

Jensen, Bjarne S.; Pedersen, Peder J.; Guest, Ross

**Working Paper**

## Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

IZA Discussion Papers, No. 15127

**Provided in Cooperation with:**

IZA – Institute of Labor Economics

*Suggested Citation:* Jensen, Bjarne S.; Pedersen, Peder J.; Guest, Ross (2022) : Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes, IZA Discussion Papers, No. 15127, Institute of Labor Economics (IZA), Bonn

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**Bjarne S. Jensen**

*University of Southern Denmark*

**Peder J. Pedersen**

*Aarhus University and IZA*

**Ross Guest**

*Griffith University*

MARCH 2022

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ISSN: 2365-9793

IZA – Institute of Labor Economics

Schaumburg-Lippe-Straße 5–9  
53113 Bonn, Germany

Phone: +49-228-3894-0  
Email: [publications@iza.org](mailto:publications@iza.org)

[www.iza.org](http://www.iza.org)

## ABSTRACT

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# Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

This paper analyzes the impact on age group wage differentials in a setting of imperfect labor substitution at different ages (years) of working life. We examine the wage prospect of assuming medium, high, and low levels of fertility during the population projection period (2020-2090). Main focus is on comparisons of selected Calendar year Age wage profiles and the comparisons of selected Cohort Lifetime wage profiles. The analytical results come from applying a CRESH Labor Aggregator to Age-group Labor supplies with a parametric calibration to register based micro data for Denmark. The results show Calendar year wage effects and Cohort wage effects from ageing that will not exist without non-zero Labor Complementarity elasticities, and are new contributions demonstrating the economic effects of large/small generations and cohort sizes. The impact of cohort size on the lifetime wage profile of its own cohort does depend on sizes of other cohorts, which are affected by the fertility rates underlying many cohorts. Hence, economic advantages of being a small cohort depend on fertilities and the sizes of many other existing cohorts.

**JEL Classification:** J1, O4, E2

**Keywords:** labor substitution, CRESH, demographic cohorts, lifetime wage incomes

**Corresponding author:**

Peder J. Pedersen  
Department of Economics and Business Economics  
Aarhus University  
Fuglesangs Alle 4  
DK-8210 Aarhus V  
Denmark

E-mail: ppedersen@econ.au.dk

# 1 Introduction

Demographic changes (projections) affect the *Population Age Distribution* as well as size and *Age Composition* (absolutely and relatively) of the available *Labor supplies* from the relevant working age groups. This paper address economic implications of *imperfect substitution* and *complementarity* of the *Labor services* from different *Labor Age groups*.

The standard assumption in demographic *macro* modelling is that the aggregate labor variable is a simple sum of the *homogeneous* labor services of different age groups - which implies perfect substitution and *same wage*. This means that the influence (size effect) of aggregate labor supply by an *increase* in workers of a *particular* age-group is *not* affected by the *Age distribution* (relative number) of workers already in the labor force. For example, when younger workers are becoming relatively scarce, it makes sense to allow for their *age-specific* contribution to an Aggregate measure (**Aggregator**) of Labor supplies. Hence we allow for *labor heterogeneity* by specifying a parametric CRESH<sup>1</sup> labor aggregator. This analytic **Labor Aggregator** function has implications for *relative wages* of both younger and older workers. In particular, if labor income is higher at younger ages and lower at older ages, then total *Lifetime* wage income of some generations (or cohorts) may be higher, while others are lowered. Various wage impacts are defined and calculated with lower/higher fertility of current and future calendar year generations.

In this paper, the *purpose* is to offer an *analytic globally regular* labor supply function (CRESH **Labor Aggregator** formed by any finite number of *labor supply* variables) with an *empirical applications* (parameter calibrations) to Danish *micro data* - and potential use for any country, where application of the principle of imperfect labor substitution is warranted. Our *focus* is next on investigating various *micro* and *macro implications* of projected **demographic** changes in this **century** (2010-2100) upon *relative* and *absolute annual wages* of *11 five-year Age groups* of *all* working ages (15-69) in *selected Calendar years* and then give the *lifetime labor incomes* of some *proper defined Cohorts*.

Among the *main new results* with our analytically extended CRESH wage model formulations are the extensive CRESH demonstrations (scenarios) of *comparative wages* (relative/absolute) of *all age groups* for some calendar **years** ( $t$ ) in period (2020-2090),

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<sup>1</sup>CRESH stands for Constant Ratio of Elasticities of Substitution, Homothetic, (Hanoch, 1971).

as well as obtaining the *Life time* wage incomes for *selected Cohorts* (Generations) of different *sizes*, entering the Labor market in the **year**, T= 2010, 2015, 2020, 2030, 2035.

Design and estimations of Labor aggregator (supply) functions have a long history. Only a short literature review is given here. Dougherty (1972, p.1110-16) discuss Labor aggregation structures based on 8 non farm *occupations* or 8 *educational* (length) categories. The aggregation functional forms are single-level CES functions or many *two-level* CES aggregations.<sup>2</sup> *Leontief* forms (fixed manpower requirements,  $\sigma = 0$ ), *Linear* aggregation,  $\sigma = \infty$ ), were extreme (invalid) forms, and *CD* function ( $\sigma = 1$ ) implied too little scope for substitution (inappropriate for aggregating labor). CES function were seen as improvements on these special forms of aggregation. Chiswick (1985, p.503) adhered to CES with moderately *high elasticity* ( $\sigma = 2.5$ ) between each pair of factors (including labor (*human capital*) of at least two quality levels of salaried employees). For US, UK, Canada, Card and Lemieux (2001, p.709,725) *estimated* ( $\sigma$ ) in the *range*: 4-6 (1/0.23, 1/0.17) for two CES *subaggregates* (High School, College) of workers from 7 *age groups*. Recently, Guest and Parr (2020, p.509) used,  $\sigma = 1$ ,  $\sigma = 2$ , for Australian CES labor aggregates of 11 age groups. However, long ago Berndt and Christensen (1973, p.407) proved that a consistent CES aggregators at all points in factor space is equivalent to *equality* and *constancy* of *all* Allen-Uzawa *partial* elasticities of *substitution* (AUES,  $\sigma_{ij}$ ). Evidently, the substitution elasticities between many labor services of different *age-groups* have *never* been the same or strictly constant (independent of labor supplies) anywhere.

Clearly, more sophisticated aggregator functions than CES are then to be considered. But functional complexity must be restrained to preserve sensible theoretical and empirical robust patterns of the substitution elasticities ( $\sigma_{ij}$ ). It is here that the CRESH function of Hanoch (1971, p.697) enter as a proper aggregator of different (*heterogeneous*) labor services, since it allows *relative substitution patterns* of  $\sigma_{ij}$  between services to be *preserved* (remain constant). Moreover, with our focus on the consequences for the *wage structure* of sizeable changes in the *age composition* of the (exogenous, demographic) *labor supplies*, we need to see for CRESH functions also the (dual) **partial complementarity elasticities** ( $c_{ij}$ ), Sato and Koizumi (1973, p.47), which **link** the *relative* and *absolute*

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<sup>2</sup>Bowles (1970, p.77) gave *Labor Supply Aggregates* with two-level CES functions of Sato (1967, p.202).

**wage** changes to variety in **Labor supplies of Age-groups**. These structure-analytic issues are probed jointly with CRESH calculations of  $(\sigma_{ij})$  and  $(c_{ij})$  in **Appendices A-B**.

Already Freeman (1979, p.301-303,313) estimated *complementarity elasticities*  $(c_{ij})$  by the *Trans-Log* production function, using *CPS* (Current Population Survey) data tapes of individual (*Micro*) age-earnings (age-wage profiles). Our *Micro* (Personal Register) data on *Danish labor supplies* (annual full time *equivalents*) of *age groups* and annual *wages* are provided by *Statistics Denmark* (Department of Labor and Income).

The paper is organized as follows.

Section 2 presents in **Table 1** and **Figure 1** already known demographic trends in the *Age composition of Populations* in this century as the background for our economic analysis of relative wages and life time labor incomes. It describes *Labor supplies* from *Microlevel* (Register) data for year 2010 in **Table 2**, within a *Macro* framework - National Income Product Accounts (*NIPA*) for year 2010, shown in **Table 3**.

Section 3 presents the CRESH labor aggregator and the implications for age-wage profiles. It explains the methodology of calibrating the **CRESH** parameters to Labor *Micro* data of 2010 ; the calibrated CRESH model is validated on *Micro* data for 2013 in **Table 4**.

Section 4 calculates *demographic* - for Medium, Low, and High fertility from *Table 1 - projected* Labor Supplies of eleven Age groups, spanning working life of 55 years (15-69).

It applies the **CRESH** model for the projected Labor supplies by showing the *comparative wages* (relative/absolute) of *all age groups* for *calendar years* ( $t$ ) in period (2020-2090).

The *main results* are collected in **Tables (5a-5c)**, **Table 6**, and exhibited in **Fig. 2-6**.

Section 5 demonstrates the **CRESH** calculations of *Life time* wage incomes of *selected Cohorts*, *entering* the Labor market in *particular years* ( $T$ ) of this century. The *main results* are explained and demonstrated by **Table 7** and illustrated in **Figures 7-12**.

Further micro and macro aspects of Labor aggregation and CRESH Age-wage profiles are discussed in section 6 in reference to the literature on Wage structure from 'Division of Labor' by labor of various levels of experience, skills - *Canonical Model*, Appendix C.

Section 7 offer final comments/suggestions for teaming up Demography and Economics.

**Appendices** (A,B,C) derive the basic CRESH Labor substitution elasticities and the *new CRESH complementarity* (Hicks-Sato) *elasticities* for *all* the Labor *age-group wages*.

## 2 Population age groups, Labor supplies and Wages

### 2.1 Demographic outlook, assumptions and future age groups

Let us briefly give the demographic outlook. Denmark, like most other developed countries, faces demographically further population ageing for some decades. **Table 1** shows Danish Population *age shares*,  $\mathbf{n}_i = \mathbf{N}_i/\mathbf{N}$ , of *three age* groups: children (0-14), working age (15-69), and old age (70+) - under **three Fertility** ‘variants’ (Medium, Low, High), cf. **Table 1a**, as published by the United Nations Population Division (United Nations, 2015). For the three age shares ( $n_i$ ), males and females are combined. *Life expectancy* is the same under all fertility variants. The ‘*Medium*’ variant projections assume that the Total Fertility rate (**TFR**) slightly and monotonically *increases* from **1.730** to **1.876**.

In the **Medium** variant scenario, the *Working Population share* ( $\mathbf{n}_{15-69}$ ) declines monotonically to a *minimum* (0.521) in year 2050, after which it monotonically recovers to (0.715), similar to its present size. The ‘Medium’ variant share numbers ( $n_i$ ) indicate an population “*ageing*” or “*burden*” problem for the next 20-40 years.

The **Low** variant numbers ( $n_i$ ) suggest in contrast that population ageing or “*burden*” problems will occur *after 2050*. The **High** variant numbers ( $n_i$ ) show a *remarkable stable* population composition *after 2025* - with even the old age (70+) share in balance.

When Fertility rate (TFR) is permanently less than **2.0**, there would be long-run tendency for the total size of population ( $N$ ) to decline. However, if *Life Expectancy* is steadily *increasing*, then population ( $N$ ) may still increase, despite (TFR) < 2.0. Population  $\mathbf{N}(t)$  for 2010-2100 in Medium Variant, Table 1, does *not decline* in *any* year.

The Danish **Population**  $\mathbf{N}(2010) = 5.551$  million. For *Medium* Variant, projected numbers are:  $\mathbf{N}(2020) = 5.776$ ,  $\mathbf{N}(2030) = 6.003$ ,  $\mathbf{N}(2050) = 6.299$ ,  $\mathbf{N}(2100) = 6.838$  million. *Low* Variant, Table 1, population eventually does *decline*. For Low Variant, projected numbers are:  $\mathbf{N}(2020) = 5.732$ ,  $\mathbf{N}(2030) = 5.792$ ,  $\mathbf{N}(2050) = 5.603$ ,  $\mathbf{N}(2100) = 4.599$  million. *High* Variant, Table 1, population certainly does *increase*. For High Variant, projected numbers are:  $\mathbf{N}(2020) = 5.819$ ,  $\mathbf{N}(2030) = 6.214$ ,  $\mathbf{N}(2050) = 7.154$ ,  $\mathbf{N}(2100) = 9.843$  million.

The *three* population *age* shares,  $n_i = N_i/N$ , define a *dependency rate* of *young/children* (0-14) to working age population : ( $\mathbf{d}_y$ ), and an *old/age* (70+) dependency rate to working



age population: ( $\mathbf{d}_o$ ), and hence give the **total dependency ratio**: ( $\mathbf{d}$ ), defined as:

$$d_y = \frac{n_{0-14}}{n_{15-69}} = \frac{N_{0-14}}{N_{15-69}}, \quad d_o = \frac{n_{70+}}{n_{15-69}} = \frac{N_{70+}}{N_{15-69}}; \quad d = d_y + d_o; \quad \frac{1}{1+d} = n_{15-69} \quad (1)$$

which, as calculated in **Table 1b**, are exhibited in **Fig. 1**. Note that in Table 1b, , e.g., *Medium* variant, **2010** :  $1/(1+d) = (1/1.411) = \mathbf{0.709} = n_{15-69}$ , (age share), **Table 1**. Thus, total dependency *ratio* ( $\mathbf{d}$ ) is *uniquely* related to  $\mathbf{n}_{15-69}$ , i.e., the columns ( $\mathbf{d}$ ), **Table 1b**, tell essentially, for every variant, a *similar* story as  $\mathbf{n}_{15-69}$  in **Table 1**, e.g. *Medium* variant, **2050** : *small*,  $\mathbf{0.643} = n_{15-69}$ , and *high* value of  $\mathbf{d} = 0.555$  ; but monotonicity of ( $\mathbf{d}$ ) in the *Low* variant seems more "dramatic" than the monotonicity of  $\mathbf{n}_{15-69}$ .

The projected *dependency ratios*,  $\mathbf{d}_y$ ,  $\mathbf{d}_o$ , (1), are *dominated* by the *paths* of  $n_{15-69}$ , although  $n_{70+}$  exerts significant influence on ( $d_o$ ) in the *Low* variant. It is projections of the dependency ratio, ( $d_o$ ), that has attracted attention in the literature, Rojas (2005, p.466), Hu et al. (2000, p.117), Kitao (2015, p.38). When dependency ratio ( $d_o$ ) is seen *redefined* as:  $\bar{\mathbf{d}}_o = N_{65+}/N_{15-64}$  (retirement age, 65), projected sizes of these numbers ( $\bar{d}_o$ ) appear in the literature more spectacular than  $\mathbf{d}_o$  in **Table 1b** for *Denmark*: 2010-2100.

The Labour Force *Participation rate*, **LFP**, is defined by,  $L_{15-69}/N_{15-69} = l_{15-69}$ :

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} \quad (2)$$

The **Support ratio** ( $L/N$ ), defined as the ratio of total **Labor** force (Labor supply) ( $L$ ) (**Employment**) to total **Population** ( $N = N_{0-70+}$ ) is obtained from the *dependency ratio* ( $d$ ), cf. (1), and the Labour Force Participation (**LFP**) rate,  $l_{15-69}$ , (2), as follows :

$$\frac{L}{N} = \frac{L_{15-69}}{N} = \frac{L_{15-69}}{N_{15-69}} \cdot \frac{N_{15-69}}{N} = l_{15-69} \cdot n_{15-69} = l_{15-69} \cdot \frac{1}{1+d} \quad (3)$$

Hence with e.g., **LFP** for 2010 :  $l_{15-69} = 0.536$ , we get by (3) for the *Medium variant*, the Support ratio in 2010 :  $L/N = 0.536 \cdot 0.709 = \mathbf{0.38}$ . Evidently, for a given,  $l_{15-69}$ , (constant LFP), the **Support** (Employment/Population) **ratio** ( $L/N$ ), (3) gives - for every fertility variant - the *same scenario* as the projected Working Population share:  $\mathbf{n}_{15-69}$  in **Table 1** - or *inversely* with the projected dependency ratio : ( $\mathbf{d}$ ) in **Table 1b**.

The **rising** *dependency ratio*, ( $d_o$ ), implies that the Danish *support ratio*, ( $L/N$ ), **falls** from 0.38 (2010) to a level around 0.34 *after* 2050. The support ratios ( $L/N$ ) are now declining in many countries and are expected to continuously *fall* in the years *until* 2050.

	<b>Medium variant</b>	<b>Low variant</b>	<b>High variant</b>	<b>Life</b>
	<b>Fertility*</b>	<b>Fertility*</b>	<b>Fertility*</b>	<b>Expectancy**</b>
2010-2015	1.730	1.730	1.730	8.507
2015-2020	1.761	1.511	2.011	8.779
2020-2025	1.785	1.385	2.185	9.053
2025-2030	1.804	1.304	2.304	9.344
2030-2035	1.817	1.317	2.317	9.653
2035-2040	1.829	1.329	2.329	9.987
2040-2045	1.841	1.341	2.341	10.304
2045-2050	1.848	1.348	2.348	10.609
2050-2055	1.854	1.354	2.354	10.903
2055-2060	1.858	1.358	2.358	11.205
2060-2065	1.862	1.362	2.362	11.505
2065-2070	1.865	1.365	2.365	11.798
2070-2075	1.868	1.368	2.368	12.091
2075-2080	1.870	1.370	2.370	12.396
2080-2085	1.872	1.372	2.372	12.715
2085-2090	1.874	1.374	2.374	13.028
2090-2095	1.875	1.375	2.375	13.353
2095-2100	1.876	1.376	2.376	13.691

Notes:

Variants differ only with respect to fertility assumptions.

\* Fertility refers to number of children per woman.

\*\* Life expectancy at age 80 for both sexes combined (number of years).

	<b>Medium Variant (<math>n_i</math>)</b>			<b>Low Variant (<math>n_i</math>)</b>			<b>High Variant (<math>n_i</math>)</b>		
	<b>0-14</b>	<b>15-69</b>	<b>70+</b>	<b>0-14</b>	<b>15-69</b>	<b>70+</b>	<b>0-14</b>	<b>15-69</b>	<b>70+</b>
2010	0.180	0.709	0.111	0.180	0.709	0.111	0.180	0.709	0.111
2015	0.173	0.702	0.125	0.173	0.702	0.125	0.173	0.702	0.125
2020	0.163	0.689	0.148	0.157	0.695	0.149	0.169	0.684	0.147
2025	0.160	0.683	0.157	0.143	0.697	0.160	0.176	0.670	0.154
2030	0.165	0.669	0.166	0.135	0.693	0.172	0.193	0.646	0.160
2035	0.168	0.656	0.176	0.132	0.682	0.186	0.201	0.631	0.168
2040	0.167	0.644	0.189	0.130	0.667	0.202	0.200	0.622	0.178
2045	0.164	0.642	0.194	0.128	0.661	0.211	0.195	0.626	0.180
2050	0.161	0.643	0.196	0.125	0.659	0.216	0.193	0.629	0.178
2055	0.160	0.639	0.201	0.120	0.664	0.216	0.197	0.633	0.170
2060	0.161	0.637	0.202	0.116	0.661	0.223	0.204	0.629	0.167
2065	0.163	0.636	0.201	0.114	0.646	0.239	0.209	0.621	0.170
2070	0.162	0.629	0.209	0.113	0.631	0.256	0.208	0.619	0.173
2075	0.160	0.625	0.215	0.112	0.618	0.270	0.204	0.622	0.174
2080	0.157	0.621	0.222	0.110	0.603	0.287	0.201	0.625	0.174
2085	0.155	0.623	0.222	0.108	0.595	0.297	0.200	0.630	0.170
2090	0.155	0.621	0.224	0.106	0.593	0.301	0.202	0.628	0.170
2095	0.155	0.617	0.228	0.105	0.591	0.304	0.203	0.622	0.175
2100	0.155	0.611	0.234	0.105	0.588	0.307	0.203	0.615	0.182

Source: United Nations (2015) 'World Population Prospects, 2015 Revision', United Nations, New York, 2015.

Total population (both sexes combined) by five-year age group.

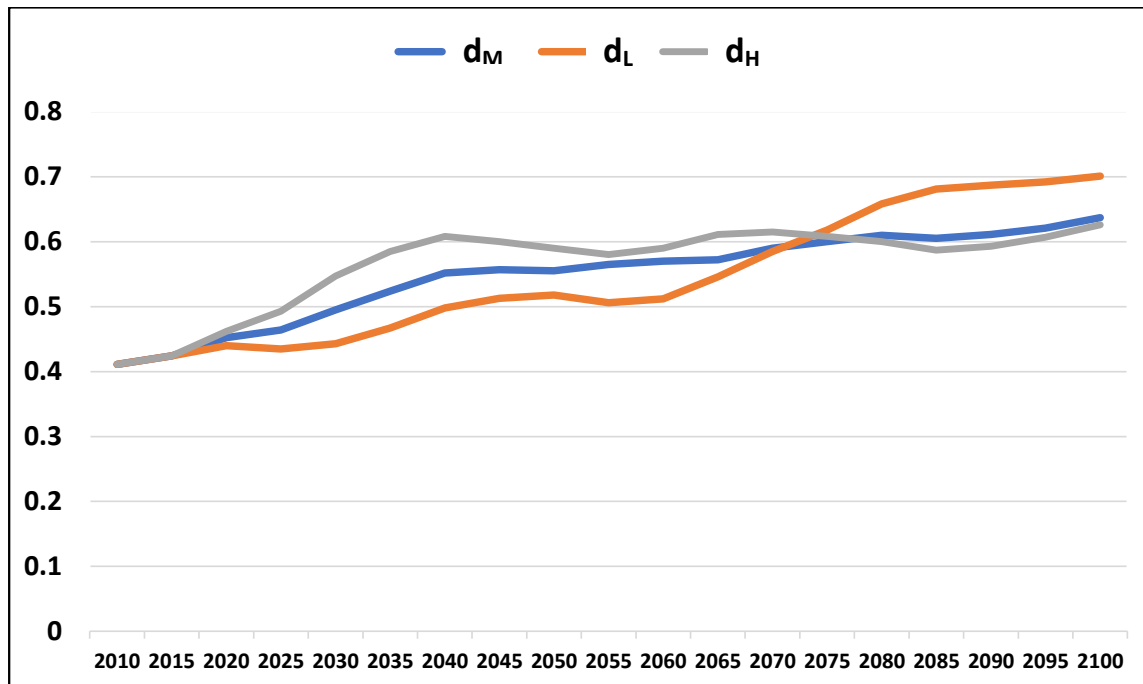
The graphics of the *Danish dependency ratio* (d) in **Table 1b** is exhibited in **Figure 1** for the *three* UN demographic<sup>3</sup> *variants* (Medium, High, Low).

