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#### Article

# Structural stability of coalitions: A formal model highlighting the role of participants positioned between members and neutral actors

Games

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*Suggested Citation:* Monsuur, Herman; Janssen, René (2022) : Structural stability of coalitions: A formal model highlighting the role of participants positioned between members and neutral actors, Games, ISSN 2073-4336, MDPI, Basel, Vol. 13, Iss. 1, pp. 1-12, https://doi.org/10.3390/g13010017

This Version is available at: https://hdl.handle.net/10419/257593

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### Article Structural Stability of Coalitions: A Formal Model Highlighting the Role of Participants Positioned between Members and Neutral Actors

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**Abstract:** Given a fixed network that links various actors, we introduce a formal model that describes the structural stability of coalitions. To this end, we used the partition of the set of all actors in three distinct positions: members, i.e., those who are central to a coalition; participants, i.e., those who are not actively engaged with the issues at hand; neutral actors, i.e., non-members, those who are not party to any interests and outside the coalition. Using the language of networks, we formulated three assumptions that may be used to characterize the stability of this partition. We paid particular attention to the role of participants as they facilitate or complicate extensions of a given coalition. Given the fixed network, we (1) illustrate the verification of the stability of a coalition, (2) provide existence results of stable coalitions, and (3) set the limits to their credible extensions. Our three formal assumptions may serve as a platform for discussions regarding the stability of coalitions.

Keywords: network science; stability; coalition formation



Citation: Monsuur, H.; Janssen, R.H.P. Structural Stability of Coalitions: A Formal Model Highlighting the Role of Participants Positioned between Members and Neutral Actors. *Games* **2022**, *13*, 17. https://doi.org/10.3390/g13010017

Academic Editors: Ulrich Berger and Stefano Moretti

Received: 27 October 2021 Accepted: 26 January 2022 Published: 28 January 2022

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#### 1. Introduction

In the literature, many models can be found that describe the various processes related to the formation and the bargaining of a coalition, as well as its maintenance and possible collapse. These models try to shed light on strategies and processes by assessing benefits and costs, incentives, and constraints imposed by or experienced by the actors. Many of these models are designed as formal models, where concepts are formalized via mathematical expressions. Formal models share a long history. Well-known examples are the concept of minimal winning coalitions in the political arena, discussed by Riker [1], or power indices proposed by Shapley and Shubik [2], where the strategic positioning of actors and their pivotal roles are taken into consideration. In organizational research also, models representing coalitions are plentiful: this can be seen, for example, in the overview presented in Mithani and O'Brien [3]. These models are useful when one is confronted with the complexity of the social, organizational, and political world, as their concepts illuminate phenomena related to coalitions and represent "slices of the world", either at the empirical or conceptual level (see Lenine [4]). In his article, Lenine mentioned that these formal models are a description of an imaginary, but credible world. Using them as a platform for discussion and instruments for investigation, one may navigate between coalitions, generating patterns in the model world that can possibly be linked to features that can be observed in the real world (cf. Sugden [5]). These patterns illustrate that the models' deductive power moves beyond description to inferences from the assumptions (see Snidal [6]). Then, observing the similarity of effects, one may infer a similarity of causes.

The aim of this paper was to introduce a formal model that informs us about the (structural) stability of coalitions and their possible extensions, where we used the mathematical concepts of networks linking the various actors. Networks identify the interactions between actors that are basic for understanding behavior, decisions, and performance. We

considered a fixed, generally incomplete network consisting of a number of links between the various actors, as elaborated in the work of Myerson [7], Bistaffa et al. [8], and Dutta and Jackson [9]. This means that we deviated from many approaches that investigate coalition dynamics through the lens of a link formation game. In this type of approach, the network structure with its links between actors is the result of actions of (autonomous) actors trying to maximize utility. Examples of such approaches can be found, for instance, in Harmsen et al. [10], Goyal [11], Jackson and Wolinsky [12], Janssen et al. [13], Van Deemen and Rusinowska [14], van Deemen [15], Ray [16], Shenoy [17], and Auer et al. [18]. An overview can be found in Jackson [19]. Instead, we investigated networks that we may consider to be fixed; no links are added or severed (at least for some time). To illustrate: let us consider countries that are engaged in mutual trade. Here, a link between two countries indicates that the volume of trade exceeds a certain threshold. The non-existence of a link indicates that the trade between the two is negligible. As a second example, let us consider political actors. A link between two actors indicates that the set of relevant themes and shared visions is large enough to establish a fruitful cooperation. Generally speaking, given a fixed network, the existence of a link between two actors indicates that fruitful communication and cooperation are possible. It generates opportunities in the coalition's decision-making process. If two actors lack such a link, their visions, missions, and important themes apparently differ too much for these actors to work together as an isolated pair. It is within the context of this fixed network that we studied the stability of a coalition. Given a coalition, we may partition the set of all actors into three distinct positions, in line with Mithani and O'Brien [3], who discussed this partition for organizational coalition theory. The first group of actors is formed by members, who are central to a coalition. The second group includes participants, who are part of a coalition, but not actively engaged with the issues at hand. The third group is formed by neutral actors, non-members who are not party to any interests and who are outside the coalition. Although participants are inside a coalition, they are not at the table shaping the agenda.

