

Marín-Sánchez, Freddy H.; Pareja-Vasseur, Julián A.; Manzur, Diego

Article

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Quadrinomial trees with stochastic volatility to value real options

Freddy H. Marín-Sánchez

Mathematical Sciences, EAFIT University, Medellín, Colombia

Julián A. Pareja-Vasseur

Finance, EAFIT University, Medellín, Colombia, and

Diego Manzur

Economics and Finance, EAFIT University, Medellín, Colombia

Abstract

Purpose – The purpose of this article is to propose a detailed methodology to estimate, model and incorporate the non-constant volatility onto a numerical tree scheme, to evaluate a real option, using a quadrinomial multiplicative recombination.

Design/methodology/approach – This article uses the multiplicative quadrinomial tree numerical method with non-constant volatility, based on stochastic differential equations of the GARCH-diffusion type to value real options when the volatility is stochastic.

Findings – Findings showed that in the proposed method with volatility tends to zero, the multiplicative binomial traditional method is a particular case, and results are comparable between these methodologies, as well as to the exact solution offered by the Black–Scholes model.

Originality/value – The originality of this paper lies in try to model the implicit (conditional) market volatility to assess, based on that, a real option using a quadrinomial tree, including into this valuation the stochastic volatility of the underlying asset. The main contribution is the formal derivation of a risk-neutral valuation as well as the market risk premium associated with volatility, verifying this condition via numerical test on simulated and real data, showing that our proposal is consistent with Black and Scholes formula and multiplicative binomial trees method.

Keywords Real options, Stochastic volatility, Diffusion processes, GARCH-diffusion, Quadrinomial numerical method

Paper type Research paper

1. Introduction

Authors such as Trigeorgis (1996), Mun (2002) and Damodaran (2019) recognize there is a difference between the theoretical concepts of valuation of capital assets and their empirical approximation, a discrepancy that is reflected in a gap not deeply explained and with sufficient academic rigour between the market value and its theoretical estimate. Therefore, different methodological alternatives have been proposed to mitigate this difference, but few consider key factors that generate value, such as those that try to measure and estimate uncertainty through the use of volatility, which usually has been moderately studied and ends up being relevant at the moment of making any type of valuation on a capital asset (Dixit and Pindyck, 1994; Valencia Herrera and Martínez Gándara, 2009; Álvarez Echeverría *et al.*, 2012).

In academia, the use of discounted cash flow (DCF) as a method used to value capital assets is common, but there are numerous criticisms about its use because it does not include



elements such as contingent events, risk present in the cash flows and volatility (Trigeorgis, 1996). To overcome some of these problems, the real options approach (ROA) was developed, which complements the DCF and allows it to include volatility as a fundamental parameter to quantify the risk and collect some elements associated with uncertainty (Keswani and Shackleton, 2006; Sabet and Heaney, 2017). The ROA model was derived from its simile in the theory of financial options by modelling the value of assets as a call or put options, considering that it is applied to investments in assets or real markets, although there is not a tradable market (Cobb and Charnes, 2004).

Since the Black and Scholes's (1973) seminal work, which is considered generally used as a method for estimating the value of European call and put options, multiple models and extensions have appeared that allow the valuation of other types of options in different contexts, using of different estimation techniques, such as closed analytical solutions, finite difference method, Adomian decomposition method and numerical method through multiplicative and additive binomial trees, but the problem of considering volatility constant persists. Recently, some models have appeared to include stochastic volatility and were created, fundamentally, to avoid considering this fixed variable in terms of evaluation time horizon. Motivated by this empirical evidence, several authors, such as Hull and White (1987), Scott (1987), Scott (1987), Chesney and Scott (1989), Stein and Stein (1991) and Heston (1993), have proposed models that involve stochastic volatility as a parsimonious extension of the Black–Scholes model (Black and Scholes, 1973). Amongst previously mentioned models, the GARCH-diffusion type proposed by Drost and Werker (1996), Duan (1997) and Duan (1996) offers the first approximation between a GARCH process and a stochastic volatility model, supported by multiple research theoretical (Ritchken and Trevor, 1999; Barone-Adesi *et al.*, 2005; Christoffersen *et al.*, 2010; Chourdakis and Dotsis, 2011) as well as empirical (Figà-Talamanca, 2009; Plienpanich *et al.*, 2009; Wu *et al.*, 2012, 2014, 2018). This paper proposes and motivates the inclusion of stochastic volatility in the ROA model, considering a rigorous mathematical development from a GARCH-diffusion stochastic differential equation (SDE) that has a numerical solution using multiplicative trees; besides, a prior estimate of a GARCH (1,1) process is required to allow an adequate real options valuation in real-world and risk-neutral situations.

This paper is organized as follows: Section 2 summarizes the stochastic volatility models and describes the GARCH-diffusion model used for the development of this research. Section 3 presents, in summary, the multiplicative quadrinomial tree method to assess real options. Section 4 offers a brief description of real options valuation, specifically, its use according to the method proposed. In Section 5, we present a series of numerical experiments and some examples. Finally, Section 6 is the discussion and conclusions.

2. Literature review

One of the strongest assumptions considered in the Black–Scholes model (Black and Scholes, 1973) is that volatility (σ) is constant; many studies have demonstrated that logarithm of returns on asset prices have leptokurtosis and conditional variance that changes randomly as a function of time (Grajales Correa and Pérez Ramírez, 2008) and the assumption of conceiving a normal distribution does not apply (Fernández Castaño, 2007), additionally, implied volatilities are considered non-constant and differ between exercise prices and time to maturity. For these reasons, several extensions have been proposed in the literature in which volatility is considered a function of time as well as the price of the underlying asset, whereupon the linearity of drift and diffusion components of the asset price are maintained but incorporate a second equation that allows modelling the variance behaviour of S_t .