<sup>3</sup>The **declining** *fertility* in recent decades and hence the *falling* *dependency ratio*, ( $d_y$ ), have in several countries *dominated* the *rising* ( $d_o$ ), such that **Support ratios** ( $L/N$ ), (3), in some countries have in certain **periods until 2010** actually **increased** - and been called "demographic (*fertility*) *dividends*."

We shall in **Table 6** see a few economic illustrations of this "dividend" in the Danish *Medium* fertility *variant* for *some* years after the '*minima*' of the 'Working Age Population' *share*,  $n_{15-69}$ , in **2050**.

Some illustrations of US *dependency ratios* and *Support ratios* are seen in, Cutler et al. (1990, p.5,8).

Fig. 1. Danish Dependency ratio -  $d$  - for Medium, Low, High fertility, 2010-2100.



Source: Total dependency ratio ( $d$ ), (1), with the numbers from Table 1b.

	Medium Variant			Low Variant			High Variant		
	$d_y$	$d_o$	$d$	$d_y$	$d_o$	$d$	$d_y$	$d_o$	$d$
2010	0.254	0.157	0.411	0.254	0.157	0.411	0.254	0.157	0.411
2015	0.246	0.178	0.424	0.246	0.178	0.424	0.246	0.178	0.424
2020	0.237	0.215	0.452	0.226	0.214	0.440	0.247	0.215	0.462
2025	0.234	0.230	0.464	0.205	0.230	0.435	0.263	0.230	0.493
2030	0.247	0.248	0.495	0.195	0.248	0.443	0.299	0.248	0.547
2035	0.256	0.268	0.524	0.194	0.273	0.467	0.319	0.266	0.585
2040	0.259	0.293	0.552	0.195	0.303	0.498	0.322	0.286	0.608
2045	0.255	0.302	0.557	0.194	0.319	0.513	0.312	0.288	0.600
2050	0.250	0.305	0.555	0.190	0.328	0.518	0.307	0.283	0.590
2055	0.250	0.315	0.565	0.181	0.325	0.506	0.311	0.269	0.580
2060	0.253	0.317	0.570	0.175	0.337	0.512	0.324	0.266	0.590
2065	0.256	0.316	0.572	0.176	0.370	0.546	0.337	0.274	0.611
2070	0.258	0.332	0.590	0.179	0.406	0.585	0.336	0.279	0.615
2075	0.256	0.344	0.600	0.181	0.437	0.618	0.328	0.280	0.608
2080	0.253	0.357	0.610	0.182	0.476	0.658	0.322	0.278	0.600
2085	0.249	0.356	0.605	0.182	0.499	0.681	0.317	0.270	0.587
2090	0.250	0.361	0.611	0.179	0.508	0.687	0.322	0.271	0.593
2095	0.251	0.370	0.621	0.178	0.514	0.692	0.326	0.281	0.607
2100	0.254	0.383	0.637	0.179	0.522	0.701	0.330	0.296	0.626

Source: The dependency ratios,  $d_y, d_o, d$ , are defined in (1) and calculated by Table 1.

The overall dependency ratio,  $d$ , (1), is *rising* for each fertility scenario in Figure 1. But in the *high* fertility scenario, the dependency ratio ( $d$ ) is 'stationary' in the 50 years from 2040 to 2090. In the *low* fertility scenario, we find a significant *increase* in the dependency ratio ( $d$ ) from 2060 onwards, as the delayed impact of prior low fertilities.

As well-known, a *Fertility* rate of **2.1** (children per woman) on average is usually necessary for reproduction of population levels; increasing Life Expectancy modifies the requirement. As mentioned above Total Danish Population size,  $\mathbf{N}(t)$ , never declines, but slowly *increases* during projection *period* 2020 – 2090 under *Medium* Variant of **Tables (1a, 1)**, cf. **Table 6** (Row 4) below - that also shows that  $\mathbf{N}(t)$  *declines* after 2040 in the *Low* Variant, and clearly  $\mathbf{N}(t)$  *increases* (nearly doubles by 2090) for the *High* Variant.

Having presented the United Nations projections of the **evolution** of the Danish *demographic structure* 2010-2100 in **Tables (1,1b,6)**, **Fig. 1**, with the *general concepts* and *terminology*, (1-3) - similar descriptions apply to any UN country - we restate (for later use) overall **LFP**, (2), in terms of **11** *age-specific* labor participation rates,  $l_i = \frac{L_i}{N_i}$  :

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = \sum_{i=1}^{11} \frac{L_i}{N_i} \cdot \frac{N_i}{N_{15-69}} \equiv \sum_{i=1}^{11} l_i \cdot \tilde{n}_i ; L_i(t) = l_i \cdot N_i(t), i = 1, , 11 \quad (4)$$

With proper chosen  $\mathbf{l}_i$  as *exogenous* parameters, we can derive *age-specific* Labor supplies in any *calendar* year ( $t$ ),  $\mathbf{L}_i(t)$ , (4), from the evolutions of,  $N_i(t) = n_i(t) \cdot N(t)$ , and  $N(t)$ .

Introducing imperfect<sup>4</sup> labor *substitution/complementarity* between *age-specific* Labor group supplies  $\mathbf{L}_i(t)$ , (4), significantly *changes* the *relative wages* within the Total Labor force (supply),  $\mathbf{L}_{15-69}(t)$ , over time. We will use a suitable **Labor** economic **model** *analytically designed* to generate/explain such **Age Group Wage Differentials**. Section 3 presents a **CRESH** model with distinct *parameters* for labor age group supplies.

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<sup>4</sup>The assumption of perfect substitution of labor among age groups has been challenged, tested empirically and relaxed in a variety of modelling approaches, Prskawetz et al. (2008), Guest (2007), Creedy and Guest (2007), Guest and Shacklock (2005), Hamermesh (1993), Lam (1989).

## 2.2 Labor supplies of age groups, Micro wage data, and NIPA

To apply the CRESH labor *aggregator* for empirically analyzing age-wage profiles, the CRESH parameters must be properly related to specific *labor supplies* and *wage data*. Here we use *Micro* and *Macro* data for Denmark in the year 2010 - to be explained and shown in **Tables 2-3**. Similar Micro data 2013 are used for CRESH aggregator validation.

As provided from the United Nations World Population Prospects, 2015 Revision, total **Population** (both sexes) by *five year age groups* (United Nations, 2015), the Danish Population numbers,  $N_i$  (column 2, **Table 2**) are : The *demographic* sizes of our *eleven* 5-year *working* age groups (15-69), and the *young* (0-14) and *old* age (70+) groups, i.e., the *absolute* sizes of the age groups in 2010, corresponding to *age shares* ( $n_i$ ) in **Table 1**.

Within the *eleven* 5-year working age groups, the **oldest**  $N_i$ , (65-69), soon fully retired, are **born** in (1941-1945). In (1945,1946), the number of *births peaked* with (95-96.000). The *post-war* (1946-1950), *generation* are seen in  $\mathbf{N}_i$  (60-64). Birth rates started to slowly decline in the 1950's ; the Danish economy was stagnating until 1957, and net *emigration occurred*, as can be seen from the  $N_i$  (55-59) numbers, which also partly reflect a *negative 'echo'* of the smaller depression year generations of 1930's. In contrast, a *positive 'echo'* of two *war-postwar* generations above and prosperous *full-employment* years of 1960's are reflected in sizes  $\mathbf{N}_i$  (45-49),  $\mathbf{N}_i$  (40-44) of the *two generations*, (1961-1965), (1966-1970). The European oil-shock *recession* and unemployment *years*, (1976-1980), (1981-1985), are reflected in the *small Danish* numbers of the,  $N_i$  (30-34),  $N_i$  (25-29).

From the beginning of 1990's, revenues from North Sea oil - as in UK and Norway - contributed to remove deficits of Danish international and public sector accounts. *Child benefits* were subsequently increased; significant *immigration* also began to matter in these years. They are explanatory population elements of a *turn-around*, seen in sizes of both  $N_i$  (20-24), and the **youngest** age group,  $N_i$  (15-19), **born** in (1991-1995).

We must next explain the **Labor supplies** used and their associated *wages* in 2010. The *Labor age group* numbers (Labor years)  $L_i$  (column 3, **Table 2**) are Danish *full time* workers (*equivalents*, **1924 hours**) - with *age distribution* ( $\lambda_i$ ), (col. 4), and their *average annual wages* ( $\mathbf{w}_i$ ), (column 6). These Microlevel data (personal register) were provided to the authors by Statistics Denmark. These *Register data*, however, were excluding

agriculture, fishery, and all firms with less than 10 full-time employees.

We calibrate our model to *National Accounting* data for Denmark in **2010**, which implies that the *sum* of  $\mathbf{L}_i$  (col. 5) must equal *aggregate employment* (**Table 3**, row 1) of **2112472** full time workers (*Labor years*). We use the *age-specific* labour fractions ( $\lambda_i$ ) of *Register* data (column 4) to gross up the values of  $\mathbf{L}_i$  such that the *total* of  $\mathbf{L}_i$  (col. 5) is equal to :  $\mathbf{L} = \mathbf{2112472}$ . The *age-specific* wages  $\mathbf{w}_i$  in **Table 2** (col.6), of Register data are multiplied by the adjusted (Total) Labor numbers,  $\mathbf{L}_i$ , (col.5), and summed to give the *aggregate Wage Bill*, which is **924.3** Billion DKK (col. 8). The *aggregate Annual wage*,  $\mathbf{w}$ , per unit of  $\mathbf{L}$  is then found by dividing the aggregate Wage Bill (col. 8) by  $\mathbf{L}$ , which gives :  $\mathbf{w} (2010) = \mathbf{437552}$  DKK  $\equiv \mathbf{W}_A$  ; cf. (41), **Tables (5a, 5b)**.

<b>Table 2. Age-specific Data: Labor groups, Wages, Wage shares, Participation rates - Denmark 2010</b>									
<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>
<b>Age (i)</b>	$N_i$	$L_i$	$\lambda_i$	$L_i$	$w_i$	$w_i/w_4$	$w_i L_i$	$\varepsilon_i$	$l_i = L_i/N_i$
<b>15-19</b>	353109	21138	0.0143	30199	187005	0.4554	5.647	0.0061	0.0855
<b>20-24</b>	331419	77504	0.0524	110731	273220	0.6653	30.254	0.0327	0.3341
<b>25-29</b>	310515	117177	0.0792	167411	358262	0.8724	59.977	0.0649	0.5391
<b>30-34</b>	347261	170826	0.1155	244059	410668	1.0000	100.227	0.1084	0.7028
<b>35-39</b>	388101	202853	0.1372	289816	449679	1.0950	130.324	0.1410	0.7468
<b>40-44</b>	408902	214618	0.1451	306624	471118	1.1472	144.456	0.1563	0.7499
<b>45-49</b>	405079	211983	0.1434	302860	472491	1.1505	143.099	0.1548	0.7477
<b>50-54</b>	366102	188472	0.1275	269270	471381	1.1478	126.929	0.1373	0.7355
<b>55-59</b>	350020	169527	0.1147	242203	461729	1.1243	111.832	0.1210	0.6920
<b>60-64</b>	368451	88874	0.0601	126974	478708	1.1657	60.783	0.0658	0.3446
<b>65-69</b>	309369	15626	0.0106	22325	483248	1.1767	10.789	0.0117	0.0722
<b>Total 15-69</b>	3938328	1478598	1.0000	2112472	437552		924.317	1.0000	0.5364
<b>0-14</b>	997084			0	0				0
<b>70+</b>	615547			0	0				0
<b>Total</b>	5550959			2112472	166515		924.317		0.3806

Source: UNITED NATIONS (UN), see **Table 1** ; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

Column 1: **Age groups**,  $i = 1, \dots, 11$ ;  $i = 1$ : 15-19, ...,  $i = 11$ : 65-69 ;  $i = 12$ : 0-14,  $i = 13$ : 70+.

Column 2: Population (totals) in age groups; UN Population Data.

Column 3: *Full time workers* (Annual equivalents, **1924 hours**), Labor services in **Labor years** ; *Microlevel* (personal register) data.

Column 4: Labor age group distribution - Fractions,  $\lambda_i$ , same in column 3 and column 5.

Column 5: **Total** full time workers in labor age groups,  $L_i = \lambda_i L$ , ( $L = \mathbf{2112472}$  = Total full time workers) ; stat.bank, **DB07, ERHV1**.

Column 6: Average annual wages of labor age groups ( $w_i$ ) in column 3 and 5;  $w = \mathbf{437552}$  DKK = **924.317** Billion DKK / **2112472**.

Annual wage income per capita,  $wL/N = 924.317$  Billion / 5550959 = 166515 DKK (Danish Kingdom Kroner).

Column 7: Relative annual wages, **age group wage profile** - generated by the *Microlevel* (personal register) data in **column 6**.

Column 8: Total wage incomes of age group, ( $i$ ), Billion DKK; ( stat.bank, **DB07, ERHV1, Total** wage sum : **930.286** Billion DKK ).

Column 9: Age group wage income shares  $\varepsilon_i$  (shares in the total wage bill, 924.317 Billion DKK).

Column 10: Labor participation rates (**LPR**) of age groups - derived from column 2 and column 5.

**Table 3. National Income Accounts - Data for Denmark, 2010**

Descriptions		Symbols	Values	
1	Total (equivalent) full time workers	$L$	2112472	Labor years
2	Average (aggregate) wage per labor year (man-year)	$w$	437552	DKK
3	Total wage incomes	$wL$	924.3	Billion DKK
4	Net capital (rental) incomes	$rK$	303.8	Billion DKK
5	Net Factor Incomes (NFI) – Net Domestic Value Added	$Y = wL + rK$	1228.1	Billion DKK
6	Capital consumption/depreciation	$\delta K$	318.1	Billion DKK
7	Gross Factor Incomes (GFI) – Gross Domestic Value Added	$GFI$	1546.2	Billion DKK
8	Net capital stock	$K$	5741.5	Billion DKK
9	Net capital-output ratio	$v = K/Y$	4.67	
10	Net capital-labour ratio	$k = K/L$	2.72	Million DKK
11	Average labour productivity	$y = Y/L$	581972	DKK
12	Depreciation rate	$\delta = \delta K/K$	0.055	percent
13	Net real interest rate	$r = rK/K$	0.053	percent
14	Gross real interest rate	$r + \delta$	0.108	percent
15	Wage share of net factor income, $w/(Y/L)$	$\varepsilon_l = wL/Y$	0.752	
16	Capital share of net factor income, $r/(Y/K)$	$\varepsilon_k = rK/Y$	0.248	
17	Factor compensation, Asset income (net), to rest of world	$O$	29.6	Billion DKK
18	Net National Income, in factor prices	$Y + O = NNI$	1257.7	Billion DKK
19	Gross National Income, in factor prices	$C + Tr + S = GNI$	1575.8	Billion DKK
20	Consumption ( private + public ), in factor prices	$C$	1104.9	Billion DKK
21	Transfers to rest of world, net	$Tr$	36.6	Billion DKK
22	Gross National Saving	$S$	434.3	Billion DKK
23	Gross Domestic Investment, in factor prices	$I$	331.3	Billion DKK
24	Balance of payment, current account, Asset accumulation	$S - I = BP$	103.0	Billion DKK
25	Consumption ratio, NNI	$C/NNI$	0.879	
26	Consumption ratio, NFI	$C/Y$	0.900	
27	Consumption per capita, in factor prices	$C/N$	199350	DKK
28	Net National Income per capita, in factor prices	$NNI/N$	226573	DKK
29	Annual wage income per capita, in factor prices	$wL/N$	166515	DKK
30	Support Ratio	$L/N$	0.3806	
31	Net Factor Income (NFI) per capita, $Y/N = (Y/L)(L/N)$	$Y/N$	221499	DKK
32	Annual wage income-consumption ratio	$wL/C$	0.837	
33	Gross national saving rate	$S/GNI$	0.276	
34	Net national saving rate	$(S - \delta K)/NNI$	0.092	
35	Gross domestic investment rate	$I/GFI$	0.214	

Source : Rows 1-3, *Microlevel* (register) data from Table 2. Rows 5-9, 12, 17-26, *Macro* (aggregate) data from NIA (National Income Accounting) : *Statistical Ten-Year Review, (STR) 2015*, p. 101-102, 104-105, 120 ; STATISTICS DENMARK, Copenhagen.  
Row 1: Full time **Labor years**,  $L = 2.112.472 = 4064.4$  Million labor hours (1 **Labor year** = 37 hours per week x 52 = **1924 labor hours**) ; STR (2015, p.121) gives in NIA : 3606.6 Million labor hours for 2010.  
Row 3: STR (2015, p.104) gives in NIA : Wage Sum = **953.7** Billion DKK – a bit more (½ %) more than,  $wL = 924.3$  Bill. DKK.  
Row 4:  $rK = \text{Row 5 } (Y) - \text{Row 3 } (wL)$  ; STR (2015, p.104) gives in NIA : Net Capital Income = 274.4 – a bit less than,  $rK = 303.8$ .  
Rows 5-7: STR (2015, p.104), with same  $Y = 1228.1$ , as in Row 5. Row 8: STR (2015, p.128) . Row 17: STR (2015, p.102).  
Row 18: Row 5 + Row 17.  
Row 19: STR (2015, p.104) gives :  $GNI$  in market prices =  $GNI$  in factor prices + indirect taxes = 1578.8 + 252.5 = 1828.3 Billion DKK.  
Row 20: STR (2015, p.104) gives :  $C$  in market prices =  $C$  in factor prices + indirect taxes = 1104.9 + 252.5 = 1357.4 Billion DKK.  
STR (2015, p.102) gives : Indirect commodity (production) taxes/subsidies, net : 248.2 + 4.3 = 252.5 Billion DKK.  
Row 21: STR (2015, p.102) ; Row 22: STR (2015, p.104) ; Row 23: STR (2015, p.103) ; Row 24: STR (2015, p.106).  
Rows 25-26: Derived from rows above; Row 27:  $C/N = (C/Y)(L/N)(Y/L) = (0.9)(0.3806)581972 = 199350$  DKK, cf. Table 2.

*Labour Force Participation (LFP) rate* (4), *Support ratio* (3) - cf. Table 2, col.10,1 - are:

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = 0.5364 ; \quad \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i = \sum_{i=1}^{11} \frac{L_i}{N_i} \frac{N_i}{N} = l_{15-69} \cdot n_{15-69} \quad (5)$$

$$\frac{L}{N} = l_{15-69} \cdot n_{15-69} = 0.5364 \cdot 0.7095 = 0.3806 ; \quad \frac{Y}{N} = \frac{Y}{L} \cdot \frac{L}{N} ; \quad \frac{C}{N} = \frac{C}{Y} \cdot \frac{Y}{N} \quad (6)$$

Support ratio  $L/N$  (5-6) is  $n_i$ -weighted age-specific,  $l_i$ . Support ratio: a multiple of **LFP**. The *per capita sizes* of National Income, Consumption ratios,  $Y/N$ ,  $C/N$ , and their *decomposition* in (6) are seen in NIPA, Table 3 (row 31,27) - summarized in Table 3a.

Thus **Micro** based *employment* - full time equivalents - and *wage* data in **Table 2** (col. 5,6,8), correspond exactly to **Macro** (National Income) data in **Table 3** (Row 1-3, 29-31), for calendar *year*, 2010.<sup>5</sup> Short version of **Table 3** is seen in **Table 3a** - template to **Table 6** - as calendar year summary of labor model results in **Tables (5a, 5b)**.

<b>Table 3a. Population Age Groups, Labor Supply, Support Ratios, Incomes per capita: Denmark</b>			
2010	1.	$N_{15-69}$	3938328
	2.	$N_{0-14}$	997084
	3.	$N_{70+}$	615547
	4.	$N = Total$	5550959
	5.	$L = L_{15-69}$	2112472
	6.	$L/N_{15-69}$	0.5364
	7.	$L/N$	0.3806
	8.	$WL Bill.$	924.317
	9.	$WL/N,$	166615
	10.	$Y/N$	221499

**Table 3a/Table 6**, Rows 1-4 show - in *absolute* quantitative form (Total numbers) for calendar years - the consequences of the *demographic changes* described in **Tables (1,1a,1b)**. Rows 5-7 show, respectively, the absolute sizes of the *total* labor force,  $L(t)$ , (supply), the sizes of **LFP** (t) rate, (5), and the sizes of Support Ratio,  $\mathbf{L}(t)/\mathbf{N}(t)$ , (6). Row 8 shows the Total **Wage Income** of  $L(t)$ , [All age groups, in calendar year (t)], working in any year (t) with **11** *age-specific* participation rates :  $l_i(t) = l_i(2010)$ , **Table 2** (col. 10). Row 9 shows the Total **Wage Income** per capita,  $N(t)$ , which - with the *macroeconomic* structure, technology levels, productivity conditions,  $(Y/L)$ ,  $(K/L)$ , of **Table 3**, (2010) - is *equivalent* to :  $\mathbf{L}(t)/\mathbf{N}(t) \cdot \mathbf{w}(2010) = \text{Support ratio} \cdot \mathbf{W}_A$ <sup>6</sup>. Row 10 shows **National Income** per capita,  $Y(t)/N(t)$ , which is here a simple *proportionality* of Row 9, cf. (6).

The labor market equilibrium model with CRESH *Aggregator* functions - underlying all results in **Tables 4-7** - must now be established and justified, theoretically and empirically, in Section 3 and Appendix A : Labor *Substitution* and *Complementarity*.