We shall now introduce three assumptions that may be seen as slices of an imaginary, but credible world. These assumptions underscore the stability of the coalition's dichotomous nature, i.e., with members and participants, while using the language of networks. Further research may be performed to test each of the assumptions and their combined predictability of the stability in real-world coalitions. Such investigations fall outside the scope of the current paper, but some remarks on the topic can be found in Section 5 below. Our first assumption was that the partition discussed above is confirmed by the fixed network through which the actors are connected. A precise definition can be found in Section 3. We demand that two members, if not directly linked in the network, are connected indirectly using links between other members only; this is illustrated in Figure 1.



**Figure 1.** A fixed network consisting of dotted, as well as solid links. In (**A**), we let  $M = \{1, 2, 3, 4\}$  be the members of the coalition and let  $P = \{5, 7, 8\}$  be the participants; Actor 6 is neutral. In (**B**), we let  $M = \{1, 2, 4, 5\}$ , so  $P = \{3, 6, 7, 8\}$ . In (**C**), we let  $M = \{1, 4\}$  and  $P = \{2, 3, 5, 7, 8\}$ ; again, Actor 6 is neutral. In all cases, the coalition-induced network that is considered eligible in studying the stability of the coalition is represented by the solid links, a subset of the fixed network.

This is in line with the idea that members are gathered at the table, shaping the coalition's agenda. Participants are characterized by the existence of direct links to at least one member. All these links, that is links between members and links between members

and participants, generate what we call the (coalition-) induced network. It highlights the relevant patterns of interactions within the coalition: links between members can be initiated to optimize gains, and this can also be done for links between members and participants. These latter links are initiated and exploited by members. Links between participants are not in the induced network as they do not play a role from the point of view of members. In Figure 1, the induced network is indicated through the use of solid links. Our second assumption was that members care about their (network) position within the coalition (cf. van Deemen [15] or Bistaffa et al. [8]). Cases where this assumption may be true are consensus decision-making situations. Here, each member of the coalition wants to be heard, demands that his/her opinion matters, and wants his/her themes to be put on the agenda. Therefore, given the fixed network, each member wants his/her network position to be sufficiently unique, thereby creating possibilities for profiling and decision-making. To reformulate our second assumption, we demand that only members of the coalition have an equal, yet distinguished position within the induced network: they are uncovered. To determine whether or not an actor has such a position, we used the concept of covering. A detailed explanation of this concept can be found in Section 2. Covering has been applied to social network theory and the identification of key players (see Janssen and Monsuur [20]). Broadly speaking, an escape from being covered can be seen as a generalization of the tendency of actors to form connected coalitions with partners on the left- and the right-hand side in a uni-dimensional policy space in order to obtain a pivotal position. More specifically, if an actor somehow perceives that he/she is in a subdued position in a coalition, the notion of covering can be used to guide the search for alternative coalitions where the actor's position is evaluated as equal, pivotal, or uncovered. We shall illustrate the use of covering with the help of the economic example that we introduced earlier. Let us consider two countries, *a* and *b*. Country *b* may try to put pressure on Country *a* by mobilizing all its trade partners to, for example, restrict trade with a. Now, let us suppose that Country a has at least one trade partner that is not a trade partner of *b*. This country therefore does not feel any pressure and may remain loyal to Country *a*, thus providing a kind of escape trade route for Country *a*. We then say that Country *a* is uncovered. Generally speaking, we argue that the structural stability of a coalition is guaranteed if only members are uncovered in the induced network. In that case, only members are regarded as equal in the coalition, enabling them to participate in the cooperative decision-making process. This is our final and third assumption. In such a coalition, all members' opinions matter, enabling the orchestrated use of the induced network to achieve their goals. Only members are assumed to be uncovered and hence to have a unique position. This means that each participant is covered by at least one member that is able to initiate its links to subdue the participants' opinion, if necessary. This way, the above-mentioned partition and the boundary of a coalition consisting of participants are confirmed and supported by the induced network structure.

