{aligned} SCR_i &= P(i, T)VaR_{99,5\%}[I(x, i + 1)E_{\mathbb{P}}(I(x + i + 1, T - i - 1)) - I(x, i)E_{\mathbb{P}}(I(x + i, T - i))] \\ &= P(i, T)I(x, i)[VaR_{99,5\%}(I(x + i, 1))E_{\mathbb{P}}(I(x + i + 1, T - i - 1)) - E_{\mathbb{P}}(I(x + i, T - i))]. \end{aligned} \tag{23}$$

Therefore, the estimation of $SCR_i |_0$ can be given by:

$$SCR_i |_0 = P(i, T)E_{\mathbb{P}}(I(x, i))[VaR_{99,5\%}(I(x + i, 1)) - E_{\mathbb{P}}(I(x + i, 1))]E_{\mathbb{P}}(I(x + i + 1, T - i - 1)). \tag{24}$$

The risk margin at time 0 is then equal to:

$$RM_0 = 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I(x, i))[VaR_{99,5\%}(I(x + i, 1)) - E_{\mathbb{P}}(I(x + i, 1))]E_{\mathbb{P}}(I(x + i + 1, T - i - 1))P(0, i + 1)P(i, T). \tag{25}$$

Finally, the price of the S-forward with the COC approach is:

$$\begin{aligned} V_{COC}(0, T) &= P(0, T) (E_{\mathbb{P}}[I(x, T) - {}_T\hat{p}_x]) \\ &+ 6\% \sum_{i=0}^{T-1} [E_{\mathbb{P}}(I(x, i))[VaR_{99,5\%}(I(x + i, 1)) - E_{\mathbb{P}}(I(x + i, 1))] \\ &E_{\mathbb{P}}(I(x + i + 1, T - i - 1))P(0, i + 1)P(i, T). \end{aligned} \tag{26}$$

Let us apply these methods in two particular mortality models.

HW Model

In order to determine the explicit expression of the price, we need to compute:

$$VaR_{99,5\%}(I(x + i, 1)) = VaR_{99,5\%}(e^{-\int_i^{i+1} \mu_x(s) ds}).$$

In general, we have:

$$VaR(I(x + t, T - t)) = VaR_{99,5\%}(e^{-\int_t^T \mu_x(s) ds}) = VaR(e^{X(t,T)}). \tag{27}$$

We know that:

$$I(x + t, T - t) = e^{-\int_t^T \mu_x^{\mathbb{P}}(u, \omega) du} = e^{X(t,T)}. \tag{28}$$

One can show that, under the real-world risk measure \mathbb{P} , we have :

$$X(t, T) \sim N(m(t, T), n(t, T)^2). \tag{29}$$

Thus, the survival index I is log-normally distributed with:

$$\begin{cases} m(t, T) = \mu_x(t) \frac{(e^{-b(T-t)} - 1)}{b} - \frac{Ae^{Bt}}{B(b+B)} (e^{B(T-t)} - 1) - \frac{Ae^{Bt}}{b(b+B)} (e^{-b(T-t)} - 1), \\ n^2(t, T) = \frac{\sigma^2}{b^2} [T - t - \frac{1 - e^{-b(T-t)}}{b} - \frac{(1 - e^{-b(T-t)})^2}{2b}]. \end{cases} \tag{30}$$

Then, we have:

$$VaR_{99,5\%}(I(x + t, T - t)) = e^{m(t, T) + z_\eta n(t, T)}, \tag{31}$$

where $z_\eta = \eta$ is the quantile of a normal distribution $N(0, 1)$.

We consider $\eta = 0,995, z_\eta = 2,58$ (Solvency II standard). Finally, the price of the S-forward at time 0 becomes:

$$\begin{aligned} V_{COC}^{HW}(0, T) &= P(0, T)(e^{\alpha_{\mathbb{P}}^{HW}(0, T) - \beta_{\mathbb{P}}^{HW}(0, T)\mu_x(0)} - {}_T\hat{p}_x) \\ &+ 6\% \sum_{i=0}^{T-1} [(e^{\alpha_{\mathbb{P}}^{HW}(0, i) - \beta_{\mathbb{P}}^{HW}(0, i)\mu_x(0)} [e^{m(i, i+1) + z_\eta n(i, i+1)} \\ &- e^{\alpha_{\mathbb{P}}^{HW}(i, i+1) - \beta_{\mathbb{P}}^{HW}(i, i+1)\overline{\mu_x(i)}}] e^{\alpha_{\mathbb{P}}^{HW}(i+1, T) - \beta_{\mathbb{P}}^{HW}(i+1, T)\overline{\mu_x(i+1)}}] P(0, i + 1)P(i, T), \end{aligned} \tag{32}$$

with $\overline{\mu_x(i)} = E_{\mathbb{P}}(\mu_x(i))$ and $\alpha_{\mathbb{P}}^{HW}(\cdot, T), \beta_{\mathbb{P}}^{HW}(\cdot, T)$ are given by Equation (7).

CIR Model

Let us here consider that the intensity of mortality follows the CIR process. The same as in the HW model, we will need to determine $VaR(e^{-\int_i^{i+1} \mu_x(s) ds})$, but as it is not easy to find an analytical solution, we will simulate it in Section 4. The price of the S-swap is then given by:

$$\begin{aligned} V_{COC}^{CIR}(0, T) &= P(0, T)(e^{\alpha_{\mathbb{P}}^{CIR}(0, T) - \beta_{\mathbb{P}}^{CIR}(0, T)\mu_x(0)} - {}_T\hat{p}_x) \\ &+ 6\% \sum_{i=0}^{T-1} [(e^{\alpha_{\mathbb{P}}^{CIR}(0, i) - \beta_{\mathbb{P}}^{CIR}(0, i)\mu_x(0)} [VaR(e^{-\int_i^{i+1} \mu_x(s) ds}) \\ &- e^{\alpha_{\mathbb{P}}^{CIR}(i, i+1) - \beta_{\mathbb{P}}^{CIR}(i, i+1)\overline{\mu_x(i)}}] e^{\alpha_{\mathbb{P}}^{CIR}(i+1, T) - \beta_{\mathbb{P}}^{CIR}(i+1, T)\overline{\mu_x(i+1)}}] P(0, i + 1)P(i, T), \end{aligned} \tag{33}$$

where $\alpha_{\mathbb{P}}^{CIR}(\cdot, T)$ and $\beta_{\mathbb{P}}^{CIR}(\cdot, T)$ are given by Equation (8).

2.3.3. Pricing at Any Time t

Following our same reasoning, we give the expression of SCR_i at any time t ($0 < t < i$) under the COC approach (Figure 5):

$$\begin{aligned} V_{COC}(t, T) &= P(t, T)I(x, t)E_{\mathbb{P}}(I(x + t, T - t)) \\ &+ 6\% \sum_{i=t}^{T-1} I(x, t)E_{\mathbb{P}}(I(x + t, i - t)) [VaR_{99,5\%}(I(x + i, 1)) - E_{\mathbb{P}}(I(x + i, 1))] \\ &E_{\mathbb{P}}(I(x + i + 1, T - i - 1))P(i, T)P(t, i + 1). \end{aligned} \tag{34}$$

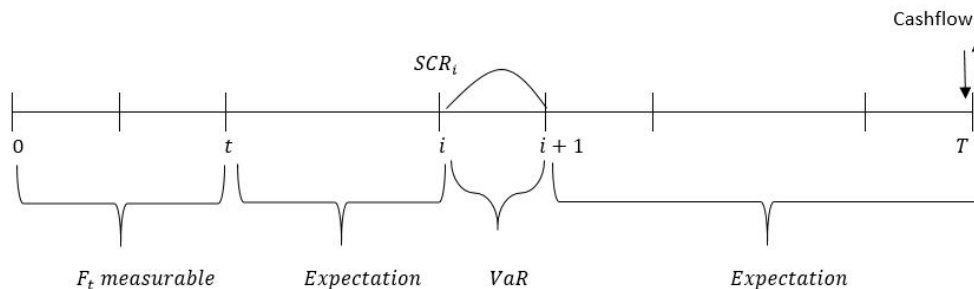


Figure 5. Computation of the successive SCR_t at any time t in our specification.

HW Model

We determine under our philosophy the price of an S-forward at time t ($0 < t < T$) using the Hull and White model:

$$\begin{aligned}
 V_{COC}^{HW}(t, T) &= P(t, T)[{}_t p_x^{obs} \cdot e^{\alpha_{\mathbb{P}}^{HW}(t, T) - \beta_{\mathbb{P}}^{HW}(t, T)\mu_x(t)} - {}_T \hat{p}_x] \\
 &+ 6\% \sum_{i=t}^{T-1} [{}_t p_x^{obs} \cdot e^{\alpha_{\mathbb{P}}^{HW}(t, i) - \beta_{\mathbb{P}}^{HW}(t, i)\mu_x(t)} [e^{m(i, i+1) + z_{\eta} n(i, i+1)} \\
 &- e^{\alpha_{\mathbb{P}}^{HW}(i, i+1) - \beta_{\mathbb{P}}^{HW}(i, i+1)\overline{\mu_x(i)}}] e^{\alpha_{\mathbb{P}}^{HW}(i+1, T) - \beta_{\mathbb{P}}^{HW}(i+1, T)\overline{\mu_x(i+1)}}] P(i, T)P(t, i + 1).
 \end{aligned}
 \tag{35}$$

CIR Model

Let us here consider that the intensity of mortality follows a CIR process. The price of the S-forward at time t ($0 < t < T$) becomes:

$$\begin{aligned}
 V_{COC}^{CIR}(t, T) &= P(t, T)[{}_t p_x^{obs} \cdot e^{\alpha_{\mathbb{P}}^{CIR}(t, T) - \beta_{\mathbb{P}}^{CIR}(t, T)\mu_x(t)} - {}_T \hat{p}_x] \\
 &+ 6\% \sum_{i=t}^{T-1} [{}_t p_x^{obs} \cdot e^{\alpha_{\mathbb{P}}^{CIR}(t, i) - \beta_{\mathbb{P}}^{CIR}(t, i)\mu_x(t)} [VaR(e^{-\int_i^{i+1} \mu_x(s) ds}) \\
 &- e^{\alpha_{\mathbb{P}}^{CIR}(i, i+1) - \beta_{\mathbb{P}}^{CIR}(i, i+1)\mu_x(i)}] e^{\alpha_{\mathbb{P}}^{CIR}(i+1, T) - \beta_{\mathbb{P}}^{CIR}(i+1, T)\overline{\mu_x(i+1)}}] P(i, T)P(t, i + 1).
 \end{aligned}
 \tag{36}$$

2.4. Comparison with Classical Methods

We develop in this section the pricing of the S-forward contract, following other alternative theoretical formulas to compare with formula (34):

- The risk-neutral approach (Cairns et al. 2006a),
- The Wang transform (Wang 2002),
- Sharpe ratio (Milevsky et al. 2005).

Numerical comparisons will be given in Section 4.

2.4.1. Risk-Neutral Approach

The risk-neutral pricing is a technique widely used in quantitative finance to compute the values of derivatives, and could be adapted to price longevity-linked securities.

As discussed in some papers, (see for example (Cairns et al. 2006b)), this approach is not ideal to price longevity derivatives as the longevity market is immature, and this method requires the availability of a considered amount of data. However, it is interesting to explore the risk-neutral pricing formula for comparison purpose.

Under the risk-neutral probability measure \mathbb{Q}_λ , the price of an S-forward at time t is given by:

$$V_{\mathbb{Q}_\lambda}(t, T) = P(t, T)[{}_t p_x^{obs} E_{\mathbb{Q}_\lambda}(I(x + t, T - t)) - {}_T \hat{p}_x \mid \mathcal{F}_t],
 \tag{37}$$

where λ is the market price of longevity risk.

We first consider the HW model to get an explicit expression of the S-forward price. We assume that $\mu_x(t)$ follows the HW process given by (2).

Let's introduce the market prices $\lambda(t, \mu_x(t))$ associated with longevity risk, and let $w^*(t, \omega) = w(t, \omega) - \int_0^t \lambda(t, \mu_x(s)) ds$.

We consider a model with a constant market price of risk $\lambda(t, \mu_x(t)) = \lambda$.

The SDE of the force of mortality under the risk-neutral measure becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\zeta(t) - b\mu_x(t) + \sigma\lambda)dt + \sigma dw^*(t). \tag{38}$$

From Equation (38), we see that a positive value of λ would increase the drift of the force of mortality. To follow the same insurer's logic, by overestimating the survival process expected value and underestimating the mortality, we have to choose a negative value for λ .