<sup>5</sup>The year 2010 - as benchmark for our projected population variants, annual labor supply and wage income comparisons - is chosen for various reasons. It takes several years before the final revision of National Income Accounting (NIA) is completed. The Financial Crisis years (2008-2009) were unsuitable as benchmark years. For year 2010 the final revision came out in 2015. The processing of the Micro register databases for corresponding employment and wage data for 2010 began after NIA revision 2015. We cannot wait for getting reliable revised NIA for 2015 or later - as our model benchmark year.

**Remark.** The *overall* sizes of LFP =  $l_{15-69}$  in (5), **Tables (2,4,6)** look small ; the *age-specific* LFP,  $l_i$ , **Table 2** show that  $l_1, l_{11}$  give small  $l_{15-69}$  (Age group 65-69,  $l_{11}$ , is *seldom* included in reported LFP).

<sup>6</sup>As will be explained by Factor (Labor) cost (income) functions and *duality* theory in *Appendix A*.



### 3 CRESH Labor supply, Relative wages, Annual wage

Hanoch (1971, p. 697) introduced a *globally regular* CRESH *implicit* production (aggregator) function,  $F(Y, X_1, X_2, \dots, X_M) = 0$ . Our CRESH function,  $\mathbf{F}(\mathbf{L}_A, \mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , as seen in equation (7), is *homogeneous* of degree **zero** - and  $\mathbf{F}$  determines **implicitly** the **Labor Aggregate** variable,  $\mathbf{L}_A$ , (*Total Labor Supply*), from the *distinct* (heterogeneous) Labor services,  $(L_1, L_2, \dots, L_M)$ , (*M Labor Supplies*), as stated in the expression:

$$F(L_A, L_1, L_2, \dots, L_M) = \gamma \sum_{i=1}^M \alpha_i \left[ \frac{L_i}{L_A} \right]^{\rho_i} - 1 = 0 \quad (7)$$

with

$$\gamma > 0; \quad \forall i : \alpha_i > 0, \quad \sum_{i=1}^M \alpha_i = 1; \quad \forall i : 0 < \rho_i \leq 1 \quad \text{or} \quad \rho_i < 0 \quad (8)$$

*Labor services* (flows),  $(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , may be *measured* in hours, working-weeks, or labor years. As in **Table 2-3**, we use as Labor **unit**: *Labor years*; the total flow variable ( $\mathbf{L}_A$ ) is also measured in Labor years. Thus *ratios*,  $(\frac{L_i}{L_A})$ , in (7) are *unit-free* (pure numbers), implying, too, that all **parameters** in (8) are *unit-free* (pure numbers).

For  $\forall i: \rho_i = \rho$ , we get CES functions by (7-8), and CD as the limit function ( $\rho = 0$ ),

$$\rho_i = \rho : L_A = \gamma^{\frac{1}{\rho}} \left[ \sum_{i=1}^M \alpha_i L_i^\rho \right]^{\frac{1}{\rho}}; \quad \rho = 1, L_A = \gamma \sum_{i=1}^M \alpha_i L_i; \quad \rho = 0, L_A = \bar{\gamma} \prod_{i=1}^M L_i^{\alpha_i} \quad (9)$$

Parameter *restrictions* (8) ensure that **CRESH** equation (7) represents a **unique** implicit **Labor Aggregator** function,  $\mathbf{L}_A = \mathbf{f}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , that is *homogeneous* of degree **one**, and is globally regular, i.e. for all  $\mathbf{L}_i > 0$ ,  $\mathbf{f}(\dots)$  is *positive*, *non-decreasing*, **concave**, with a negative semi-definite *Hessian* matrix,  $\frac{\partial^2 \mathbf{f}}{\partial \mathbf{L}_i \partial \mathbf{L}_j}$ .

$$\forall L_i > 0 : L_A = f(L_1, L_2, \dots, L_M) > 0; \quad \frac{\partial f}{\partial L_i} > 0, \quad \frac{\partial^2 f}{\partial L_i^2} < 0; \quad L_A = \sum_{i=1}^M \frac{\partial f}{\partial L_i} L_i \quad (10)$$

The CRESH function,  $F(L_A, L_1, L_2, \dots, L_M)$ , in (7) has the first-order derivatives,

$$\frac{\partial F}{\partial L_i} = \frac{\gamma \alpha_i \rho_i (L_i/L_A)^{\rho_i - 1}}{L_A}, \quad i = 1, \dots, M; \quad \frac{\partial F}{\partial L_A} = - \frac{\gamma \sum_{i=1}^M \alpha_i \rho_i (L_i/L_A)^{\rho_i}}{L_A} \quad (11)$$

**Marginal contributions** of  $L_i$  to  $L_A$ :  $\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i}$ , and *marginal rates of substitution* (**MRS**) are given by implicit differentiation of  $F(L_A, L_1, L_2, \dots, L_M)$ , i.e. we get by (10-11):

$$\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i} = - \frac{\partial F / \partial L_i}{\partial F / \partial L_A} = \frac{\alpha_i \rho_i (L_i/L_A)^{\rho_i - 1}}{\sum_{i=1}^M \alpha_i \rho_i (L_i/L_A)^{\rho_i}} > 0, \quad i = 1, \dots, M \quad (12)$$

$$\frac{dL_j}{dL_i} = MRS = \frac{\partial f / \partial L_i}{\partial f / \partial L_j} = \frac{\alpha_i \rho_i (L_i / L_A)^{\rho_i - 1}}{\alpha_j \rho_j (L_j / L_A)^{\rho_j - 1}}, \quad i \neq j \quad (13)$$

The elasticities,  $E(L_A, L_i)$ , *shares* ( $\varepsilon_i$ ), add up to 1 by the degree of homogeneity in (10),

$$\varepsilon_i = E(L_A, L_i) \equiv \frac{\partial L_A}{\partial L_i} \frac{L_i}{L_A} = \frac{\alpha_i \rho_i (L_i / L_A)^{\rho_i}}{\sum_{i=1}^M \alpha_i \rho_i (L_i / L_A)^{\rho_i}}; \quad \sum_{i=1}^M \varepsilon_i = 1 \quad (14)$$

**Relative wages** - relative factor prices, (13) - must reflect their **MRS**. Hence CRESH *relative wages*,  $(\frac{w_i}{w_j})$ , CRESH *relative wage income shares*,  $(\frac{\varepsilon_i}{\varepsilon_j} = \frac{w_i L_i}{w_j L_j})$ , become by (13-14):

$$\frac{w_i}{w_j} = \frac{\alpha_i \rho_i (L_i / L_A)^{\rho_i - 1}}{\alpha_j \rho_j (L_j / L_A)^{\rho_j - 1}} = \frac{\alpha_i \rho_i L_i^{\rho_i - 1}}{\alpha_j \rho_j L_j^{\rho_j - 1}} L_A^{\rho_j - \rho_i}, \quad i \neq j; \quad \frac{\varepsilon_i}{\varepsilon_j} = \frac{\alpha_i \rho_i (L_i / L_A)^{\rho_i}}{\alpha_j \rho_j (L_j / L_A)^{\rho_j}} \quad (15)$$

These CRESH expressions emphasize the *relative wage effects* of particular labor *supplies* (pair),  $L_i, L_j$ , the *substitution* parameters,  $\rho_i, \rho_j$ , and the relative *intensity* parameters,  $\alpha_i, \alpha_j$ . Via **Total** variable  $L_A$ , *all* variables  $L_i$ , and *all* parameters in (7) affect  $(\frac{w_i}{w_j})$ , (15).

The special cases of (15) for the CD-CES family (9) become  $(1 - \rho = \frac{1}{\sigma})$  :

$$CD : \frac{w_i}{w_j} = \frac{\alpha_i L_j}{\alpha_j L_i}; \quad CES : \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \left[ \frac{L_i}{L_j} \right]^{\rho - 1}; \quad Linear : \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j}; \quad i \neq j \quad (16)$$

If (9) takes the linear form, *relative wages* (16) depend *only* on relative intensity *parameters*,  $\alpha_i, \alpha_j$ , whereas,  $L_i, L_j$ , *also* affect CD, CES, (16). On CRESH aggregator, see Conlon (1993); for discussions of empirical estimation of CRESH, see Weiss (1977, p.765).

*Changes* in relative wages,  $w_i/w_j$ , (15), are *smaller*, the *higher* is the value of  $\rho_i$ . Intuitively, the *more flexible* a labor supply is (higher value of  $\rho_i$ ), the *smaller* change in its *relative wage* is required to *clearing* the *labor markets* (supply-demand equilibrium) for the given change in supply of the labor service,  $L_i$ .

The CRESH elasticity of the wage ratio,  $(w_i/w_j)$ , with respect to the labor supply ( $L_i$ ), is simply obtained from, (15), (14), (by elasticity rules for composite functions):

$$E\left[\frac{w_i}{w_j}, L_i\right] = \rho_i - 1 + (\rho_j - \rho_i) \varepsilon_i < 0; \quad E\left[\frac{w_j}{w_i}, L_i\right] = 1 - \rho_i + (\rho_i - \rho_j) \varepsilon_i > 0 \quad (17)$$

where  $\varepsilon_i$  is *labor income share* of  $L_i$ . Thus by (17), *increasing*  $L_i$  will always *decrease* the **CRESH** *relative wage* of  $L_i$ ; but the higher  $\rho_i$  is, the smaller is the percentage decline in  $(w_i/w_j)$ ; a higher  $\rho_j$  has a similar effect on diminishing the decline in  $(w_i/w_j)$  as  $\rho_i$ . Moreover, a larger  $L_i$  will always *increase* the CRESH *relative wages* of  $L_j$  (other labor groups compared to  $L_i$ ); the higher  $\rho_i$  is, the larger is the relative increase in  $(w_j/w_i)$ ; the effect of higher  $\rho_j$  gives a smaller increase in  $(w_j/w_i)$ , as a result of larger  $L_i$ .

By (15) - and using the same elasticity rules above - we also here note that,

$$E\left[\frac{\varepsilon_i}{\varepsilon_j}, L_i\right] = \rho_i + (\rho_j - \rho_i)\varepsilon_i > 0 \quad ; \quad E\left[\frac{\varepsilon_j}{\varepsilon_i}, L_i\right] = -\rho_i + (\rho_i - \rho_j)\varepsilon_i < 0 \quad (18)$$

Thus, in *contrast* to their *relative wages* in (17), the *relative labor shares* of  $L_i$  in (18), always *increases* with larger  $L_i$  ; moreover, the **CRESH** *relative labor income shares* of the *other* labor groups  $L_j$  *decline*, when  $L_i$  is increased.

The labor *services*,  $(L_1, L_2, \dots, L_M)$ , can refer to **any** disaggregation of *labor supply*. Our services relate to **labor age groups** ; hence CRESH (15) relative wages will represent : **Age Group Wage Differentials** - to be linked up to *demographic labor supply projections*.

Since logically,  $E\left(\frac{w_i}{w_j}, L_i\right) = E(w_i, L_i) - E(w_j, L_i)$ ,  $E\left(\frac{\varepsilon_i}{\varepsilon_j}, L_i\right) = E(\varepsilon_i, L_i) - E(\varepsilon_j, L_i)$ , we should *embed* the pairwise CRESH *relative annual wage* relations and **ratio** elasticities (13-18) into a **complete** CRESH framework of **comparative statics** for the **absolute 'own-price'**,  $E(w_i, L_i)$ , **'cross-price'**,  $E(w_j, L_i)$ , *wage* elasticities, factor *share* (distributional) elasticities, and hereto labor *substitution* and labor *complementarity* elasticities.

All these elasticities and the **basic economic** implications of CRESH function (7-8) are revealed and derived below by using **duality** theory for *implicit* CRESH *Aggregator* function, (10) :  $f(L_1, L_2, \dots, L_M)$ , *Wage Cost* function,  $C(w_1, w_2, \dots, w_M, L_A)$ , and *Wage Income* function,  $W(L_1, L_2, \dots, L_M, W_A)$ . Our **new** and important expressions for *labor complementarity elasticities* ( $c_{ij}$ ) are derived for CRESH, (7), (10), in **Appendix B**<sup>7</sup>.

### 3.1 CRESH model calibration and validation : 2010, 2013

In **Table 4** (Col.2,5,6c) is collected the **2010 data** of wage shares,  $(\varepsilon_i)$ , relative wages,  $(w_i/w_4)$ ,  $w_i$ ,  $i=1, \dots, M$ . The **intensity** (weight) parameters  $(\alpha_i)$  in CRESH Labor supply (7) - are obtained by calibrations, as described below ; cf. Guest and Jensen (2016, p.30).

From (15), we get :

$$\frac{\alpha_i}{\alpha_j} = \frac{\varepsilon_i \rho_j (L_j/L_A)^{\rho_j}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} = \frac{\varepsilon_i \rho_j (L_j)^{\rho_j}}{\varepsilon_j \rho_i (L_i)^{\rho_i}} L_A^{\rho_i - \rho_j} \quad ; \quad i \neq j \quad (19)$$

In (19), 2010 wage **shares**  $(\varepsilon_i)$ ,  $i=1, \dots, M$ , are known (Col.2), and so by making particular **assumptions** (choices) of the *substitution* parameters  $(\rho_i)$ ,  $i=1, \dots, M$  in Col.3, and by using also the 2010 **data**,  $L_i$ ,  $i=1, \dots, M$ ,  $L_A = L$ , from **Table 2** (Col.5), the **ratios** of the intensity parameters,  $(\alpha_i/\alpha_j)$ , can then be derived (**calculated**) from the equation (19).

<sup>7</sup>Ratio elasticity,  $E\left[\frac{w_i}{w_j}, L_i\right]$ , in (17), comes from (87-88) & *complementarity elasticities*,  $c_{ij}$ , (78-79).

Table 4. Wage income shares, ( $\hat{t}$ ), CRESH parameter values, ( $\hat{t}$ ), Relative wages, ( $\hat{t}$ ), Absolute wages, ( $\hat{t}$ ) : Data and Model - Denmark 2010, 2013																	
2013																	
1	2	3	4	5	6a	6b	6c	7	8	9	10	11	12	13	14	15	16
Age ( $\hat{t}$ )	$\hat{\epsilon}_i$	$\hat{\rho}_i$	$\hat{\alpha}_i$	$w_i/w_A$	$w_i/w_A$	$w_i$	$w_i$	$N_i$	$L_i$	$\lambda_i$	$L_i$	$w_i$	$w_i/w_A$	$w_i/w_A$	$w_i$	$w_i$	$L_i/N_i$
	data			data	model	data	data					data	data	model*	model**		data
15-19	0.0061	0.8	0.0316	0.4554	0.4557	187061	187005	358224	17514	0.0123	25929	187815	0.4432	0.4572	193754	201755	0.0724
20-24	0.0327	0.8	0.0598	0.6653	0.6650	272987	273220	359625	67404	0.0472	99498	272254	0.6425	0.6612	280194	291765	0.2767
25-29	0.0649	0.7	0.0756	0.8724	0.8727	358235	358262	322222	111329	0.0780	164435	364345	0.8598	0.8538	361782	376723	0.5103
30-34	0.1084	0.7	0.0970	1.0000	1.0000	410492	410668	328063	150950	0.1057	222817	423743	1.0000	1.0000	423743	441243	0.6792
35-39	0.1410	0.6	0.1070	1.0950	1.0953	449623	449679	371971	188695	0.1322	278680	466584	1.1011	1.0824	458667	477609	0.7492
40-44	0.1563	0.5	0.1134	1.1472	1.1471	470892	471118	388543	199024	0.1394	293857	493182	1.1639	1.1397	482955	502901	0.7563
45-49	0.1548	0.5	0.1131	1.1505	1.1512	472556	472491	427351	217491	0.1523	321051	503375	1.1879	1.0875	460827	479858	0.7513
50-54	0.1373	0.6	0.1090	1.1478	1.1491	471699	471381	374370	189628	0.1328	279944	495068	1.1683	1.1007	466395	485656	0.7478
55-59	0.1210	0.7	0.1089	1.1243	1.1253	461908	461729	353381	172363	0.1207	254437	484135	1.1425	1.0789	457161	476041	0.7200
60-64	0.0658	0.8	0.1078	1.1657	1.1664	478708	478708	341481	92507	0.0648	136599	499596	1.1790	1.1188	474077	493656	0.4000
65-69	0.0117	0.8	0.0768	1.1767	1.1765	482944	483248	352035	20949	0.0146	30777	509897	1.2033	1.0738	455026	473818	0.0874
Total	1.0000		1.0000			437552	437552	3977266	1427854	1.0000	2108014	459463			441241	459463	0.5300
15-69								977596									
0-14								647766									
70+								5602628			2108014	172875					0.3763
Total																	

Source: Year 2010, see Table 2 ; Year 2013 : UNITED NATIONS ; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

- Column 1: Age groups,  $i = 1, \dots, 11$ ;  $i = 1$ : 15-19, ...,  $i = 11$ : 65-69.
- Column 2: Age group wage income shares (wage cost shares in the total wage bill), cf. column 9, Table 2.
- Column 3: Substitution parameters in CRESH function equation.
- Column 4: Intensity parameters in CRESH function equations,  $L_A = L = 2112472$ .
- Column 5: Actual relative wages (Age group annual wage profile), given by data, Table 2.
- Column 6a: CRESH Model (optimal) Relative annual wages (Age group annual wage profile), 2010:  $\gamma = 4.378$ ,  $L_A = L = 2112472$ .
- Column 6b: CRESH Model Absolute annual wages (Age groups) with,  $W(2010) = 437552$ , and with calculated  $w_A(2010) = 410492 = (w_A \text{ tilde})$ .
- Column 6c: The observed Annual wages 2010 : Data, cf. column 6, Table 2.
- Column 7: Population (totals) in age groups, *Statistical Ten-Year Review* (STR), STR (2013, p.19).
- Column 8: Full time (annual equivalent) workers, Labor services, (Labor years), *Micro* (individual, register) data.
- Column 9: Labor age groups, relative sizes - fractions ( $\lambda_i$ ) - obtained from column 8.
- Column 10: Total full time workers, Labor services,  $L_i = \lambda_i L$ ; 2013 : ( $L = 2108014$  : statbank, DB07, ERHV1, full-time employees, total).
- Column 11: Average annual wages of labor age groups; Total Wages,  $W/L = 968.555$  Billion DKK ;  $W = 459463$ ; (statbank, DB07, ERHV1, Annual wage sum,  $W/L = 964.998$  Billion DKK).
- Column 12: Actual relative annual wages ('Age annual wage profile'), given by data from column 11.
- Column 13: CRESH Model (optimal) Relative annual wages ('Age annual wage profile'), by  $w_i/w_A$ ; 2013 :  $\gamma = 4.365$ ,  $L_A = L = 2108014$ .
- Column 14: CRESH Model Absolute annual wages of labor age groups,  $w_i = w_A \times$  column 13, with observed  $w_A(2013) = 423743$  - giving :  $W/L = 930.142$  Billion ;  $W = 441241$ .
- Column 15: CRESH Model Absolute annual wages of labor age groups,  $w_i = w_A \times$  column 13, with calculated  $w_A(2013) = 441243$  - giving :  $W/L = 968.555$  Billion ;  $W = 459463 = W(2013) = \text{Data}$ .
- Column 16: Labor participation rates (LPR) of age groups - derived from columns (7, 10).
- Remark: As mentioned in Note 5, both the Overall LFP and some of the Age-specific LFP,  $L_i = L_i/N_i$  in Table (2,4) (column 10,16) are smaller than often reported as LFP rates. Labor Force in LFP (ILO statistics) consists of : Employed, and Unemployed,  $L_i$  in Table (2,4), (column 5,10), are Full-time equivalents. Unemployment rate, 2010-13: 6 %, (approx. 100.000 persons).

By next *summing* the equation (19), and using,  $\sum_{i=1}^M \alpha_i = 1$ , cf. (8), we have,

$$\sum_{i=1}^M \frac{\alpha_i}{\alpha_j} \equiv \frac{1}{\alpha_j} \equiv \sum_{i=1}^M \frac{\varepsilon_i \rho_j (L_j/L_A)^{\rho_j}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} = \frac{\rho_j [L_j]^{L_j}}{\varepsilon_j [L_A]^{L_A}} \sum_{i=1}^M \frac{\varepsilon_i}{\rho_i (L_i/L_A)^{\rho_i}} \equiv \frac{\rho_j [L_j]^{L_j}}{\varepsilon_j [L_A]^{L_A}} \mu \quad (20)$$

Thus the **size** of  $\alpha_j$  is *determined* by the **RHS** expression of (20). Next (19-20) give :

$$\alpha_i = \frac{\varepsilon_i \rho_j (L_j/L_A)^{\rho_j}}{\varepsilon_j \rho_i (L_i/L_A)^{\rho_i}} \alpha_j = \frac{\varepsilon_i [L_A]^{L_A}}{\rho_i [L_i]^{L_i}} \frac{1}{\mu}, \quad i = 1, \dots, M \quad (21)$$

Hence **all absolute** values of  $\alpha_i$  *parameters* are obtained by (21) - [with  $\mu$  as seen in (20)].

By this **calibration** procedure, (21), such associated values of 11 CRESH **intensity** parameters,  $(\alpha_i)$ , **Table 4**, (Col.4), for 2010 can - with  $L_i, i=1, \dots, M, L_A = L$ , from **Table 2** - be *calculated* for *any* assumptions (*pattern*) of these 11 *parameters*,  $\rho_i$ , Col.3.

