Chirinko, Robert S.; Mallick, Debdulal

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THE ELASTICITY OF DERIVED DEMAND, FACTOR SUBSTITUTION AND PRODUCT DEMAND: CORRECTIONS TO HICKS’ FORMULA AND MARSHALL’S FOUR RULES

ROBERT S. CHIRINKO
DEBDULAL MALLICK

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Abstract

Nearly 75 years ago, John Hicks introduced and formalized the concept of the elasticity of substitution between capital and labour and its relation to derived demand. The resulting formula has proven very useful in understanding the derived demand for productive factors, the distribution of factor incomes, and Marshall's Four Rules. This short paper notes that a slip occurred in the original derivation, presents a modified formula, and shows that Marshall's First Rule is no longer generally valid.

JEL Code: D21, D33, J23, A2.

Keywords: derived demand, substitution elasticity, John Hicks.

Robert S. Chirinko
Department of Economics
Emory University
Atlanta, Georgia 30322-2240
USA
robert.chirinko@emory.edu

Debdulal Mallick
Department of Economics
Emory University
Atlanta, Georgia 30322-2240
USA

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The Elasticity Of Derived Demand, Factor Substitution, And Product Demand:
Corrections To Hicks' Formula And Marshall's Four Rules

Nearly 75 years ago, John Hicks introduced and formalized the concept of the elasticity of substitution between capital and labour. Hicks' substantial achievement in *The Theory of Wages* (1932/1963) was to develop a serviceable formal framework with which to analyze the concept and its implications for demand theory. These efforts yielded an important formula linking the derived demand for a factor of production to the extent to which it substituted with other factors, its importance in production, the availability of competing factors, and the ultimate demand for the good produced. This formula proved valuable in evaluating Marshall's Four Rules of derived demand and continues in use today. Leading U.S. labour economics texts discuss Hicks' formula and Marshall's Four Rules.¹ Molina (2005) relies on Hicks formula in analyzing capital theory debates, and Chirinko and Mallick (2006) use it for inferring macro production function parameters from estimates based on micro data. Hamermesh (1993, Chapter Two, especially equation (2.7a')) evaluates extant empirical labour demand studies with the Hicks formula, which he refers to as "the fundamental law of factor demand" (p. 24).

However, a slip occurred in deriving the original formula. Hicks and later Allen (1938/1964) assumed that factor shares are constant. While this assumption is appropriate when the substitution elasticity is unity, we now know with the benefit of the celebrated article by Arrow, Chenary, Minhaus, and Solow (1961/1985) that factor shares vary with relative factor prices when the substitution

elasticity differs from unity. This note presents a modified formula that recognizes variation in factor shares. Section 1 introduces the formula and notation used originally by Hicks. Section 2 offers a new derivation based on the CES production function. Given its wide currency, the constant elasticity of substitution (CES) production function is used to establish the required modification to the Hicks formula for a range of values of the substitution elasticity. When factor shares are constant, the original and modified formulas are identical. In light of the modified formula, Section 3 re-evaluates Marshall's Four Rules, and we show that the First Rule no longer holds in general when the substitution elasticity differs from unity.

1. Hicks' Formula

The original formula was presented by Hicks (1932/1963) in Appendix (iii) and is based on a neoclassical production function, $x = f[a, b]$, relating output ($x$) to two inputs ($a$, $b$). The formula is stated in terms of four elasticities and the factor share for $a$:²

The elasticity of substitution between factors $a$ and $b$,

$$\sigma \equiv \left( \frac{f_a}{f_b} \right) \left( \frac{f_{ab}}{x} \right), \quad (1a)$$

$$= -\left( \frac{(da/a) - (db/b)}{(dp_a/p_a) - (dp_b/p_b)} \right), \quad (1b)$$

$$= -\left( \frac{d\log(a/b)}{d\log(p_a/p_b)} \right), \quad (1c)$$

² The only changes in notation from Hicks are to introduce explicit notation for the production function ($f[a, b]$), replace the partial derivatives $x_a$, $x_b$, and $x_{ab}$ by $f_a$, $f_b$, and $f_{ab}$, respectively, and substitute factor prices for their marginal products.
The price elasticity of demand for factor a,
\[ \lambda = -(\partial a / a) / (\partial p_a / p_a), \]  
(2)

The price elasticity of demand for the product x,
\[ \eta = -(\partial x / x) / (\partial p_x / p_x), \]  
(3)

The price elasticity of supply of the substitute factor b,
\[ e = (\partial b / b) / (\partial p_b / p_b), \]  
(4)

The factor shares for a and b, respectively,
\[ \kappa = (p_a a) / (p_x x), \]  
(5a)
\[ 1 - \kappa = (p_b b) / (p_x x). \]  
(5b)

The elasticity of substitution between factors a and b can be represented in three equivalent ways. Equation (1a) is the original formulation by Hicks. Robinson (1933/1959, p. 256) independently introduced the substitution elasticity as specified in equations (1b) and (1c). Hicks (1963, Section VII, "Notes on the Elasticity of Substitution", sub-section 1) showed that the two formulations are equivalent, though Robinson's definition has proven the more convenient and popular. Note that equations (5a) and (5b) state that factor shares are constant.