Under the risk-neutral measure, the survival probability takes the form:

$$E_{\mathbb{Q}_\lambda}(I(x+t, T-t)) = e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(t,T) - \beta_{\mathbb{Q}_\lambda}^{HW}(t,T)\mu_x(t)}, \tag{39}$$

where $\alpha_{\mathbb{Q}_\lambda}^{HW}(t, T)$ and $\beta_{\mathbb{Q}_\lambda}^{HW}(t, T)$ are given by:

$$\begin{cases} \alpha_{\mathbb{Q}_\lambda}^{HW}(t, T) = \frac{A}{b} \left[e^{-bT} \frac{e^{(B+b)T} - e^{(B+b)t}}{B+b} - \frac{e^{BT} - e^{Bt}}{B} \right] - \frac{\sigma^2}{2b^2} \left[\frac{1}{b} (1 - e^{-b(T-t)}) - T + t \right] - \frac{\sigma^2}{4b^3} (1 - e^{-b(T-t)})^2 - \frac{\sigma\lambda}{b} (1 - \exp(-b(T-t))), \\ \beta_{\mathbb{Q}_\lambda}^{HW}(t, T) = \frac{1}{b} (1 - e^{-b(T-t)}). \end{cases} \tag{40}$$

Hence, the price of the S-forward at time t with the risk-neutral approach is as follows:

$$V_{\mathbb{Q}_\lambda}^{HW}(t, T) = P(t, T) ({}_tP_x^{obs} e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(t,T) - \beta_{\mathbb{Q}_\lambda}^{HW}(t,T)\mu_x(t)} - {}_T\hat{p}_x). \tag{41}$$

CIR Model

Let us now assume that $\mu_x(t)$ follows the CIR process given by (3).

We introduce the market prices $\lambda(t, \mu_x(t))$ associated with longevity risk, $w^*(t, \omega) = w(t, \omega) - \int_0^t \lambda(t, \mu_x(s)) ds$

Under the real-world probability measure \mathbb{P} , w^* is a Brownian motion with drift, and therefore is not a martingale. According to Girsanov's theorem, $w^*(t, \omega)$ is a standard Brownian motion under the risk-neutral measure \mathbb{Q} .

The SDE of the force of mortality becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\zeta(t) - b\mu_x(t) + \lambda\sigma\sqrt{\mu_x(t)})dt + \sigma\sqrt{\mu_x(t)}dw^*(t). \tag{42}$$

The survival probability also takes the form: $E_{\mathbb{P}}(I(x+t, T-t)) = e^{\alpha_{\mathbb{Q}_\lambda}^{CIR}(t,T) - \beta_{\mathbb{Q}_\lambda}^{CIR}(t,T)\mu_x(t)}$.

We assume that the market price of longevity risk has the following classical form: $\lambda = \kappa\sqrt{\mu_x(t)}$ with κ is a constant.

Equation (42) then becomes:

$$d\mu_x^{\mathbb{Q}_\lambda}(t) = (\zeta(t) - (b - \sigma\kappa)\mu_x(t))dt + \sigma\sqrt{\mu_x(t)}dw^*(t). \tag{43}$$

The parameters $\alpha_{\mathbb{Q}_\lambda}^{CIR}(t, T)$ and $\beta_{\mathbb{Q}_\lambda}^{CIR}(t, T)$ are given by:

$$\begin{cases} \alpha_{\mathbb{Q}_\lambda}^{CIR}(t, T) = 2A \int_t^T \frac{e^{Bs}(e^{\gamma(T-s)} - e^{-\gamma(T-s)})}{(2\gamma + b - \sigma\kappa)e^{\gamma(T-s)} + (2\gamma - b + \sigma\kappa)e^{-\gamma(T-s)}} ds, \\ \beta_{\mathbb{Q}_\lambda}^{CIR}(t, T) = \frac{\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t) + \frac{1}{2}(b - \sigma\kappa) \sinh(\gamma(T-t)))}, \end{cases} \tag{44}$$

with $\gamma = \frac{((b - \sigma\kappa)^2 + 2\sigma^2)^{\frac{1}{2}}}{2}$.

Finally, the price is given by:

$$V_{\mathbb{Q}_\lambda}^{CIR}(t, T) = P(t, T) [{}_t p_x^{obs} e^{\alpha_{\mathbb{Q}_\lambda}^{CIR}(t, T) - \beta_{\mathbb{Q}_\lambda}^{CIR}(t, T) \mu_x(t)} - {}_T \hat{p}_x]. \tag{45}$$

2.4.2. Wang Transform Approach

The Wang transform method is a distortion approach that is based on a distortion operator (Leung et al. 2018). In the insurance context, this operator converts the best estimate of the survival index into its risk equivalent using a specific price of risk. This procedure was presented by (Dowd et al. 2006) as a pricing method used in the pricing of several over-the-counter longevity swaps in practice, and used for instance by (Denuit et al. 2007).

The Wang distortion risk measure is given by:

$$\rho_{g_\delta}(x) = \int_0^\infty g_\delta(\bar{F}_x(s)) ds, \tag{46}$$

where x is a continuous non-negative random variable.

$\bar{F}_x(s)$ is its decumulative function, and g_δ is the distortion function associated with the distortion parameter δ and given by:

$$g_\delta(s) = \Phi(\Phi^{-1}(s) + \delta), \tag{47}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution

Then:

$$\rho_{g_\delta}(I(x + t, T - t)) = \int_0^\infty \Phi(\Phi^{-1}(\bar{F}_{I(x+t, T-t)}(s)) + \delta) ds. \tag{48}$$

The price of the S-forward at time t is then given by:

$$V_{Wang}(t, T) = P(t, T) [{}_t p_x^{obs} \rho_{g_\delta}(I(x + t, T - t)) - {}_T \hat{p}_x | \mathcal{F}_t]. \tag{49}$$

HW Model

To obtain an explicit expression of the S-forward price with the HW model, we need to determine the formula of $\rho_{g_\delta}(I(x + t, T - t))$.

The Wang distortion of a log-normal distribution is still log-normal with a corrected drift parameter:

$$e^{N(m(t, T), n^2(t, T))} \longrightarrow e^{N(m(t, T) + \delta n(t, T), n^2(t, T))}. \tag{50}$$

Thus:

$$\rho_{g_\delta}(I(x + t, T - t)) = e^{m(t, T) + \delta n(t, T) + \frac{n(t, T)^2}{2}}. \tag{51}$$

The price of S-forward with Wang approach at time t is finally given by:

$$V_{Wang}^{HW}(t, T) = P(t, T)[{}_t p_x^{obs} e^{m(t,T)+\delta n(t,T)+\frac{n(t,T)^2}{2}} - {}_T \hat{p}_x]. \tag{52}$$

CIR Model

Because of its complicated distribution, it is not easy to find an explicit formula for $\rho_{g\delta}(I(x + t, T - t))$. Hence, we will need an SDE discretization scheme.

For the CIR model, the Euler discretization scheme is as follows:

$$\mu_x(t + \Delta t) = \mu_x(t) + (\xi(t) - b\mu_x(t))\Delta t + \sigma\sqrt{\mu_x(t)}\Delta w(t). \tag{53}$$

In order to simulate the survival index $I(x + t, T - t)$, we will make a classical assumption often encountered in life insurance theory: we will suppose that the force of mortality is piecewise constant between each point computed by the above Euler discretization.

We will use an R numerical simulation to determine the price of the S-forward; the algorithm will be described in Section 4.

2.4.3. The Sharpe Ratio Approach

The Sharpe ratio method was used for instance by (Milevsky et al. 2005) and by (Loeys et al. 2007) to price q-forwards. It is a standard deviation principle that does not use a transformation of survival probabilities.

In liquid financial markets, the Sharpe ratio provides a benchmark for the determination of risk premia. The Sharpe ratio is given by:

$$S^{market} = \frac{r_p - r_f}{\sigma_p}, \tag{54}$$

where r_p is the expected return of the portfolio, r_f is the risk free rate and σ_p is the standard deviation of the portfolio return.

Using the analogy to the Sharpe ratio in financial market, the Sharpe ratio in the insurance context (Barrieu and Veraart 2014) is equal to the excess payoff on the longevity-linked securities over its best estimate divided by its standard deviation:

$$S^{Insurance} = \frac{V(X) - E_{\mathbb{P}}(X)}{\sigma_{\mathbb{P}}(X)}. \tag{55}$$

According to this approach, the price of the S-forward is given by:

$$V_{Sharpe}(t, T) = P(t, T)({}_t p_x^{obs} E_{\mathbb{P}}(I(x + t, T - t)) - {}_T \hat{p}_x + S {}_t p_x^{obs} \sqrt{var_{\mathbb{P}}(I(x + t, T - t))}), \tag{56}$$

where S is the chosen fixed Sharpe ratio, and $var_{\mathbb{P}}(I(x + t, T - t))$ is the variance of the survival index.

HW Model

The variance of the survival process is as follows:

$$var_{\mathbb{P}}(I(x + t, T - t)) = V_{\mathbb{P}}(e^{X(t,T)}) = V_{\mathbb{P}}(e^{N(m(t,T), n(t,T)^2)}). \tag{57}$$

Then:

$$var_{\mathbb{P}}(I(x + t, T - t)) = (e^{n(t,T)^2} - 1)e^{2m(t,T)+n(t,T)^2}. \tag{58}$$

The price of the S-forward at time t is then given by:

$$V_{Sharpe}^{HW}(t, T) = P(t, T)({}_t p_x^{obs} E_{\mathbb{P}}[I(x + t, T - t)] - {}_T \hat{p}_x - S_t p_x^{obs} \sqrt{(e^{n(t,T)^2} - 1)e^{2m(t,T)+n(t,T)^2}}). \tag{59}$$

CIR Model

We have:

$$var_{\mathbb{P}}(I(x + t, T - t)) = E_{\mathbb{P}}[I(x + t, T - t)] - E_{\mathbb{P}}[I(x + t, T - t)]^2, \tag{60}$$

where:

$$E_{\mathbb{P}}[I(x + t, T - t)^2] = E_{\mathbb{P}}(e^{-\int_t^T 2\mu_x(s)ds}) = e^{\alpha_{\mathbb{P}}^{CIR^*}(t,T) - \beta_{\mathbb{P}}^{CIR^*}(t,T)2\mu_x(t)} \tag{61}$$

and:

$$d2\mu_x(t) = ((2\zeta(t) - 2b\mu_x(t))dt + \sqrt{2\sigma} \sqrt{2\mu_x(t)}dw(t)). \tag{62}$$

$\alpha_{\mathbb{P}}^{CIR^*}(t, T)$ and $\beta_{\mathbb{P}}^{CIR^*}(t, T)$ are then given by:

$$\begin{cases} \alpha_{\mathbb{P}}^{CIR^*}(t, T) = 4A \int_t^T \frac{e^{Bs}(e^{\gamma^*(T-s)} - e^{-\gamma^*(T-s)})}{(2\gamma^* + b)e^{\gamma^*(T-s)} + (2\gamma^* - b)e^{-\gamma^*(T-s)}} ds, \\ \beta_{\mathbb{P}}^{CIR^*}(t, T) = \frac{\sinh(\gamma^*(T-t))}{\gamma^* \cosh(\gamma^*(T-t) + \frac{1}{2}b \sinh(\gamma^*(T-t)))} \end{cases} \tag{63}$$

with $\gamma^* = \frac{(b^2 + 4\sigma^2)^{\frac{1}{2}}}{2}$.

$\alpha_{\mathbb{P}}^{CIR}(t, T)$ and $\alpha_{\mathbb{P}}^{CIR^*}(t, T)$ can be solved numerically. The variance of the survival index under the real world probability is:

$$var_{\mathbb{P}}(I(x + t, T - t)) = e^{\alpha_{\mathbb{P}}^{CIR^*}(t,T) - \beta_{\mathbb{P}}^{CIR^*}(t,T)2\mu_x(t)} - e^{2[\alpha_{\mathbb{P}}^{CIR}(t,T) - \beta_{\mathbb{P}}^{CIR}(t,T)\mu_x(t)]}. \tag{64}$$

Finally, the price of the S-forward is as follows:

$$V_{Sharpe}^{CIR}(t, T) = P(t, T)({}_t p_x^{obs} e^{\alpha_{\mathbb{P}}^{CIR}(t,T) - \beta_{\mathbb{P}}^{CIR}(t,T)\mu_x(t)} - {}_T \hat{p}_x + {}_t p_x^{obs} S \sqrt{e^{\alpha_{\mathbb{P}}^{CIR^*}(t,T) - \beta_{\mathbb{P}}^{CIR^*}(t,T)2\mu_x(t)} - e^{2[\alpha_{\mathbb{P}}^{CIR}(t,T) - \beta_{\mathbb{P}}^{CIR}(t,T)\mu_x(t)]}}). \tag{65}$$

3. Survival-Swaps

As we mentioned previously, the S-swap is considered as the sum of S-forward contracts.