The actual **selected** CRESH *substitution* parameters  $\rho_i$  in **Table 4**, (Col.3) were determined as follows. An *initial* set of 11 ( $\rho_i$ ) determines 11 ( $\alpha_i$ ), as described by (19) and (21). The *aim* is to find a *pattern* of  $\rho_i$  which generates CRESH *relative* wages (15) by  $(\frac{w_i}{w_4})$ , (j=4), (22) that *best fit*, (Col.6a), the *actual relative* wages, (2010),  $(\frac{w_i}{w_4})$ , (Col.5).

$$\frac{w_i}{w_4} = \frac{\alpha_i \rho_i L_i^{\rho_i - 1}}{\alpha_4 \rho_4 L_4^{\rho_4 - 1}} L_A^{\rho_4 - \rho_i}, \quad i = 1, \dots, M \quad \Leftrightarrow \quad \frac{w_i}{w_4} = \frac{\alpha_i \rho_i \lambda_i^{\rho_i - 1}}{\alpha_4 \rho_4 \lambda_4^{\rho_4 - 1}} \quad (22)$$

Various patterns of  $\rho_i$  have been tested in this way for *Australia* as discussed in Guest & Jensen (2016). The best fit for *Denmark* is found to be the approximate **U-shape pattern** of  $\rho_i$ , shown in **Table 4**, (Col.3). This U-shape pattern of  $\rho_i$  implies that *middle age workers*, who have relatively *low values* for  $\rho_i$  have a mix of labor attributes ('qualities') that make them *harder to substitute* (replace) [lower  $\rho_i$  give smaller **substitution elasticities**,  $\sigma_{ij}$ , (49)] than the younger or older workers. This pattern has also important consequences for **wages**, relative and absolute [lower  $\rho_i$  give larger labor **complementarity elasticities**,  $c_{ij}$ , (57), for *age group* (i), and so group (i) have larger *annual wage elasticities*,  $E(w_i, L_j) = \varepsilon_j c_{ij}$ , (86), and hence gain larger wage increases by bigger labor supplies of other age groups  $L_j$ ]. The *combined* set (sizes) for  $\rho_i, \alpha_i$ , Col.3-4 fitted best **2010** : (Col.5, 6a); or (Col.6b, 6c) by (28-29),  $\lambda_i$  (22), RHS , cf. footnotes 8-9 below.

To **validate** the calibrated year 2010 CRESH parameter *values*,  $(\rho_i, \alpha_i)$ , **Table 4**, Col.3-4, we corroborate these parameter sizes  $(\rho_i, \alpha_i)$  upon another data set, year 2013. Thus the **2013 data** seen in the six columns, **Table 4**, Col.7-12, correspond (with same content/explanations) exactly to earlier six columns for year 2010, **Table 2**, Col.2-7.

In order to *validate* the calibrated parameters outside the base year (2010), we insert the *calibrated (2010) values*,  $\rho_i$ ,  $\alpha_i$ , (Col.3-4), into our formula of *relative wage*, (22), together with using the observed **2013 data** for :  $L_i$ ,  $i=1,\dots,M$ ,  $L_A = L$ , (Col.10). Thus columns (Col.3-4,10) give by (22) the CRESH *results* for relative wages,  $\frac{w_i}{w_j}$ , (j=4), **Col.13** for year 2013<sup>8</sup> - to be compared with *observed* relative wages,  $\frac{w_i}{w_j}$ , (j=4), **Col.12** for 2013. Apart from age groups (60-64, 65-69), the **Col.12-13** are concurring pretty well for all age groups. So the calibrated CRESH **parameters**,  $\rho_i$ ,  $\alpha_i$ , (Col.3-4), are essentially **confirmed** (validated) on the **new data** set (**2013**).

By (22), only the **relative wage** numbers ( $\frac{w_i}{w_4}$ ) were calculated for 2010 og 2013. But how to get the **absolute** sizes of the *Annual wages* ( $w_i$ ) for *Ages*,  $i=1,\dots,M$ , in **Table 4** ? **Absolute wages**. *Total Wage Income* for all Age groups ( $i$ ) is by definition, **wL**, i.e,

$$wL = \sum_{i=1}^M w_i L_i ; \quad w_4 = \frac{wL}{L_4 + \sum_{i \neq 4} \frac{w_i}{w_4} L_i} ; \quad w_i = w_4(2013) \cdot \frac{\alpha_i \rho_i (L_i)^{\rho_i - 1}}{\alpha_4 \rho_4 (L_4)^{\rho_4 - 1}} L_A^{\rho_4 - \rho_i} \quad (23)$$

Dividing **LHS** of (23) by  $w_4$ , and using  $[\frac{w_i}{w_4}]$ ,  $i \neq 4$ , rearranging, gives  $w_4$ , stated above. With  $w_4$  (23) allows all  $w_i$  for 2013 to be calculated by **RHS** (23), by using *observed*,  $w_4(2013) = 423743$ , **Table 4** (Col.14). However, by  $w_4(2013)$  as 'scaling factor' for  $w_i$ , (23), generates for  $L_A = L(2013) = 2108014$ , the *Total Wages* : **wL** (2013) = 930142 Billion, *Average Annual wage*,  $w(2013) = 441241$ . But Col.11, 15, give for  $L_A = L(2013)$  the **actual Total Wage sum** : **wL** (2013) = 968555 Billion, *Average Annual wage*,  $w(2013) = 459463 \equiv \mathbf{W}_A(2013)$ , i.e.,  $\mathbf{W}_A(2013)$  gives by  $w_4 = \widetilde{w}_4$  as in (24) a **consistent scaling wage** of  $\frac{w_i}{w_4}$  to use in computing **absolute annual wages** ( $\mathbf{w}_i$ ) for age,  $i=1,\dots,M$ , cf. Col.15:

$$w_i = \widetilde{w}_4(2013) \cdot \frac{\alpha_i \rho_i (L_i)^{\rho_i - 1}}{\alpha_4 \rho_4 (L_4)^{\rho_4 - 1}} L_A^{\rho_4 - \rho_i}, \quad i=1,\dots,M ; \quad \widetilde{w}_4(2013) = \frac{\mathbf{W}_A(2013) L}{L_4 + \sum_{i \neq 4} \frac{w_i}{w_4} L_i} \quad (24)$$

All  $w_i$  (24) were still calculated with *chosen*  $\alpha_i$  and  $\rho_i$  parameters from **2010, Table 4**. By using (22), (24), we have a *consistent* CRESH formula for **absolute Age wage** ( $\mathbf{w}_i$ ) *calculations* to be applied *any year* ( $t$ ) - also with  $\mathbf{W}_A$ ,  $\mathbf{w}_i$ , defined by (25), (26) below.

<sup>8</sup>CRESH  $\gamma$ , (7), a "total productivity" (efficiency) *parameter* was *not* involved in *relative wages*, (15). For given values of  $\alpha_i$ ,  $\rho_i$ , *Table 4* (Col.3-4), the  $\gamma$  *size* can be adapted so that *aggregate variable*, (7),  $L_A = L$  (Total Labor force, Labor supply) =  $L(t) = \sum_{i=1}^M L_i$  ; for  $t=2010, 2013$ , see  $\gamma$ , **Table 4** (Col.6,13). Such  $\gamma$  values are to be used for *any year*, if as in *all Tables 4-7*,  $L_A = L(t) = \sum_{i=1}^M L_i(t)$ .

**3.1.1.** From dual CRESH Labor Cost function (52), or dual **Wage Income** function (81) in **App.B**, we have for CRESH -  $\mathbf{F}(\mathbf{L}_A, \mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M) = 0$ , (7);  $\mathbf{L}_A = \mathbf{f}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , **Aggregator** (10) - the *basic duality* relations, cf. (12), (14-15), (52), (67-68), (81), (84):

$$W_A L_A = \sum_{i=1}^M w_i L_i \equiv wL \equiv W \equiv C \equiv c(w_1, w_2, \dots, w_M) L_A; \quad W_A = c(w_1, w_2, \dots, w_M) \quad (25)$$

$$w_i = w_i(L_1, L_2, \dots, L_M, W_A) = \frac{\partial W(L_1, L_2, \dots, L_M, W_A)}{\partial L_i} = W_A \frac{\partial L_A(L_1, L_2, \dots, L_M)}{\partial L_i} \quad (26)$$

$$= W_A \cdot \frac{\partial f(L_1, L_2, \dots, L_M)}{\partial L_i} = W_A \frac{\alpha_i \rho_i (L_i/L_A)^{\rho_i-1}}{\sum_{i=1}^M \alpha_i \rho_i (L_i/L_A)^{\rho_i}} = W_A \frac{\alpha_i \rho_i \lambda_i^{\rho_i-1}}{\sum_{i=1}^M \alpha_i \rho_i \lambda_i^{\rho_i}} \quad (27)$$

$\mathbf{W}_A$  (25): *Arithmetic Average* of all *money age wages* ( $\mathbf{w}_i$ ) - or 'shadow values', (26-27).

For demographic *projection period*,  $t$ : **2015-2090**, we *don't* have - as (24) in **Table 4** - *empirical* values of  $\mathbf{W}_A(t)$ . Throughout the projection period the **exogenously** imputed size to  $\mathbf{W}_A$  (25-27) is  $\mathbf{W}_A(2010)$ , **Tables (2,3)**. Thus our **absolute** Annual wages are:

$$w_i = \widetilde{w}_4 \cdot \frac{\alpha_i \rho_i (L_i)^{\rho_i-1}}{\alpha_4 \rho_4 (L_4)^{\rho_4-1}} L_A^{\rho_4-\rho_i}, \quad \widetilde{w}_4 = \frac{W_A(2010) L}{L_4 + \sum_{i \neq 4}^M \frac{w_i}{w_4} L_i}; \quad i = 1, \dots, M; \quad M = 11 \quad (28)$$

By our  $\gamma$  *calibration*<sup>9</sup>, hence  $\lambda_i \equiv \frac{L_i}{L_A}, \sum_{i=1}^M \lambda_i = 1$ ,  $w_i$  (28) is *equivalent* to, cf. (22-24), (5):

$$w_i = \widetilde{w}_4 \cdot \frac{\alpha_i \rho_i \lambda_i^{\rho_i-1}}{\alpha_4 \rho_4 \lambda_4^{\rho_4-1}}, \quad \widetilde{w}_4 = \frac{W_A(2010)}{\lambda_4 + \sum_{i \neq 4}^M \frac{w_i}{w_4} \lambda_i}, \quad \lambda_i \equiv \frac{L_i}{L} = \frac{l_i n_i}{l_{15-69} n_{15-69}}; \quad i = 1, \dots, M \quad (29)$$

Note that (26-29) give the **same** wages  $\mathbf{w}_i$ , but CRESH *duality* formulas (25-27) provide economic content and intuition. We saw an illustration of (28-29) in **Table 4**, (Col.6b). For  $\mathbf{W}_A(2010) = 437552$  and,  $\mathbf{L}_A = \mathbf{L}(2010) = \sum_{i=1}^M \mathbf{L}_i(2010)$ , with all  $\mathbf{L}_i(2010)$  in **Table 2**, (Col.5), the calculation of  $\widetilde{w}_4$  by (28-29) gives,  $\widetilde{w}_4(2010) = \mathbf{410492}$ ; applying this  $\widetilde{w}_4$  as '*scaling multiplier*' to all wage *ratios*, ( $\frac{w_i}{w_4}$ ), **Table 4** (Col.6a), gives CRESH **absolute** (money) **annual wages**,  $\mathbf{w}_i(2010)$  (Col.6b) - actual observed ( $w_i$ ), **data** are in (Col.6c).

Finally, note CRESH formulas (26-29) in **2010** give *higher wages* for  $w_{60-64}$ ,  $w_{65-69}$  than to  $w_{55-59}$ ,  $w_{45-49}$  (despite *lower* substitution *parameters*:  $\rho_{55-59}$ ,  $\rho_{45-49}$ , (Col.5). The influence of much *smaller* Labor *supplies* (scarcity) of  $L_{60-64}$ ,  $L_{65-69}$ , **Table 2** (Col.5), *dominate* (22), (27-28), and explain the *high*,  $w_{60-64}$ ,  $w_{65-69}$ , in both model/data 2010.

CRESH **Age wage profiles**, (26-29), of the age-groups *over time* are complex, but versatile - as will be seen in projected **calendar years**, and over entire **cohort life times**.

<sup>9</sup>See footnote 8 - where for year 2010:  $\gamma = 4.378$ . Using (29),  $\gamma$ 's to **Tables 5-7** are not needed.

## 3.2 Disaggregations of Labor Supply - CRESH Subaggregators

The Register based Columns (5-6), **Table 2**, of *age-specific* (labor, wage) *data*,  $(L_i, w_i)$ ,  $i=1, \dots, 11$ , [making 75 % of GDP (Value Added),  $wL/Y = 924.3/1228.1 = 0.75$ , cf. **Table 3**] form directly by *Column (7) empirical points* outlining a *shape*, seen below in **Fig. 2d**.

The *Age-wage profile* in Columns (6-7) refers to the **complete** Danish **Labor supply** (age 15-69), year 2010 (in *full time equivalents*) : men, women, every occupation, private and public sector, all lengths of schooling, educations, etc.<sup>10</sup>

Standard *Human capital* (Labor quality levels) models posit that earnings (wages) rise with *levels* (years) of *Schooling* (5-7, 9-11, 13-15), or with *Education levels* (High school, College, Graduate school), or with *Occupational* classifications [blue-collar (skilled/crafts-men, unskilled) workers, white-collar (professionals, administrators, clerical) employees].

If available *data* of *age-specific* Labor inputs and wages,  $(\mathbf{L}_i, \mathbf{w}_i)$ , are *disaggregated* (by subscript) :  $(\mathbf{L}_{iJ}, \mathbf{w}_{iJ})$ , into e.g., **8 quality levels (J)**, we may construct **8 CRESH Subaggregators**,  $\mathbf{L}_{AJ} = \mathbf{f}_J(\mathbf{L}_{1J}, \mathbf{L}_{2J}, \dots, \mathbf{L}_{MJ})$ , cf. (10), and hence analogous to (15) get **wage ratios**,  $\frac{w_{iJ}}{w_{jJ}}$  ; by analogous **duals** of, (25-27),  $W_{AJ} L_{AJ} = \sum_{i=1}^M w_{iJ} L_{iJ} \equiv w_J L_J \equiv W_J$ , the **money wages** ( $w_{iJ}$ ) of *ages* and *qualities* of Labor input/supply ( $L_{iJ}$ ) become :

$$w_{iJ}(t) = W_{AJ} \frac{\partial f_J(L_{1J}, L_{2J}, \dots, L_{MJ})}{\partial L_{iJ}} = W_{AJ} \frac{\alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}-1}}{\sum_{i=1}^M \alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}}}, \quad i = 1, \dots, M, \quad J = I, II, \dots, VIII \quad (30)$$

For each **year** (t), disaggregated data  $(\mathbf{L}_{iJ}, \mathbf{w}_{iJ})$  can for each **level (J)** be organized by *age* (i) as in **Table 2**, Col.(5-6), and the analogous CRESH wage formulas, (29), for each level (J) can be implemented for  $w_{iJ}(t)$ , (30), as in **Table 4**, Col.(6a, 6b, 6c), (11,13,15).

Estimating *different Age-wage profiles*,  $\mathbf{w}_{iJ}$ ,  $i=1, \dots, M$ , (30), corresponding to *each* school level (J), Hanoch (1967, p.315-319) obtained **8 Age-wage** profiles of essentially *similar* shape, but stacked *vertically* above each other with higher school level (J).

Although Hanoch (1967) did not formally use Labor subaggregator functions, but *vertical*

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<sup>10</sup>The high wages of two *age groups*, (60-64, 65-69), in the Danish Age-wage profiles, **Fig. (2d, 2e)**, **Fig. (12, 13)**, a puzzle, are to some extent, partly due to their high proportions of Public Sector employees with seignior wage systems (Medical profession in Public Hospitals, other Academics in Government Services (including Universities, Secondary Schools). Lower paid Public Sector employees in Primary School and Hospitals have mostly retired by age 65 in 2010 - as in the Private Sector.

Age-wage profiles on *Disaggregated* Labor data should re-establish global concavity of age-wage profiles.



shifting<sup>11</sup>- on *disaggregated* data - of age-wage profiles by a *specific level* (exogenous) variable is seen as an extension of parametric CRESH age-wage formula (27), where  $W_A$  (26) is an *exogenous* vertical shift variable of all (*one* quality level) wages,  $w_i$ ,  $i=1,\dots,M$ . Thus all *disaggregated* CRESH Age-wage profiles,  $w_{iJ}(t)$ , (30), are for any year (t) stacked vertically<sup>12</sup> through their quality level *Average wage*,  $W_{AJ}$ , cf. (25).

To see clearly how **Subaggregators**,  $L_{AJ} = f_J(L_{1J}, L_{2J}, \dots, L_{MJ})$ , work in (30) without attention to Total Labor supply numbers ( $L_{AJ}$ ), we may recall that CRESH function, (7), is *homogeneous* of degree **zero** :  $F(L_A, L_1, L_2, \dots, L_M) = F(1, \lambda_1, \lambda_2, \dots, \lambda_M)$ , and that *Aggregator*,  $L_A = f(L_1, L_2, \dots, L_M)$ , (10), *Subaggregators*, are *homogeneous* of degree **one** - implying that all their *partial derivatives* :  $\frac{\partial f_J(L_{1J}, L_{2J}, \dots, L_{MJ})}{\partial L_{iJ}}$ , are *homogeneous* functions of degree **zero**, such that as stated in (12), (30), we have partial derivatives :

$$\frac{\partial f_J(L_{1J}, L_{2J}, \dots, L_{MJ})}{\partial L_{iJ}} = \frac{\partial f_J(\lambda_{1J}, \lambda_{2J}, \dots, \lambda_{MJ})}{\partial \lambda_{iJ}} = \frac{\alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}-1}}{\sum_{i=1}^M \alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}}}, \quad i = 1, \dots, M, \quad J = I, \dots, VIII \quad (31)$$

By CES (9), the derivatives (31) are much simplified, as denominator above drops out:

$$\forall i : \rho_{iJ} = \rho_J, \quad \frac{\partial f_J(\lambda_{1J}, \lambda_{2J}, \dots, \lambda_{MJ})}{\partial \lambda_{iJ}} = \alpha_{iJ} \lambda_{iJ}^{\rho_J-1}, \quad i = 1, \dots, M, \quad J = I, \dots, VIII; \quad \sum_{i=1}^M \alpha_{iJ} \lambda_{iJ}^{\rho_J} = 1 \quad (32)$$

Only *derivatives* (31-32) of *Aggregator* functions,  $L_{AJ} = f_J$ , are used in *imputing wages*  $w_{iJ}$ , (30), to the *age-groups* (i) of Total Labor supply,  $L_{AJ} = \sum_{i=1}^M L_{iJ}$ .

The *derivatives* (27), (31-32) are **not** *marginal products* (output) of  $L_i$  in age group (i), but *marginal contributions* of  $L_i$  to  $L_A$  by the *Aggregator* function,  $\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i}$ , cf. (12); *marginal contributions*,  $\frac{\partial L_A}{\partial L_i}$ , do *not* depend on (invariant to) the *absolute* sizes of ( $L_A, L_i$ ), but only upon the *size* of  $\lambda_i$ , in CES, (32) - and upon **all**  $\lambda_i$  with CRESH, (31).

As in (15) efficient utilization of Labor supplies - within  $L_{AJ} = \sum_{i=1}^M L_{iJ}$ ,  $J = I, II$  - requires that the ratio (relative) of age-wages were equated to the ratio (relative) of their marginal contribution :  $\frac{w_{iJ}}{w_{jJ}} = \frac{\partial f_J(\lambda_{1J}, \lambda_{2J}, \dots, \lambda_{MJ}) / \partial \lambda_{iJ}}{\partial f_J(\lambda_{1J}, \lambda_{2J}, \dots, \lambda_{MJ}) / \partial \lambda_{jJ}}$ ,  $i \neq j$ . With data and the accounting identities,  $W_{AJ} L_{AJ} = \sum_{i=1}^M w_{iJ} L_{iJ} \equiv W_J$ ,  $J = I, \dots, VIII$ , we have the *Average wages* :  $W_{AJ} = W_J / L_{AJ}$ ,  $J = I, \dots, VIII$ . Thus, by (31) and,  $W_{AJ}$ , we have also the *absolute money wages* ( $w_{iJ}$ ) for *all* *Age groups* (M) in *all* the *Labor categories* (qualities), (VIII) :

$$w_{iJ}(t) = W_{AJ} \frac{\partial f_J(\lambda_{1J}, \lambda_{2J}, \dots, \lambda_{MJ})}{\partial \lambda_{iJ}} = W_{AJ} \frac{\alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}-1}}{\sum_{i=1}^M \alpha_{iJ} \rho_{iJ} \lambda_{iJ}^{\rho_{iJ}}}, \quad i = 1, \dots, M, \quad J = I, \dots, VIII \quad (33)$$

which are the CRESH **Calendar year** (t) **Age-wage profiles**, (30), , restated in  $\lambda_{iJ}$ .

<sup>11</sup>The disaggregated age-wage profiles not only shift vertically, but they may also twist/rotate.

<sup>12</sup>Age-earnings (wage) profiles from education have a long economic history, Blaug (1967, p.337).

Thus analytic **wage structure** description for *different* Labor *qualities* (education levels) require the analytic tools of CRESH **Subaggregator**,  $L_{AJ} = f_J(L_{1J}, L_{2J}, \dots, L_{MJ})$ , *derivatives* (31) as used in (33)<sup>13</sup>. See hereto the *Canonical Model* in **Appendix C**. Finally, we note that changes in *wage structure* (distributions) can be *analyzed* in calendar years (section 4) by the apparatus of CRESH Labor *Aggregators* [not using solely total Labor supplies, but only *age* distributions,  $\lambda_{iJ}$ ,  $i = 1, \dots, M$ ] - *without* production functions.

### 3.3 Outputs and Multi-factor CRESH Production Functions

For a long time, the scope of *Macro* (Y) models has been enlarged by increasing the number of *primary factors*. But here a problem has also existed for years, viz. that with more than two factors the multi-factor CES function has the *same constant* substitution *elasticity* ( $\sigma$ ) between any and all factors - *severe* restriction that we *removed* by CRESH Labor *aggregator*,  $F(L_A, L_1, L_2, \dots, L_M) = 0$ , (7), (10), and the *Sub-aggregators* above.