Here, we shall illustrate the implications and predictions of our three assumptions in a less formal way; Section 3 presents further details. Let us consider the network depicted in Figure 1A. We observe that Actor 3 is in a covered position compared to Actor 1: all members and participants to which Actor 3 is linked are also linked to Actor 1, which in addition has a link to Participant 7. In our example situation, this may give Actor 1 the possibility to put into position Actors 2, 4, and 5 by convincing them to postpone cooperation or trading with Actor 3. Because Actor 1 has an extra link (to Participant 7), Actor 3 cannot initiate a comparable countermove towards Actor 1. This invalidates the claim on membership of Actor 3. In addition, Participant 5 is not covered by other actors/members: compared to 1, it has an escape for cooperating or trading with 3; compared to 4, it has a link to 2, and its position is comparable to that of Member 2. This supports the claim that Actor 5 should be at the table and should be a member. Altogether, the induced network does not support the suggested partitioning into members, participants, and neutrals. The coalition is invalidated when it comes to serving as a credible and stable cooperation pursuing its interests, and it cannot be called a stable coalition. As

an example of a stable coalition, let us consider the coalition with members  $M = \{1, 2, 4, 5\}$ depicted in Figure 1B. Its members are uncovered and have a unique position within the coalition. This enables decision-making as a coalition. At the same time, members are able to cover participants  $P = \{3, 6, 7, 8\}$ , providing the means to subdue the participants' opinions if necessary. This means that this coalition can justify its claim on structural stability. Another example of a stable coalition is given in Figure 1C. To illustrate the role of participants in the structural stability of a coalition, as well as their (indirect) role in possible extensions of a coalition with neutral actors, we take Figure 1C. Here, Member 1 needs to exploit and initiate the link to Participant 2 and/or 7 to maintain its uncovered position: it would otherwise be covered by Member 4, who in that case only has to point to his/her link with Participant 3 to pressure Member 1. Moreover, some of the participants, from a structural point of view, will be eligible as potential entrants to the class of members (of course, after an unambiguous commitment is communicated). For example, the entrance of Participant 5 and the realization of its structural potential invokes a new partitioning of the set of all actors, thus changing the status of neutral Actor 6 to the status of participant in this new and stable coalition. Actor 2 may also join, resulting in the stable coalition depicted in Figure 1B. After that, no further extensions are possible.

Relevant terminology and the notion of covering are presented and elaborated in Section 2. In Section 3, we introduce structurally stable coalitions. Coalition dynamics and the limits on extensions of coalitions are studied in Section 4. Finally, Section 5 presents directions for further research.

#### 2. Terminology and Covering: A Technical Introduction

It is possible to distinguish several levels of analysis for positions within networks. First, one may study networks at the atomic level of individual actors. For example, the positioning of an actor in a network can be investigated by counting the number of its neighbors. A second level of analysis is the evaluation of the network at a more global level; this can be done by looking at degree distributions, connected components, or clusters of structurally equivalent actors. For a particular actor in a network, the notion of covering that we used here combines these two levels and captures the idea of being outperformed by another actor in terms of its local neighborhood structure. Being uncovered expresses the notions of visibility, status, centrality, and usefulness for a coalition. Our notion is characterized by a dichotomy of the set of all actors: uncovered or covered.

In order to give a precise definition, we need terminology: we considered a fixed and connected network *G*. A *network G* is a pair (*V*, *E*), where  $V = \{v_1, v_2, ..., v_n\}$  with  $n \ge 1$  is a *finite* set of *nodes* and *E* is a subset of  $\{\{x, y\} : x, y \in V, x \ne y\}$ , the set of *non-ordered* pairs of *V*. An element  $\{a, b\}$  of *E* is called a *link* between *a* and *b* and is written as (a, b), meaning that (a, b) = (b, a). If  $(a, b) \in E$ , then we call the nodes *a* and *b neighbors*. In Figure 1, for example, Nodes (representing actors) 1 and 4 are linked and act as neighbors. Nodes 1 and 3 are not neighbors, but are connected through a path via, for example Node 4. A *path*  $< c_0, c_1, ..., c_k >$  in *G* from node  $c_0$  to  $c_k$  is a set  $\{c_0, c_1, ..., c_k\} \subset V$ , such that  $(c_i, c_{i+1}) \in E$  for  $i \in \{0, ..., k-1\}$ . A *cycle* is a path  $< c_0, ..., c_k, c_0 >$  of at least  $k \ge 2$  distinct nodes. A network *G* is complete if for any pair of nodes  $a \ne b$ , we have  $(a, b) \in E$ . A network is *connected* if between any two nodes, there exists at least one path that connects them. We assumed our fixed network to be connected. Given a network G = (V, E) and subset *S* of *V*, we denote  $G_{|S} = (S, E_{|S})$ , which is the network restricted to the set *S*. Here,  $E_{|S} = E \cap (S \times S)$ .

We shall now present the formal definition of covering. This cover relation coincides with the cover relation proposed in Monsuur and Storcken [21] and Janssen and Monsuur [20], which in turn is a generalization of the cover relation defined for tournaments as introduced in Miller [22] and discussed in Laslier [23].

**Definition 1.** *Fix a network* G = (V, E)*, let a and b be nodes in* V*, a*  $\neq$  *b. Then, a covers b in* G = (V, E)*, or equivalently, b is covered by a in* G = (V, E)*, if:* 

- 1. For all nodes  $x \in V \setminus \{a, b\}, (b, x) \in E$  implies  $(a, x) \in E$ ;
- 2. At least one node  $c \in V \setminus \{a, b\}$  exists such that  $(a, c) \in E$ , while  $(b, c) \notin E$ .