3.1. Definition and Structure

A Survival swap (or S-swap) is a collection of S-forwards exchanged at a predetermined calendar times $[t_1, \dots, t_n]$. Schematically, an S-swap can be represented as follows (Figure 6):

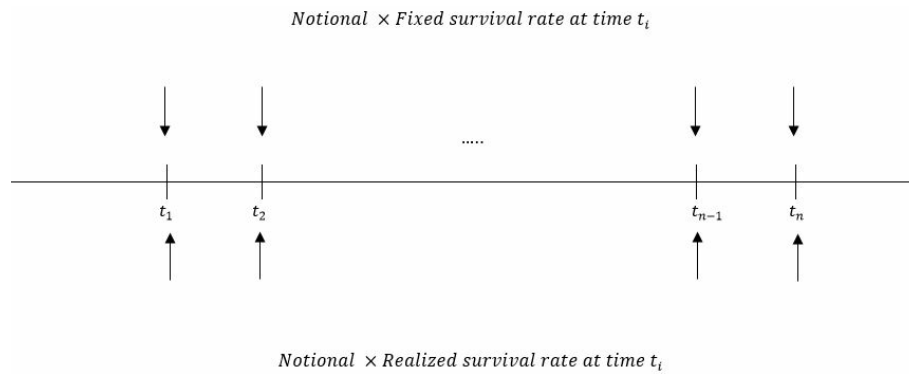


Figure 6. S-swap structure.

3.2. S-Swap Pricing under the Cost of Capital Approach

Let $P(s, t_1, t_2, \dots, t_n)$ denote the price of an S-swap at time s (with $s < t_1 < t_2 < \dots < t_n$). Under the assumption of non-existence of arbitrage opportunities, the price at time s of an S-Swap with t_1, t_2, \dots, t_n as exchange dates is given by:

$$P(s, t_1, t_2, \dots, t_n) = \sum_{i=1}^n V(s, t_i), \tag{66}$$

where $V(s, t_i)$ is the S-forward price in an approach and a model that have to be specified.

Let us consider an S-swap with n cash flows. The price at time s of an S-swap is given by:

$$P_{\text{COC}}(s, t_1, t_2, \dots, t_n) = \sum_{j=1}^n BE_{s,t_j} + \sum_{j=1}^n RM_{s,t_j}, \tag{67}$$

where BE_{s,t_j} and RM_{s,t_j} are as follows:

$$BE_{s,t_j} = P(s, t_j) [I(x, s) E_{\mathbb{P}}(I(x + s, t_j - s) - t_j \hat{p}_x)], \tag{68}$$

$$RM_{s,t_j} = 6\% \sum_{i=s}^{t_j-1} SCR_{i,t_j} |_s P(s, i + 1). \tag{69}$$

We finally find the price of the S-swap at time s :

$$\begin{aligned} P_{\text{COC}}(s, t_1, t_2, \dots, t_n) &= \sum_{j=1}^n [P(s, t_j) (I(x, s) E_{\mathbb{P}}(I(x + s, t_j - s) - t_j \hat{p}_x) \\ &+ 6\% \sum_{i=s}^{t_j-1} P(i, t_j) P(s, i + 1) I(x, s) E_{\mathbb{P}}(I(x + s, i - s) [VaR_{99,5\%}(I(x + i, 1)) \\ &- E_{\mathbb{P}}(I(x + i, 1))] E_{\mathbb{P}}(I(x + i + 1, t_j - i - 1))]. \end{aligned} \tag{70}$$

HW Model

If we consider that the force of mortality follows the HW process, the price under the Cost of Capital approach is given by:

$$\begin{aligned} P_{\text{COC}}^{\text{HW}}(s, t_1, t_2, \dots, t_n) &= \sum_{i=1}^n [P(s, t_j) ({}_s p_x^{\text{obs}} e^{\alpha_{\mathbb{P}}^{\text{HW}}(s,t_i) - \beta_{\mathbb{P}}^{\text{HW}}(s,t_i) \mu_x(s)} - t_j \hat{p}_x) \\ &+ 6\% \sum_{i=s}^{t_j-1} P(i, t_j) P(s, i + 1) {}_s p_x^{\text{obs}} [e^{\alpha_{\mathbb{P}}^{\text{HW}}(s,i) - \beta_{\mathbb{P}}^{\text{HW}}(s,i) \mu_x(s)} (e^{m(i,i+1) + z_{\alpha} n(i,i+1)} \\ &- e^{\alpha_{\mathbb{P}}^{\text{HW}}(i,i+1) - \beta_{\mathbb{P}}^{\text{HW}}(i,i+1) \mu_x(i)}) e^{\alpha_{\mathbb{P}}^{\text{HW}}(i+1,t_i) - \beta_{\mathbb{P}}^{\text{HW}}(i+1,t_i) \overline{\mu_x(i+1)}}]. \end{aligned} \tag{71}$$

CIR Model

If we consider that the force of mortality follows the CIR process, the price under the Cost of Capital approach is given by:

$$\begin{aligned}
 P_{COC}^{CIR}(s, t_1, t_2, \dots, t_n) &= \sum_{i=1}^n [P(s, t_j)({}_s p_x^{obs} e^{\alpha_{\mathbb{P}}^{CIR}(s,t_i) - \beta_{\mathbb{P}}^{CIR}(s,t_i)\mu_x(s)} - t_j \hat{p}_x) \\
 &+ 6\% \sum_{i=s}^{t_j-1} P(i, t_j) P(s, i+1) {}_s p_x^{obs} [e^{\alpha_{\mathbb{P}}^{CIR}(s,i) - \beta_{\mathbb{P}}^{CIR}(s,i)\mu_x(s)} (VaR_{99,5\%}(e^{-\int_i^{i+1} \mu_x(s) ds}) \\
 &- e^{\alpha_{\mathbb{P}}^{CIR}(i,i+1) - \beta_{\mathbb{P}}^{CIR}(i,i+1)\mu_x(i)}) e^{\alpha_{\mathbb{P}}^{CIR}(i+1,t_i) - \beta_{\mathbb{P}}^{CIR}(i+1,t_i)\mu_x(i+1)}].
 \end{aligned} \tag{72}$$

4. Numerical Illustration

In this section, we give the numerical results of the prices of different S-forwards, using the approaches studied previously for the Belgian population.

We use the following assumptions:

- An insurer with a portfolio of pure endowment contracts paying a lump sum of 1 € at maturity T , and who is interested in hedging its longevity risk.
- $N_0 = 10,000$ initial policyholders for each cohort
- Individuals aged 65, 70, or 75 years old in 2015
- Payment of 1 € to each policyholder alive at time T
- Calibration and fixed leg based on projected data from the IABE unisex projected generational mortality table¹.
- The riskless interest rate is constant (equals to 1%).

The fixed rates used for different ages and maturities are reported in Table 1:

Table 1. The fixed survival rates ${}_t \hat{p}_x$ for different ages and maturities.

T/Age	65	70	75
$\mu_x(0)$	0.0105677	0.01608859	0.02633591
$T = 5$	0.9419321	0.9101241	0.850733508
$T = 10$	0.8658090	0.7865578	0.647397474

4.1. Calibration

We calibrate our models to the data, using the Least Square Estimation method.

HW Model

In order to determine the parameters of the survival function, we calibrate the HW model on projected mortality data for each cohort as explained in the beginning of this section.

The optimal parameters are given by Table 2:

Table 2. Optimal parameters values for the survival function in the HW model.

Age	A	B	b	σ
65	0.002317753	0.115622207	0.250629489	0.017700069
70	0.00517446	0.11645594	0.32024870	0.02352572
75	0.01008112	0.11656453	0.32687591	0.06456541

CIR Model

The optimal parameters for the three generations are summarized in Table 3:

¹ IA | BE Mortality projection for the Belgian population, 2015. Available from: <https://www.iabe.be/nl/iabe-mortality-tables>.

Table 3. Optimal parameters values for the survival function in the CIR model.

Age	A	B	b	σ
65	0.002398110	0.115379365	0.261814487	0.001864268
70	0.005079817	0.116501598	0.311927223	0.006213681
75	0.01573756	0.11389749	0.55079961	0.02816582

4.2. S-Forwards COC Prices

Table 4 gives the the prices found under the COC approach, using HW and CIR models for mortality, the different values of the best estimates (BE), and the Risk Margins (RM):

Table 4. Comparison between the prices of the different S-forwards with HW and CIR models.

	HW Model					CIR Model				
	BE	RM	Price	BE%	RM%	BE	RM	Price	BE%	RM%
$x_0 = 65, T = 5$	45.87512	6.756132	52.63125	87.16%	12.83%	44.28595	10.68733	54.97327	80.55%	19.44%
$x_0 = 65, T = 10$	76.53061	11.86875	88.39936	86.57%	13.42%	74.69407	22.70057	97.39464	76.69%	23.30%
$x_0 = 70, T = 5$	28.58322	21.38396	49.96718	57.20%	42.79%	28.47494	16.53768	45.01261	63.25%	36.74%
$x_0 = 70, T = 10$	26.35731	35.16336	61.52067	42.84%	57.15%	24.18322	33.45732	57.64054	41.95%	58.04%
$x_0 = 75, T = 5$	7.247361	75.30577	82.5531	91.22%	8.78%	-4.960416	55.56708	50.60666	-9.80%	109.8%
$x_0 = 75, T = 10$	-11.20288	112.8399	101.6371	-11.02%	111.02%	-14.28582	90.50064	76.21482	-18.74%	118.74%

In order to have a better idea of the magnitude of these S-forward prices, we can link them with the original price of the pure endowment contract received by the insurer.

The premium P of this pure endowment contract is given by: ${}_T E_x = \frac{{}_T \hat{P}_x}{(1+i)^T}$, with i the technical interest rate.

Table 5 reports a comparison between the premium P and the price of the S forward S_price , considering a technical rate of 0.5% and using COC approach for both models of longevity. HC is the the hedging cost by $HC = \frac{S_price}{P}$

Table 5. Comparison between the prices of the pure endowment and S-forward contracts.

	HW Model			CIR Model		
	P	S_price	HC	P	S_price	HC
$x_0 = 65, T = 5$	9187.329	52.63125	0.57%	9187.329	54.97327	0.6%
$x_0 = 65, T = 10$	8444.847	88.39936	1.04%	8444.847	97.39464	1.15%
$x_0 = 70, T = 5$	8297.805	49.96718	0.6%	8297.805	45.01261	0.54%
$x_0 = 70, T = 10$	7671.854	61.52067	0.8%	7671.854	57.64054	0.75%
$x_0 = 75, T = 5$	8877.084	82.5531	0.93%	8877.084	50.60666	0.57%
$x_0 = 75, T = 10$	6314.525	101.6371	1.61%	6314.525	76.21482	1.2%

We can see that HC is around 1%.

4.3. Consistency with Other Pricing Methods

In this section, we compare numerically the COC approach with other methods developed in Section 2.4. In order to check the consistency between COC and each of these methods, we compute the value of the parameters of the three methods; these parameters are computed such that the COC price is equal to the price given by the considered methods. We can say that a method is consistent with the COC approach if its parameters are quiet stable for different ages and maturities.

4.3.1. Risk Neutral Approach

In this method, we determine the market price of longevity risk, generating the same price as the one found with the COC method.

HW Model

Under the risk-neutral method, and under the assumptions considered previously, the price at time 0 is:

$$V_{\mathbb{Q}_\lambda}^{HW}(0, T) = e^{-rT} (e^{\alpha_{\mathbb{Q}_\lambda}^{HW}(0,T)} - \beta_{\mathbb{Q}_\lambda}^{HW}(0,T)\mu_x(0) - {}_T\hat{p}_x). \tag{73}$$

The market price of risk λ is determined such that this risk-neutral price is equal to the cost of capital price. If this coefficient is stable, it means that the two methods are consistent. Table 6 gives the values of λ when we use the HW model for longevity:

Table 6. Equivalent market price of longevity risk in the risk-neutral approach, HW model.

S-Forward	COC Price	Market Price of Longevity Risk
$x_0 = 65, T = 5$	52.63125	$\lambda = -14.40\%$
$x_0 = 65, T = 10$	88.39936	$\lambda = -22.61\%$
$x_0 = 70, T = 5$	49.96718	$\lambda = -15.53\%$
$x_0 = 70, T = 10$	61.52067	$\lambda = -25.47\%$
$x_0 = 75, T = 5$	82.5531	$\lambda = -19.33\%$
$x_0 = 75, T = 10$	101.6371	$\lambda = -37.53\%$

CIR Model

Under the risk-neutral approach, and considering that the intensity of mortality follows the CIR model, the price of the S-forward is:

$$V_{\mathbb{Q}_\lambda}^{HW}(0, T) = e^{-rT} (e^{\alpha_{\mathbb{Q}_\lambda}^{CIR}(0,T)} - \beta_{\mathbb{Q}_\lambda}^{CIR}(0,T)\mu_x(0) - {}_T\hat{p}_x). \tag{74}$$

Table 7 represents the different values of the market price of longevity risk.

Table 7. Equivalent market price of longevity risk in the risk-neutral approach, CIR model.

S-Forward	COC Price	κ	Market Price of Longevity Risk $\lambda = \kappa\sqrt{\mu_x(0)}$
$x_0 = 65, T = 5$	54.97327	-0.743	$\lambda = -7.64\%$
$x_0 = 65, T = 10$	97.39464	-0.533	$\lambda = -5.48\%$
$x_0 = 70, T = 5$	45.01261	-0.609	$\lambda = -7.72\%$
$x_0 = 70, T = 10$	57.64054	-0.436	$\lambda = -5.53\%$
$x_0 = 75, T = 5$	50.60666	-0.468	$\lambda = -7.06\%$
$x_0 = 75, T = 10$	76.21482	-0.297	$\lambda = -4.83\%$

Clearly, the two methods are not completely consistent, even if the parameters seem to be more stable in the CIR model.