The CRESH functional form can also be used to CRESH implicit *production* functions:

$$G(Y, X_I, X_{II}, \dots, X_V) = G(Y, L_I, L_{II}, K_{III}, K_{IV}, K_V) = \gamma \sum_{J=I}^V \alpha_J \left[ \frac{X_J}{Y} \right]^{\rho_J} - 1 = 0 \quad (34)$$

$$\gamma > 0; \quad \forall J: \alpha_J > 0, \quad \sum_{J=I}^V \alpha_J = 1; \quad \forall J: 0 < \rho_J \leq 1 \quad \text{or} \quad \rho_J < 0 \quad (35)$$

where the *parameters* (35) again preserve the important *global* regularity properties.

As in (10), a **unique** implicit **production function**,  $\mathbf{Y} = \mathbf{g}(\mathbf{X}_I, \mathbf{X}_{II}, \dots, \mathbf{X}_V)$  exists and  $g$

$$\forall X_J > 0 : Y = g(X_I, X_{II}, \dots, X_V) > 0; \quad \frac{\partial g}{\partial X_J} > 0, \quad \frac{\partial^2 g}{\partial X_J^2} < 0; \quad Y = \sum_{J=I}^V \frac{\partial g}{\partial X_J} X_J \quad (36)$$

having all the globally regularity properties as the Labor Aggregator,  $L_A = f$ , (10).

All expressions and illustrations of the *Substitution* elasticities and the *Complementarity* elasticities in **Appendix A-B** carry over to (34-36).

Old problems with *different* substitution elasticities between **two** Labor categories,  $L_I$ ,  $L_{II}$ , and various **nonlabor** inputs such as services of *Capital* goods<sup>14</sup>, (34), can be

<sup>13</sup>Labor Aggregator derivatives, (27), (30), (33), are analogous to 'Inverse factor (consumer) demand functions' by derivatives of production (utility) functions; first-order and second-order derivatives define *complementarity* elasticities,  $c_{ij}$ , (56), (83), giving *wage* elasticities, (63), (86-88), w.r.t Labor *supplies*.

<sup>14</sup>See Berndt and Cristensen (1974, p.391-92) ; cf. skill-biased technological change in footnote 25.

resolved with proper *Macro* wage numbers assigned to  $\mathbf{W}_{AJ}$ ,  $J = I, II$  - and subsequently used for the *Age-wage profiles* of the two Labor Subaggregates,  $\mathbf{w}_{iJ}$ , in (33-34).

Analogously to (27), Macro money wages  $\mathbf{W}_{AJ}$ ,  $J = I, II$ , are simply derived from **CRESH** macro *production function*, (34-36) (single output,  $Y$ ) and the *output price* ( $P$ ):

$$W_{AI} = P \cdot \frac{\partial g(L_I, L_{II}, K_{III}, K_{IV}, K_V)}{\partial L_I} = P \cdot \frac{\alpha_I \rho_I (L_I/Y)^{\rho_I - 1}}{\sum_{J=I}^V \alpha_J \rho_J (X_J/Y)^{\rho_J}} \quad (37)$$

$$W_{AII} = P \cdot \frac{\partial g(L_I, L_{II}, K_{III}, K_{IV}, K_V)}{\partial L_{II}} = P \cdot \frac{\alpha_{II} \rho_{II} (L_{II}/Y)^{\rho_{II} - 1}}{\sum_{J=I}^V \alpha_J \rho_J (X_J/Y)^{\rho_J}} \quad (38)$$

Depending on the evolution (time series) of factor productivities (unit requirements),  $(L_I/Y, L_{II}/Y, K_{III}/Y, K_{IV}/Y, K_V/Y)$ , the sizes of the two **Macro wages** (37), (38), are changing, which **shift** the **Calendar year Subaggregate** (Micro) *Age-wage profiles*, (33). Shifting of (33) by  $W_{AJ}$ , (37-38), does not alter the *shape* of (33) and its *relative* wages.

### 3.3.1. Inverse Labor demands - Wage functions - Age-wage profiles, and Empiric methods

Standard labor demand analyses have estimated various *explicit* production functions,  $Y = G(L_I, L_{II}, L_{III}, L_{IV}, L_V, K_I, K_{II})$ , as e.g., Trans-Log, Freeman (1979), cf. Introduction, Hamermesh & Grant (1979, p.538; 1981, p.357), or Generalized Leontief production function, Borjas (1986, p.59), to obtain relevant Labor demand functions, complementarity elasticities and partial wage elasticities - survey in Hamermesh (1993). In Multi-factor *Production functions*, many *classifications* into Labor & wage sub-groups were used: various occupations, educations (length of schooling), gender (male, female), age (young, middle age, old). However, we are *not* using production functions at all ; we have no proper data for capital inputs (quantities or their factor prices). Instead, we have for our purposes a complete data set of Danish Labor supplies and wages, seen in sections 2.2, 3.1. Hence we used (constructed) and estimated (calibrated) the CRESH *Labor Aggregator function*,  $L_A = f(L_1, L_2, \dots, L_M)$ , (7-8), (10), and accordingly here get its *Inverse Labor demand* system as,  $w_i = W_A \cdot \frac{\partial f}{\partial L_i}(L_1, \dots, L_M)$ ,  $i = 1, 2, \dots, M$ , cf. (92), or as *wage functions* also called *Age-Annual Wage profiles*, which in *explicit parametric CRESH form* is stated in the *equation*, (27). The CRESH *Age-Annual wage* formula (27) is used in sections 4-5 to perform analytic '**controlled experiments**' of *Demographic impacts* upon Calendar year wages and Cohort life cycle wages. In these scenarios, the benchmark value of  $W_A$  is  $\mathbf{W}_A(2010)$  - and hence (27) becomes '*operative*' as (28-29).

## 4 Demographics, Labor supplies, and Calendar wages

### 4.1 Projected labor age groups, relative wages, annual wages

Danish Population sizes,  $\mathbf{N}_i(\mathbf{t})$ , for the **11** *age-groups* ( $i$ ) of *working life* (15-69) - obtained from United Nations (2015) source, cf. **Table 1** - are seen in **Tables (5a, 5b)**, Col.1.

Danish Labor Supplies (full-time workers),  $\mathbf{L}_i(\mathbf{t})$ , (39), in *age-groups* ( $i$ ) - calculated by  $\mathbf{N}_i(\mathbf{t})$ , (4), and *Labor Participation* rates,  $\mathbf{l}_i(2010)$ , **Table 2** - are **Tables (5a,5b)**, Col.2.,

$$L_i(t) = l_i(2010) \cdot N_i(t), \quad t = 2020, 2030, 2040, 2050, 2070, 2090; \quad L(t) = \sum_{i=1}^M L_i(t) \quad (39)$$

e.g.,  $L_{15-19}(2020) = 0.0855 \cdot 338740 = 28962$ ,  $L_{35-39}(2030) = 0.7468 \cdot 412710 = 308212$ .

The Participation rates as  $\mathbf{l}_i(2010)$  are held **constant** through the whole *demographic projection period* (2020-2090), and for **all** (*Medium, Low, High*) demographic variants.

In the *three* Fertility variants described in **Table 1a**, the *Fertility* change *commences* in **2015**, cf. **Fig.1**. This implies that *Labor Supplies*,  $L_i(t)$ , starting  $i = 15$  - are *equal* for all fertility variants **until** 2030 ; hence 2035 is the *first* five year period in which labor supplies,  $L_i(t)$  *differ* across the *three* fertility variants. Hence we report population, labor supplies for 2020 and 2030 separately in **Table 5a**, since these are common to all variants.

But age-specific *wages*  $w_i(t)$  are *not* constant for 2020, 2030, as they have different  $L_i(t)$ .

**Table 5b** extends **Table 5a** for 2040 to 2090 for the *three* fertility variants. In **Tables (5a, 5b)**, last Col. are shown in all years/variants the *Age wage profile* of 2010,  $\mathbf{w}_i(2010)$ .

The **relative** age-group *wages*,  $w_i(t)/w_4(t)$  - calculated by inserting  $L_i(t)$  and total  $L(t)$  from (39) into CRESH, (22), with  $L(t) = L_A$  - are exhibited in **Tables (5a, 5b)**, Col.3.

$$\frac{w_i(t)}{w_4(t)} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\rho_i - 1}}{L_4(t)^{\rho_4 - 1}} L(t)^{\rho_4 - \rho_i}, \quad t = 2020, 2030, 2040, 2050, 2070, 2090 \quad (40)$$

The values for  $w_i(t)/w_4(t)$  in **Table 5b** differ for each variant, as  $\mathbf{L}_i(\mathbf{t})$  (column 2) differ.

The conforming *absolute/money age-group wages*,  $w_i(t)$  - by multiplying (40) with *money wage*,  $\widetilde{w}_4(t)$ , with  $W_A(2010) = 437552$ , cf. (28), (84) - are in **Tables (5a, 5b)**, Col.4 :

$$w_i(t) = \widetilde{w}_4(t) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\rho_i - 1}}{L_4(t)^{\rho_4 - 1}} L(t)^{\rho_4 - \rho_i}, \quad i = 1, \dots, 11; \quad \widetilde{w}_4(t) = \frac{W_A(2010) L(t)}{L_4(t) + \sum_{i \neq 4} \frac{w_i(t)}{w_4(t)} L_i(t)} \quad (41)$$

Thus **Tables (5a, 5b)** present 'comparative' *age-group annual wages* in *two forms* : directly as **ratio**:  $\frac{w_i(t)}{w_4(t)}$ , (40), and on *absolute* income scale as : **money wage**,  $w_i(t)$ , (41).

Both these two forms (40-41) are necessary for calculating and understanding the influence of demographic *projections*  $N_i(t)$  via  $L_i(t)$ , (39), (4), upon *any* and **all** (11) 'age group (i) *annual wages*',  $w_i(t)$ , (41), (29) - **Age-wage profile** - in **Calendar years** ( $t$ ), **Tables (5a, 5b)**.<sup>15</sup> Later *translated*  $w_i(t)$  of (41) are used in *summing* annual wages of *Cohorts*, (43-44), **Table 7**, during their whole *working life* (employment years,  $i : 15-69$ ).

<b>Table 5a. Population and Employment projections, Relative wages, Annual wages - Denmark</b>						
		<b>All Variants</b>				
	<b>Age (<math>i</math>)</b>	$N_i$	$L_i$	$w_i/w_4$	$w_i$	$w_i(2010)$
<b>2020</b>	<b>15-19</b>	338740	28970	0.4617	188933	187005
	<b>20-24</b>	371920	124262	0.6529	267205	273220
	<b>25-29</b>	398400	214793	0.8141	333172	358262
	<b>30-34</b>	353410	248380	1.0000	409251	410668
	<b>35-39</b>	316830	236594	1.1949	489015	449679
	<b>40-44</b>	359770	269781	1.2309	503731	471118
	<b>45-49</b>	377910	282547	1.1996	490917	472491
	<b>50-54</b>	424970	312567	1.0889	445644	471381
	<b>55-59</b>	383500	265371	1.1006	450428	461729
	<b>60-64</b>	345940	119216	1.1868	485695	478708
	<b>65-69</b>	310130	22380	1.1814	483504	483248
<b>Total</b>		3981520	2124861		437552	437552
<b>2030</b>	<b>15-19</b>	309940	26507	0.4858	191356	187005
	<b>20-24</b>	359050	119962	0.6797	267744	273220
	<b>25-29</b>	365930	197287	0.8632	340041	358262
	<b>30-34</b>	394580	277315	1.0000	393930	410668
	<b>35-39</b>	412710	308193	1.1110	437674	449679
	<b>40-44</b>	361800	271303	1.2685	499693	471118
	<b>45-49</b>	320320	239490	1.3465	530442	472491
	<b>50-54</b>	356350	262097	1.2076	475705	471381
	<b>55-59</b>	368240	254811	1.1515	453630	461729
	<b>60-64</b>	407810	140537	1.1871	467615	478708
	<b>65-69</b>	358960	25904	1.1860	467214	483248
<b>Total</b>		4015690	2123406		437552	437552

Source: See **Table 5b**.

<sup>15</sup>For  $t = 2020$  :  $N_{15-19}$  are born in the (*Generation* from) *calendar years* ( $t$ ): 2001-2005 ; similarly,  $N_{20-24}$  born 1996-2000;  $N_{25-29}$  born 1991-1995;  $N_{30-34}$  born 1986-1990;  $N_{35-39}$  born 1981-1985;  $N_{40-44}$  born 1976-1980;  $N_{45-49}$  born 1971-1975;  $N_{50-54}$  born 1966-1970;  $N_{55-59}$  born 1961-1965;  $N_{60-64}$  born 1956-1960;  $N_{65-69}$  born 1951-1955. For  $t = 2030$  :  $N_{15-19}$  are born in *calendar years* ( $t$ ): 2011-2015, and,  $N_{20-24}$  born 2006-2010;  $N_{25-29}$  born 2001-2005;  $N_{30-34}$  born 1996-2000; , ;  $N_{65-69}$  born 1961-1965. For  $t = 2040$  :  $N_{15-19}$  are born in *calendar years* ( $t$ ): 2021-2025; , ;  $N_{65-69}$  born 1971-1975. For  $t = 2050$  :  $N_{15-19}$  are born in *calendar years* ( $t$ ): 2031-2035; , ;  $N_{65-69}$  born 1981-1985. For  $t = 2070$  :  $N_{15-19}$  are born in *calendar years* ( $t$ ): 2051-2055; , ;  $N_{65-69}$  born 2001-2005. For  $t = 2090$  :  $N_{15-19}$  are born in *calendar years* ( $t$ ): 2071-2075; , ;  $N_{65-69}$  born 2021-2025. The *pure* impact of increased *Life Expectancy*, cf. **Table 1a**, upon *Population* numbers  $N_i$  may noted by comparing for  $t=2020$ ,  $t=2030$ , the *corresponding sizes* of the *age group* of *same* birth years - e.g.,  $N_{25-29}(2030) = 365930 > N_{15-19}(2020) = 338740$  ;  $N_{35-39}(2030) = 412710 > N_{25-29}(2020) = 398400$ . The so-called *Millennial* Generation (**Y**), born (**1981-1995**), is seen,  $t = 2020$  as :  $N_{35-39} + N_{30-34} + N_{25-29}$ . The *Generation* (**Z**), born (1996-2010), is seen above,  $t = 2030$  as :  $N_{30-34} + N_{25-29} + N_{20-24}$ .

Table 5b. Population and Employment projections (Labor supplies), Relative annual wages (Age wage profiles), Annual wages - Denmark													
Age (t)	Medium Variant				Low Variant				High Variant				
	$N_i$	$L_i$	$w_i/w_4$	$w_i$	$N_i$	$L_i$	$w_i/w_4$	$w_i$	$N_i$	$L_i$	$w_i/w_4$	$w_i$	$w_i(2010)$
<b>2040</b>													
15-19	345610	29558	0.4708	188390	272370	23294	0.4943	196878	418860	35823	0.6526	181947	187005
20-24	336720	112502	0.6820	272882	293580	98088	0.7017	279471	337320	293580	0.6651	267355	273220
25-29	337320	181862	0.8759	350480	381930	268424	1.0000	348891	381930	268424	0.8759	352103	358262
30-34	381930	268424	1.0000	400119	381930	268424	1.0000	398306	381930	268424	1.0000	401972	410668
35-39	380600	284214	1.1361	454562	380600	284214	1.1350	452058	380600	284214	1.1372	457112	449679
40-44	403130	302296	1.1892	475811	403130	302296	1.1868	472725	403130	302296	1.1915	477049	471118
45-49	403130	310995	1.1693	467867	415960	310995	1.1670	464833	415960	310995	1.1716	470949	472491
50-54	359430	264362	1.1913	476666	359430	264362	1.1901	474040	359430	264362	1.1925	479340	471381
55-59	314110	217355	1.1961	478566	314110	217355	1.1961	476396	314110	217355	1.1961	480782	461729
60-64	345740	119147	1.2154	486304	345740	119147	1.2166	484575	345740	119147	1.2142	488081	478708
65-69	349650	25232	1.1811	472981	349650	25232	1.1823	470901	349650	25232	1.1800	474308	483248
<b>Total</b>	3970200	2115947		437552	3853820	2095269		437552	4086590	2136625		437552	437552
<b>2050</b>													
15-19	358920	30696	0.4578	187440	265200	22681	0.4705	196405	452750	38721	0.4498	181473	187005
20-24	373010	124627	0.6545	268022	278730	93127	0.6712	280211	467300	156130	0.6441	259847	273220
25-29	373030	201115	0.8348	341834	299900	161688	0.8578	358107	446170	240548	0.8184	330170	358262
30-34	359790	252864	1.0000	409479	316730	222601	1.0000	417453	402850	283127	1.0000	403417	410668
35-39	352260	263051	1.1541	472578	352260	263051	1.1051	461321	352260	263051	1.1998	484008	449679
40-44	390850	293087	1.1927	488388	390850	293087	1.1362	474304	390850	293087	1.2460	502659	471118
45-49	384650	287586	1.2009	491733	384650	287586	1.1440	477552	384650	287586	1.2545	506101	472491
50-54	401150	295048	1.1229	459807	401150	295048	1.0752	448854	401150	295048	1.1673	470928	471381
55-59	409140	283113	1.0853	444392	409140	283113	1.0445	436048	409140	283113	1.1227	452915	461729
60-64	351200	121029	1.1869	485996	351200	121029	1.1482	479335	351200	121029	1.2218	492895	478708
65-69	302090	21800	1.1913	487822	302090	21800	1.1526	481136	302090	21800	1.2264	494747	483248
<b>Total</b>	4056090	2174016		437552	3751900	2064811		437552	4360410	2283240		437552	437552
<b>2070</b>													
15-19	355310	30387	0.4675	187910	224120	19168	0.4813	196251	515840	44117	0.4556	182023	187005
20-24	357800	119545	0.6727	270394	246230	82268	0.6805	277619	485980	162371	0.6644	265435	273220
25-29	368850	198861	0.8541	343271	271430	146539	0.8618	351421	472460	254722	0.8468	338317	358262
30-34	383870	269788	1.0000	401926	291060	204560	1.0000	407792	477950	335908	1.0000	399510	410668
35-39	396630	296185	1.1226	451193	303220	226431	1.1277	459879	490150	366021	1.1204	447622	449679
40-44	401890	301366	1.2001	482346	308040	230990	1.2125	494466	495750	371749	1.1939	476972	471118
45-49	390560	292005	1.2159	488720	317960	237725	1.1921	486123	463160	346285	1.2337	492890	472491
50-54	367500	270298	1.1862	476753	324960	239010	1.1242	458453	410040	301586	1.2333	492711	471381
55-59	352640	244016	1.1570	465032	352640	244016	1.0648	434226	352640	244016	1.2357	493656	461729
60-64	383430	132136	1.1887	477769	383430	132136	1.1159	455043	383430	132136	1.2481	498625	478708
65-69	369080	26634	1.1666	468895	369080	26634	1.0951	446591	369080	26634	1.2249	489364	483248
<b>Total</b>	4127560	2181221		437552	3392170	1789277		437552	4916480	2585545		437552	437552
<b>2090</b>													
15-19	356750	30511	0.4678	187931	197880	16923	0.4733	195744	572910	48997	0.4643	182664	187005
20-24	370270	123711	0.6691	268794	214020	71506	0.6714	277672	575620	192321	0.6683	262964	273220
25-29	382930	206453	0.8472	340305	228220	123366	0.8574	354595	581740	313639	0.8404	330648	358262
30-34	387900	276220	1.0000	401696	241260	169560	1.0000	413565	573670	403381	1.0000	393454	410668
35-39	384620	287216	1.1419	458701	253740	189481	1.1280	466488	547600	406801	1.1517	453129	449679
40-44	380400	285251	1.2415	498691	269200	201865	1.1904	492318	508140	381040	1.2835	505013	471118
45-49	382680	286113	1.2363	496622	285770	213658	1.1540	477273	485760	363182	1.3112	515911	472491
50-54	389970	268825	1.1639	467530	297960	219151	1.0841	448345	483250	355433	1.2383	487208	471381
55-59	396800	274574	1.1203	450011	304750	210878	1.0516	434898	488960	338346	1.1833	465578	461729
60-64	396730	136719	1.1824	474956	305110	105145	1.1205	463405	488350	168292	1.2374	486862	478708
65-69	379450	27383	1.1619	466731	309620	22343	1.0881	450019	449280	32422	1.2255	482162	483248
<b>Total</b>	4208500	2217376		437552	2908130	1543876		437552	5752440	3003654		437552	437552

Source:  $N_i$ , United Nations (2015), cf. Tables (1, 2);  $L_i = l_i(2010) \cdot N_i(t)$ , (4), (39);  $w_i(t)$ , (40);  $w_4(t)$ , (41).

Table 5c. Labor supplies and Annual wages of Younger (30-34), Middle aged (45-49), and Older (55-59) workers in Calendar years : 2020, 2030, 2040, 2050, 2070, 2090 – Three Fertility (M, L, H) variants.							
Year		$L_{30-34}$	$w_{30-34}$	$L_{45-49}$	$w_{45-49}$	$L_{55-59}$	$w_{55-59}$
2020	-	248380	409251	282547	490917	265371	450428
2030	-	277315	393930	239490	530442	254811	453630
2040	M	268424	400119	310995	467867	217355	478566
	L	268424	398306	310995	464833	217355	476396
	H	268424	457112	310995	470949	217355	480782
2050	M	252864	409479	287586	491733	283113	444392
	L	222601	417453	287586	477552	283113	436048
	H	283127	403417	287586	506101	283113	452915
2070	M	269788	401926	292005	488720	244016	465032
	L	204560	407792	237725	486123	244016	434226
	H	395908	399510	346285	492890	244016	493656
2090	M	272620	401696	286113	496622	274574	450011
	L	169560	413565	213658	477273	210878	434898
	H	403181	393454	363182	515911	338346	465578

Source: Tables (5a,5b), rows, age (i) : 30-34, 45-49, 55-59.

Based on Table 5c we show in Figures (2a-2c) the annual wages  $w_i(t)$  for *younger* (30-34 years), *middle aged* (45-49 years) and *older* (55-59 years) workers in each of the *calendar years* 2020, 2030, 2040, 2050, 2070, 2090, assuming *three different fertility levels*.