Node *b* in *V* is called covered in G = (V, E) if for at least one node *a* in *V*,  $a \neq b$ , *b* is covered by *a* in *G*. It is called uncovered otherwise. The set of uncovered nodes is denoted uc(G) = uc(V, E).

A node *a* covers a node *b* if each neighbor of *b* (minus *a* if  $(a, b) \in E$ ) is also a neighbor of *a* and Node *a* has at least one extra neighbor different from *b*. Equivalently, the set of all neighbors of *b* (minus *a* if  $(a, b) \in E$ ) is a strict subset of the set of all neighbors of *a* (minus *b* if  $(a, b) \in E$ ). Note that we do not require that  $(a, b) \in E$ . In the fixed network of Figure 1A (dotted, as well as solid links), Node 1 does not cover Node 2 as Node 2 has a link to Node 3, which is not a neighbor of Node 1. At the same time, Node 1 covers Node 3, as all neighbors of Node 3 are also neighbors of 1, while Node 1 has an extra link to Node 7. We therefore argue that Node 3 is in a subdued position (cf. the economic case discussed in Section 1). Node 2 is not covered by any node, and therefore, it occupies a distinguishable position: compared to any node, it has an "escape" to a unique neighbor.

There are several properties of the covering relation. Apart from asymmetry (if *a* covers *b*, then *b* does not cover *a*), we have transitivity:

**Theorem 1** ([21]). *The cover relation is a transitive relation: if a covers b and b covers c, then a also covers c.* 

As the set of nodes is finite, Theorem 1 implies that for each *G*, the set  $uc(G) \neq \emptyset$ . Note that we may also use this theorem to conclude that if some *y* is covered, it is also covered by at least one  $x \in uc(G)$ . Next, we state that the uncovered nodes form a connected sub-network, so that pairs of uncovered nodes are directly or indirectly linked through other uncovered nodes.

**Theorem 2** ([21]). Let G = (V, E) be a connected network, with uncovered nodes uc(G). Then,  $G' = (uc(G), E_{|uc(G)})$  is a connected network.

We refer to [20,24] for a comprehensive analysis of the concept of covering in, for example, social networks. In these networks, an uncovered position captures the idea of a visible position. In this paper, it was used to "uncover" a primitive notion of not being subdued, i.e., being regarded equal in the coalition.

#### 3. Structural Stability of a Coalition

We shall now introduce the structural perspective on the stability of a coalition S = (M, P). Here, M is the set of members and P the set of participants. All other actors are neutral. The participants are positioned in between the coalition's members and neutral actors. The following definition formalizes our Assumption 1 of Section 1:

**Definition 2.** Let G = (V, E) be a fixed connected network. S = (M, P) with  $M \cup P \subset V$  is a coalition if, and only if:

- All  $p \in P$  have at least one link to an  $m \in M$ ;
- No  $n \in N = V \setminus (M \cup P)$  has a link to any  $m \in M$ ;
- The network  $(M, E_{|M})$  is connected.

Given a network G = (V, E), we then verify whether the distinction between members and participants of a particular coalition S = (M, P) will be confirmed and concurred on by a particular part of the underlying network, the so-called coalition-induced network. This induced network (V, E') is a sub-network of G and has vertex set V, while E' consists of links in E between members themselves and, in addition, links connecting members to participants (solid links in Figure 1, or stated equivalently: eliminate the dotted links). **Definition 3.** Given is a network G = (V, E) and a coalition S = (M, P). The coalitioninduced network is given by  $G^S = (V, \{(x, y) \in E : x \text{ and/or } y \in M\})$ . We also call it the *S*-induced network.

Here, the perspective is that the coalition considers the induced network relevant. The induced network counts, and its links are actively utilized in order to maintain an uncovered position. In this network, members have to be uncovered, while all other actors are covered. The following definition formalizes Assumptions 2 and 3 of Section 1:

**Definition 4.** Given is a network G = (V, E). A coalition S = (M, P) is (structural) stable if  $uc(G^S) = M$ .

The members therefore are regarded as equal in the coalition, enabling them to participate in the cooperative decision-making process. As in general,  $uc(M, E_{|M}) \neq M$ , participants may be needed to assist the justification of a member's claim of being uncovered (see the examples hereafter). At the same time, the collection of members is able to cover the participants. Using the transitivity of the cover relation, this implies that participants should be covered by members. In sum, to stand as a coalition, the induced network has to confirm the status of both members and participants. We note that in the definition of  $G^S$ , given a coalition S = (M, P), we may replace V with  $M \cup P$ .