4.3.2. Consistency with Sharpe Ratio

We will now compute the value of the Sharpe ratio corresponding to the S-forward price found with the COC method.

HW Model

The price with this approach using HW model is given by:

$$V_{Sharpe}^{HW}(0, T) = e^{-rT} (e^{\alpha_{\mathbb{P}}^{HW}(0,T)} - \beta_{\mathbb{P}}^{HW}(0,T)\mu_x(0) - {}_T\hat{p}_x + S \sqrt{(e^{n(0,T)^2} - 1)e^{2m(0,T)+n(0,T)^2}}). \tag{75}$$

Table 8 gives the equivalent Sharpe ratio generating the prices found with the COC approach.

Table 8. Equivalent Sharpe parameters, HW model.

S-Forward	COC Price	Sharpe Parameter
$x_0 = 65, T = 5$	52.63125	$S = 9.62\%$
$x_0 = 65, T = 10$	88.39936	$S = 9.55\%$
$x_0 = 70, T = 5$	49.96718	$S = 10.13\%$
$x_0 = 70, T = 10$	61.52067	$S = 10.43\%$
$x_0 = 75, T = 5$	82.5531	$S = 11.47\%$
$x_0 = 75, T = 10$	101.6371	$S = 13.91\%$

CIR Model

With the CIR model, the price of the S-forward at time 0 is given by:

$$V_{Sharpe}^{CIR}(0, T) = e^{-rT} (e^{\alpha_{\mathbb{F}}^{CIR}(0, T) - \beta_{\mathbb{F}}^{CIR}(t, T)\mu_x(t)} - {}_T\hat{p}_x + S \sqrt{e^{\alpha_{\mathbb{F}}^{CIR^*}(0, T) - \beta_{\mathbb{F}}^{CIR^*}(0, T)2\mu_x(0)} - e^{2[\alpha_{\mathbb{F}}^{CIR}(0, T) - \beta_{\mathbb{F}}^{CIR}(0, T)\mu_x(0)]}}). \tag{76}$$

Table 9 presents the different values of the Sharpe parameter under the CIR model:

Table 9. Equivalent Sharpe parameters, CIR model.

S-Forward	COC Price	Sharpe Parameter
$x_0 = 65, T = 5$	54.97327	$S = 15.35\%$
$x_0 = 65, T = 10$	97.39464	$S = 17.44\%$
$x_0 = 70, T = 5$	45.01261	$S = 16.17\%$
$x_0 = 70, T = 10$	57.64054	$S = 17.55\%$
$x_0 = 75, T = 5$	50.60666	$S = 16.45\%$
$x_0 = 75, T = 10$	76.21482	$S = 17.61\%$

We can see that, for both mortality models, Sharpe parameters have a better stability.

4.3.3. Consistency with Wang Transform

We will now compute the value of the Wang transform parameter corresponding to the COC S-forward price.

HW Model

Table 10 gives the different values of the Wang parameter under the CIR model:

Table 10. Equivalent Wang parameters, HW model.

S-Forward	COC Price	Wang Parameter
$x_0 = 65, T = 5$	52.63125	$\delta = 9.62\%$
$x_0 = 65, T = 10$	88.39936	$\delta = 9.55\%$
$x_0 = 70, T = 5$	49.96718	$\delta = 10.12\%$
$x_0 = 70, T = 10$	61.52067	$\delta = 10.41\%$
$x_0 = 75, T = 5$	82.5531	$\delta = 11.44\%$
$x_0 = 75, T = 10$	101.6371	$\delta = 13.48\%$

CIR Model

For the CIR model, we use numerical simulation to compute the the Wang Transform parameters corresponding to the COC prices. We get the following results:

Table 11 gives the different values of the Wang parameter under the CIR model:

Table 11. Equivalent Wang parameters, CIR model.

S-Forward	COC Price	Wang Parameter
$x_0 = 65, T = 5$	54.97327	$\delta = 16.29\%$
$x_0 = 65, T = 10$	97.39464	$\delta = 16.49\%$
$x_0 = 70, T = 5$	45.01261	$\delta = 16.89\%$
$x_0 = 70, T = 10$	57.64054	$\delta = 17.91\%$
$x_0 = 75, T = 5$	50.60666	$\delta = 16.55\%$
$x_0 = 75, T = 10$	76.21482	$\delta = 17.75\%$

4.3.4. Synthesis

The Cost of Capital approach is parameterized by one variable: the Cost of Capital rate, which is fixed by regulation (6% in Solvency II nowadays). This legal and unique parameter value can be seen as a benchmark, and allows us to find equivalent values of the parameters in the three other methods. The simulations presented above show that:

- None of the classical pricing methods is completely consistent with the COC approach, which is not surprising (the COC price (Formula (26)) is based on a multiperiod formula).
- Sharpe ratio and Wang transform methods seem closer to COC than risk-neutral approach.
- The absolute values found for Sharpe and Wang methods are quite similar. In Appendix A, we present for the HW model complete results for ages between 55 and 85, and maturities from $T = 1$ to $T = 15$.

4.3.5. Remark: Smoothing

Taking into account the irregular form of all these prices due to the use of gross mortality parameters for each age (Formulas (2) and (3)), we can smooth these parameters. Appendix B gives the methods used for this smoothing. Appendix B gives then the smoothed prices with COC method in the HW model.

5. Conclusions

In this paper, we have developed in continuous time a new way to price longevity linked securities based on the Cost of Capital approach, consistent with the directive of Solvency II. We considered two examples of stochastic time continuous processes for longevity: Hull and White and CIR extended. This COC method has been compared with three classical methods used in finance: the risk-neutral, Wang and Sharpe methods. More precisely, we have found the parameters of these three methods giving equivalent prices to the COC approach with 6% COC charge, as mentioned in the Solvency II regulation. Even if none of these methods is consistent with the COC approach, it appears that Wang transform and Sharpe ratio methods are more similar to COC than the risk-neutral method. In this paper, we didn't take into account the basis risk. We could use this proposed COC method to price other longevity linked securities that take into account this risk—for instance, the exchange forwards and exchanged options.

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Appendix A

COC prices of different S-forwards, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	2.1768706	7.645950536	16.33313671	27.66480999	40.64993112	55.44692673	70.66711363	86.36881849	101.2928691	114.9826224	127.4397041	139.2642165	147.1956821	153.7583705	158.1290551
56	1.804452614	7.141070861	15.44932364	25.64332178	37.97904649	50.97117035	64.7213424	77.95695133	90.13311681	101.3221848	112.1431302	119.2556828	125.2720068	129.352999	129.8719672
57	1.811826103	7.648725775	16.05012504	27.16993677	39.2369969	52.26699877	64.86198942	76.29292926	86.64487884	96.5100457	102.4248649	107.0729366	109.5819853	108.2974216	104.4799611
58	1.921457249	7.920167398	18.05871237	30.2542374	44.42009981	59.00887794	73.04967574	86.6096054	100.1962391	110.158004	119.1750683	126.2555655	129.6155118	130.4268428	129.2984779
59	2.642948583	9.943257288	19.7747718	32.17534381	45.62022971	59.01751402	72.52673467	86.66302449	97.63954671	108.2011827	117.2772806	122.9959048	126.4789669	128.3496366	126.0893723
60	2.215083941	8.711412336	19.14726772	31.62991976	44.62915064	58.19946652	72.70600368	84.11846129	95.18929064	104.7365807	110.7931071	114.4571608	116.4118338	114.0732334	108.1408484
61	4.735216291	14.2038521	27.11310808	42.0329614	59.2169846	78.9896111	97.05021834	116.068152	134.6011976	150.41367	164.388671	177.0825281	185.1151056	188.6430333	187.5581031
62	6.587732845	21.23944835	39.32925734	59.44172877	80.89562805	98.73259696	115.5543037	129.8827373	139.5706718	145.7215307	149.2412294	147.4938866	141.2423595	130.6609384	117.3882424
63	5.482982277	13.92642322	25.71882976	41.11183479	55.43115179	71.53940415	87.62181391	100.9920245	112.4788675	122.7808907	128.8743127	131.1887049	129.5609005	125.440654	118.4013969
64	1.508059137	7.612287022	18.09791488	27.79522377	39.48672657	51.37172298	60.92901863	68.98958396	76.37911249	80.02344759	80.35525739	77.20176768	72.15024695	64.74788032	55.37888244
65	2.696896474	12.93272397	24.06172239	38.11826228	52.61632773	64.56731471	74.55632498	83.3903719	87.85452468	88.34422041	84.67728092	78.63810515	69.82053955	58.74166476	44.13148303
66	4.614823032	14.70019181	30.17790943	47.12023085	61.68631284	74.04400261	84.92702183	90.96105054	92.49951057	89.34859731	83.49987579	74.57252128	63.19150347	48.08872992	34.05617679
67	4.645759801	18.00554077	34.873336	50.55239268	64.70876231	77.89160123	86.41877207	90.45424822	89.6479275	86.09414769	79.31815963	69.95160513	56.6227264	44.20118202	29.39681558
68	2.842969562	11.86213427	22.31493623	33.65537116	46.25839593	55.97295051	62.53705859	65.18465349	65.89000355	63.8451608	59.47580749	51.1013859	43.49644597	33.00643291	20.33470013
69	2.423335339	9.828537047	20.1979862	33.14436205	43.82134841	51.53842873	55.23865342	56.99509509	55.8901631	52.37936921	44.69019465	37.7324747	27.66516155	15.31667731	5.200264764
70	3.66017856	12.72158932	26.38434097	39.15575211	49.83012195	56.89394113	62.38635228	65.09191902	65.33527725	61.08193416	57.24732846	49.6641262	39.18943547	30.4493177	19.14663223
71	6.437697912	19.36139302	34.51532037	50.71416162	65.91566247	81.8433937	96.38994075	107.9478129	113.8963906	118.9405067	118.4270687	113.184392	107.9791853	98.28643241	93.46497565
72	9.626547277	25.60957253	44.22020037	61.77712352	79.62024634	95.0679164	106.9139386	112.5812831	117.1533352	115.9158426	109.9975096	104.5087689	94.71621509	90.2923888	85.77119758
73	0.708690787	7.745575089	14.88434237	22.8014924	28.79186491	32.74639203	31.8996754	31.51384471	26.57459596	18.33593112	12.12373136	2.818518646	0.335971796	-1.107867918	2.703995403
74	1.324452393	6.790093136	15.94248958	24.83985865	32.66650309	35.92712688	39.71020253	38.41220462	33.27380515	29.79448071	22.42079287	21.40947446	20.73149664	24.48855191	27.4313317
75	8.96517587	26.27183403	46.65248463	67.41543229	82.55313459	96.81027227	103.9153274	105.1664864	106.3800282	101.6370614	101.7319246	100.5097345	102.0912462	101.5189943	105.4746949
76	11.37193116	31.32023918	53.85898223	71.1704332	87.66077103	96.48468211	99.14833263	101.8914638	98.35389107	99.85919291	99.95691176	102.592183	102.9073232	107.5287843	101.8696323
77	9.658499145	28.94395364	49.73262865	74.55628636	92.14289136	102.8797711	112.9217768	115.101614	121.2465898	124.5760643	128.9285846	129.7131393	133.6854071	126.1820471	116.7579353
78	-0.447350425	4.836355716	15.07039139	19.82820347	20.03731989	22.47408563	19.39108984	23.30984615	26.94946238	33.74686107	39.01944187	49.16110779	48.80895259	47.43346191	43.88340405
79	5.622704323	27.57268564	52.85582276	76.69639617	102.9355754	121.3566417	144.5677921	164.4761678	183.5556911	194.3075811	204.0164902	197.9108021	186.88201	170.7733475	150.5928846
80	-13.91536234	-22.21973085	-27.24202708	-23.81954003	-23.54800789	-13.59881353	-3.355733165	9.357538472	19.83428968	34.13432718	35.90610776	35.53945197	31.97247623	25.53022891	16.8675239
81	6.156511574	21.67351482	52.33034683	80.3326296	116.3830377	148.6407925	174.412026	190.1447675	203.0038736	197.2817564	185.2499442	167.0297243	144.14982	118.3409479	91.46531186
82	-23.40305252	-30.8710047	-38.56783304	-32.12126964	-23.02208561	-10.24862174	1.873272233	19.04155593	23.24779093	25.8751564	25.36567381	21.80115739	15.6396906	7.693179186	-0.966380885
83	-21.99351947	-37.28919886	-32.59446789	-22.3467536	-7.36699892	7.618202923	27.66777175	33.50429934	37.19812275	36.98416167	32.99747555	25.8239165	16.45087398	6.146465409	-3.73060085
84	-24.35691878	-21.96495376	-11.31965378	4.5534146	20.72940225	41.78248461	47.12300994	49.90577873	48.24442713	42.41805974	33.19167911	21.74362393	9.519340755	-1.981304238	-11.45051649
85	-31.8519033	-41.192253	-36.04810097	-23.36502	-0.732724167	8.299605391	17.03516395	22.32444309	23.62348514	21.04500877	15.29301645	7.52509245	-0.861874474	-8.49566101	-14.28804438

Figure A1. Prices of S-forward prices under the Cost of Capital approach, HW model.