Computing a series of total derivatives, exploiting the linear homogeneity of the production function, and using the above relations, Hicks (1963, pp. 242-244) derives the following formula for the elasticity of the derived demand for factor a,\(^3\)

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\(^3\) See Allen (1938/1964, Section 14.8) for an alternative derivation.
To better focus attention on the slip that has occurred in deriving equation (6), we examine the simpler formula based on an infinitely elastic supply of the substitute factor of production. Letting $e \to \infty$, equation (6) can be rewritten as follows,

\[
\lambda = \sigma - \kappa \sigma + \kappa \eta, \quad (7a)
\]
\[
\lambda = \sigma + (\eta - \sigma) \kappa, \quad (7b)
\]

Equation (7a) captures in a succinct manner the substitution and scale effects associated with a decline of the factor price of $a$ on its derived demand. As represented by the first term, there is a direct substitution effect holding output price and output constant. The second term represents an additional indirect substitution effect driven by the lower marginal cost of production. Under competitive conditions, the decline in marginal cost translates into a decline in the output price. The extent of this decline is determined by the relative importance of factor $a$ in production represented by its factor share ($\kappa$). The decline in output price raises the relative price of and lowers the demand for factor $a$. The third effect occurs because the lower factor price allows the firm to slide down the product demand curve and increase output. This scale effect is represented by the product of $\kappa$ and the demand elasticity ($\eta$) in the third term of equation (7a).

2. A New Derivation and A Modification

To highlight the roles of the substitution elasticity and factor shares, we develop our modified formula from the following CES production function,
where $\phi$ is the distribution parameter. Assuming that the firm is maximizing profits subject to this CES production function and a vector of prices, the first-order condition for factor $a$ is as follows,

$$a = \phi^\sigma (p_a / p_x)^{-\sigma} x.$$  \hfill (9)

Since output ($x$) and output price ($p_x$) will vary with the factor price ($p_a$), they must be restated in terms of factor prices and parameters describing the technology and the output market. If the output market is competitive and the production function is linear homogeneous, output price equals marginal cost that, in turn, equals average cost. The latter is specified by starting with the basic cost function as the sum of the purchase costs of each factor and using equation (9) and the kindred relation for factor $b$ to obtain the following expression for average cost qua output price,

$$p_x = \left( \phi^\sigma p_a^{(1-\sigma)} + (1-\phi)^\sigma p_b^{(1-\sigma)} \right)^{(1/(1-\sigma))}.$$  \hfill (10)

We assume that industry product demand is described by the following constant elasticity function,

$$x = p_x^{-\eta} W,$$  \hfill (11)

where $W$ represents a set of exogenous variables that affect the demand.
Substituting equation (11) into (9) and equation (10) into the resulting expression, we obtain the following equation for the derived demand for factor a,

\[ a = \phi^\sigma p_a ^{-\sigma} \left( \phi^\sigma (1-\sigma) p_a + (1-\phi)^\sigma p_b (1-\sigma) \right)^{(\sigma-\eta)/(1-\sigma)} \]

(12)

Differentiation of equation (12) with respect to the factor price of a and some transformations yield the following modified formula for the price elasticity of demand for factor a, \( \lambda^* \),

\[ \lambda^* = \sigma + (\eta - \sigma) \mu[\phi,(p_a / p_x) , \sigma] \]

(13a)

\[ \mu[\phi,(p_a / p_x) , \sigma] \equiv 1 \]

(13b)

\[ \mu[\phi,(p_a / p_x) , \sigma = 1] = \phi \]

(13c)

\[ \partial\mu / \partial\sigma = (\ln[\phi] - \ln[p_a / p_x]) \mu >\leq 0 \]

(13d)

The difference between the original (cf. equation (7b)) and modified formulas is represented by \( \mu[\phi,(p_a / p_x) , \sigma] \) defined in equation (13b), which depends on the CES distribution and substitution parameters and the factor price ratio. When \( \sigma \) takes on the restrictive Cobb-Douglas value of unity, \( \mu[\phi,(p_a / p_x) , \sigma] \) reduces to \( \phi \) (equation (13c)), and there is no discrepancy between Hicks' original formula and \( \lambda^* \). For general values of the substitution