2.1 Stochastic volatility models

A model with stochastic volatility describes its change over time and generalizes the Black–Scholes model, defined in a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, which generally satisfies a system of SDEs (Hull and White, 1987; Scott, 1987; Wiggins, 1987; Chesney and Scott, 1989; Nelson, 1990a, b; Stein and Stein, 1991; Heston, 1993; Hilliard and Schwartz, 1996; Drost and Werker, 1996; Duan, 1996, 1997; Wilmott, 1998; Ritchken and Trevor, 1999; Barone-Adesi *et al.*, 2005; Chang and Fu, 2009; Figà-Talamanca, 2009; Moretto *et al.*, 2010; Christoffersen *et al.*, 2010; Chourdakis and Dotsis, 2011; Wu *et al.*, 2012; Wu *et al.*, 2014; Wu and Zhou, 2016; Peng and Peng, 2016; Wu *et al.*, 2018; Wu *et al.*, 2020, see Table 1). We considered an SDE system described as follows:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^{(1)} \\ d\sigma_t &= f(S_t, \sigma_t, t)dt + g(S_t, \sigma_t, t)dW_t^{(2)} \end{aligned} \quad (1)$$

where μ is constant and volatility σ_t is considered as a dynamic variable in the price S_t . Studies like Wu *et al.* (2018) showed that the implied volatility, captured from historical data from the market, takes a relevant role in fitting the prices of the options as Kim and Ryu (2015) explained as well. Functions f and g correspond to the tendency and diffusion of volatility, respectively. The model incorporates two sources of randomness $W_t^{(1)}$ and $W_t^{(2)}$ that correspond to Wiener's standard processes with a correlation coefficient ρ . The price process $\{S_t, 0 \leq t \leq T\}$ is not completely described by equation (1), and the value S_t will be conditioned to the information of S_0, σ_0 and to the trajectory followed by volatility $\{\sigma_s, 0 \leq s \leq t\}$.

The GARCH-diffusion model was introduced by Wong in 1964, but its popularity only grew following the works of Nelson (1990a, b). An important condition was discovered by Christoffersen *et al.* (2010) who demonstrated empirically, through the use of realized volatilities on the S&P500 returns with an option's data panel, that the Heston model was poorly specified because, in the diffusion model presented by the author, volatility was found in the square root instead of being considered linear; these conclusions were reaffirmed by Chourdakis and Dotsis (2011) although they also suggested that the model should consider a nonlinear drift against a linear one.

Recent studies have indicated that this model allows a better description of the behaviour and dynamics of financial series than other types of models, such as Heston (1993), Ait-Sahalia and Kimmel (2007), Jones (2003) and Wu *et al.* (2018).

Also, it has been used as a good model for adjusting financial option data (Christoffersen *et al.*, 2010; Chourdakis and Dotsis, 2011; Kaeck and Alexander, 2012; Wu *et al.*, 2012, 2014, 2018, 2020). The most recent applied research related to this model is summarized in Table 2.

In general terms, this type of model is usually characterized by not having a closed solution and is in the class of non-affine models; also, their solutions must be achieved with simulation, numerical methods (Wu *et al.*, 2012) or the use of integrals in SDEs (Barone-Adesi *et al.*, 2005; Florescu and Viens, 2008; Vellekoop and Nieuwenhuis, 2009). The system of traditional equations presented by this model has the following functional structure (Barone-Adesi *et al.*, 2005):

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \\ dV_t &= (c_1 - c_2 V_t)dt + c_3 V_t dW_t^{(2)} \end{aligned} \quad (2)$$

where c_1, c_2 and c_3 are positive constants; μ is the tendency parameter; c_2 is the mean-reversion rate; $(c_1)/(c_2)$ is the mean long-term variance and c_3 models the random behaviour of volatility and corresponds usually to the volatility of volatility. For its part, $W_t^{(1)}$ and $W_t^{(2)}$ correspond to Wiener processes with a correlation coefficient ρ .

Method proposed	Author(s)	Comments
$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}$ $dV_t = \delta V_t dt + \xi V_t dW_t^{(2)}$	Hull and White (1987)	The major contribution was that these parameters δ and ξ were declared and considered constant An analytical solution was also presented to assess a European call or put options The authors concluded that Black–Scholes model generally overestimates the option value, and the degree of this overvaluation increases with time at expiration
$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}$ $d\sigma_t = \beta(\sigma^* - \sigma_t)dt + \gamma dW_t^{(2)}$	Scott (1987)	As a novelty, σ_t which is the instantaneous volatility of the price asset, follows a mean-reversion process, the Ornstein–Uhlenbeck type It was concluded that it was not possible to find an analytical solution of the proposed SDE, but that its results were satisfactory when the estimated prices of options were contrasted by the Monte Carlo simulation method
$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}$ $d\sigma_t = f(\sigma_t)dt + \theta \sigma_t dW_t^{(2)}$	Wiggins (1987)	The author decided to use $f(\sigma_t)$ as a function that describes the volatility tendency σ_t The results showed that there are not many differences between the valuation of his proposal and the one obtained through the Black–Scholes model
$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}$ $d\sigma_t = \sigma_t \left[\frac{1}{2} \gamma^2 + \beta(\bar{a} - \ln \sigma_t) \right] dt$ $+ \gamma \sigma_t dW_t^{(2)}$	Chesney and Scott (1989)	In this paper, it was possible to demonstrate that the wealth variable can be eliminated if the utility function includes constant relative risk aversion (CRRA), and the claim payments do not depend directly on wealth
$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}$ $d\sigma_t = -\delta(\sigma_t - \theta)dt + \kappa dW_t^{(2)}$	Stein and Stein (1991)	In this case, they used analytical techniques to derive a closed explicit solution when volatility follows an Ornstein–Uhlenbeck process (or AR1 process), with a reversion tendency to the long-term mean θ
$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}$ $dv_t = \kappa(\theta - v_t)dt + \sigma \sqrt{v_t} dW_t^{(2)}$	Heston (1993)	The author used a technique to derive a closed-form solution for pricing a European call option on an asset with stochastic volatility. The technique is based on characteristic functions
$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)}$ $dV_t = (c_1 - c_2 V_t)dt + c_3 V_t dW_t^{(2)}$	Nelson (1990a); Nelson (1990b)	The first approach between a GARCH process and a stochastic volatility model (Duan, 1997), (Duan, 1996), (Drost and Werker, 1996). This model is known as GARCH-diffusion An algorithm was used to evaluate options, both European and American, based on the construction of trinomial trees (Ritchken and Trevor, 1999) Barone-Adesi <i>et al.</i> (2005) analytically derived the first four moments of the model and obtained a closed solution to easily value an option
Source(s): Own elaboration		