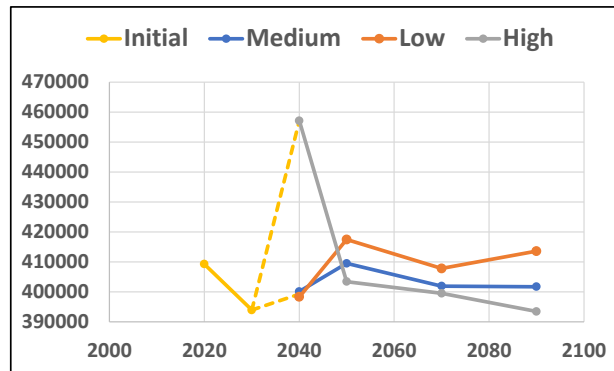
First, it is possible from Table 5c, Fig. (2a-2c), to get an impression of the importance for the *annual wages* of belonging to a *small* (respectively big) *Birth group* (*Generation*), Tables (5a, 5b) and hence their subsequent *Labor supplies* in Table 5c. The clearest example of this is found by looking at  $w_{30-34}$ , Fig. 2a, of the 30-34 years old in 2020,  $L_{30-34}$ , born as a very *small* generation in 1986-1990<sup>16</sup>, next at the *highest*  $w_{45-49}$ , Fig. 2b, of the 45-49 years old in 2030,  $L_{45-49}$ , born as the even *smaller* generation in 1981-1985<sup>17</sup>, and finally at the high  $w_{55-59}$ , Fig. 2c, of the 55-59 years old in 2040,  $L_{55-59}$ , born also in same years, 1981-1985, (Generation, Y). In all cases, the wage-supply response to belonging to **small** generations (early *Millennial*) is a **high** annual wage.

Secondly, Fig.2a shows how *Low* fertility permanently from 2050 creates a scarcity of workers 30-34 years old, resulting in higher wages  $w_{30-34}$  from 2050. Higher fertility does *not* help wages of *younger members* ( $L_{30-34}$ ) of *Labor Supply*,  $L(t)$ ; Fig. (2b-2c), 2050-2090, show clearly how *Higher* fertility *raise* wages of *middle-aged* and *older* workers.

<sup>16</sup>Belonging to (Y),  $N_{30-34}$ ,  $t=2020$ , in footnote 10.

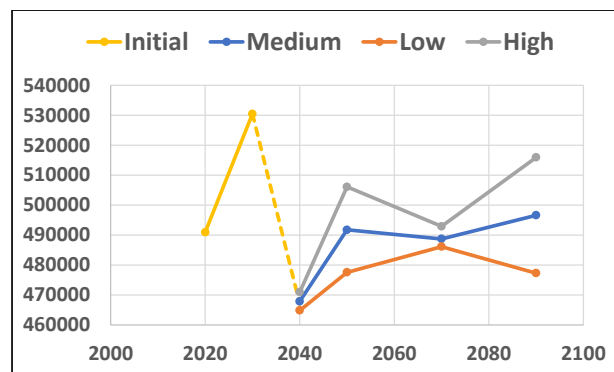
<sup>17</sup>Belonging to (Y),  $N_{45-49}$ ,  $t=2030$ , in footnote 10.

**Fig. 2a.** Annual wages for younger workers  $w_{30-34}(t)$  in calendar years  $(t)$ : 2020, ,2090  
 - in three variants : Medium, Low, High fertility.



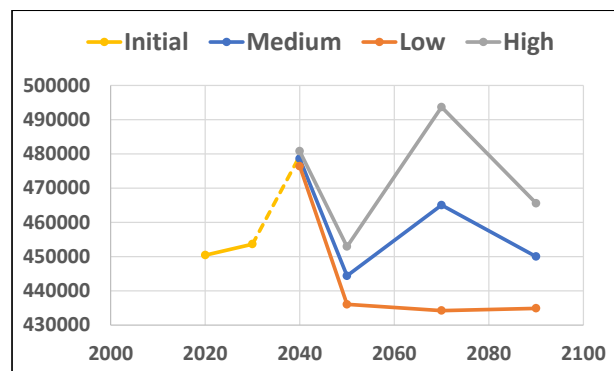
Source: The six numbers of  $w_{30-34}(t)$  - for each variant - are seen in **Table 5c**.

**Fig. 2b.** Annual wages for the age group  $w_{45-49}(t)$  in calendar years  $(t)$ : 2020, ,2090  
 - in three variants : Medium, Low, High fertility.



Source: The six numbers of  $w_{45-49}(t)$  - for each variant - are seen in **Table 5c**.

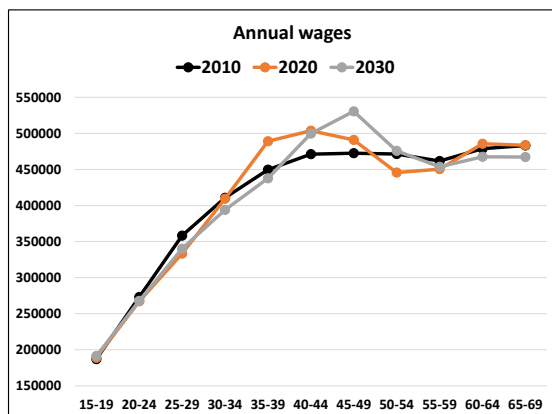
**Fig. 2c.** Annual wages for older workers  $w_{55-59}(t)$  in calendar years  $(t)$ : 2020, ,2090  
 - in three variants : Medium, Low, High fertility.



Source: The six numbers of  $w_{55-59}(t)$  - for each variant - are seen in **Table 5c**.

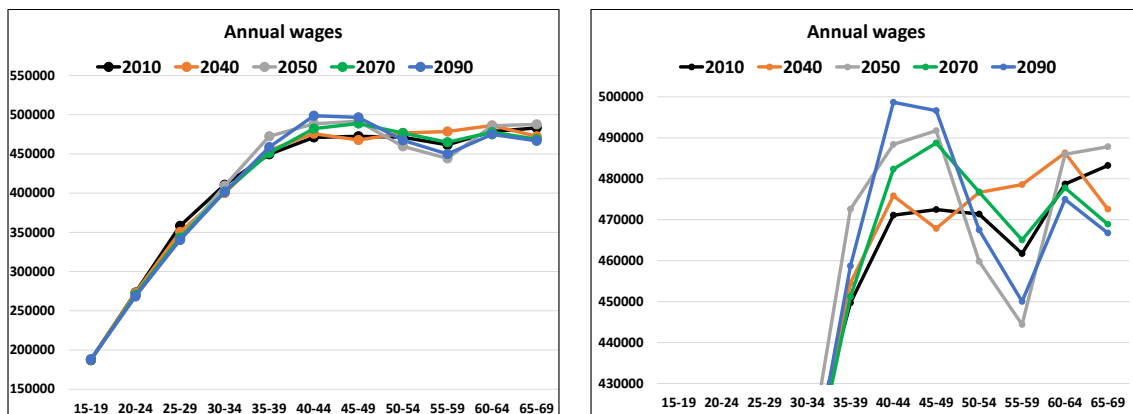


**Fig. 2d.** Age-wage profiles,  $w_i(t)$ ,  $i = 1, 2, \dots, M$ , age group :  $i = 1 = 15-19$ ,  $i = 11 = 65-69$  - for the **calendar** years,  $t = 2010, 2020, 2030$ .



Source: Sizes of annual wages,  $w_i(t)$ , (41), (29),  $t = 2010, 2020, 2030$ , see **Table 5a**.

**Fig. 2e.** Age-wage profiles,  $w_i(t)$ ,  $i = 1, 2, \dots, M$ ,  $i = 1 = 15-19$ ,  $i = 11 = 65-69$  - for the **calendar** years,  $t = 2010, 2040, 2050, 2070, 2090$  : **Medium** fertility.



Source: Annual wages,  $w_i(t)$ , (41), (29),  $t = 2010, 2040, 2050, 2070, 2090$ , see **Table 5b**.

Whereas **Fig. (2a-2c)** focused on the *fertility* variants and showing their wage impacts upon particular age groups at selected time points, we exhibit in **Fig. (2d)** annual *wages*  $w_i(t)$  of *all* (overlapping) 11 age groups ( $i$ ) - *Age wage profile* - for 3 **calendar** years ( $t$ ). The observed *Age wage profile* of 2010,  $w_i(2010)$ , **Table 2** (col.6), is seen (black) in **Fig. (2d)** - and in **Fig. (2e)** (two panels), where  $w_i(t)$  is shown for 5 **calendar** years ( $t$ ).

Apart from the last two age groups (60-64, 65-69), the *shape* of the *calendar* year ( $t$ ) Danish *Age-wage profiles*, **Fig. (2d, 2e)**, are *qualitatively* the same (**concave**) - a shape of Age-wage profiles that we shall later see extended to **Cohorts** ( $T$ ) in **Fig. (12, 13)**.

## 4.2 Age groups, LFP, Support ratios, Annual wages: 2020-2090

**Table 6** provides *demographic summary* variables and *wage incomes* of national accounts. *Row 1* gives the total *working age population size*,  $N_{15-69}$ , which corresponds to the **Totals** of  $N_i$  in **Tables 5a, 5b**, (column 2). *Row 2-4* give the *population sizes*,  $N_{0-14}$ ,  $N_{70+}$ ,  $N$ , from which the Danish *Dependency ratios*, (1) in **Table 1b** were derived.

*Row 5* gives the Labor Force (*Labor Supply*),  $L_{15-69}$ , the **Totals** of  $L_i$  in **Tables 5a, 5b**, (column 3), where the  $L_i$  were generated by (39). *Row 6* (ratio of row 5/row 1) give *sizes of the macro* (endogenous) *Labor Force Participation rate* ( $l_{15-69}$ ), (**LFP**), (2).

The Danish macro **LFP** ( $l_{15-69}$ ), (2), (39), for population projection *period 2010-2090* are shown in **Fig. 3** below. In the *High* fertility scenario, the **LFP** ( $l_{15-69}$ ) is close to *stationary* in the 50 years from 2040 to 2090 as the population  $N_{15-69}$  grows at the same rate as  $L_{15-69}$ . In the *Low* fertility scenario, we find some changes from 2030 to 2050, as the Population 15-69 years old, ( $N_{15-69}$ ), falls more than the Labor Supply, ( $L_{15-69}$ ), while both magnitudes fall at the same rate from 2050.

**Fig. 4** illustrates (based on **Table 6**), the *dramatic long-run* consequences regarding the *composition* of the *population* by age groups *outside* the labor force. For the 0-14 years old ( $N_{0-14}$ ), the **range** is between 10 and 20 percent of the population ( $N$ ) for the *Low*, respectively the *High* fertility case. An even bigger *range* is found for the *share* of the population 70 years and older,  $N_{70+}$ . Until 2070, the *upper part* of **Fig. 4** shows a low fertility '*dividend*'. The *shift* in the *last 20 years* (2070-2090) is due to large increase in the *dependency rate* for the 70+ group,  $d_o$ , (1), cf. **Fig. 1**, in the *Low* fertility projection.

*Row 7* gives the **Support ratios**,  $L/N$ , (3-5), for the period 2020-2090. The *Support ratio* (3) for 2010, **Table 3a**, was given in (6). While Support ratios for Denmark are widely available (World Bank and OECD, for example<sup>18</sup>), our calculations in **Tables (5a, 5b)** show how  $L = L_{15-69}$  in the *Support ratio*, **Table 6** (row 5), are obtained as the *Total* of the *same 11 age-specific* Labor supplies  $L_i(t)$  that are used for calculating the CRESH *relative* wages, (40), CRESH *absolute* wages, (41), seen in **Tables (5a, 5b)**.

Danish **Support ratios** for the whole population,  $L/N$ , (3), (6), for 2010-2090 are shown in **Fig. 5**. Compared with the **LFP** ( $l_{15-69}$ ) in **Fig. 3**, the only difference is - as

<sup>18</sup>See, <http://data.worldbank.org/> ; <https://data.oecd.org/society.htm>.

expected - found in the *terminal* year **2090**, where the *Support* ratio ( $L/N$ ) in the *High* fertility case is *higher* than in the *Low* fertility case, as a consequence of the increasing *share* of 0-14 years old,  $N_{0-14}/N$ , cf. **Fig. 4**.

Row 8 gives **Total Wage** Income ( $wL$ ), as *sum* of all the **11** age groups [ $L_i(t) \cdot w_i(t)$ ] in **Tables (5a, 5b)**, (columns 3, 5), in *calendar year* ( $t$ ).

Row 9 gives (as explained for **Table 3a**) similarly, *Total Wage Income per capita*,  $wL/N$ , decomposed as :  $\mathbf{W}_A \cdot \text{Support ratio}$ , with  $\mathbf{W}_A = \mathbf{w}(2010) = 437552$ . The  $\mathbf{w} \cdot \mathbf{L}(t)/\mathbf{N}(t)$  for 2020-2090 are shown in **Fig. 6**. Over *30 years*, 2020-2050, wage income *per capita* in **Fig. 6** is significantly *lower* - the *higher* the fertility is - as high growth first in the *younger* parts ( $L_i$ ) of the Labor supplies,  $L(t)$ , due to imperfect substitution with their CRESH *substitution/complementarity* parameters, ( $\rho_i$ ), **Table 4**, (Col.3) - implies *lower* productivity/wages. Over the next decades (*after 2050*) this effect is stabilised (stopped), as *Higher* fertility results in larger *increases* in the *Labor supplies* at *all* ages ( $i$ ). Moreover, at the *Macro* (aggregate) level we note the simple *proportionality* relation -  $\mathbf{W}_A \cdot \text{Support ratio}$  - between **Fig. 5** and **Fig. 6**.

The *average annual wage*  $\mathbf{W}_A$  for our Labour aggregator (Aggregated Labour supplies),  $L_A(t) = L(t)$ , is an *exogenous constant* for any *Calendar year* ( $t$ ) by assumption:

$$W_A(2010) = W_A(2020) = W_A(2030) = W_A(2040) = W_A(2050) = W_A(2070) = W_A(2090) \quad (42)$$

Aggregate wages  $wL$  is *any* year allocated to workers by *age* according to (12-15),(25-29). Despite (42) and **Fig. (5,6)**, *annual wages*,  $w_i(t)$ , of particular *Age groups* ( $i$ ) or *Generations* are certainly *non-constant* for *Calendar* years, as seen in **Fig. (2a-2e)**. More on this below; cf. **Table 8D**, and Age group wages,  $w_i(t)$ , as "*shadow values*" (*marginal value-added* :  $W_A \partial f / \partial L_i$ ) in (84).

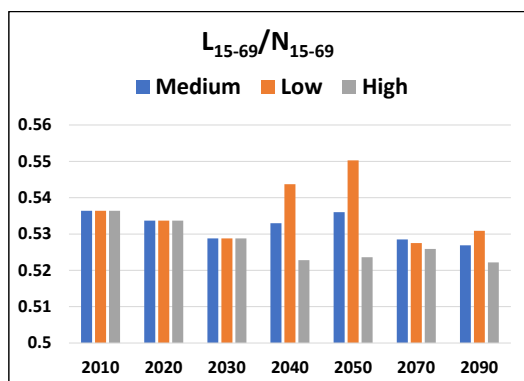
Row 10 provides the *macro* values of ( $Y/N$ ), (6), in accordance with *Support ratio*, ( $L/N$ ) (row 7), and *macro* Labour productivity,  $Y/L(2010) = 581972$  DKK, **Table 3** (row 11). We have not shown ( $Y/N$ ) graphically over time - being just *proportional* to **Fig. 5** with the *constant*,  $Y/L(2010)$ . Thus, e.g., **2030**, ( $Y/N$ ) is 213128 DKK in the *Low* variant, and 205629 DKK in the *Medium* variant, cf. **Table 6** (row 10).

<b>Table 6. Population Age Groups, Labor Supply, LFP and Support Ratios, Incomes per capita</b>						
		<b>Medium Variant</b>	<b>Low Variant</b>	<b>High Variant</b>		
2020	1.	$N_{15-69}$	3981520	3981520	3981520	3981520
	2.	$N_{0-14}$	941060	897840	984290	984290
	3.	$N_{70+}$	853060	853060	853060	853060
	4.	$N = Total$	5775640	5732420	5818870	5818870
	5.	$L = L_{15-69}$	2124861	2124861	2124861	2124861
	6.	$L/N_{15-69}$	0.5337	0.5337	0.5337	0.5337
	7.	$L/N$	0.3679	0.3707	0.3652	0.3652
	8.	$wL, Bill.$	929.739	929.739	929.739	929.739
	9.	$wL/N$	160976	162190	159780	159780
	10.	$Y/N$	213884	215512	212314	212314
2030	1.	$N_{15-69}$	4015690	4015690	4015690	4015690
	2.	$N_{0-14}$	990530	779610	1201460	1201460
	3.	$N_{70+}$	997060	997060	997060	997060
	4.	$N = Total$	6003280	5792360	6214210	6214210
	5.	$L = L_{15-69}$	2123406	2123406	2123406	2123406
	6.	$L/N_{15-69}$	0.5288	0.5288	0.5288	0.5288
	7.	$L/N$	0.3537	0.3666	0.3417	0.3417
	8.	$wL, Bill.$	929.102	929.102	929.102	929.102
	9.	$wL/N$	154766	160401	149513	149513
	10.	$Y/N$	205629	213128	198652	198652
2040	1.	$N_{15-69}$	3970200	3853820	4086590	4086590
	2.	$N_{0-14}$	1033640	752420	1316240	1316240
	3.	$N_{70+}$	1168780	1168780	1168780	1168780
	4.	$N = Total$	6172620	5775020	6571610	6571610
	5.	$L = L_{15-69}$	2115947	2095269	2136625	2136625
	6.	$L/N_{15-69}$	0.5330	0.5437	0.5228	0.5228
	7.	$L/N$	0.3428	0.3628	0.3251	0.3251
	8.	$wL, Bill.$	925.839	916.791	934.887	934.887
	9.	$wL/N$	149991	158751	142261	142261
	10.	$Y/N$	199292	210919	189002	189002
2050	1.	$N_{15-69}$	4056090	3751900	4360410	4360410
	2.	$N_{0-14}$	1011340	709050	1337760	1337760
	3.	$N_{70+}$	1231760	1231760	1231760	1231760
	4.	$N = Total$	6299190	5692710	6929930	6929930
	5.	$L = L_{15-69}$	2174016	2064811	2283240	2283240
	6.	$L/N_{15-69}$	0.5360	0.5503	0.5236	0.5236
	7.	$L/N$	0.3451	0.3627	0.3295	0.3295
	8.	$wL, Bill.$	951.247	903.464	999.038	999.038
	9.	$wL/N$	151011	158705	144163	144163
	10.	$Y/N$	200629	210861	191560	191560
2070	1.	$N_{15-69}$	4127560	3392170	4916480	4916480
	2.	$N_{0-14}$	1065040	607470	1655720	1655720
	3.	$N_{70+}$	1375060	1375060	1375060	1375060
	4.	$N = Total$	6567660	5374700	7947260	7947260
	5.	$L = L_{15-69}$	2181221	1789277	2585545	2585545
	6.	$L/N_{15-69}$	0.5285	0.5275	0.5259	0.5259
	7.	$L/N$	0.3321	0.3329	0.3253	0.3253
	8.	$wL, Bill.$	954.400	782.903	1131.313	1131.313
	9.	$wL/N$	145318	145665	142353	142353
	10.	$Y/N$	193071	193536	189118	189118
2090	1.	$N_{15-69}$	4208500	2908130	5752440	5752440
	2.	$N_{0-14}$	1049000	521510	1846430	1846430
	3.	$N_{70+}$	1515320	1475420	1555220	1555220
	4.	$N = Total$	6772820	4905060	9154090	9154090
	5.	$L = L_{15-69}$	2217376	1543876	3003654	3003654
	6.	$L/N_{15-69}$	0.5269	0.5309	0.5222	0.5222
	7.	$L/N$	0.3274	0.3148	0.3281	0.3281
	8.	$wL, Bill.$	970.219	675.528	1314.258	1314.258
	9.	$wL/N$	143252	137721	143571	143571
	10.	$Y/N$	190339	183014	190746	190746

Table 2, 2010:  $w = 437552$  DKK,  $L = 211472$  Labor years,  $wL = 924.317$  Bill. DKK,  $wL/N = 166615$  DKK,  $L/N = 0.3806$   
 $Y/L = 581972$  DKK,  $Y/N = (Y/L)(L/N) = 221499$  DKK.

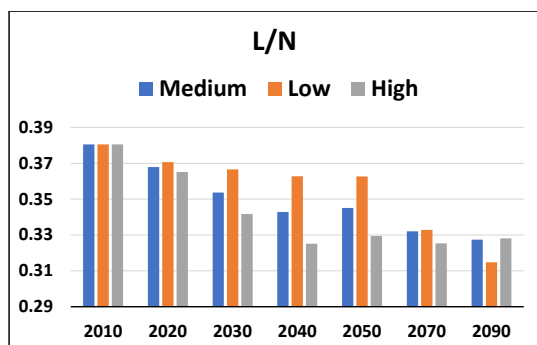
Source: Rows 1-4: United Nations (2015), cf. **Tables (1,2)**; Rows 5-9: **Tables (5a,5b)**.

**Fig. 3.** Labor Force Participation rates: Medium, Low, High fertility, 2010-2090.



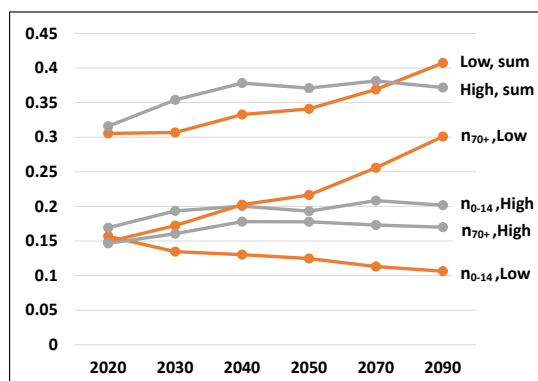
Source: Danish LFP =  $\frac{L_{15-69}}{N_{15-69}} = l_{15-69}(t)$ , (2), (4), (39), Tables (6, 3a, 2).