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4 Consistent with the assumption that the supply of factor b is infinitely elastic, we assume that variations in \( p_a \) do not affect \( p_b \). If this assumption is relaxed, then the derivation becomes more complicated; we need to also analyze the derived demand for factor b (similar to equation (12)), differentiate this equation with respect to \( p_a \), and use the resulting relations to eliminate the cross-price elasticity. Details are provided in the Appendix. The end result is that equation (13a) is replaced by equation (6) with the \( \kappa \)'s removed in favour of the \( \mu[.] \) in equation (13b).
parameter, however, the elasticity formula must account for the variability in factor shares due to the relative price term and the substitution elasticity.\(^5\)

This analysis of the CES production function suggests a "shortcut" method for modifying the elasticity formula. The slip in the derivations by Hicks and Allen occurred in treating the factor share of a as a constant. This assumption was imposed toward the end of their derivations and did not affect the evaluation of their differentials. Consequently, it is valid to merely use the CES production function to derive the appropriate expression for the factor share of a and to insert it into the original formula. We begin with the factor share of a (equation (5a)),

\[
\kappa \equiv \left( \frac{p_a}{p_x} \right) = \frac{(p_a \cdot a)}{(p_x \cdot x)},
\]

and use the first-order condition for factor a (equation (9)) to eliminate \((a / x)\),

\[
\kappa = \phi^{\sigma} \left( \frac{p_a}{p_x} \right)^{(1-\sigma)} = \mu[\phi, \left( \frac{p_a}{p_x} \right), \sigma].
\]

Replacing \(\kappa\) in equation (7b) with \(\mu[\phi, \left( \frac{p_a}{p_x} \right), \sigma]\) in equation (15) yields the modified formula for \(\lambda^*\) in equations (13).

\(^5\) These newly defined factor shares sum to unity: \(\mu[.] + (1-\mu[.])\)

\[
= \phi^{\sigma} \left( \frac{p_a}{p_x} \right)^{1-\sigma} + (1-\phi)^{\sigma} \left( \frac{b}{p_x} \right)^{1-\sigma} = \left( \phi^{\sigma} p_a^{1-\sigma} + (1-\phi)^{\sigma} p_b^{1-\sigma} \right) p_x^{-(1-\sigma)}. \]

The expression in parentheses can be related to \(p_x\) with equation (10); hence, \(p_x^{(1-\sigma)} p_x^{-(1-\sigma)} = 1\).
3. Re-evaluating Marshall's Four Rules

Hicks used his formula to evaluate Marshall's Four Rules of derived demand. He cast the Four Rules in terms of the response of $\lambda$ (equation (6)) to four parameters ($\sigma, \kappa, \epsilon, \eta$), and he confirmed that three of the four rules were valid. Hicks' formal analysis did highlight that Marshall's Second Rule was not generally true and depended on the sign of ($\eta-\sigma$). Since the correction factor, $\mu[\phi, (p_a/p_x), \sigma]$, is always positive and only involves the parameter $\sigma$, Hicks' analysis of three of Marshall's Rules remain valid. However, the modified formula demonstrates that the first of Marshall's Four Rules is problematic. The First Rule (Hicks, 1932/1963, p. 242) is as follows,

I. "The demand for anything is likely to be more elastic, the more readily substitutes for that thing can be obtained."

Following Hicks, we evaluate this Rule in terms of the derivative of $\lambda^*$ with respect to $\sigma$,

$$\frac{\partial \lambda^*}{\partial \sigma} = 1 - \mu[.] + (\eta - \sigma) (\ln[\phi] - \ln[p_a/p_x])\mu[.], \quad (16)$$

When this derivative is based on Hicks' original formula, only the first two terms appear and, since $\mu[.]$ lies between 0 and 1, the derivative is always positive, thus confirming Marshall's First Rule. However, under the modified formula and as captured in the third term, the derivative is also affected by variation in the factor share. The sign of this additional term is indeterminate and depends on the relations among $\eta$, $\sigma$, $\phi$, and $(p_a/p_x)$. Even if quantities and prices are defined such that the relative price entering equation (16) is unity, the indeterminacy remains, and Marshall's First Rule is not generally valid.
Appendix

This appendix derives the modified Hicks formula when the price elasticity of supply of the substitute factor \( b \), \( e \), is finite. We begin with the equation for the derived demand for factor \( a \) (equation (12),

\[
a = \phi^\sigma p_a^{-\sigma} \left( \phi^\sigma p_a^{(1-\sigma)} + (1-\phi)^\sigma p_b^{(1-\sigma)} \right)^{(\sigma-\eta)/(1-\sigma)} W. \tag{A1}
\]