Consistency COC / Risk neutral, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	-8.99%	-9.51%	-10.04%	-10.59%	-11.15%	-11.73%	-12.33%	-12.94%	-13.57%	-14.21%	-14.87%	-15.54%	-16.23%	-16.93%	-17.64%
56	-8.97%	-9.44%	-9.92%	-10.42%	-10.92%	-11.44%	-11.97%	-12.51%	-13.07%	-13.64%	-14.22%	-14.81%	-15.42%	-16.03%	-16.66%
57	-9.01%	-9.64%	-10.30%	-10.98%	-11.68%	-12.41%	-13.16%	-13.94%	-14.74%	-15.56%	-16.40%	-17.26%	-18.15%	-19.05%	-19.97%
58	-9.01%	-9.65%	-10.32%	-11.01%	-11.73%	-12.47%	-13.24%	-14.04%	-14.85%	-15.69%	-16.55%	-17.44%	-18.34%	-19.27%	-20.21%
59	-8.99%	-9.50%	-10.02%	-10.55%	-11.10%	-11.67%	-12.24%	-12.84%	-13.44%	-14.06%	-14.69%	-15.34%	-16%	-16.67%	-17.36%
60	-9.03%	-9.71%	-10.43%	-11.18%	-11.95%	-12.76%	-13.59%	-14.45%	-15.34%	-16.25%	-17.19%	-18.15%	-19.13%	-20.14%	-21.16%
61	-4.15%	-4.45%	-4.78%	-5.14%	-5.54%	-5.98%	-6.49%	-7.06%	-7.71%	-8.44%	-9.28%	-10.23%	-11.21%	-12.17%	-13.12%
62	-9.28%	-10.96%	-12.81%	-14.80%	-16.94%	-19.20%	-21.57%	-24.03%	-26.56%	-29.16%	-31.81%	-34.50%	-37.22%	-39.96%	-42.73%
63	-7.07%	-7.64%	-8.28%	-8.99%	-9.80%	-10.71%	-11.67%	-12.60%	-13.53%	-14.45%	-15.38%	-16.32%	-17.27%	-18.22%	-19.19%
64	-9.04%	-9.77%	-10.53%	-11.33%	-12.16%	-13.03%	-13.92%	-14.85%	-15.81%	-16.80%	-17.81%	-18.85%	-19.91%	-21.00%	-22.11%
65	-9.15%	-10.33%	-11.60%	-12.96%	-14.40%	-15.91%	-17.49%	-19.14%	-20.85%	-22.61%	-24.41%	-26.26%	-28.15%	-30.06%	-32.01%
66	-9.29%	-11.06%	-13.02%	-15.14%	-17.41%	-19.81%	-22.33%	-24.94%	-27.63%	-30.38%	-33.18%	-36.03%	-38.90%	-41.80%	-44.71%
67	-9.30%	-11.11%	-13.10%	-15.27%	-17.58%	-20.03%	-22.60%	-25.26%	-28.00%	-30.80%	-33.65%	-36.54%	-39.46%	-42.40%	-45.36%
68	-9.16%	-10.37%	-11.66%	-13.04%	-14.50%	-16.04%	-17.65%	-19.32%	-21.05%	-22.84%	-24.67%	-26.54%	-28.45%	-30.39%	-32.36%
69	-9.23%	-10.73%	-12.36%	-14.11%	-15.98%	-17.96%	-20.04%	-22.20%	-24.43%	-26.72%	-29.06%	-31.45%	-33.87%	-36.32%	-38.79%
70	-9.22%	-10.63%	-12.16%	-13.79%	-15.53%	-17.36%	-19.28%	-21.28%	-23.35%	-25.47%	-27.64%	-29.86%	-32.11%	-34.38%	-36.69%
71	-5.42%	-6.34%	-7.39%	-8.61%	-10.01%	-11.63%	-13.46%	-15.26%	-17.07%	-18.89%	-20.72%	-22.57%	-24.44%	-26.33%	-28.22%
72	-6.34%	-7.93%	-9.84%	-12.11%	-14.79%	-17.78%	-20.82%	-23.91%	-27.05%	-30.23%	-33.43%	-36.66%	-39.89%	-43.14%	-46.39%
73	-9.43%	-11.77%	-14.40%	-17.28%	-20.36%	-23.62%	-27.02%	-30.51%	-34.08%	-37.69%	-41.35%	-45.03%	-48.73%	-52.44%	-56.15%
74	-9.40%	-11.51%	-13.86%	-16.43%	-19.18%	-22.08%	-25.12%	-28.25%	-31.46%	-34.72%	-38.03%	-41.36%	-44.72%	-48.09%	-51.47%
75	-7.51%	-9.85%	-12.70%	-15.96%	-19.33%	-22.82%	-26.41%	-30.07%	-33.78%	-37.53%	-41.31%	-45.09%	-48.88%	-52.67%	-56.46%
76	-8.54%	-11.96%	-16.03%	-20.36%	-24.92%	-29.64%	-34.45%	-39.32%	-44.22%	-49.13%	-54.04%	-58.95%	-63.85%	-68.74%	-73.63%
77	-7.08%	-9.56%	-12.59%	-16.25%	-20.08%	-24.02%	-28.05%	-32.13%	-36.26%	-40.41%	-44.56%	-48.72%	-52.87%	-57.02%	-61.16%
78	-9.78%	-13.45%	-17.66%	-22.28%	-27.17%	-32.23%	-37.39%	-42.61%	-47.85%	-53.11%	-58.38%	-63.64%	-68.90%	-74.16%	-79.41%
79	-4.08%	-5.83%	-7.98%	-10.57%	-13.60%	-17.11%	-21.10%	-25.61%	-30.61%	-35.57%	-40.51%	-45.43%	-50.31%	-55.16%	-59.99%
80	-9.29%	-11.01%	-12.91%	-14.96%	-17.15%	-19.48%	-21.91%	-24.44%	-27.04%	-29.71%	-32.42%	-35.18%	-37.97%	-40.78%	-43.60%
81	-5.58%	-7.70%	-10.30%	-13.41%	-17.07%	-21.28%	-25.51%	-29.76%	-34.02%	-38.28%	-42.52%	-46.75%	-50.96%	-55.15%	-59.31%
82	-9.35%	-11.41%	-13.70%	-16.20%	-18.88%	-21.72%	-24.68%	-27.74%	-30.89%	-34.09%	-37.34%	-40.62%	-43.92%	-47.24%	-50.57%
83	-9.34%	-11.24%	-13.32%	-15.59%	-18.01%	-20.58%	-23.26%	-26.04%	-28.90%	-31.82%	-34.78%	-37.78%	-40.80%	-43.85%	-46.90%
84	-9.44%	-11.67%	-14.17%	-16.90%	-19.82%	-22.91%	-26.13%	-29.44%	-32.83%	-36.27%	-39.74%	-43.25%	-46.77%	-50.30%	-53.83%
85	-9.34%	-11.33%	-13.53%	-15.93%	-18.51%	-21.23%	-24.08%	-27.02%	-30.05%	-33.14%	-36.27%	-39.44%	-42.63%	-45.84%	-49.06%

Figure A2. Consistency between risk-neutral and Cost of Capital method, HW model.

Consistency COC / Sharpe, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	15.34%	11.31%	9.62%	8.67%	8.06%	7.64%	7.33%	7.11%	6.94%	6.81%	6.72%	6.65%	6.59%	6.55%	6.53%
56	15.34%	11.27%	9.55%	8.57%	7.94%	7.50%	7.18%	6.94%	6.76%	6.62%	6.50%	6.42%	6.35%	6.30%	6.26%
57	15.34%	11.41%	9.78%	8.88%	8.32%	7.95%	7.68%	7.50%	7.37%	7.28%	7.22%	7.18%	7.16%	7.16%	7.16%
58	15.34%	11.42%	9.80%	8.90%	8.35%	7.97%	7.72%	7.54%	7.41%	7.33%	7.27%	7.23%	7.22%	7.22%	7.23%
59	15.35%	11.31%	9.61%	8.65%	8.03%	7.60%	7.29%	7.06%	6.89%	6.76%	6.65%	6.57%	6.52%	6.47%	6.44%
60	15.34%	11.46%	9.86%	8.99%	8.46%	8.10%	7.86%	7.70%	7.58%	7.51%	7.47%	7.45%	7.44%	7.45%	7.48%
61	7.05%	5.25%	4.52%	4.14%	3.92%	3.81%	3.76%	3.77%	3.82%	3.91%	4.03%	4.20%	4.36%	4.50%	4.62%
62	15.36%	12.30%	11.27%	10.88%	10.76%	10.80%	10.92%	11.10%	11.32%	11.55%	11.81%	12.07%	12.34%	12.61%	12.88%
63	12.01%	9.00%	7.82%	7.23%	6.92%	6.79%	6.74%	6.70%	6.67%	6.66%	6.66%	6.67%	6.69%	6.72%	6.75%
64	15.34%	11.50%	9.93%	9.08%	8.56%	8.22%	8.00%	7.85%	7.75%	7.69%	7.66%	7.66%	7.67%	7.69%	7.73%
65	15.34%	11.88%	10.58%	9.94%	9.62%	9.46%	9.40%	9.41%	9.47%	9.55%	9.66%	9.79%	9.93%	10.08%	10.23%
66	15.35%	12.36%	11.39%	11.04%	10.97%	11.04%	11.20%	11.41%	11.65%	11.92%	12.20%	12.49%	12.78%	13.07%	13.37%
67	15.36%	12.40%	11.44%	11.11%	11.04%	11.13%	11.30%	11.52%	11.77%	12.05%	12.33%	12.63%	12.93%	13.23%	13.53%
68	15.36%	11.91%	10.61%	9.99%	9.67%	9.51%	9.46%	9.47%	9.53%	9.62%	9.73%	9.86%	10.01%	10.16%	10.31%
69	15.35%	12.14%	11.02%	10.53%	10.34%	10.30%	10.36%	10.48%	10.63%	10.82%	11.02%	11.23%	11.45%	11.68%	11.91%
70	15.38%	12.09%	10.90%	10.37%	10.13%	10.06%	10.07%	10.15%	10.28%	10.43%	10.60%	10.78%	10.98%	11.18%	11.38%
71	9.07%	7.26%	6.69%	6.55%	6.62%	6.83%	7.14%	7.39%	7.63%	7.85%	8.05%	8.26%	8.45%	8.65%	8.83%
72	10.39%	8.72%	8.43%	8.61%	9.06%	9.61%	10.13%	10.61%	11.07%	11.52%	11.95%	12.38%	12.79%	13.18%	13.57%
73	15.36%	12.81%	12.16%	12.07%	12.23%	12.52%	12.87%	13.26%	13.68%	14.10%	14.52%	14.95%	15.37%	15.78%	16.19%
74	15.39%	12.66%	11.87%	11.67%	11.73%	11.93%	12.20%	12.52%	12.86%	13.22%	13.58%	13.95%	14.32%	14.68%	15.05%
75	12.18%	10.64%	10.62%	11.02%	11.47%	11.95%	12.44%	12.93%	13.42%	13.91%	14.39%	14.86%	15.32%	15.77%	16.21%
76	13.55%	12.43%	12.75%	13.31%	13.98%	14.68%	15.40%	16.11%	16.81%	17.49%	18.16%	18.80%	19.43%	20.04%	20.63%
77	11.40%	10.17%	10.33%	10.99%	11.65%	12.29%	12.92%	13.53%	14.13%	14.71%	15.27%	15.82%	16.34%	16.86%	17.35%
78	15.43%	13.83%	13.87%	14.36%	15.02%	15.75%	16.50%	17.25%	17.99%	18.71%	19.42%	20.10%	20.77%	21.42%	22.06%
79	6.44%	5.99%	6.26%	6.79%	7.48%	8.30%	9.23%	10.26%	11.38%	12.38%	13.30%	14.15%	14.94%	15.68%	16.37%
80	15.36%	12.33%	11.33%	10.95%	10.86%	10.91%	11.05%	11.24%	11.47%	11.72%	11.98%	12.25%	12.53%	12.81%	13.10%
81	8.92%	8.11%	8.32%	8.90%	9.70%	10.65%	11.49%	12.25%	12.95%	13.60%	14.21%	14.79%	15.34%	15.86%	16.35%
82	15.34%	12.58%	11.77%	11.56%	11.60%	11.79%	12.04%	12.35%	12.68%	13.03%	13.38%	13.74%	14.10%	14.45%	14.81%
83	15.39%	12.48%	11.57%	11.27%	11.23%	11.34%	11.54%	11.78%	12.05%	12.35%	12.65%	12.97%	13.28%	13.60%	13.91%
84	15.41%	12.76%	12.04%	11.90%	12.01%	12.25%	12.56%	12.91%	13.29%	13.68%	14.07%	14.47%	14.86%	15.25%	15.64%
85	15.35%	12.53%	11.68%	11.43%	11.44%	11.60%	11.83%	12.11%	12.42%	12.74%	13.08%	13.42%	13.76%	14.10%	14.44%

Figure A3. Consistency between Sharpe and Cost of Capital method, HW model.