Table 1.
Summary of the SDE
systems with
stochastic volatility

An alternative way to present the structural form of equation (2) is an SDE; following (Wu *et al.*, 2012):

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \quad (3)$$

Table 2.
Summary of the
empirical or applied
research of GARCH-
diffusion model

Author	Main conclusions
Figà-Talamanca (2009) Plienpanich <i>et al.</i> (2009)	Use multiple stock indexes to compare theoretical and empirical autocovariance. The conclusion was that this model captures autocovariance observed in data Integrated into the diffusion model a disturbance through a fractional noise, their results showed that the estimation of stock price of a commercial bank is better using this model than the traditional Black–Scholes model
Wu <i>et al.</i> (2012) Wu <i>et al.</i> (2014)	Used the Hang Seng index (HSI) and concluded that using this model was better for predicting the price of warrants than the classic model Studied Hong Kong stock market through American options and found the same advantage using this model against the traditional one
Wu and Zhou (2016) Wu <i>et al.</i> (2018) Wu <i>et al.</i> (2020)	Used the Chinese volatility index (iVIX); their findings indicated that the risk of volatility market values and risk premium volatility were negative, which implied that investors in the Shanghai stock exchange are risk averse They proposed an estimation procedure that uses joint data on the underlying asset and options prices. They used HSI data and index warrant prices. Their findings indicated that market prices of the volatility risk play an important role in fitting option prices They extended the realized EGARCH-MIDAS (REGARCH-MIDAS) model, which incorporates the implied volatility (IV) derived from option prices. Through an empirical study with the S&P 500 index, they found that IV contains valuable information for predicting volatility
Source(s): Own elaboration	

$$dV_t = \alpha(\theta - V_t)dt + \sigma V_t dW_t^{(2)} \tag{4}$$

where parameters α , θ and σ are constant and are equivalent to the mean-reversion speed, the mean long-term volatility or tendency and the volatility of volatility, respectively. Consequently, $W_t^{(1)}$ and $W_t^{(2)}$ correspond to independent one-dimensional standard Wiener motion processes.

This paper focuses on using [equations \(3\) and \(4\)](#) to summarize the numerical method for a multiplicative quadrinomial tree that includes stochastic volatility, which will allow the valuation of a derivative instrument, such as real options, when the volatility of the underlying asset is stochastic. Hence, we use the equivalence proposed by [Hull \(2003\)](#) and described by others ([Pareja-Vasseur *et al.*, 2020](#)) that relates the conditional variance from GARCH (1,1) and the stochastic variance from differential equation proposed in [\(4\)](#), i.e. V_t is equivalent to h_t^2 , with $\alpha = \frac{(1-\alpha_1-\beta_1)}{\Delta t}$, $\theta = \frac{\alpha_0}{1-\alpha_1-\beta_1}$ and $\sigma = \frac{\alpha_1\sqrt{2}}{\sqrt{\Delta t}}$. These set of parameters of the conditional variance could be estimated by the maximum likelihood method, through computational tools such as EvIEWS or MatLab, previously checking the assumption of conditional heteroskedasticity ([Bollerslev, 1986](#)).

2.2 Quadrinomial recombination

The multiplicative quadrinomial trees methodology is summarized below, including all formulas for probabilities and transitions factors, and discounted rates. Considering the GARCH-diffusion model, over the time interval $[t_i, t_k]$, where $\mu \in \mathbb{R}$, $\alpha \geq 0$, $\theta > 0$ and $\sigma > 0$ are constant, while $\{W_t^{(1)}\}_{t \geq 0}$ and $\{W_t^{(2)}\}_{t \geq 0}$ are independent one-dimensional standard Wiener motions, supposing further that $S_t = S_i$ and $V_t = V_i$, probabilities of the transition and growth factors for the processes V_t and S_t , respectively, are defined as [Pareja-Vasseur and Marín-Sánchez \(2019\)](#) and [Marín Sánchez \(2010\)](#):

$$u_j^{(i)} = \exp\left(\sigma\sqrt{\Delta t}\right), d_j^{(i)} = \frac{1}{u_j^{(i)}}, A_{jV}^{(i)} = \exp\left(\alpha\left(\frac{\theta}{V_j^{(i)}} - 1\right)\Delta t\right) \text{ and}$$

$$p_j^{(i)} = \frac{1}{2} + \frac{\left(\frac{\alpha\theta}{V_j^{(i)}} - \alpha - \frac{1}{2}\sigma^2\right)\sqrt{\Delta t}}{2\sigma}$$

$$h_j^{(i)} = \exp\left(\sqrt{V_j^{(i)}}\sqrt{\Delta t}\right), l_j^{(i)} = \frac{1}{h_j^{(i)}}, A_S = \exp(\mu\Delta t) \text{ and } q_j^{(i)} = \frac{1}{2} + \frac{\left(\mu - \frac{V_j^{(i)}}{2}\right)\sqrt{\Delta t}}{2\sqrt{V_j^{(i)}}}$$