Taking logarithms of equation (A1), we obtain the following equation,

\[
\ln a = \sigma \ln \phi - \sigma \ln p_a + \left( \frac{\sigma - \eta}{1 - \sigma} \right) \ln \left[ \phi^\sigma p_a^{1-\sigma} + (1-\phi)^\sigma p_b^{1-\sigma} \right] + \ln W. \tag{A2}
\]

Differentiating equation (A2) w.r.t. \( p_a \) and recognizing the relations among \( p_a \), \( b \), and \( b_p \), we obtain the following derivative,

\[
\frac{\partial a}{\partial p_a} = -\sigma + (\sigma - \eta) \left\{ \phi^\sigma \left( \frac{p_a}{p_x} \right)^{1-\sigma} \right\} + (\sigma - \eta) \left\{ (1-\phi)^\sigma \left( \frac{p_b}{p_x} \right)^{1-\sigma} \frac{\partial p_b}{\partial b} \frac{\partial b}{\partial p_a} \right\}. \tag{A3}
\]

Defining the cross-price elasticity of factor \( b \),

\[
\varepsilon_c = \left( \frac{\partial b}{b} \right) / \left( \frac{\partial p_a}{p_a} \right), \tag{A4}
\]
and using the definitions in equations (2), (4), and (13b), we can rewrite equation (A3) as follows,

\[ \lambda^* = \sigma - (\sigma - \eta)\mu[i] - (\sigma - \eta)(1 - \mu[i])(\varepsilon_c/e). \]  

(A5)

The problematic element in equation (A5) is \( \varepsilon_c \). We eliminate this cross-price elasticity by analyzing the derived demand for factor b w.r.t. variations in \( p_a \). Paralleling the above analysis of the derived demand for factor a, we start with the equation for the derived demand for factor b (similar to equation (12)),

\[ b = (1 - \phi)^\sigma p_b^{-\sigma}\left(\phi \sigma p_a^{(1-\sigma)} + (1 - \phi)^\sigma p_b^{(1-\sigma)}\right)^{(\sigma - \eta)/(1-\sigma)}W, \]  

(A6)

take logarithms of equation (A6), and obtain the following equation,

\[ \ln b = \sigma \ln(1 - \phi) - \sigma \ln p_b + \left(\frac{\sigma - \eta}{1 - \sigma}\right) \ln \left[\phi^\sigma p_a^{1-\sigma} + (1 - \phi)^\sigma p_b^{1-\sigma}\right] \]  

(A7)

+ \ln W.

Differentiating equation (A7) w.r.t. \( p_a \) and recognizing the relations among \( p_a \), b, and \( p_b \), we obtain the following derivative,

\[ \frac{\partial b / b}{\partial p_a / p_a} = -\sigma \frac{\partial p_b}{\partial b} \frac{\partial b}{\partial p_a} p_a + (\sigma - \eta)\left\{\phi^\sigma (p_a/p_x)^{1-\sigma}\right\} \]  

(A8)

\[ + (\sigma - \eta)\left\{(1 - \phi)^\sigma (p_b/p_x)^{1-\sigma} \frac{\partial p_b}{\partial b} \frac{\partial b}{\partial p_a} p_a\right\} \]
Again using the definitions in equations (2), (4) and (13b), equation (A8) can be rewritten as follows,

\[
\varepsilon_c = -\sigma (\varepsilon_c/e) + (\sigma - \eta) \mu (\cdot) + (\sigma - \eta)(1 - \mu (\cdot))(\varepsilon_c/e).
\]  \hspace{1cm} (A9)

Equation (A9) can be solved for \( (\varepsilon_c/e) \),

\[
(\varepsilon_c/e) = \frac{(\sigma - \eta) \mu (\cdot)}{\eta + e + (\sigma - \eta) \mu (\cdot)}.
\]  \hspace{1cm} (A10)

Returning to the price elasticity of demand for factor a, we use equation (A10) to eliminate the cross-price elasticity term in equation (A5) and, after substantial manipulation, obtain the following equation,

\[
\lambda = \frac{\sigma(\eta + e) + e(\eta - \sigma) \mu (\cdot)}{\eta + e - (\eta - \sigma) \mu (\cdot)}.
\]  \hspace{1cm} (A11)

Equation (A11) is equation (6) in the text with the sole modification that our \( \mu (\cdot) \) replaces Hicks's \( \kappa \).
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