Consistency COC / Wang, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	15.34%	11.31%	9.62%	8.66%	8.05%	7.63%	7.33%	7.11%	6.94%	6.81%	6.71%	6.64%	6.59%	6.55%	6.52%
56	15.34%	11.27%	9.55%	8.57%	7.94%	7.50%	7.18%	6.94%	6.75%	6.61%	6.50%	6.41%	6.35%	6.29%	6.26%
57	15.34%	11.41%	9.78%	8.88%	8.32%	7.94%	7.68%	7.50%	7.37%	7.27%	7.21%	7.18%	7.16%	7.15%	7.16%
58	15.34%	11.41%	9.79%	8.90%	8.34%	7.97%	7.72%	7.54%	7.41%	7.32%	7.26%	7.23%	7.21%	7.21%	7.22%
59	15.35%	11.31%	9.60%	8.64%	8.03%	7.60%	7.29%	7.06%	6.88%	6.75%	6.65%	6.57%	6.51%	6.47%	6.43%
60	15.34%	11.46%	9.86%	8.99%	8.45%	8.10%	7.86%	7.69%	7.58%	7.51%	7.46%	7.44%	7.44%	7.45%	7.47%
61	7.05%	5.25%	4.52%	4.14%	3.92%	3.81%	3.76%	3.77%	3.82%	3.91%	4.04%	4.21%	4.37%	4.52%	4.65%
62	15.36%	12.29%	11.27%	10.87%	10.76%	10.79%	10.91%	11.09%	11.30%	11.54%	11.79%	12.05%	12.31%	12.58%	12.85%
63	12.00%	9%	7.82%	7.22%	6.92%	6.78%	6.73%	6.69%	6.67%	6.66%	6.66%	6.67%	6.69%	6.72%	6.75%
64	15.34%	11.49%	9.93%	9.08%	8.56%	8.22%	7.99%	7.84%	7.75%	7.69%	7.66%	7.65%	7.66%	7.68%	7.72%
65	15.34%	11.88%	10.57%	9.94%	9.62%	9.46%	9.40%	9.41%	9.46%	9.55%	9.66%	9.78%	9.92%	10.07%	10.22%
66	15.34%	12.36%	11.39%	11.04%	10.96%	11.03%	11.19%	11.40%	11.65%	11.91%	12.19%	12.47%	12.76%	13.06%	13.35%
67	15.35%	12.39%	11.44%	11.10%	11.04%	11.12%	11.29%	11.51%	11.76%	12.03%	12.32%	12.61%	12.91%	13.20%	13.50%
68	15.36%	11.91%	10.61%	9.98%	9.66%	9.50%	9.45%	9.46%	9.52%	9.61%	9.72%	9.85%	9.99%	10.14%	10.30%
69	15.35%	12.14%	11.01%	10.53%	10.33%	10.30%	10.35%	10.47%	10.62%	10.80%	11.00%	11.22%	11.44%	11.66%	11.89%
70	15.38%	12.08%	10.90%	10.36%	10.12%	10.04%	10.06%	10.14%	10.26%	10.41%	10.58%	10.76%	10.95%	11.15%	11.35%
71	9.06%	7.25%	6.69%	6.54%	6.61%	6.83%	7.13%	7.39%	7.63%	7.85%	8.07%	8.28%	8.48%	8.68%	8.88%
72	10.38%	8.71%	8.42%	8.60%	9.04%	9.59%	10.10%	10.58%	11.04%	11.48%	11.91%	12.33%	12.74%	13.13%	13.52%
73	15.35%	12.81%	12.15%	12.06%	12.22%	12.51%	12.86%	13.25%	13.66%	14.08%	14.50%	14.92%	15.34%	15.75%	16.16%
74	15.38%	12.65%	11.86%	11.66%	11.72%	11.91%	12.18%	12.49%	12.83%	13.19%	13.55%	13.91%	14.27%	14.64%	14.99%
75	12.17%	10.62%	10.59%	11.00%	11.44%	11.91%	12.39%	12.88%	13.36%	13.84%	14.32%	14.78%	15.23%	15.68%	16.11%
76	13.53%	12.41%	12.72%	13.27%	13.93%	14.62%	15.33%	16.02%	16.71%	17.38%	18.02%	18.65%	19.26%	19.86%	20.43%
77	11.39%	10.16%	10.31%	10.95%	11.61%	12.25%	12.87%	13.47%	14.06%	14.63%	15.19%	15.72%	16.24%	16.75%	17.24%
78	15.42%	13.82%	13.85%	14.34%	14.99%	15.71%	16.45%	17.19%	17.92%	18.64%	19.33%	20.01%	20.67%	21.30%	21.93%
79	6.44%	5.99%	6.26%	6.80%	7.51%	8.34%	9.28%	10.33%	11.46%	12.48%	13.42%	14.28%	15.09%	15.85%	16.56%
80	15.36%	12.33%	11.33%	10.95%	10.85%	10.90%	11.04%	11.23%	11.45%	11.70%	11.96%	12.23%	12.51%	12.79%	13.07%
81	8.91%	8.10%	8.31%	8.90%	9.71%	10.67%	11.53%	12.30%	13.02%	13.70%	14.33%	14.94%	15.51%	16.06%	16.58%
82	15.34%	12.58%	11.77%	11.55%	11.60%	11.78%	12.04%	12.34%	12.67%	13.02%	13.37%	13.73%	14.08%	14.44%	14.79%
83	15.39%	12.48%	11.56%	11.25%	11.22%	11.33%	11.52%	11.76%	12.03%	12.32%	12.62%	12.93%	13.24%	13.55%	13.87%
84	15.40%	12.76%	12.03%	11.88%	11.99%	12.23%	12.53%	12.88%	13.25%	13.64%	14.03%	14.42%	14.81%	15.19%	15.57%
85	15.35%	12.53%	11.68%	11.42%	11.44%	11.59%	11.82%	12.10%	12.41%	12.73%	13.06%	13.40%	13.74%	14.08%	14.42%

Figure A4. Consistency between Sharpe and Cost of Capital method, HW model.

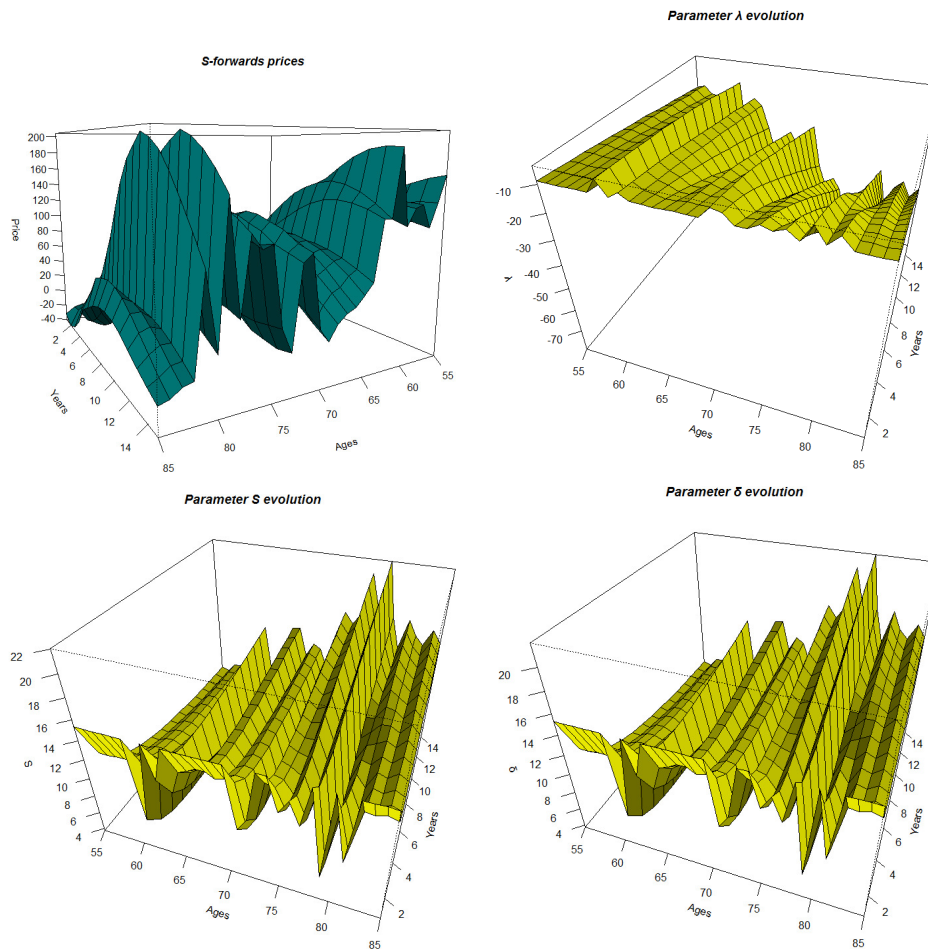


Figure A5. 3D presentation of the S-forwards prices and the corresponding parameters with the other methods.

Appendix B

Table A1. Nonlinear regression results, HW model.

Parameter	Fitted Model	Error	Pseudo- R^2
A	$a + bx + cx^2$ $a = 2.675 \times 10^{-1}$ $b = -8.947 \times 10^{-3}$ $c = 7.481 \times 10^{-5}$	0.004228	0.9300884
B	$a + bx + cx^2$ $a = 1.968 \times 10^{-2}$ $b = 2.872 \times 10^{-3}$ $c = -2.154 \times 10^{-5}$	0.001877	0.5578862
b	ax^4 $a = 1.968 \times 10^{-2}$	0.1431	0.5408143
σ	ae^{-bx} $a = 0.0002570$ $b = -0.0560466$	0.02138	0.1448673