To consider the recombination of two binomial trees $S^{(i)} = (S_j^{(i)})$, $j \in J^{(i)}$ and $V^{(i)} = (V_k^{(i)})$, $k \in K^{(i)}$ which possess the same number of nodes along their time axes, that is, $J^{(i)} = K^{(i)}$ for all i . Denote the transition probabilities of $S^{(i)}$ by $q_j^{(i)}$ with increases up and down defined by $h_j^{(i)}$ and $l_j^{(i)}$, and for $V^{(i)}$ by $p_k^{(i)}$, $u_k^{(i)}$ and $d_k^{(i)}$, respectively. The values of the two variables are being modelled as a direct product $S^{(i)} \times V^{(i)}$, defined by a tree $T^{(i)}$ with a node described by $T_{j,k}^{(i)} = (S_j^{(i)}, V_k^{(i)})$ at the time i . In the next step, $T_{j,k}^{(i)}$ generates four nodes: $T_{j+1,k+1}^{(i+1)} = (S_j^{(i)}h_j^{(i)}, V_k^{(i)}u_k^{(i)})$, $T_{j,k+1}^{(i+1)} = (S_j^{(i)}l_j^{(i)}, V_k^{(i)}u_k^{(i)})$, $T_{j+1,k}^{(i+1)} = (S_j^{(i)}h_j^{(i)}, V_k^{(i)}d_k^{(i)})$ and $T_{j,k}^{(i+1)} = (S_j^{(i)}l_j^{(i)}, V_k^{(i)}d_k^{(i)})$, and respective probabilities are $q_j^{(i)}p_k^{(i)}$, $(1-q_j^{(i)})p_k^{(i)}$, $q_j^{(i)}(1-p_k^{(i)})$ and $(1-q_j^{(i)})(1-p_k^{(i)})$, respectively (Lari-Lavassani *et al.*, 2001) (Pareja-Vasseur and Marín-Sánchez, 2019).

After all the corresponding parameters have been defined for V_t and for S_t , it is possible to construct a tree that emulates the behaviour of the price S_t , where each discrete step has a value of branches equal to 4^n , where n corresponds to the number of steps, but when the respective recombination is performed, the number of branches decreases to n^2 , as can be seen in Figure 1. Also, the value of each position for the first step of the tree is presented in Figure 1 (Pareja-Vasseur and Marín-Sánchez, 2019).

3. Method

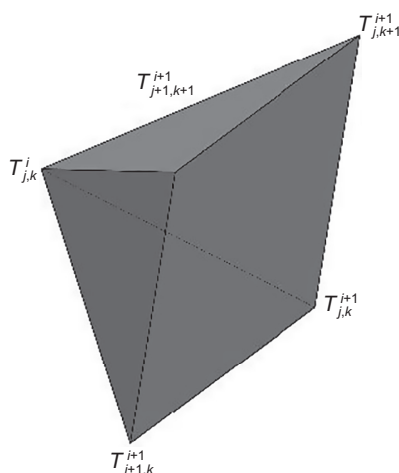
It is common knowledge that an option is considered a financial instrument that grants a right to the buyer and an obligation to the seller, to buy or sell a certain asset or underlying asset at a price established on a fixed date; the exchange price that is obtained for acquiring the option is known as premium. Amongst the methods used in its valuation, the multiplicative binomial method is the most often used and appropriate because it is easy to understand and construct (Cox *et al.*, 1979). After all, it allows us to emulate the price of the underlying asset in discrete time and offer the respective valuation of the derivative.

3.1 Real options valuation

One of the recent development techniques for valuing capital assets is known as ROA; according to this method, which is complementary to the DCF model, the aim is to introduce the volatility present in cash flows and the occurrence of contingent events. The method emerges from the theory of financial options, except the valuation is performed for capital assets in real markets. This type of options, such as financial options, can be assessed using different techniques, where the most appropriate is the numerical method with multiplicative binomial trees (Cox *et al.*, 1979), because of its simple and intuitive handling, in the sense that

Figure 1.

The first and second step of the proposed multiplicative quadrinomial tree model to stochastic asset price and volatility



Note(s): The figure was made with Solid Edge

Source(s): Own elaboration

continuous-time price of an underlying asset approximates to a geometric Brownian motion (GBM) (Smith, 2005; Siña and Guzmán, 2019), emulated through a process in discrete time organized as a tree, where it is possible to analyse, in graphical and numerical forms, the anticipated execution (or not) of the option. The ROA technique allows us to assess the strategic Net Present Value (NPV), through the appropriated estimation of the premium value of call/put options and comparing this value to the traditional DCF method, which is also called static NPV. According to this theory, options are used to, for example, defer, contract, expand or abandon that correspond to American call and put options adapted or modified to that context (Sabet and Heaney, 2017; Siña and Guzmán, 2019).