**Fig. 5.** Danish Support ratios - L/N -



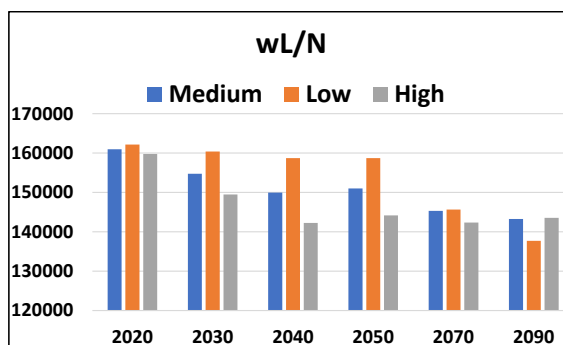
Source: Danish  $\frac{L(t)}{N(t)}$ , (3-6), Tables (6,3a).

**Fig. 4.** Shares,  $N_{0-14}/N$ ,  $N_{70+}/N$ , for Low, High fertility scenario, 2020-2090.



Source:  $n_{0-14}$ ,  $n_{70+}$ , (1), Tables (1, 6).  
 $sum \equiv n_{0-14} + n_{70+} = 1 - n_{15-69} = \frac{d}{1+d}$

**Fig. 6.** Wage Income per capita, 2020-2090.



Source: Danish  $W_A(2010) \cdot \frac{L(t)}{N(t)}$ , Table 6.

## 5 Projected lifetime wage incomes of special cohorts

From the **calendar** (time,  $t$ ) annual wages,  $\mathbf{w}_i(\mathbf{t})$ , of labor age groups,  $\mathbf{L}_i(\mathbf{t})$ , in **Tables (5a,5b)**, we can for a particular **cohort** ( $T$ ), *extract* the *cohort annual wages*,  $\mathbf{w}_i^*(\mathbf{T})$ , at each *life-cycle age*,  $i = 1, 2, \dots, M = 11$ , where, age  $i = 1 = 15-19$ , age  $i = 2 = 20-24$ , etc.

The **Labor supply** (Labor inputs),  $\mathbf{L}_i^*(\mathbf{T})$ , of **cohort** ( $T$ ) at life-cycle *age* ( $i$ ) is related to the **calendar** Labor supplies,  $\mathbf{L}_i(\mathbf{t})$  of *age* group ( $i$ ), **Tables (5a,5b)**, as follows :

$$L_i^*(T) = L_i(T + [i - 1] 5) = L_i(t), \quad L_1^*(T) = L_1(t); \quad L_3^*(2020) = L_3(2030) = L_{25-29}(2030)$$

Similarly for  $\mathbf{w}_i^*(\mathbf{T})$ ,<sup>19</sup> *Cohort* ( $T$ ) annual wage *age* ( $i$ ) and *Calendar* annual wage  $\mathbf{w}_i(\mathbf{t})$ :

$$w_i^*(T) = w_i(T + [i - 1] 5) \equiv w_i(t), \quad i = 1, \dots, 11; \quad w_1^*(T) = w_1(t), \quad w_3^*(2020) = w_3(2030) \quad (43)$$

All the results for,  $L_i^*(T)$ ,  $w_i^*(T)$ , for every **Cohort**  $\mathbf{T}$  are collected in **Table 7**.

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<sup>19</sup>By **Tables (5a,5b)** and (43), a few examples of *extracting*,  $w_i^*(T)$ ,  $L_i^*(T)$ , **Table 7**, are :

**Table 7. Medium Variant** : Example - Cohort wages,  $w_i^*(T)$ , of Cohort,  $\mathbf{T} = \mathbf{2010}$ ,

Table 5a:  $w_{15-19}^*(2010) = w_{15-19}(2010) = 187005$ ;  $w_{25-29}^*(2010) = w_{25-29}(2020) = 333172$

Table 5b:  $w_{45-49}^*(2010) = w_{45-49}(2040) = 467867$ ;  $w_{55-59}^*(2010) = w_{55-59}(2050) = 444392$  **Table 7.**

**Medium Variant** : Example - Cohort wages,  $w_i^*(T)$ , of Cohort,  $\mathbf{T} = \mathbf{2015}$ ,

Table 5a:  $w_{15-19}^*(2015) = w_{15-19}(2015) = 187053$ ;  $w_{30-34}^*(2015) = w_{30-34}(2030) = 393930$

Table 5b:  $w_{40-44}^*(2015) = w_{40-44}(2040) = 475811$ ;  $w_{50-54}^*(2015) = w_{50-54}(2050) = 459807$

**Table 7. Medium Variant** : Example - Cohort wages,  $w_i^*(T)$ , of Cohort,  $\mathbf{T} = \mathbf{2020}$ ,

Table 5a:  $w_{15-19}^*(2020) = w_{15-19}(2020) = 188933$ ;  $w_{25-29}^*(2020) = w_{25-29}(2030) = 340041$

Table 5b:  $w_{35-39}^*(2020) = w_{35-39}(2040) = 454562$ ;  $w_{65-69}^*(2020) = w_{65-69}(2070) = 468895$

**Table 7. High Variant** : Example - Cohort wages,  $w_i^*(T)$ , of Cohort,  $\mathbf{T} = \mathbf{2030}$ ,

Table 5a:  $w_{15-19}^*(2030) = w_{15-19}(2030) = 191356$ ;

Table 5b:  $w_{35-39}^*(2030) = w_{35-39}(2050) = 484008$ ;  $w_{55-59}^*(2030) = w_{55-59}(2070) = 493656$

**Table 7. Low Variant** : Example - Cohort wages,  $w_i^*(T)$ , of Cohort,  $\mathbf{T} = \mathbf{2035}$ ,

Table 5b:  $w_{20-24}^*(2035) = w_{20-24}(2040) = 279471$ ;  $w_{50-54}^*(2035) = w_{50-54}(2070) = 459807$

Thus **Medium** variant size of  $w_i^*(2010)$  for the **Cohort 2010** at age 25-29 is 333172 (**Table 7**, row 6 column 2). This number was seen as  $w_i(2020)$  for 25-29 year old in 2020 (**Table 5a**, column 5, row 6).

For the **Cohort 2020**, the sizes of  $w_{15-19}^*(2020)$ ,  $w_{25-29}^*(2020)$ ,  $w_{35-39}^*(2020)$ ,  $w_{45-49}^*(2020)$  in **Table 7** are the sizes seen in **Tables 5a, 5b**, (column 5) for,  $w_{15-19}(2020)$ ,  $w_{25-29}(2030)$ ,  $w_{35-39}(2040)$ ,  $w_{45-49}(2050)$ , respectively.

The *cohort* annual wages,  $w_i^*(T)$ , and *cohort* labor supplies,  $L_i^*(T)$ , in **Table 7** can be *summed* to generate the *Total Life Wage Income* of **Cohort** ( $T$ ) :  $w^*(T)L^*(T)$ , where Labor Supply,  $L^*(T)$ , is the *Total Life Time* Labor Supply of **Cohort** ( $T$ ), and  $w^*(T)$  is the *Average (Life) Annual Wage* of **Cohort** ( $T$ ),. i.e., as defined in accordance with:

$$w^*(T)L^*(T) = \sum_{i=1}^M w_i^*(T)L_i^*(T); L^*(T) = \sum_{i=1}^M L_i^*(T); w^*(T) = \frac{\sum_{i=1}^M w_i^*(T)L_i^*(T)}{L^*(T)} \quad (44)$$

These *Longitudinal* (Cohort) Labor *supplies*,  $L_i^*(T)$ , [life-cycle ages ( $i$ )], *Longitudinal* annual *wages*,  $w_i^*(T)$ , and *Life Time* Cohort Labor supply,  $L^*(T)$ , making the **Average Annual Wage**,  $w^*(T)$ ,<sup>20</sup>, (44), are shown in **three** demographic *Variants* for **six** Labor **Cohorts**,  $\mathbf{T}$ , (45), in **Table 7** - where,  $\mathbf{T} - 15 = \mathbf{t}$ , is *Birth year* ( $\mathbf{t}$ ) of the *youngest*

$$T = 2010, 2015, 2020, 2025, 2030, 2035; t = 1995, 2000, 2005, 2010, 2015, 2020 \quad (45)$$

*Generation* ( $\mathbf{t}$ ), which enter the Labor **Cohort** ( $\mathbf{T}$ ). Thus Cohort  $\mathbf{T}=2035$  (born 2020) (45) starts working  $t= 2035$  and is retired in year,  $t = 2090$  - the *end year* of **Table 5b**.

We gave above a few examples on how to extract (translate) information from Tables (5a,5b) to Cohorts in Table 7. Similarly many other numbers,  $w_i^*(T)$ ,  $L_i^*(T)$ , in Table 7 can be traced back to Tables (5a,5b). However, for *every Cohort* ( $T$ ), (45) - all longitudinally Cohort variables of  $w_i^*(T)$ ,  $L_i^*(T)$ , exhibited in **Table 7**, contain *much more* information (numbers) than available in **Tables (5a,5b)**, as evidently many *needed intermittent* calendar *years* (2015, 2025, 2035, 2045, etc.) are *not* shown in **Tables (5a,5b)**. Hence the entire **Table 7** has been obtained by *completing* all the additional *calculations* needed (but not shown) to **extend** the **Tables (5a,5b)**.

It is important to fully realize, however, that it is **all** 11 *overlapping age-group* (cross-section) calendar wages,  $w_i(\mathbf{t})$ , of *many Calendar* years ( $\mathbf{t}$ ) that **generate** - by *equation*, (43) - the relevant **longitudinal** annual wages,  $w_i^*(\mathbf{T})$ ,  $i= 1, ,11$ , of Labor **Cohort**  $\mathbf{T}$  (and their **Generation**) through a *working life-cycle* of **55 years** in 11 *age-groups* ( $i$ ).

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<sup>20</sup> The **Average** lifetime **Wage** (Earnings) of **Cohort**  $\mathbf{T}$ ,  $w^*(\mathbf{T})$ , (44), differs from Lifetime Earnings of a *Cohort worker*, optimally *accumulating* human capital (education, experience) and *rentals (wages)* during fixed lengths of working-life (*ages/years*). On shape of the life-cycle (*Age-wage profiles*) of such Cohort, see Rosen (1972, p.330; 1976, p.52), Welch (1979. p.79), and Berger (1984, p.590; 1985, p.572).

**Table 7. Age-group Wages and Lifetime Wage Incomes of Cohorts, entering Labor market at age 15, size  $N_{15-19}$ , in year : 2010, --, 2035 ; Three Variants : Denmark**

Medium	Cohort 2010 ( $N_{15-19}, 353109$ )		Cohort 2015 ( $N_{15-19}, 358260$ )		Cohort 2020 ( $N_{15-19}, 338740$ )		Cohort 2025 ( $N_{15-19}, 345300$ )		Cohort 2030 ( $N_{15-19}, 309940$ )		Cohort 2035 ( $N_{15-19}, 322890$ )	
	$w_t^*$	$L_t^*$	$w_t^*$	$L_t^*$	$w_t^*$	$L_t^*$	$w_t^*$	$L_t^*$	$w_t^*$	$L_t^*$	$w_t^*$	$L_t^*$
15-19	187005	30199	187053	30640	188933	28970	187560	29531	5339	191356	26507	5072
20-24	265673	128659	267205	124262	269161	117761	267744	119962	32119	273955	108172	29659
25-29	333172	214793	335522	207736	340041	197287	338724	200851	68033	348891	181862	29634
30-34	390937	286458	393930	277315	400498	263751	398306	268424	106915	408972	243728	99678
35-39	437974	308193	444123	298552	452058	284214	447615	289121	129458	461321	263051	121351
40-44	466068	311857	472725	302296	482722	288018	474304	293087	133012	489117	266774	130484
45-49	464833	310995	470441	301650	477552	287586	465765	292640	136301	478578	265052	127542
50-54	469933	303977	478854	295048	485221	284556	480909	286523	126315	450383	261075	117552
55-59	436048	283113	435060	275079	435721	262658	427892	267571	114491	434226	244016	105958
60-64	462208	319335	460730	135561	460833	129609	455043	133136	60128	460166	120667	55527
65-69	448299	28526	446666	27807	446591	26634	441658	27192	12010	446861	24874	11115
Total	2,346,105	984,690	2,275,946	962,114	2,167,944	927,807	2,207,135	930,337	2,007,228	867,343	2,007,228	867,343
Lifetime	$w^* = 419713$	$w^* = 427371$	$w^* = 427371$	$w^* = 428173$	$w^* = 436983$	$w^* = 438740$	$w^* = 438740$	$w^* = 438740$	$w^* = 438740$	$w^* = 432110$	$w^* = 432110$	$w^* = 441289$
High	Cohort 2010 ( $N_{15-19}, 353109$ )		Cohort 2015 ( $N_{15-19}, 358260$ )		Cohort 2020 ( $N_{15-19}, 338740$ )		Cohort 2025 ( $N_{15-19}, 345300$ )		Cohort 2030 ( $N_{15-19}, 309940$ )		Cohort 2035 ( $N_{15-19}, 366070$ )	
15-19	187005	30199	187053	30640	188933	28970	187560	29531	5339	191356	26507	5072
20-24	265673	128659	267205	124262	269161	117761	267744	119962	32119	273955	108172	29659
25-29	333172	214793	335522	207736	340041	197287	338724	200851	68033	348891	181862	29634
30-34	390937	286458	393930	277315	400498	263751	398306	268424	106915	408972	243728	99678
35-39	437974	308193	444123	298552	452058	284214	447615	289121	129458	461321	263051	121351
40-44	466068	311857	472725	302296	482722	288018	474304	293087	133012	489117	266774	130484
45-49	464833	310995	470441	301650	477552	287586	465765	292640	136301	478578	265052	127542
50-54	469933	303977	478854	295048	485221	284556	480909	286523	126315	450383	261075	117552
55-59	436048	283113	435060	275079	435721	262658	427892	267571	114491	434226	244016	105958
60-64	462208	319335	460730	135561	460833	129609	455043	133136	60128	460166	120667	55527
65-69	448299	28526	446666	27807	446591	26634	441658	27192	12010	446861	24874	11115
Total	2,346,105	984,690	2,275,946	962,114	2,167,944	927,807	2,207,135	930,337	2,007,228	867,343	2,007,228	867,343
Lifetime	$w^* = 419713$	$w^* = 427371$	$w^* = 427371$	$w^* = 428173$	$w^* = 436983$	$w^* = 438740$	$w^* = 438740$	$w^* = 438740$	$w^* = 438740$	$w^* = 432110$	$w^* = 432110$	$w^* = 441289$
High	Cohort 2010 ( $N_{15-19}, 353109$ )		Cohort 2015 ( $N_{15-19}, 358260$ )		Cohort 2020 ( $N_{15-19}, 338740$ )		Cohort 2025 ( $N_{15-19}, 345300$ )		Cohort 2030 ( $N_{15-19}, 309940$ )		Cohort 2035 ( $N_{15-19}, 366070$ )	
15-19	187005	30199	187053	30640	188933	28970	187560	29531	5339	191356	26507	5072
20-24	265673	128659	267205	124262	269161	117761	267744	119962	32119	273955	108172	29659
25-29	333172	214793	335522	207736	340041	197287	338724	200851	68033	348891	181862	29634
30-34	390937	286458	393930	277315	400498	263751	398306	268424	106915	408972	243728	99678
35-39	437974	308193	444123	298552	452058	284214	447615	289121	129458	461321	263051	121351
40-44	466068	311857	472725	302296	482722	288018	474304	293087	133012	489117	266774	130484
45-49	464833	310995	470441	301650	477552	287586	465765	292640	136301	478578	265052	127542
50-54	469933	303977	478854	295048	485221	284556	480909	286523	126315	450383	261075	117552
55-59	436048	283113	435060	275079	435721	262658	427892	267571	114491	434226	244016	105958
60-64	462208	319335	460730	135561	460833	129609	455043	133136	60128	460166	120667	55527
65-69	448299	28526	446666	27807	446591	26634	441658	27192	12010	446861	24874	11115
Total	2,346,105	984,690	2,275,946	962,114	2,167,944	927,807	2,207,135	930,337	2,007,228	867,343	2,007,228	867,343
Lifetime	$w^* = 425771$	$w^* = 433651$	$w^* = 433651$	$w^* = 445933$	$w^* = 447984$	$w^* = 449984$	$w^* = 449984$	$w^* = 449984$	$w^* = 449984$	$w^* = 470117$	$w^* = 470117$	$w^* = 450325$

Source : United Nations (2015), New York 2015 ; Cohorts ( $N_{15-19}$ ) in years, 2010, 2015, 2020, 2025, 2030, 2035. Source : Tables (5a,5b).

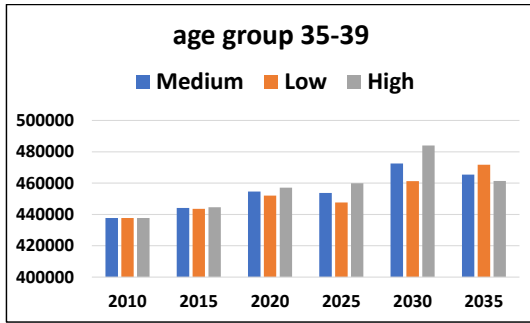
$w_t^*$  : Cohort wages, age ( $t$ ), DKK ;  $L_t^*$  : Cohort Labor supply ;  $w_t^* L_t^*$  : Cohort Wage Income, Billion DKK ;  $w^* L^*$  = Cohort Life Wage Income ;  $w^*$  : Average Cohort Life Annual Wage, DKK. Comparison, e.g., with Table 2, 2010 :  $L(2010) = 2.112472 = \text{Total Labor Supply (Total Labor years)}$ , Million ; Total Wage sum,  $wL = 924.317$  Billion DKK ;  $w(2010) = 437552$  DKK.

For **T = 2010** :  $L_{15-19}^*$  are born in the calendar years ( $t$ ) : 1991-1995 (*Generation*). Then :  $L_{20-24}^*$  refer to 2015-2020;  $L_{25-29}^*$  refer to 2021-2025;  $L_{30-34}^*$  refer to 2026-2030 ;  $L_{35-39}^*$  refer to 2031-2035;  $L_{40-44}^*$  refer to 2036-2040;  $L_{45-49}^*$  refer to 2041-2045;  $L_{50-54}^*$  refer to 2046-2050;  $L_{55-59}^*$  refer to 2051-2055;  $L_{60-64}^*$  refer to 2056-2060;  $L_{65-69}^*$  refer to 2061-2065. For **T = 2015** :  $L_{15-19}^*$  born calendar years ( $t$ ) : 1996-2000 (*Generation Z*). Then  $L_{20-24}^*$  refer to 2005-2010;  $L_{25-29}^*$  refer to 2010-2015. Then  $L_{30-34}^*$  refer to 2015-2020;  $L_{35-39}^*$  refer to 2020-2025;  $L_{40-44}^*$  refer to 2025-2030;  $L_{45-49}^*$  refer to 2030-2035;  $L_{50-54}^*$  refer to 2035-2040;  $L_{55-59}^*$  refer to 2040-2045;  $L_{60-64}^*$  refer to 2045-2050;  $L_{65-69}^*$  refer to 2050-2055. For **T = 2020** :  $L_{15-19}^*$  born calendar years ( $t$ ) : 2001-2005. Then  $L_{20-24}^*$  refer to 2006-2010;  $L_{25-29}^*$  refer to 2011-2015. Then  $L_{30-34}^*$  refer to 2016-2020;  $L_{35-39}^*$  refer to 2021-2025;  $L_{40-44}^*$  refer to 2026-2030;  $L_{45-49}^*$  refer to 2031-2035;  $L_{50-54}^*$  refer to 2036-2040;  $L_{55-59}^*$  refer to 2041-2045;  $L_{60-64}^*$  refer to 2046-2050;  $L_{65-69}^*$  refer to 2051-2055.



**Fig. 7.**

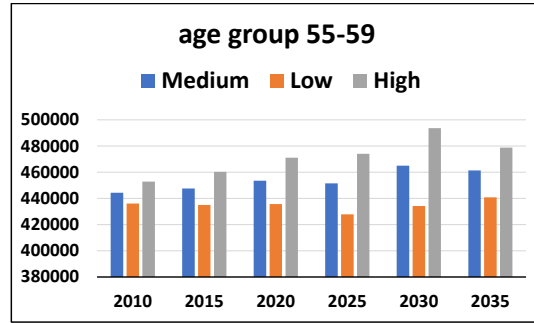
Annual wages of age group 35-39,  $w_{35-39}^*(T)$ , for 6 Cohorts 2010-2035  
Medium, Low, High fertility



Source:  $w_i^*(T)$ , (43),  $i = 35 - 39$ , T, (45), in Table 7.

**Fig. 8.**

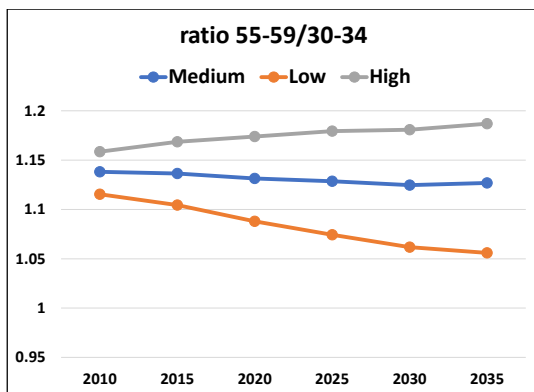
Annual wages of age group 55-59,  $w_{55-59}^*(T)$ , for 6 Cohorts 2010-2035  
Medium, Low, High fertility



Source:  $w_i^*(T)$ , (43),  $i = 55 - 59$ , T, (45), in Table 7.