Appendix C

COC prices of different S-forwards, with fixed rate and smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	12.8234315	15.7275763	9.46135275	-5.93161936	-31.8114337	-67.7758301	-115.017976	-173.362047	-243.998295	-327.332617	-423.298498	-531.181737	-654.04103	-789.020279	-936.434438
56	16.8249269	30.8657931	42.0911495	49.7938578	54.4892988	54.8564764	51.0464156	41.2684067	24.9334242	2.03595937	-26.9038495	-65.3330101	-110.739929	-164.042497	-226.906674
57	20.6706857	43.9019461	68.7301591	95.6801782	123.259435	151.694298	179.747187	206.811731	233.066827	259.175765	281.727279	303.440817	323.267406	339.25663	352.626129
58	23.8942982	53.6825583	89.7883655	130.419677	175.742577	224.446051	275.793804	330.095105	388.112894	446.467021	508.125776	572.400834	637.824128	705.891018	777.534527
59	25.615872	61.0889654	104.178249	154.861326	211.622125	273.452482	340.671966	414.030239	490.045902	571.835435	658.7603	749.434681	845.513061	948.188741	1055.54022
60	27.8000384	65.7858178	113.635374	169.504159	231.96268	301.231015	377.909733	458.274334	545.451106	638.695729	736.536526	840.618061	952.211603	1069.35263	1193.37817
61	27.8232919	67.394053	116.45885	173.048508	237.19974	309.2667	385.165214	467.926794	556.579108	649.438411	747.998768	853.466593	963.671435	1079.7855	1202.38476
62	28.0385395	66.751118	113.569412	168.310745	231.026523	297.200484	369.718448	447.309914	528.005642	613.081728	703.620048	797.161616	894.636687	996.360617	1104.06446
63	26.8286353	62.0545012	105.322839	156.383085	210.262895	269.692657	333.080681	398.154642	465.963457	537.464668	609.907813	683.986025	759.765352	838.864343	920.916413
64	23.6235722	55.1238222	94.0419676	134.985575	180.572631	228.905494	277.42873	326.989781	378.466198	428.854145	478.530334	526.810575	575.254254	623.327626	671.259767
65	21.1186963	48.9193328	77.7423973	110.207724	144.175282	176.858376	208.955504	241.144764	269.676037	294.874429	316.461664	336.098103	353.221207	368.153484	379.38559
66	19.3975551	38.3026818	59.6014804	81.0571117	99.7527066	115.872108	129.989896	138.719454	142.326018	140.493419	135.08204	125.58689	112.526291	94.5507928	76.4028509
67	12.8860933	26.2416496	38.0577897	45.2135255	47.8198873	46.9347294	39.1208875	24.6501729	3.22369582	-23.0166521	-54.4650015	-90.3487728	-131.816814	-173.676598	-218.780579
68	10.2479045	16.3515347	15.4055055	7.99146678	-4.45013811	-25.2925341	-54.1836983	-91.3545703	-134.312192	-183.313273	-237.307964	-297.245716	-357.485811	-420.660199	-484.873773
69	6.20514836	2.15612113	-10.9402356	-30.9100994	-60.8702773	-100.274941	-149.21856	-204.790874	-267.092037	-334.790644	-408.665394	-482.649939	-559.235068	-636.032226	-706.75984
70	-0.22643065	-13.5563745	-36.39077	-71.2188617	-117.088427	-173.852184	-238.077272	-309.693161	-387.082132	-470.893861	-554.674963	-640.861086	-726.650469	-805.264365	-878.672374
71	-6.15629342	-26.1319218	-61.0168287	-109.043656	-169.601169	-238.584883	-315.702417	-399.037077	-489.147579	-579.234469	-671.767346	-763.613262	-847.575717	-925.65344	-985.930385
72	-9.40784681	-38.4906599	-83.810547	-143.864811	-213.591445	-292.378487	-377.966222	-470.849398	-563.907542	-659.737362	-754.944599	-841.986416	-923.023167	-985.739402	-1033.00426
73	-15.5267247	-52.2949612	-107.0043	-173.133688	-249.538914	-333.517677	-425.506969	-518.07298	-614.025595	-709.772718	-797.48009	-879.605306	-943.483456	-992.176081	-1018.93073
74	-21.0221179	-64.999492	-122.966978	-192.845879	-271.306533	-358.69474	-447.251002	-540.081225	-633.448253	-719.275749	-800.447613	-863.991208	-913.272683	-941.728267	-952.534973
75	-27.0699421	-72.055917	-131.138867	-200.034325	-278.929519	-359.712104	-445.87031	-533.571532	-614.535461	-692.217745	-753.371516	-801.782224	-831.166505	-844.690881	-834.447043
76	-27.4893354	-71.8936346	-127.398247	-193.939972	-263.049445	-338.723235	-417.102655	-489.752991	-560.862335	-616.938263	-662.318591	-691.106466	-706.51227	-700.736314	-687.845697
77	-27.4178803	-66.6335899	-117.286029	-170.69225	-231.630245	-296.375475	-356.429968	-416.9238	-464.159336	-503.187584	-528.627282	-543.807436	-541.114304	-534.892469	-515.935341
78	-23.9757624	-57.4460439	-92.151053	-134.372568	-180.96304	-223.610977	-268.725256	-302.475101	-330.834879	-349.104593	-360.824524	-358.678058	-357.434148	-347.464003	-330.285301
79	-21.7842984	-38.3055277	-59.3321392	-83.5616207	-103.602158	-127.780029	-142.333572	-154.478923	-160.416347	-164.032605	-158.449527	-159.063365	-155.850415	-150.092367	-142.157957
80	-9.892879	-12.8490813	-13.0264398	-6.0582118	-3.62214956	7.37876422	17.9426225	30.6121492	40.9692128	55.2586919	57.2052363	57.173457	53.9924806	47.8213268	39.1261703
81	-3.64370231	14.7385746	50.4633248	84.8545887	128.675908	169.983588	209.398916	241.63232	272.016281	283.044568	285.347579	277.687273	260.630732	235.436049	204.030103
82	8.09331233	63.4477879	130.484178	212.83905	292.881388	367.85992	430.930489	486.029555	513.970875	525.533456	519.18905	495.755357	457.175152	406.4751	347.569656
83	34.6923827	121.659527	243.335499	369.13773	489.349494	593.91095	684.409771	739.264484	769.206516	772.118337	748.864413	701.919925	635.320419	554.419173	465.434207
84	53.2660323	198.925645	372.488185	547.489146	706.242742	845.767231	940.726251	1001.40126	1024.64128	1011.07913	963.522201	886.88462	787.901934	674.594121	555.506757
85	98.3229728	299.281898	530.525075	754.167258	956.875531	1106.64508	1212.29961	1268.92233	1276.51913	1237.97809	1158.93113	1047.41265	913.239885	767.12949	619.647558

Figure A6. Prices of S-forward prices under the Cost of Capital approach with smooth parameters, HW model.

Consistency COC / Risk neutral with smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	-4.04%	-5.10%	-6.29%	-7.62%	-8.68%	-9.59%	-10.43%	-11.22%	-11.98%	-12.73%	-13.48%	-14.23%	-14.98%	-15.73%	-16.48%
56	-3.83%	-4.45%	-5.14%	-5.91%	-6.78%	-7.75%	-8.83%	-9.81%	-10.75%	-11.64%	-12.52%	-13.39%	-14.25%	-15.10%	-15.96%
57	-3.74%	-4.05%	-4.38%	-4.76%	-5.18%	-5.65%	-6.18%	-6.78%	-7.46%	-8.22%	-9.09%	-10.07%	-11.12%	-12.14%	-13.15%
58	-3.77%	-3.87%	-3.97%	-4.09%	-4.21%	-4.36%	-4.52%	-4.70%	-4.91%	-5.15%	-5.42%	-5.74%	-6.10%	-6.52%	-7.00%
59	-3.89%	-3.88%	-3.87%	-3.84%	-3.81%	-3.78%	-3.74%	-3.70%	-3.65%	-3.61%	-3.56%	-3.52%	-3.48%	-3.44%	-3.40%
60	-4.10%	-4.07%	-4.02%	-3.97%	-3.90%	-3.83%	-3.74%	-3.65%	-3.55%	-3.44%	-3.32%	-3.20%	-3.06%	-2.92%	-2.77%
61	-4.38%	-4.40%	-4.41%	-4.42%	-4.42%	-4.43%	-4.44%	-4.45%	-4.47%	-4.50%	-4.53%	-4.58%	-4.65%	-4.73%	-4.83%
62	-4.74%	-4.86%	-5%	-5.15%	-5.32%	-5.52%	-5.74%	-6.00%	-6.30%	-6.65%	-7.04%	-7.50%	-8.03%	-8.63%	-9.32%
63	-5.15%	-5.44%	-5.76%	-6.13%	-6.55%	-7.03%	-7.58%	-8.21%	-8.94%	-9.76%	-10.71%	-11.79%	-13.02%	-14.41%	-15.99%
64	-5.62%	-6.11%	-6.68%	-7.32%	-8.07%	-8.92%	-9.89%	-11.01%	-12.28%	-13.74%	-15.36%	-16.98%	-18.60%	-20.23%	-21.87%
65	-6.13%	-6.87%	-7.72%	-8.70%	-9.83%	-11.12%	-12.61%	-14.24%	-15.88%	-17.53%	-19.19%	-20.86%	-22.56%	-24.28%	-26.01%
66	-6.69%	-7.70%	-8.88%	-10.23%	-11.80%	-13.44%	-15.08%	-16.74%	-18.42%	-20.13%	-21.88%	-23.65%	-25.45%	-27.27%	-29.12%
67	-7.28%	-8.60%	-10.13%	-11.79%	-13.41%	-15.06%	-16.74%	-18.47%	-20.24%	-22.06%	-23.91%	-25.80%	-27.72%	-29.66%	-31.63%
68	-7.92%	-9.55%	-11.20%	-12.81%	-14.46%	-16.17%	-17.95%	-19.78%	-21.66%	-23.60%	-25.58%	-27.59%	-29.64%	-31.71%	-33.81%
69	-8.58%	-10.26%	-11.85%	-13.50%	-15.22%	-17.03%	-18.91%	-20.87%	-22.88%	-24.95%	-27.06%	-29.21%	-31.40%	-33.61%	-35.84%
70	-9.27%	-10.76%	-12.35%	-14.07%	-15.89%	-17.81%	-19.81%	-21.90%	-24.05%	-26.26%	-28.51%	-30.81%	-33.13%	-35.49%	-37.86%
71	-9.30%	-10.87%	-12.57%	-14.40%	-16.35%	-18.40%	-20.55%	-22.78%	-25.08%	-27.44%	-29.84%	-32.28%	-34.75%	-37.25%	-39.76%
72	-9.33%	-10.99%	-12.80%	-14.75%	-16.83%	-19.03%	-21.32%	-23.70%	-26.15%	-28.66%	-31.22%	-33.81%	-36.44%	-39.08%	-41.74%
73	-9.35%	-11.11%	-13.04%	-15.12%	-17.34%	-19.68%	-22.13%	-24.67%	-27.28%	-29.94%	-32.66%	-35.40%	-38.18%	-40.97%	-43.78%
74	-9.38%	-11.24%	-13.29%	-15.50%	-17.87%	-20.36%	-22.97%	-25.67%	-28.44%	-31.27%	-34.14%	-37.05%	-39.98%	-42.92%	-45.88%
75	-9.41%	-11.38%	-13.55%	-15.90%	-18.42%	-21.08%	-23.85%	-26.71%	-29.65%	-32.65%	-35.68%	-38.75%	-41.84%	-44.94%	-48.04%
76	-9.44%	-11.52%	-13.82%	-16.32%	-19.00%	-21.82%	-24.76%	-27.80%	-30.91%	-34.07%	-37.27%	-40.50%	-43.75%	-47.00%	-50.27%
77	-9.47%	-11.67%	-14.10%	-16.76%	-19.60%	-22.59%	-25.71%	-28.93%	-32.21%	-35.54%	-38.91%	-42.30%	-45.71%	-49.12%	-52.54%
78	-9.50%	-11.82%	-14.40%	-17.21%	-20.22%	-23.40%	-26.70%	-30.09%	-33.55%	-37.05%	-40.59%	-44.15%	-47.72%	-51.30%	-54.87%
79	-9.54%	-11.98%	-14.71%	-17.69%	-20.88%	-24.23%	-27.72%	-31.29%	-34.93%	-38.61%	-42.32%	-46.04%	-49.78%	-53.52%	-57.26%
80	-9.57%	-12.14%	-15.02%	-18.18%	-21.55%	-25.10%	-28.77%	-32.53%	-36.34%	-40.20%	-44.08%	-47.98%	-51.88%	-55.78%	-59.68%
81	-9.61%	-12.31%	-15.36%	-18.69%	-22.25%	-25.99%	-29.85%	-33.80%	-37.80%	-41.83%	-45.89%	-49.95%	-54.02%	-58.09%	-62.16%
82	-9.65%	-12.49%	-15.70%	-19.22%	-22.97%	-26.91%	-30.96%	-35.10%	-39.28%	-43.50%	-47.73%	-51.97%	-56.21%	-60.44%	-64.68%
83	-9.69%	-12.68%	-16.06%	-19.77%	-23.72%	-27.85%	-32.10%	-36.43%	-40.80%	-45.20%	-49.60%	-54.02%	-58.43%	-62.83%	-67.23%
84	-9.73%	-12.87%	-16.43%	-20.33%	-24.49%	-28.82%	-33.27%	-37.79%	-42.35%	-46.92%	-51.51%	-56.10%	-60.68%	-65.26%	-69.83%
85	-9.77%	-13.06%	-16.81%	-20.92%	-25.28%	-29.82%	-34.47%	-39.18%	-43.92%	-48.68%	-53.44%	-58.21%	-62.96%	-67.71%	-72.45%

Figure A7. Consistency between risk neutral and Cost of Capital method with smooth parameters, HW model.