The main criticism of this methodology is that the same variables are used as in financial options valuation, with any kind of adjustment respect the application's field. Specifically, the problem lies in these variables: the discount rate and volatility. The first problem concerns some adjustment measures, and it is even possible to use utility functions to represent an economic agent that models attitudes and preferences for a certain situation and thereby estimate the value as a true equivalent, using a risk-free discount rate without complications (Pareja Vasseur and Cadavid Pérez, 2016). The second problem is the precarious development of the theoretical approaches to estimate volatility; the traditional estimation methodology, starting from the logarithmic return of the cash flows (Rogers, 2002) to the conditional volatility method of Brandão *et al.* (2012), going through the normal returns of cash flows (Lewis and Spurluck, 2004), the project proxy approach, the market asset disclaimer method or Copeland and Antikarov's method (Copeland and Antikarov, 2003) the administrative estimates or made by experts, the correlated risk-free asset (Trigeorgis, 1996), the market proxy approach or historical analysis (Mun, 2002), the implied volatility (Lewis and Spurluck, 2004), the Herath and Park's method (Herath and Park, 2002) and those of Godinho (2006), basically focused on the estimation of the unconditional volatility, which retains the drawback of assuming this phenomenon as constant. Works such as Brandão *et al.* (2012) only allow us to estimate it conditionally but without exploring the benefits of modelling it stochastically.

Most of the volatility estimation methods based on the ROA technique are basic, built on a single-level SDE, which only allows modelling the random asset's price, and fails to include

the effect of volatility on this variable. A differential equation that commonly uses the following structure is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (5)$$

with $t \in [0, T]$ and the initial condition $S_0 = S$, where μ is constant and denotes the asset's average rate of return, $\sigma > 0$ corresponds to volatility and $\{W_t\}_{t \geq 0}$ is a one-dimensional standard Wiener process.

Because of that it is necessary to make progress on both sides: the estimation of the volatility phenomena and its respective modelling through numerical methods because the high number of methodologies focuses mainly on an unconditional estimation of it. Finally, some of these techniques only present basic models for estimating the variable treated, which allows more advanced extensions or versions to be modelled, both the underlying asset and the volatility behaviour, conditionally and stochastically, but none of them explore the necessary changes that must be implemented in the valuation of options with these methods, specifically numerical ones, which are the most used and are also a motivation for the current research.

3.2 Algebraic expression

Assume an option (real or financial) over an asset that does not pay dividends, with an initial price S_0 and strike price K , is divided into N sub-intervals, each one with duration Δt . Define $f_{j,k}^{(i)}$ the value of the option in the node (i, j, k) . Based on [Marín Sánchez \(2010\)](#) and [Pareja-Vasseur and Marín-Sánchez \(2019\)](#), the price of the asset has quadrinomial recombination in node (i, j, k) , which can be represented by the following expression:

$$T_{j,k}^{(i)} = S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{i-k} l_k^{i-w} * V_0 \prod_{m=1}^{j-1} u_m^m \prod_{m=1}^{i-j} d_j^{i-m} \quad (6)$$

with $T_0 = T_{1,1}^{(1)}$, $i = 2, \dots, N$, $j = 1, 2, \dots, i$ and $k = 1, 2, \dots, i$. Keep in mind that in this case, both u and d are constant, so [equation \(6\)](#) can be summarized as follows: $T_{j,k}^{(i)} = S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{i-k} l_k^{i-w} * V_0 u^{j-1} d^{i-j}$.

Below there is an algebraic approach based on the proposed methodology to assess the basic real options:

- (1) In case of an option to wait, the evaluation is performed in a similar way as the American financial call option – where the value at its maturity date is given by $\max(T_t - K)$; so,

$$f_{j,k}^{(N)} = \max \left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j} - K, 0 \right),$$

for $j = 1, 2, \dots, i$ and $k = 1, 2, \dots, i$, while its discounted value is defined as,

$$\begin{aligned} f_{j,k}^{(i)} = \max & \left[T_{j,k}^{(i)} - K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)} \right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)} \right) f_{j+1,k}^{(i+1)} \right. \right. \\ & \left. \left. + \left(1 - q_j^{(i)} \right) \left(1 - p_k^{(i)} \right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i)-1} A_S^{-1} \right] \end{aligned}$$

- (2) In case of an expansion option, where the expansion factor is defined by EF and K represents the additional investment for expansion, it is considered as a modified American call option and the value at its maturity date is given by $\max(T_t * EF - K, T_t)$; so,

$$f_{j,k}^{(N)} = \max \left(\left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j} \right) * EF - K, S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j} \right),$$

for $j = 1, 2, \dots, i$ and $k = 1, 2, \dots, i$. Also, its discounted value is defined as,

$$f_{j,k}^{(i)} = \max \left[T_{j,k}^{(i)} * EF - K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)} \right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)} \right) f_{j+1,k}^{(i+1)} \right. \right. \\ \left. \left. + \left(1 - q_j^{(i)} \right) \left(1 - p_k^{(i)} \right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i-1)} A_S^{-1} \right]$$

- (3) In case of a contraction option, the contraction factor is defined by CF , and at the same time, K represents the disinvestment or release of funds by contraction, it is considered as a modified American put option and the value at its maturity date is given by $\max(T_i, T_i * CF + K)$; so,

$$f_{j,k}^{(N)} = \max \left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}, \left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j} \right) * CF + K \right),$$

For $j = 1, 2, \dots, i$ and $k = 1, 2, \dots, i$. Also, its discounted value is defined as,

$$f_{j,k}^{(i)} = \max \left[T_{j,k}^{(i)} * CF + K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)} \right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)} \right) f_{j+1,k}^{(i+1)} \right. \right. \\ \left. \left. + \left(1 - q_j^{(i)} \right) \left(1 - p_k^{(i)} \right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i-1)} A_S^{-1} \right]$$