To return to our network in Figure 1C, the coalition with Members 1 and 4 is structurally stable, as may be easily verified. As a pair, they are able to exclude all participants from membership. However, this does not mean that it is impossible to extend the set of members with, for example, Participant 2. Then, again, we have a stable coalition consisting of Actors 1, 2, and 4. Finally, Participant 5 may also become a member of this stable coalition, promoting Actor 6 from neutral to a participant in the coalition. After that, no stable extensions are possible.

In Figure 2A, the coalition  $S = (M, P) = (\{1, 2, 3, 4\}, \{5, 6, 7, 8\})$  is stable. All members are uncovered and occupy a unique position. Through a joint membership action, Participants 5, 6, 7, and 8 are covered by at least one member, subduing their positions and securing the stability of the coalition. In (B), we encounter a different situation. The coalition  $S = (\{1, 2, 3, 4\}, \{5\})$  is not stable: the participant is covered, but Member 1 covers Member 3, invalidating the credibility of the coalition. The coalition  $S = (\{1,2,3\},\{4,5\})$ in (C) is stable. To illustrate: Member 1 escapes from being covered by Member 2 due to the link with Participant 5, illustrating the relation between covering and related notions such as centers in a uni-dimensional policy space (cf. van Deemen [25]). In (D), the coalition  $S = (\{1, 2, 3, 4, 5\}, \emptyset)$  is not stable (Members 4 and 5 are covered by Member 2). In (E), we encounter another situation where the coalition  $S = (\{1, 2, 3, 4\}, \{5, 6\})$  is not stable: Participant 5 holds an uncovered position in the induced network. It would legitimize the claim of Actor 5 of being seen as a member of the coalition, being at the table to shape the agenda. This invalidates the stability of the coalition. Note that in (E), Participant 6 is covered by Members 1, 2, as well as 3 (the neutral Actors 7 and 8 are covered by all other actors, as they have no links in the induced network). These examples illustrate the role of participants in harming or providing the stability of a coalition S = (M, P). They are in-between members and neutral actors: they may assist in covering all neutral actors and other participants, and they may also assist in securing the uncovered status of each member.



**Figure 2.** For each subfigure (A–E), the link set of the fixed underlying network is given by the union of dotted and solid links. The solid links are the links of the induced network  $G^S$ . (Again, note that links between participants are not part of the induced network. The same holds for links to neutral actors.)

The next theorem informs us about the existence of stable coalitions for any given fixed network *G*. To this end, we present a specific stable coalition:

**Theorem 3.** Let G = (V, E) be a connected network. The coalition S = (M, P) with M = uc(G) and  $P = V \setminus uc(G)$  is stable.

**Proof.** Let M = uc(G). We shall first show that the network  $(uc(G), E_{|uc(G)})$  is connected. We may refer to Theorem 2, but for the sake of clarity, we shall give a direct proof here. Take any two actors, say *a* and *b* in *V*. As *G* is connected, there exists a path  $\langle a = c_0, c_1, \ldots, c_k = b \rangle$ . Let us suppose that  $c_1 \notin uc(G)$ . Then,  $c_1$  is covered by some actor, say  $d_1 \in uc(G)$ . This means that we have a path  $\langle a = c_0, d_1, c_2, \ldots, c_k = b \rangle$ . Continuing this way, we construct a path from *a* to *b* with all intermediate actors in uc(G), proving the connectedness. This also proves that any  $v \in V$  has at least one link with some member  $m \in M$ , from which we may deduce that  $M \cup P = V$  and  $N = \emptyset$ . In the coalition-induced network, which differs from *G* only in the absence of links between participants, we clearly have  $uc(V, \{(x, y) \in E : x \text{ and }/ or y \in M\}) = M$ .  $\Box$ 

Besides coalitions of the type of Theorem 3, there may be other stable coalitions. For example, in Figure 2B, besides  $(M, P) = (\{1, 2, 4\}, \{3, 5\})$ , the coalition of Theorem 3, we also have a stable coalition with members  $M = \{1, 2\}$  and participants  $P = \{3, 4, 5\}$ . In (E) of the same figure, besides  $S = (M, P) = (\{1, 2, 3, 4, 5, 6, 7\}, \{8\})$ , we may take  $M = \{3, 4\}$  and  $P = \{1, 2, 5\}$ . (For the sake of completeness, we mention that in (A), we have  $uc(G) = \{1, 2, 3, 4\}$ , in (B), we have  $uc(G) = \{1, 2, 3, 4\}$ , in (C) and (D)  $uc(G) = \{1, 2, 3\}$ , and finally,  $uc(G) = \{1, 2, 3, 4, 5, 6, 7\}$  in (E).)

We further remark that  $\lim_{n\to\infty} f(n, p) = 1$ , where f(n, p) is the probability that for a random network of *n* nodes and link probability p ( $0 \le p \le 1$ ), we have uc(G) = V; see [20]. Therefore, with probability one, for large random networks, we have that  $(M, P) = (V, \emptyset)$  is a stable coalition. In Section 4, we use Theorem 6 to show that for cases with uc(G) = V, it is impossible to extend stable coalitions to V without sacrificing stability. Note also that the coalition with M = uc(V, E) may not be the only stable coalition with  $M \cup P = V$ .