Consistency COC / Sharpe ratio with smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	6.89%	6.06%	6.01%	6.21%	6.24%	6.21%	6.16%	6.12%	6.08%	6.05%	6.03%	6.02%	6.02%	6.02%	6.03%
56	6.52%	5.27%	4.89%	4.80%	4.85%	4.98%	5.18%	5.31%	5.40%	5.48%	5.54%	5.60%	5.66%	5.71%	5.76%
57	6.37%	4.78%	4.16%	3.84%	3.68%	3.61%	3.60%	3.63%	3.71%	3.83%	3.98%	4.17%	4.36%	4.53%	4.69%
58	6.40%	4.56%	3.76%	3.29%	2.98%	2.76%	2.61%	2.50%	2.42%	2.37%	2.35%	2.35%	2.36%	2.40%	2.46%
59	6.60%	4.56%	3.64%	3.07%	2.68%	2.38%	2.14%	1.95%	1.78%	1.64%	1.53%	1.42%	1.33%	1.25%	1.18%
60	6.94%	4.77%	3.77%	3.16%	2.72%	2.39%	2.13%	1.90%	1.72%	1.55%	1.41%	1.28%	1.16%	1.05%	0.95%
61	7.42%	5.15%	4.12%	3.50%	3.07%	2.75%	2.50%	2.30%	2.14%	2.01%	1.90%	1.81%	1.74%	1.68%	1.64%
62	8%	5.67%	4.65%	4.06%	3.67%	3.40%	3.21%	3.07%	2.99%	2.94%	2.92%	2.93%	2.97%	3.03%	3.12%
63	8.69%	6.32%	5.34%	4.80%	4.48%	4.29%	4.20%	4.17%	4.19%	4.27%	4.39%	4.55%	4.76%	5.00%	5.29%
64	9.46%	7.08%	6.15%	5.70%	5.48%	5.40%	5.42%	5.53%	5.71%	5.94%	6.23%	6.48%	6.72%	6.95%	7.16%
65	10.31%	7.93%	7.08%	6.73%	6.62%	6.68%	6.85%	7.09%	7.30%	7.50%	7.69%	7.88%	8.06%	8.24%	8.42%
66	11.22%	8.86%	8.10%	8.25%	7.86%	7.89%	8.00%	8.25%	8.38%	8.53%	8.68%	8.83%	9.00%	9.16%	9.32%
67	12.20%	9.86%	9.19%	8.99%	8.90%	8.89%	8.93%	9.01%	9.11%	9.24%	9.38%	9.53%	9.69%	9.85%	10.02%
68	13.23%	10.91%	10.11%	9.71%	9.52%	9.46%	9.48%	9.54%	9.65%	9.78%	9.92%	10.08%	10.25%	10.42%	10.60%
69	14.31%	11.67%	10.64%	10.16%	9.94%	9.87%	9.89%	9.97%	10.08%	10.22%	10.38%	10.56%	10.74%	10.93%	11.13%
70	15.43%	12.18%	11.02%	10.51%	10.29%	10.23%	10.26%	10.35%	10.48%	10.64%	10.82%	11.02%	11.22%	11.43%	11.64%
71	15.44%	12.26%	11.15%	10.68%	10.50%	10.47%	10.53%	10.65%	10.82%	11.00%	11.21%	11.43%	11.65%	11.88%	12.12%
72	15.45%	12.34%	11.28%	10.85%	10.71%	10.72%	10.82%	10.97%	11.16%	11.38%	11.61%	11.85%	12.10%	12.35%	12.61%
73	15.46%	12.42%	11.42%	11.04%	10.93%	10.98%	11.11%	11.30%	11.52%	11.77%	12.02%	12.29%	12.56%	12.84%	13.12%
74	15.46%	12.51%	11.56%	11.22%	11.16%	11.25%	11.42%	11.64%	11.89%	12.17%	12.45%	12.75%	13.04%	13.34%	13.64%
75	15.47%	12.60%	11.71%	11.42%	11.40%	11.53%	11.74%	11.99%	12.28%	12.58%	12.90%	13.22%	13.54%	13.86%	14.18%
76	15.48%	12.69%	11.86%	11.62%	11.65%	11.82%	12.07%	12.36%	12.68%	13.01%	13.35%	13.70%	14.05%	14.39%	14.74%
77	15.48%	12.78%	12.01%	11.83%	11.91%	12.12%	12.41%	12.73%	13.09%	13.45%	13.82%	14.20%	14.57%	14.94%	15.31%
78	15.49%	12.88%	12.18%	12.05%	12.18%	12.43%	12.76%	13.12%	13.51%	13.91%	14.31%	14.71%	15.11%	15.51%	15.90%
79	15.50%	12.98%	12.34%	12.28%	12.45%	12.75%	13.12%	13.52%	13.94%	14.37%	14.81%	15.24%	15.66%	16.08%	16.50%
80	15.51%	13.08%	12.52%	12.51%	12.74%	13.08%	13.49%	13.93%	14.39%	14.85%	15.31%	15.77%	16.23%	16.67%	17.11%
81	15.52%	13.19%	12.70%	12.75%	13.03%	13.42%	13.88%	14.36%	14.85%	15.34%	15.84%	16.32%	16.80%	17.27%	17.74%
82	15.53%	13.30%	12.88%	12.99%	13.33%	13.77%	14.27%	14.79%	15.32%	15.84%	16.37%	16.88%	17.39%	17.89%	18.37%
83	15.53%	13.41%	13.07%	13.25%	13.64%	14.13%	14.67%	15.23%	15.79%	16.36%	16.91%	17.46%	17.99%	18.51%	19.03%
84	15.54%	13.53%	13.26%	13.51%	13.96%	14.50%	15.08%	15.68%	16.28%	16.88%	17.47%	18.04%	18.60%	19.15%	19.69%
85	15.55%	13.65%	13.46%	13.77%	14.28%	14.87%	15.50%	16.14%	16.78%	17.41%	18.03%	18.63%	19.22%	19.80%	20.36%

Figure A8. Consistency between Sharpe and Cost of Capital method with smooth parameters, HW model.

Consistency COC / Wang with smooth parameters, HW model															
Age/Years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
55	6.89%	6.06%	6.01%	6.21%	6.23%	6.20%	6.15%	6.11%	6.07%	6.05%	6.03%	6.01%	6.01%	6.01%	6.02%
56	6.52%	5.27%	4.89%	4.80%	4.84%	4.98%	5.17%	5.30%	5.40%	5.48%	5.54%	5.60%	5.66%	5.71%	5.76%
57	6.37%	4.78%	4.16%	3.84%	3.68%	3.61%	3.60%	3.63%	3.71%	3.83%	3.98%	4.17%	4.36%	4.53%	4.69%
58	6.40%	4.56%	3.75%	3.29%	2.98%	2.76%	2.61%	2.50%	2.42%	2.37%	2.35%	2.35%	2.37%	2.41%	2.46%
59	6.60%	4.56%	3.64%	3.07%	2.68%	2.38%	2.14%	1.95%	1.78%	1.65%	1.53%	1.42%	1.33%	1.25%	1.18%
60	6.94%	4.77%	3.77%	3.16%	2.72%	2.39%	2.13%	1.91%	1.72%	1.55%	1.41%	1.28%	1.16%	1.05%	0.95%
61	7.41%	5.14%	4.12%	3.50%	3.07%	2.75%	2.50%	2.30%	2.14%	2.01%	1.90%	1.81%	1.74%	1.68%	1.64%
62	8.00%	5.67%	4.65%	4.05%	3.66%	3.40%	3.21%	3.07%	2.99%	2.94%	2.92%	2.93%	2.97%	3.03%	3.12%
63	8.68%	6.32%	5.33%	4.80%	4.48%	4.29%	4.19%	4.17%	4.19%	4.27%	4.39%	4.55%	4.76%	5.00%	5.30%
64	9.46%	7.08%	6.15%	5.69%	5.47%	5.40%	5.42%	5.53%	5.70%	5.94%	6.22%	6.48%	6.72%	6.94%	7.15%
65	10.30%	7.93%	7.07%	6.72%	6.62%	6.67%	6.84%	7.08%	7.29%	7.49%	7.68%	7.87%	8.05%	8.23%	8.41%
66	11.22%	8.86%	8.09%	7.85%	7.88%	7.99%	8.11%	8.23%	8.37%	8.51%	8.66%	8.82%	8.98%	9.14%	9.31%
67	12.19%	9.85%	9.18%	8.98%	8.89%	8.88%	8.91%	8.99%	9.10%	9.22%	9.36%	9.51%	9.67%	9.83%	10.00%
68	13.22%	10.90%	10.09%	9.70%	9.51%	9.44%	9.46%	9.52%	9.63%	9.75%	9.90%	10.05%	10.22%	10.39%	10.57%
69	14.30%	11.66%	10.62%	10.14%	9.93%	9.85%	9.87%	9.94%	10.06%	10.20%	10.36%	10.53%	10.71%	10.90%	11.09%
70	15.43%	12.17%	11.01%	10.49%	10.27%	10.20%	10.23%	10.32%	10.46%	10.61%	10.79%	10.98%	11.18%	11.39%	11.60%
71	15.44%	12.25%	11.14%	10.66%	10.47%	10.44%	10.51%	10.63%	10.79%	10.97%	11.18%	11.39%	11.61%	11.84%	12.07%
72	15.44%	12.33%	11.27%	10.83%	10.69%	10.69%	10.79%	10.94%	11.13%	11.34%	11.57%	11.81%	12.06%	12.31%	12.56%
73	15.45%	12.41%	11.40%	11.01%	10.91%	10.95%	11.08%	11.27%	11.48%	11.73%	11.98%	12.25%	12.52%	12.79%	13.06%
74	15.45%	12.49%	11.54%	11.20%	11.14%	11.22%	11.39%	11.60%	11.85%	12.12%	12.41%	12.70%	12.99%	13.28%	13.58%
75	15.46%	12.58%	11.69%	11.40%	11.38%	11.50%	11.70%	11.95%	12.23%	12.53%	12.84%	13.16%	13.48%	13.80%	14.11%
76	15.47%	12.67%	11.84%	11.60%	11.62%	11.79%	12.03%	12.31%	12.63%	12.96%	13.30%	13.64%	13.98%	14.32%	14.66%
77	15.47%	12.76%	11.99%	11.81%	11.88%	12.09%	12.37%	12.69%	13.03%	13.39%	13.76%	14.13%	14.50%	14.86%	15.22%
78	15.48%	12.86%	12.15%	12.02%	12.14%	12.39%	12.71%	13.07%	13.45%	13.84%	14.24%	14.63%	15.03%	15.42%	15.80%
79	15.49%	12.96%	12.32%	12.25%	12.42%	12.71%	13.07%	13.47%	13.88%	14.30%	14.73%	15.15%	15.57%	15.98%	16.39%
80	15.50%	13.06%	12.49%	12.48%	12.70%	13.04%	13.44%	13.87%	14.32%	14.78%	15.23%	15.68%	16.12%	16.56%	16.99%
81	15.50%	13.17%	12.67%	12.71%	12.99%	13.38%	13.82%	14.29%	14.77%	15.26%	15.74%	16.22%	16.69%	17.15%	17.60%
82	15.51%	13.28%	12.85%	12.96%	13.29%	13.72%	14.21%	14.72%	15.24%	15.75%	16.27%	16.77%	17.27%	17.75%	18.23%
83	15.52%	13.39%	13.04%	13.21%	13.59%	14.07%	14.60%	15.15%	15.71%	16.26%	16.80%	17.33%	17.86%	18.37%	18.86%
84	15.53%	13.51%	13.23%	13.47%	13.90%	14.44%	15.01%	15.60%	16.19%	16.77%	17.35%	17.91%	18.45%	18.99%	19.51%
85	15.54%	13.62%	13.43%	13.73%	14.23%	14.81%	15.42%	16.05%	16.68%	17.29%	17.90%	18.49%	19.06%	19.62%	20.17%

Figure A9. Consistency between Wang and Cost of Capital method with smooth parameters, HW model.

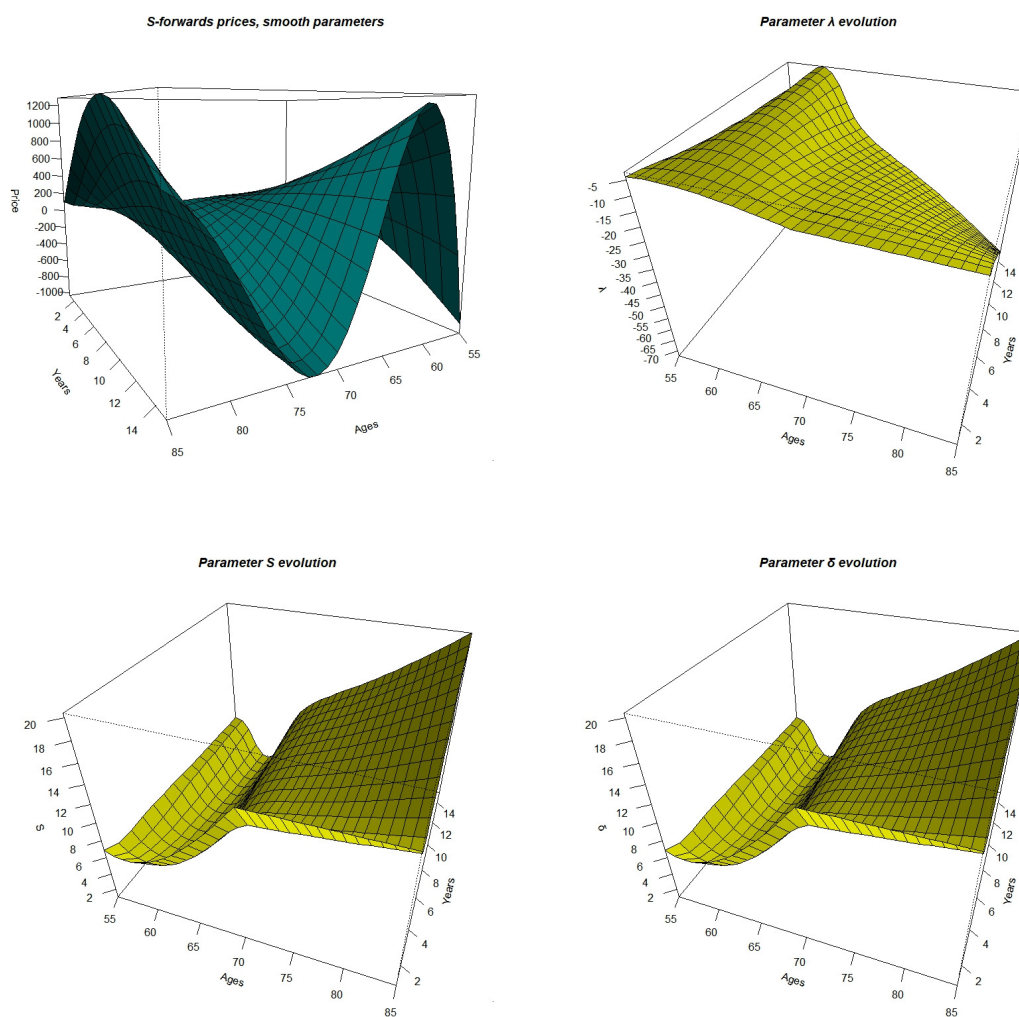


Figure A10. 3D presentation of the S-forwards prices and the corresponding parameters with the other methods.

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