- (4) In case of an abandonment option, the salvage value corresponds to K , it is considered as a modified American put option, the value at its maturity date is given by $\max(T_i, K)$; so,

$$f_{j,k}^{(N)} = \max \left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}, K \right)$$

for $j = 1, 2, \dots, i$ and $k = 1, 2, \dots, i$. Also, its discounted value is defined as follows,

$$f_{j,k}^{(i)} = \max \left[K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)} \right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)} \right) f_{j+1,k}^{(i+1)} \right. \right. \\ \left. \left. + \left(1 - q_j^{(i)} \right) \left(1 - p_k^{(i)} \right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i-1)} A_S^{-1} \right]$$

3.3 Risk-neutral valuation

To evaluate options in a risk-neutral world, two conditions must be assumed: The first is that estimated future cash flows can be determined by discounting their expected values at the risk-free rate, and the second is that the expected return on derivatives is the risk-free interest rate. To eliminate the potential for arbitrage on the discounted expected value, a single probability measure P^* must be constructed from the subjective p , such that discounted price at interest rate r is a martingale and as mentioned earlier, the discounted expected value at this rate according to the new probability P^* does not present arbitrage opportunities. The measure of the probability that is considered in a risk-neutral world is known as Martingale equivalent measure (MEM). Following Heston (1993), Marín Sánchez (2010), Pareja-Vasseur and Marín-Sánchez (2019), Wu *et al.* (2012), Wu *et al.* (2014) and Wu *et al.* (2018) and applying the theorem Cameron–Martin–Girsanov and Girsanov’s theorem (Mao, 1997, pp. 270–272),

the discounted pay based on the risk-free rate r is $S_t^* = e^{-rt} S_t$ and its stochastic differential corresponds to:

$$\begin{aligned} d(e^{-rt} S_t) &= -r S_t e^{-rt} dt + e^{-rt} (\mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)}) \\ dS_t^* &= (u - r) S_t^* dt + \sqrt{V_t} S_t^* dW_t^{(1)} \end{aligned} \quad (7)$$

As above, the discounted price S_t^* satisfies the same equation that S_t , where the return μ had been replaced too ($\mu - r$). This way the expected value under the subjective probability P from the discounted payoff of a derivative instrument will present arbitrage opportunities due S_t^* is not a Martingale. To find a probability measure where S_t^* is a Martingale, [equation \(7\)](#) must be written in a way that the tendency term could be assimilated inside the Martingale term. From there, the two previously mentioned theorems could establish that $(\mu - r)$ and $1/\sqrt{V_t}$ are bounded so $\phi(t) = \frac{u-r}{\sqrt{V_t}}$. It is clear that $\phi(t) \in L^2([0, T]; \mathbb{R})$, and therefore, $dS_t^* = [(u - r) + \sqrt{V_t} \theta] S_t^* dt + \sqrt{V_t} S_t^* dW_t^{(1)}$. From this is obtained $dS_t^* = \sqrt{V_t} S_t^* dW_t^{(1)}$, where $S_t^* = S_0^* + \int_0^t \sqrt{V_u} S_u^* dW_u^{(1)}$. Hereby could be observed that S_t^* is a Martingale respect \mathcal{F}_t to $0 \leq \tau \leq t$. Now, replacing $\phi(t)$ in [equation \(3\)](#) and $dW_t^{(1)*} = dW_t^{(1)} + \phi(t)$, we got:

$$dS_t = r S_t dt + \sqrt{V_t} S_t dW_t^{(1)*} \quad (8)$$

Note that [equation \(8\)](#) corresponds to the behaviour price in a risk-neutral world. In the same way to [equation \(4\)](#), the following expression is obtained: $\varphi(t) = \frac{-rV_t + \alpha(\theta - V_t)}{\sigma V_t}$, that in terms of market risk premium associated with volatility, it could be represented by $\varphi(t) = \frac{-\lambda(t, S, V)}{\sigma V_t}$ accordingly to $dW_t^{(2)*} = dW_t^{(2)} + \frac{-\lambda(t, S, V)}{\sigma V_t}$.

Substituting in [equation \(4\)](#) and considering that $\lambda(t, S, V)$ is proportional to volatility, $\exists \lambda > 0$ such as $\lambda(t, S, V) = \lambda V_t$, so:

$$\begin{aligned} &= [\alpha(\theta - V_t) - \lambda V_t] dt + \sigma V_t dW_t^{(2)*} \\ &= \alpha^* (\theta^* - V_t) dt + \sigma V_t dW_t^{(2)*} \end{aligned}$$

where $\alpha^* = (\alpha + \lambda)$ and $\theta^* = \frac{\alpha\theta}{(\alpha+\lambda)}$, $\{W_t^{(1)*}\}_{t \geq 0}$ and $\{W_t^{(2)*}\}_{t \geq 0}$ are independent one-dimensional standard Brownian motions in the probability space $(\Omega, \mathcal{F}, P^*)$. The term $\lambda(S, V, t)$ represents the price of volatility risk and is independent of the particular asset, and there is evidence that this term is not zero for options ([Heston, 1993](#)), while in contrast, [Hull and White \(1987\)](#) proposed to set this term at zero, based on the assumption that it is independent of the aggregate consumption. This term is the variance risk premium as a linear function of variance, namely $\lambda(V_t, t) = \lambda V_t$ ([Wu et al., 2012, 2014, 2018](#)). Following [Heston's \(1993\)](#) theory, this parameter should be determined for each particular asset ([Pareja-Vasseur and Marín-Sánchez, 2019](#)).