In the following section, we illustrate the use of our stability concept in the coalitionformation process.

#### 4. Coalition Dynamics on a Fixed Network Structure

As mentioned before, links indicate possibilities of communication and cooperation between pairs of actors. This interpretation also holds in the trading example, where a link indicates that the volume of trade between partners exceeds a certain threshold value. This section is not meant to predict or prescribe a coalition-formation process. Instead, starting with any initial stable coalition of willing actors, we map out the possible extensions of this coalition: which of the participants may join, without sacrificing stability. (Note that, due to the absence of links between members and neutral actors, neutral actors are not in a position to directly join a coalition). Although actors may be intrinsically motivated to join coalitions, only moves between stable coalitions will be accepted. For example, a participant p may want to join the members of a coalition S, as there is a link to at least one member. This means that there exists at least one member with shared opinions, or at least one trading partner. There may be two reasons why the new coalition, with M extended with  $p \in P$  and  $P \setminus p$  extended with neutral actors linked to p, is not stable. Firstly, one actor in  $M \cup p$  may be covered in the new induced network and is therefore eclipsed by another member from the coalition. The new coalition cannot be termed a stable coalition and is not suited for pursuing its interest. We show below that in this case, the covered member has to be p. Secondly, one of the (new) participants in the extended coalition may become uncovered in the (new) induced network. In that case, it would legitimize a claim of being seen as a member of the coalition and being at the table. This invalidates the stability of the coalition with  $M \cup p$ . We therefore consider two "credible moves" in a coalition-formation process that do not sacrifice stability (see Definitions 5 and 6).

#### 4.1. Credible Moves

Given a network G = (V, E), we start with an initial coalition S = (M, P) (of "willing actors") that is stable. This means that only members of *S* are uncovered in the *S*-induced network. Given  $p \in P$ , let  $N(p) \subset N$  be the neutral actors linked to *p*. Of course, this set may be empty. We introduce two credible moves, based on the promotion of participant to member, that also take into account the promotion of some neutrals to participant.

**Definition 5** (Credible moves, Part 1). *Given is a fixed network structure G. We consider the following credible move for a coalition* S = (M, P): *(addition heuristic) the entrance of a participant*  $p \in P$  *into* M, *if both* S = (M, P) *and* S' = (M', P') *with*  $M' = M \cup \{p\}$  *and*  $P' = (P \setminus \{p\}) \cup N(p)$  *are stable in their respective induced networks*  $G^S$  *and*  $G^{S'}$ .

To give an example, consider Figure 3. Here, solid links form the coalition-induced network, while dotted links are links of the fixed network between two participants, between two neutrals, or between participants and neutrals.



**Figure 3.** Starting with a stable coalition  $S = (\{6,7,8,9\}, \{1,4,5\})$  in (**A**), it may be (in just one way) sequentially extended via (**B**) with two actors, giving the coalition  $S = (\{1,2,6,7,8,9\}, \{3,4,5\})$  in (**C**). After that, no new entrance is possible.

In Network (A), the coalition with members  $S = (\{6,7,8,9\}, \{1,4,5\})$  is stable: if we consider the uncovered actors in the induced network, only members are uncovered. Indeed, Participant 1 is covered by Member 9; 4 is covered by Members 1, 7, and 9; 5 is covered by Members 6 and 8. (As in the coalition-induced network, the neutral Actors 2 and 3 have no links at all, and the neutrals are automatically covered.) It is easy to verify that the members are uncovered. If it happens that Participant 1 is allowed to join the members, then (see Network (B)) also, the link between the new Member 1 and the new Participant 2 signals possible cooperation from the point of view of the extended coalition and is included in the induced network. This new coalition is also stable, where, for example, the new Participant 2 is covered by Member 1, who itself has become uncovered. Given this new coalition, now 2 can join. However, it is subsequently impossible for 3 to join in turn, as in that case, in (C), also the link between Actors 3 and 4 would be included in the induced network. However, then, Participant 4, who was previously covered by Members 1, 7, and 9, also becomes uncovered. This implies that the uncovered set is larger than the set of members  $\{1, 2, 3, 6, 7, 8, 9\}$ , contradicting stability. Altogether, this example shows that the participants, apart from their influence on the stability of a given coalition, also influence the coalition-formation process, as they may invalidate or support specific coalitions. As may be easily verified, if for a participant p, the set  $N(p) \neq \emptyset$ , extending the induced network of S' compared to that of S, the enlarged coalition is again stable. Therefore, if such a participant "takes with him/her" a neutral actor as a new participant, it is considered a credible move. Note that the opposite is not necessarily true. These examples can serve to illustrate the following theorem:

**Theorem 4.** Suppose that the entrance of a participant  $p \in P$  into M of a stable coalition S = (M, P) gives an unstable coalition S' = (M', P') where  $M' = M \cup \{p\}$  and  $P' = (P \setminus \{p\}) \cup N(p)$ . Then, either p is covered or some other participant in  $P \setminus p$  becomes uncovered.

**Proof.** Given an actor *i* and coalition *S*, we define the *S*-local neighborhood as that part of the *S*-induced network that consists of links (i, a) for arbitrary *a*. First, note that the *S*-local neighborhood structure of a member of *S* is the same as the *S'*-local neighborhood in case *p* is promoted to a member in *S'*. Therefore, no member  $m \in M$  covers another member of *S* in the *S'*-induced network. It is also impossible that *p* covers some member  $m \in M$  in the *S'*-induced network. Indeed, if *p* were to cover some member in *M* in the *S'*-induced network, this would contradict the fact that *p* is covered by some member  $m \in M$  in the *S*-induced network. Another possibility of *S'* being unstable is that a participant becomes uncovered. Evidently, this is some actor in  $P \setminus p$ .  $\Box$ 

Suppose that it is impossible for a participant p to join. It then is conceivable that it may join at a later stage in the formation process. This is due to the fact that the *S*-local neighborhood structure (as defined in the proof of Theorem 4) may change as other participants join. Therefore, in addition to the move, we may also consider the swap to be heuristic, which provides an opportunity to pull off the coalition-formation process in certain specific situations. Before illustrating this move with Figure 3, we shall give its definition:

**Definition 6** (Credible moves, Part 2). *Given is a connected network structure G. We consider the following credible move for a coalition* S = (M, P): *(swap heuristic) the swap between member m and participant p, if both* S = (M, P) *and* S' = (M', P') *with*  $M' = (M \setminus \{m\}) \cup \{p\}$  *and* P' *consisting of non-members linked to at least one member of*  $M \setminus \{m\} \cup \{p\}$ *, is stable in their respective coalition-induced networks.* 

Let us take the coalition *S* with  $M = \{1, 6, 7, 8, 9\}$  in Figure 3B. Participant 4 cannot join this stable coalition without harming the stability; apparently, actors do care who is a member of a coalition. Suppose that Member 6 is prepared to leave the set of members, downplaying himself/herself as a participant and thus enabling Participant 4 to enter. The coalition with members  $M = \{1, 4, 7, 8, 9\}$  is stable. The swap heuristic includes situations where members leave the coalition to pave the path for others to join. After other actors have joined, they may possibly rejoin later on. This illustrates the phenomenon that two actors may be together in the set of members and active in the agenda-setting activity at the same time only if a certain third actor also is present as a member. We refer to Nikookar and Monsuur [26] for an application of the addition and swap heuristic in cooperative wireless sensor networks and to Apt and Witzel [27] for a more general approach.

#### 4.2. End-State Coalitions

To map out all possible moves starting at any stable coalition for a fixed network G, we introduce a directed (meta-)network on the set of stable coalitions, where we take into account the partition into M and P. We let  $G^*$  be the set of all stable coalitions of G = (V, E).

**Definition 7.** Let G = (V, E). Then,  $C_G$  is defined as the directed network  $C_G = (G^*, C)$  with  $C = \{((S_1, M_1), (S_2, M_2)) : (S_1, M_1), (S_2, M_2) \in G^* \text{ and the move from } (S_1, M_1) \text{ to } (S_2, M_2) \text{ is credible} \}.$ 

The arrows in Figure 3 are a subset of this relation *C*. Note that it may be possible that  $((S_1, M_1), (S_2, M_2)) \in C$ , as well as  $((S_2, M_2), (S_1, M_1)) \in C$  in case the move is a swap. To define end-state coalitions, we introduce three notions for arbitrary relations (V, R): two derived relations and a subset of *V*. The two relations are the transitive closure and asymmetric part; the subset is the set of bottom elements. The transitive closure of an arbitrary relation (V, R) is given by  $(V, \tau R)$  with  $\tau R = \{(x, y): a \text{ path exists from } x \text{ to } y \text{ in } R\}$ . For example, in Figure 3,  $(A, C) \in (G^*, \tau C)$ . The asymmetric part of an arbitrary relation is given by (V, aR) with  $aR = \{(x, y) \in R : (y, x) \notin R\}$ . Finally, the set of bottom elements is  $End(V, R) = \{x \in V : \text{ there is no } y \text{ such that } (x, y) \in R\}$ . We now can state our definition:

**Definition 8.** Let a connected network G be given. A coalition  $S = (M, P) \in G^*$  is an end-state coalition if  $S = (M, P) \in End(G^*, a(\tau C))$ . Coalitions in  $End(G^*, a(\tau C))$  are denoted by  $\overline{S}$ 

To explain this definition, it has to be noted that  $\tau C$  reveals all paths between coalitions if one uses credible moves. Now, it may occur that there is a cycle of credible moves  $((S_0, M_0), (S_1, M_1)), \ldots, ((S_{k-1}, M_{k-1}), (S_k, M_k)), ((S_k, M_k), (S_0, M_0)) \in C$ , meaning that  $\tau C|_{\{(S_0, M_0), \ldots, (S_k, M_k)\}}$  is a complete sub-network. We neutralize these complete sub-networks by considering  $a(\tau C)$ , which retains only paths in *C* without cycles. This opens the way to identify end states. These are coalitions that cannot be extended in terms of their size using credible moves starting at that coalition. Of course, to make this possible nonetheless, we would have to introduce new types of credible moves, but we shall stick to the two most intuitive ones discussed here.