4. Results

4.1 Comparison between quadrinomial method with stochastic volatility, a binomial method with constant volatility and the Black–Scholes equation

Before presenting examples applying the proposed methodology using real options, it is necessary to graphically illustrate the asymptotic behaviour of quadrinomial and binomial methods, comparing the results with those obtained with the Black–Scholes model. Suppose that the dynamic behaviour of daily prices of a given financial asset can be modelled by a GBM, using [equation \(5\)](#) with an initial condition $S_0 = 10$ and an annualized volatility $\sigma_{BS} = 0.3162$. Previous studies like [Aziz \(2017\)](#) assumed the historic price volatility capture

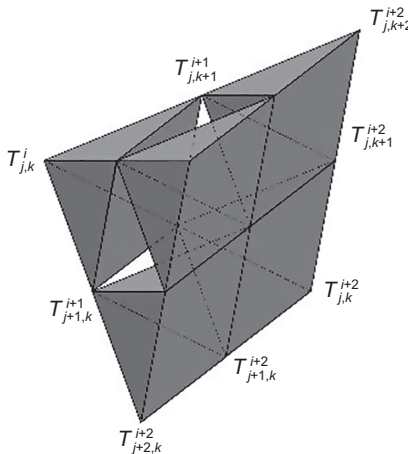
properly the annual volatility project. With that in mind, now let us assume that it is required to value a European call finance option derived from the underlying asset, whose exercise price is $K = 10$, with risk-free rate $r = 0.05$ and time to maturity $T = 1$; likewise, consider traditional multiplicative recombination in binomial trees with $N = 200$. It is possible to verify that the binomial method is a particular or special case of the multiplicative quadrinomial when $V_0 = (\sigma_{BS})^2 = \theta$ and σ tends toward zero. Let us further assume that α is an arbitrary constant.

Figure 2 presents the results, showing that quadrinomial and binomial methods have near solutions at around $N = 100$, with values that are also proximate to those of the exact solution estimated by the Black–Scholes model. Based on the previous results, we can conclude that the traditional binomial tree is a particular case of the quadrinomial proposed model, indicating that our model is not just a generalization, but also more flexible to evaluate different contracts where the underlying asset does not necessarily behave like the classic GMB. Our framework opens a broad window to build analogous procedures with a different stochastic process (e.g. as could be Heston’s and Scott’s model), which are used on financial option valuations, and few studied on real options.

4.2 Example 1: valuation of an abandonment real option and sensitivity’s analysis of the variables in the model in a real-world and risk-neutral world

Suppose that a firm in a certain sector of the economy wants to know the strategic value of a project according to ROA methodology and that its cash flows can be modelled using the SDE with equations (3) and (4). Assume in this case, the price of the commodity used to estimate the cash flows is perfectly correlated with a project, and its respective cash flows and volatility equal to the cash flows and the project’s volatility without administrative flexibility, as stated in Brandão *et al.* (2012). The estimated present value of the project is $S_0 = T_0 = 100$, which was estimated according to the traditional DCF methodology using an appropriate risk-adjusted rate. The company also determined that it has an opportunity to assign the rights and property of the project to a third party at a value of $K = 75$ when the market conditions are unfavourable, that is, this value corresponds to the salvage value and remains constant

Figure 2.
Comparison between
quadrinomial proposed
methodology,
traditional binomial
method and the Black–
Scholes method



Note(s): The figure was made with MATLAB
Source(s): Own elaboration

over the evaluation horizon of the firm's abandonment option. Assume, besides, the following arbitrary values: $V_0 = 0.1$, $r = 0.05$, $\alpha = 0.01$, $\theta = 0.1$ and $\sigma = 0.1131$, for the proposed method, and $\sigma_{BS} = 0.3162$ to the multiplicative binomial traditional methodology.

The results of the strategic NPV for the periods from Year 1–5, with steps of two years, are summarized in Table 3. Further, it was necessary to compare the results of the same methodologies, but in this case, the volatility σ tending towards zero was assumed for the proposed sample to show that it is a general case containing the particular one (caused by the traditional binomial method). In the last row of Table 3, we can see that the proposed methodology had a strategic NPV equals to 105.16, with a value of the abandonment option 5.16. If we assess them at three years, we found a result that was roughly in line with the traditional methodology, with 105.37 and 5.37, respectively. It was therefore concluded that there is a difference of less than 1% amongst the NPVs for the different evaluation periods presented. After estimating the volatility value of the volatility for the proposed method, the strategic NPV was set to a value of 108.51 for the case of $\sigma = 0.1131$, with the corresponding abandonment option corresponding to 8.51 if valued over five years. These results will increase depending on the different values estimated for parameters σ and α . It is worthy to mention that in this case, we are relaxing the constant volatility assumption, and due to the estimation of the quadrinomial tree, we can estimate properly the value of the option without this restriction. This means that a traditional estimation of an abandonment real option will be inadequately estimated, regardless it is under or overestimated.

4.3 Example 2: a valuation case of an abandonment real option in a real-world

This example describes a real options valuation case using the algebraic expression of an option of abandonment to find the value of the strategic NPV (the strategic NPV is the one estimated through the ROA, and it considers the static NPV calculated through the CDF, as the value of the real options identified and valued for a project). The numerical method utilized was our proposed methodology (a recombining quadrinomial tree with stochastic volatility and considering the values estimated for the proposed system). Finally, the solution was compared with the estimated results of the traditional binomial method.

Suppose that an oil and gas firm wants to know the strategic NPV value of a project according to the ROA methodology and that its cash flows can be modelled using SDE equations (3) and (4). For this case, the West Texas Intermediate (WTI) price series were taken from the Bloomberg platform between January 2013 and August 2018 and were used to estimate the cash flows, where those flows are perfectly correlated with the project, additionally, its respective cash flows and volatility equal those cash flows of the project

Method\time	1	2	3	4	5
<i>Example 1</i>					
(1) Quadrinomial tree $\sigma = 0.1131$	102.4	104.49	106.36	107.39	108.51
(2) Binomial tree $\sigma_{BS} = 0.3162$	101	104.92	105.37	107.21	107.51
(3) Quadrinomial tree $\sigma = 0.000000001$	100.92	104.78	105.16	106.93	107.17
<i>Example 2</i>					
(1) Quadrinomial tree	95.59	100.09	102.37	104.57	106.31
(2) Binomial tree	91.88	96.93	96.96	99.99	100.01
(3) Percentage difference	4.04%	3.26%	5.58%	4.58%	6.30%

Note(s): This exhibit shows the comparison of the value of the strategic NPV through the quadrinomial method with $T_0 = 100$, $K = 75$, $V_0 = 0.1$, $\alpha = 0.01$, $\theta = 0.1$, $\sigma = 0.1131$ and 0.000000001, and binomial method with $S_0 = 100$ and $\sigma_{BS} = 0.3162$. The table was made using MATLAB and Excel

Source(s): Own elaboration

Table 3.
Comparison of the
value of the strategic
NPV through the
quadrinomial method
and the binomial
method

It is worth noting if constant volatility of 44.06% would be used in the traditional methodology, which results from finding $\sqrt{\theta}$ in the proposed method, in this case, the difference amongst the results of the two methodologies would be narrow, an element that would indicate that real option value of the traditional method would be undervalued due to a

Note(s): Prepared by the authors using EViews and MATLAB
Source(s): Own elaboration

Source(s): Own elaboration

poor estimation of the volatility in the series because such a value was greater than estimated using the standard deviation of the yields, which is commonly used.

5. Discussion

Regarding the computational cost, it is possible to conclude that it is not significant since the objective of our research is to adopt the proposed numerical method to the conditions of the process itself. Strictly speaking, we want to structure a methodology with some time-series characteristics that are taken from the process, to find a more adjusted result, generating a logical discrepancy between the values of the traditional method and the proposed one. Future research aims are focus to analyse what occurs to the proposed model when there is a correlation between Wiener motions. In the other hand, it can used different commodities in real options applications to apply this proposed methodology.

In this paper, we study the particular case in which the correlation between the asset price and its volatility is zero, although many empirical studies establish that this correlation can be different from zero, as for example the model of [Heston \(1993\)](#).

The extension of this numerical method to models with non-zero correlation will be the subject of future work.

5.1 Practical implications

As for the practical implications, the classical methodology of real options valuation assumes that the underlying asset has constant volatility, but there is empirical and technical evidence such as that presented in Example 5.3, where it is observed that the volatility of the underlying asset is not constant and can be modelled as GARCH(1,1); not considering an adequate functional form for the volatility of any underlying asset can generate an incorrect strategic NPV estimation and consequently make unwise investment decisions.

Our methodology presents reliable results, mainly because our derivation lies in a rigorous mathematical background and regarding the implementation, the multiplicative binomial recombination tends to reduce the computational cost.

6. Conclusions

This paper extends the literature about numerical methods for valuing derivatives, specifically for ROA valuation; the main advantage of the methodological proposed is that it includes non-constant volatility, from a formal derivation of the first two moments of a GARCH-diffusion system, to determine the probabilities of transition, growth factors and discount rates. Throughout the paper was applied a rigorous mathematical technique, which offers a relatively easy method for the public to implement and use, so it can be used to assess all types of options, in both, risk-neutral and real situations, using non-constant volatility.

The findings can be summarized as follows. First, the methodological proposed was comparable to the traditional methodology using multiplicative binomial trees as well as the closed solution proposed by Black and Scholes. The above fact occurs when the volatility in the first-mentioned method tends toward zero; since this variable is defined positive, the value of the option will vary, and this change will be in function of the values assigned to the parameters α , θ and σ that are part of the methodology proposed. Second, it was possible to use an algebraic expression to depict the evolution of the price of an asset in the presence of both dynamic and constant volatility; in this manner, its effect is captured in an appropriate way, which allows better modelling of the evolution of behaviour of the underlying asset in the market. Third, the proposed method can be used in the presence of risk neutrality through the appropriate use of a market premium, which harms the value of a call option (wait and expand) and has a positive effect on a put option (contraction and abandonment). Besides, the

examples have shown that an appropriate valuation of real options is possible regardless of whether volatility is constant or not. Finally, in both examples, it was possible to verify how the inclusion of conditional and stochastic volatility in the ROA model affected the value of the real option; in the first example, it was detailed when the σ value of the proposed method changes, the strategic NPV values between the two examined methodologies differ, whereas when this variable takes values close to zero, both methodologies offer an identical solution. On another example, it was concluded that the option took a higher value in our method than the binomial method, mainly because the constant and unconditional volatility is not reflected in a time series, specifically, in the case of WTI oil commodity, the value of the option seems to be higher than its traditional simile using the multiplicative binomial method, which is explained by greater volatility obtained than estimated with standard simple yields deviation. It was also indicated, when the traditional method uses $\sqrt{\theta}$ as volatility, the values of the strategic NPV were similar.

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Corresponding author

Diego Manzur can be contacted at: dmanzurg@eafit.edu.co