The next theorem shows that in the limit, we may observe end-state-stable coalitions for which it holds that the percentage of actors being in the coalition goes to zero.

**Theorem 5.** There exist  $\overline{S} \in End(G^*, a(\tau C))$  such that  $\lim_{V \to \infty} \frac{|\overline{S}|}{|V|} = 0$ .

**Proof.** Let us take the network (V, E) with  $V = \{a_1, a_2, \ldots, a_6, b_1, b_2, c_1\}$  and consisting of the cycle  $L_1 = \langle a_1, a_2, a_3, a_4, a_5, a_6, a_1 \rangle$  and the four links  $(a_6, b_1), (a_2, b_2), (b_1, b_2), (b_1, c_1)$ . Then, the coalition  $S = (\{a_1, a_2, \ldots, a_6\}, \{b, b_2\})$  is stable, where only the links  $(b_1, b_2)$  and  $(b_1, c_1)$  are not in the induced network. If we let actor  $b_1$  join the members, the induced network is extended with the link between  $b_1$  and  $b_2$  and between  $b_1$  and  $c_1$ , letting  $c_1$  become a participant. In this network,  $b_1$  and also participant  $b_2$  become uncovered. The same situation occurs if we let actor  $b_2$  join the members. It is easy to verify that here, too, no swap is possible. Adding an arbitrary number of neighbors to  $c_1$  proves our claim.  $\Box$ 

Finally, it has to be noted that in Figure 3, the coalition consisting of Members 1,...,9 is stable. However, as may be easily verified, stability cannot be reached starting with a smaller set of members. This observation is true in general:

**Theorem 6.** Let us now consider a connected network G = (V, E) where  $S = (V, \emptyset)$  is a stable coalition. Then, there is no path in  $(G^*, C)$  to S that starts at some S' = (M', P') with  $M' \subset V$ .

**Proof.** Suppose, in contrast, that such a path would exist in  $(G^*, C)$  to  $S = (V, \emptyset)$ . Then, also,  $S' = (V \setminus i, \{i\})$  would be stable for some *i*. Due to the connectedness of *G*, *i* is a participant, implying the existence of links between *S'* and *i*. Therefore, the *S'*-induced network is equal to *G*. The stability of *S'* implies that  $V \setminus i = uc(V, \{(x, y) \in E : x \text{ and/or } y \in V \setminus i\})$ . However,  $(V, \{(x, y) \in E : x \text{ and/or } y \in V \setminus i\}) = (V, E)$  and  $(V, \emptyset)$  is stable (so uc(V, E) = V), and we obtain  $V \setminus i = V$ , a contradiction.  $\Box$ 

#### 5. Conclusions and Directions for Further Research

The perspective on stability that we introduced in this paper highlights the roles of members and participants. Only after the continuing action of members to maintain this stability will pursuing the interests of the coalition become a fruitful business. Here, members in a coalition consider mutual interaction, but they will also take into account links to participants at the border of the coalition. Starting with an initially stable coalition of willing actors, participants may try to become actively involved in the agenda-setting of the coalition, changing their status from participant to member and possibly extending the coalition with neutral actors. The precise sequence of inviting participants to become members may be important, resulting in different coalitions that reflect many myopic real-life situations. We derived existence results for stable coalitions and determined the limits of possible extensions. In future investigations, we may also consider variants of our stability criterion in Definition 3, by changing "and/or" to "and" in the definition of  $G^{5}$ . Then, instead of focusing on the stability with an eye on extension with neutrals becoming participants, we could also consider (internal) stability for members only, which may describe another credible world. This line of investigation would probably result in a different assessment regarding the stability of a given coalition.

A tacit assumption used in the current study is that coalition formation takes place in a networked environment. This means that empirical research will have to identify the underlying fixed network connecting the actors. Only after this is done will it be possible to examine the assumptions listed in Section 1. We presented a formal model and argued that each of our three assumptions is credible. Nevertheless, these assumptions have not been verified using actual data, showing the limitations of our study. Only further research can reveal to what extent our three formal assumptions may serve as a platform for discussions regarding the stability of coalitions.

**Author Contributions:** Conceptualization, H.M. and R.H.P.J.; methodology, H.M. and R.H.P.J.; writing—original draft preparation, H.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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