**Fig. 9.**

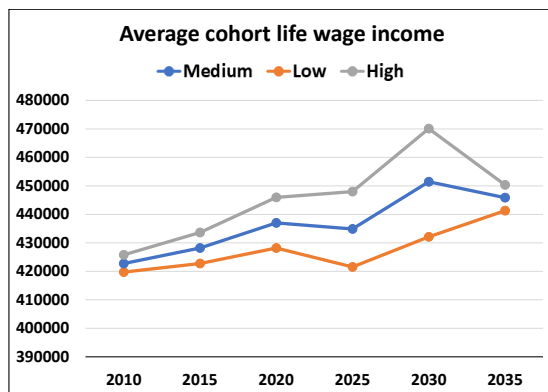
Ratio of the Annual Wages between stages (old/young) for 6 Cohorts T, 2010-2035  
Medium, Low, High fertility.



Source:  $w_{55-59}^*(T)/w_{30-34}^*(T)$  in Table 7, as obtained from :  $w_i^*(T)$  in Fig. 7 - 8.

**Fig. 10.**

Average (annual wage) Life - all ages (i) - Income,  $w^*(T)$ , for 6 Cohorts T, 2010-2035  
Medium, Low, High fertility.



Source:  $w^*(T)$ , (44), (45), in Table 7.

The *sizes* of the *Generations/Cohorts* are seen in **Table 7** (second row) as,  $\mathbf{N}_{15-19}(\mathbf{T})$ , **Table (5a,5b)**, i.e., specific *size* of *Population age group* (15-19) - *youngest* (15) entering Labor market,  $\mathbf{L}_{15-19}^*(\mathbf{T})$  - in the *years*, (45) :  $\mathbf{T} = 2010, 2015, 2020, 2025, 2030, 2035$ .

In the **Medium** variant for example,  $\mathbf{N}_{15-19}(\mathbf{2020}) = 338740$  *people* aged 15-19 in **2020**, which is also seen in **Table 5a** (row 1, column 2). For the same variant,  $\mathbf{N}_{15-19}(\mathbf{2030}) = 309940$  can be seen in **Table 5a**, (row 13, column 2). The  $\mathbf{N}_{15-19}(\mathbf{2035}) = 322890$  (not shown in **Tables 5,6**) will be 80-85 years of age in 2100, which is the last year of the United Nations population projections. The corresponding rows with  $\mathbf{N}_{15-19}(\mathbf{T})$  for the *Low* and *High* variants are given further down in **Table 7**.

Generation *sizes*  $\mathbf{N}_{15-19}(\mathbf{T})$  - and Cohort *Lifetime Labor supply*,  $\mathbf{L}^*(\mathbf{T})$ , (44) - are *equivalent* for **all** variants in all years, *except* for the last, *Cohort* (2035), since the *fertility change* commences in 2020 and takes until 2035 to be reflected in Cohort Labor,  $\mathbf{L}^*(\mathbf{T})$ .

**Figures (7, 8)** show **Cohort (T) annual wages** for workers in the second half of the 30s,  $\mathbf{w}_{35-39}^*(\mathbf{T})$ , and for workers in the second half of the 50s,  $\mathbf{w}_{55-59}^*(\mathbf{T})$ . For the youngest age group, **Fig. 7**, significant changes are found when comparing the 2030 and the 2035 **Cohorts**. Here we find a strong impact in the *High* fertility case, where relative increase in younger workers,  $\mathbf{L}_{35-39}^*(\mathbf{T})$ , has a *depressing* effect on  $\mathbf{w}_{35-39}^*(\mathbf{T})$ . The counterpart to this is shown clearly in **Fig. 8**, where *High* fertility *improves* the position of *older* workers,  $\mathbf{w}_{55-59}^*(\mathbf{T})$ , for *all 6 Cohorts*, more so for the *2030* and *2035 Cohorts*.

**Fig. 9** presents an alternative illustration of how the *ratio*, old/young *annual wages* are affected for 6 **Cohorts** in *three* Fertility scenarios. Not surprisingly, *old* workers,  $\mathbf{w}_{55-59}^*(\mathbf{T})$ , are much *better* off relatively in the *High* compared to the *Low* fertility case.

**Fig. 10** shows **Cohort Average (Life-time) annual wage**,  $\mathbf{w}^*(\mathbf{T})$ , for **Cohorts T**, 2010 to 2035, covering their *full* working life, summing up their wages at different ages (life cycle) in **Fig. 7-8**. Thus **Cohort 2035** consists of workers 15-19 years  $\mathbf{L}_{15-19}^*(\mathbf{2035})$  in year 2035, and of workers  $\mathbf{L}_{65-69}^*(\mathbf{2035})$  retiring during 2086-2090, and living as 70-74 years old,  $\mathbf{N}_{70+}(\mathbf{2015})$ , in 2090. Even though the **Cohort 2035** had *lowest*  $\mathbf{w}_{35-39}^*(\mathbf{2035})$  with High fertility, then much better wages later as e.g.,  $\mathbf{w}_{55-59}^*(\mathbf{2035})$ , ensured that the *Average* (Life time) wage,  $\mathbf{w}^*(\mathbf{2035})$ , were *highest* with High fertility. The importance for any Cohort  $\mathbf{w}^*(\mathbf{T})$  of having *many* and *large* surrounding (*cooperating*) cohorts as

*co-workers* for the particular *Cohort T* (Generation) during its full working life (period). The *explicit* wage formula of  $\mathbf{w}^*(\mathbf{T})$ , (44), with the *analytic* CRESH forms (40-41) of *wage complementarity* emphasize such interaction (mutual interdependence) behind  $\mathbf{w}^*(\mathbf{T})$ .

**Fig. 10** compared future prospects of **Cohorts** using the *Average* Lifetime annual wages of the Cohort,  $\mathbf{w}^*(\mathbf{T})$ , shown in **Table 7** (Row - Lifetime - in each variant). The value of  $\mathbf{w}^*(\mathbf{T})$  is *highest* for the *smallest Cohort*, (**2030**), in *Medium, High* variants. For the *Medium* variant:  $\mathbf{w}^*(2010)=422715$ ,  $\mathbf{w}^*(2015)=428173$ ,  $\mathbf{w}^*(2020) =436983$ ,  $\mathbf{w}^*(2025) =434895$ ,  $\mathbf{w}^*(2030)=451446$ ,  $\mathbf{w}^*(2035)=445866$  - depicted in **Fig. 10** (*blue*). Thus *Medium* it Cohort (2030) has  $\mathbf{w}^*(2030)$  as 5.2 *percent* higher<sup>21</sup> than  $\mathbf{w}^*(2015)$ . For *Low Fertility* variant, the *highest*  $\mathbf{w}^*$  is also seen for the *smallest Cohort* (T=1935).

The relationships between **Cohort size** - measured by Cohort Labor supply,  $\mathbf{L}^*(\mathbf{T})$  - and *Average* (life-time) annual *wage*,  $\mathbf{w}^*(\mathbf{T})$ , are illustrated for the three fertility scenarios in **Fig. 11**. The *slopes* are *negative* with Cohort *Average* (life-time) wages *increasing* with *decreasing Cohort size*. The *exceptions* are found for the **High** fertility cases where *Average* (life-time) annual wage is significantly *higher* for *large* 2025 and 2035 *Cohorts*, reflecting the positive wage impacts increased Labor supply of *co-workers* in other cohorts.

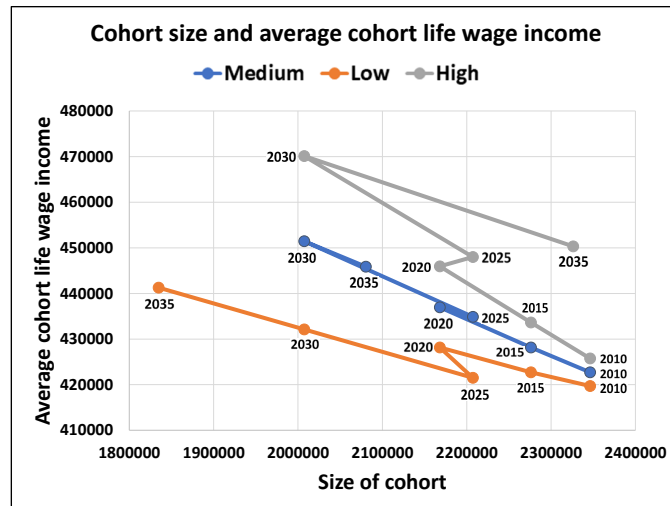
Indeed **Cohort** differences in  $\mathbf{w}^*(\mathbf{T})$ , **Fig. 10-11**, for all three demographic variants, are - with *same parametric* CRESH model for  $\mathbf{w}_i^*(\mathbf{T})$ , **Fig. 7 - 9** - fully explained by *wage complementarity differences* that Cohort Labor supplies,  $\mathbf{L}^*(\mathbf{T})$ ,<sup>22</sup>, (44) are exposed to.

In section 2.2, we saw in **Table 2** (col.2) some positive/negative Population **echo's** of the *sizes* of earlier generations, and in section 4.1, we saw for *calendar years* in **Table 5a**, **Fig. (2a-2c)**, the *annual wage*,  $\mathbf{w}_i(\mathbf{t})$ , effects of belonging to the *small* (early) *Millennial* generations, (1981-1985), (1986-1990). We have not shown (calculated) the *Life time* wage income,  $\mathbf{w}^*(\mathbf{T})$ , of first two *Millennial* generations (*Cohorts*, T=2000, T=2005), but they should be *high* as  $\mathbf{w}^*(2000)$ ,  $\mathbf{w}^*(2005)$  - not shown in **Fig. 10-11**. But  $\mathbf{w}^*(\mathbf{T})$  of the last *Millennial* generation and the first *Z* - generation (*Cohorts*, T=2010, T=2015) are as  $\mathbf{w}^*(2010)$ ,  $\mathbf{w}^*(2015)$  in **Fig. 10-11** - being *lower* than  $\mathbf{w}^*(\mathbf{T})$  of more *future* Cohorts.

<sup>21</sup>The  $\mathbf{N}_{15-19}(2030)$ , is is 13.5 *percent* smaller than  $\mathbf{N}_{15-19}(2015)$ , cf. **Table 7**.

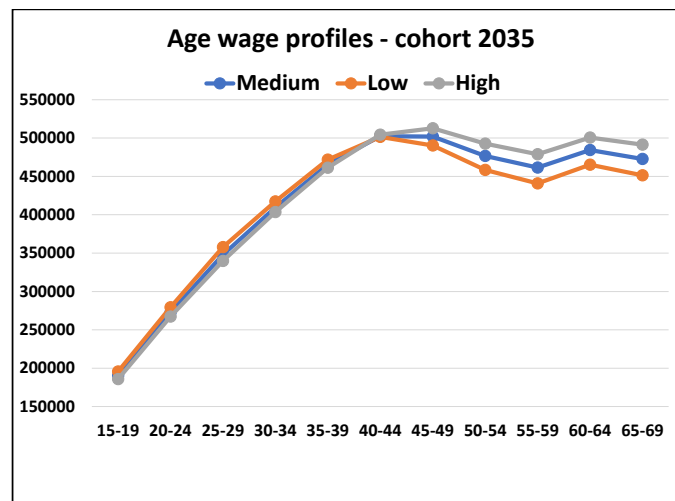
<sup>22</sup>The size of these  $\mathbf{w}^*(\mathbf{T})$  effects depends on the degree of labor substitutability reflected in the parameter  $\rho_i$ . If all  $\rho_i = 1$ , there is perfect labor substitution and *sizes* of Cohort/Labor supplies have no effect on its *own* relative (absolute) wages nor affect the relative (absolute) wage of *other* Cohorts.

**Fig. 11.** Life Time Cohort Labor Supply,  $L^*(T)$ , ("Cohort size"), and Cohort Average Life Income (*Annual wage*),  $w^*(T)$  - for the *six Cohorts*, 2010-2035, in *three* variants : Medium, Low, High fertility.



Source: Six numbers of  $L^*(T)$ , and,  $w^*(T)$ , (44), (45), seen (bottom) in Table 7.

**Fig. 12.** Annual wages - Age wage profile - for Cohort 2035,  $w_i^*(T)$ ,  $i = 1, 2, \dots, 11$ , Cohort  $T = 2035$ , Generation,  $t = 2020$ , - Medium, Low, High fertility.



Source:  $w_i^*(T)$ , (43),  $i = 1, 2, \dots, 11 \equiv i = 15-19, 20-24, \dots, 65-69$ ,  $T = 2035$ ,  $t = 2020$ , (45), from Table 7, (last Cohort, RHS).

We may trace some **echo** of small *first Millennial* Generation (1981-1985), *Cohort* (T=2005), on *Life time* wage income ( $\mathbf{w}^*$ ) of their *descendants* (progeny). Generation (1981-1985) is not exclusively - but it is the main *Progenitor* of Generation (2011-2015), Cohort T = 2030, and we do see an **echo** of first *Millennial* Cohort (T=2005) in *Progenitor* Cohort (T=2030) - as reflected in  $\mathbf{w}^*(2030)$  - which indeed is the *highest* ( $\mathbf{w}^*$ ) in **Table 7, Fig. 10-11**, with *smallest* sizes of  $\mathbf{N}_{15-19}(2030) = 309940$ , or  $\mathbf{L}^*(2030) = 2.007.228$ .

**Fig. 12** shows the **longitudinal annual wages**,  $\mathbf{w}_i^*(\mathbf{T})$ , to *all ages* (i) (*life cycle*) of the **Cohort**, T = 2035, for three fertility scenarios. Annual wages peak at *ages* 40-44, independently of the fertility scenarios. *After* this age, 40-44, *differences* in **fertility** has a clear impact with *higher* annual wages for *older* workers in the *high* fertility case, reflecting the scarcity of the older workers together with ample supplies of younger workers. The *shape* of the **Age profile** of annual wages,  $\mathbf{w}_i^*(\mathbf{T})$ , **Fig. 12**, applies *qualitatively* to any **Cohort** T in **Table 7** (all *vertical* wage columns of  $\mathbf{w}_i^*(\mathbf{T})$ , to the *left* of Cohort 2035).

In **Fig. 13**, we show the *Age-wage profiles*,  $\mathbf{w}_i^*(\mathbf{T})$ ,  $i = 1, 2, \dots, M = 11$  for the *three* Cohorts, T = 2010, 2020, 2030, *Medium* fertility - already seen with their *Life time*,  $\mathbf{w}^*(\mathbf{T})$ , for T = 2010, 2020, 2030 (on *blue* line) in **Fig. 10**. Hence **Fig. 13** demonstrate that e.g., that the largest *Life time*,  $\mathbf{w}^*(2030)$ , in **Fig. 10** also have the largest  $\mathbf{w}_i^*(\mathbf{T})$  at *any* stage (all *ages*), (i), during entire working life (15-69).

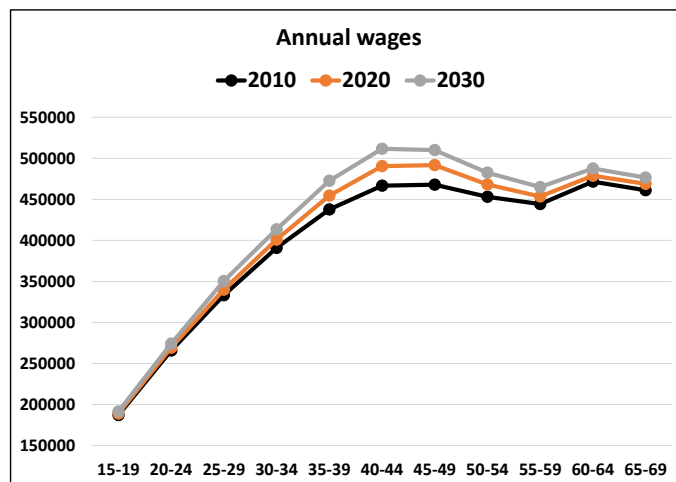
We saw in **Fig. (2d)** that the smallest Generation (1981-1985) had - as the age group (45-49) in *calendar* year 2030 - the highest wages (above normal). Such above normal wages in calendar year 2010 is not just a temporary effect - but become a permanent effect - of being a small generation as (1981-1985). Thus such *permanent* wage effect of the small<sup>23</sup> *Generation* (2011-15), **Cohort**, T = 2030, is seen for *all ages* (i) in **Fig. 13**.

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<sup>23</sup>The age composition of the labor force varies much over time due to demographic changes. The large post-war *Generations*, born 1946-1964 (defining American "baby boomers"), included *four* 5-year *age groups*, from the leading edge group (1946-50), *peak* in (1956-60), to trailing edge group (1961+), cf. Freeman (1979, p.289, Easterlin et al. (1990, p.281). Danish "baby boomers" refer to *decade* (1941-50).

The economic effects of several large "baby boom generations" (age-groups) are explored extensively in economic/demographic literature. "Twist [shift/rotation] in male *age-wage profiles* in late 1960s and early 1970s" (relative *low* earnings of younger workers) have empirically been attributed, Freeman (1979, p.315), Easterlin (1978, p.401), as impacts of *large* "baby boom" generations (1946-60) - the opposite of *small* generations (Y, 1981-90) effects, *calendar* year "twists", **Fig. 2d** - or as *life-time* impacts, **Fig. 13**.

**Figure 13. Age-wage profiles,  $w_i^*(\mathbf{T})$ ,  $i= 1, 2, \dots, M$ , age group :  $i = 1 = 15-19$ ,  $i = 11 = 65-69$  - for the **Cohorts**,  $T = 2010, 2020, 2030$  - **Medium** fertility variant.**



Source: Cohort wages,  $w_i^*(\mathbf{T})$ , (43),  $T = 2010, 2020, 2030$ , **Table 7**, *horizontal top*.

The overall **Age-wage** profiles as **Fig. (12,13)**, **Fig. (2d,2e)**, hold *generally* for Cohorts and Calendar years, and the **shape** of such *Age-wage profiles* are determined by the CRESH parametric *Labor Aggregator*, (7-8), (12-14), (15), or obtained by the dual CRESH *Age-Wage* [**Inverse Labor demand**]  $w_i$  - *form*, (25-29), (81), (84), (92-93).

## 6 Population, Division of Labor, and Wages

Demography, Population, Labor Allocation, Wages, and National Income per capita are subjects of classic fields and studies in Political Economy/Economics. Let us end with a few literature comments provided for both *inductive* and *deductive* aspects of this paper. For this purpose, it is useful to recall the macro relations and the ratios in (5-6), (25),

$$\frac{Y}{N} = \frac{Y}{L} \cdot \frac{L}{N} ; \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i = l_{15-69} \cdot n_{15-69} ; \frac{W}{N} = \frac{W}{L} \cdot \frac{L}{N} = W_A \cdot \frac{L}{N} ; W_A L_A = \sum_{i=1}^M w_i L_i \equiv W \quad (46)$$

As to *proportions* in the Per Capita National Income ("Wealth of Nation") *identity*, (46), Smith (1790,1961, p.1) opens with the statement: "The annual labour of every nation is the fund, which supplies it with all the necessaries and conveniences of life. - According therefore, this *produce* [product, output, Y] bears a greater or smaller *proportion* [Y/N] to number [N] of those who are to consume it. But this *proportion* [Y/N] must in every nation be regulated by *two* different circumstances : 1. the *skill, dexterity, and judgement*

with which its labour is generally applied [*Labor productivity*,  $Y/L$ ] 2. the *proportion* [ $L/N$ ] between the number of those who are *employed* in *useful labor* [ $L$ ] and those not so employed [ $N-L$ ]. - The abundance or scantiness of this *supply* [per-capita produce,  $Y/N$ ] seems to depend more upon the *former* [ $Y/L$ ] of those two circumstances than upon the *latter*" [ $L/N$ ] (italics ours).

The *Employment/Population* ("**Support**") ratio, ( $L/N$ ), (bounded above by one<sup>24</sup>) is always much lesser than one as the numerical *size* of ( $L/N$ ), (46), is by definition the product of  $LFP = l_{15-69}$  (interval: 0.5-0.6), and the Working population share,  $n_{15-69}$  (interval: 0.7-0.6), i.e., e.g.,  $L/N = l_{15-69} \cdot n_{15-69} = 0.38$ , cf. (5-6), and **Tables (1,2,6)**.

As to ( $Y/L$ ), Smith (1790, p.7) says: "The greatest improvements in the *productive powers* of *labour*, and the greater part of the *skill, dexterity, and judgement* with which is anywhere directed, or applied, seems to have been the *effects* of the *division of labour*. - p.11: This great increase of the *quantity* [Output] of *work* [Labor productivity,  $Y/L$ ] which, in consequence of the *division of labour*, the *same* number of people [ $L$ ] are capable of performing, is owing to *three* different circumstances 1. the increase of *dexterity* in *every* particular workman 2. the *saving* of the *time* which is commonly lost in passing from one species of work to another 3. the invention of a *great number* of *machines* [ $K, K_j$ ] which facilitate and abridge labour, and enable *one man* to do the work of many" (italics ours).

The **Productive powers** of *Labor* by the **Division of Labor** (the notions used by Smith above), is economically and conceptually expressed with : **Production functions**,  $Y = F(L, K) = L \cdot f(K/L) \equiv L \cdot f(k)$ , or as,  $F(Y, L_I, L_{II}, K_{III}, K_{IV}, K_V) = 0$ , cf. (34).

Production (Division of Labor) by *different qualities* of *workers* according to skills [education, training], dexterity and judgement [*age/maturity/experience*] provide the framework for analyzing differences in *factor prices - earnings structures*, Smith (1790, p.111)<sup>25</sup>: "Pecuniary *wages* and *profits* [rentals of machinery], indeed, are everywhere in Europe extremely different according the *different employments* of *labour* and *stock*" (italics ours).

As to Pecuniary *wage distributions* for the *complete* set of *demographic age-groups* of the *National Labor* supply - making *Total Wages* ( $W$ ), (46), in the *National income* ( $Y$ ),

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<sup>24</sup>Modern Growth Theory and Macroeconomics, cf. standard exposition, Solow (2000), Romer (2019), do not allow in any of the models for the distinctions between Labor ( $L$ ) and Population ( $N$ ).

<sup>25</sup>As to the wage structure analysis in Smith (1790, Ch. 10), see Katz and Autor (1999, p.1464).

**Table 3**, our CRESH Labor **Aggregator**,  $L_A = f(L_1, L_2, \dots, L_M)$  - and **wage generator** by its *derivatives*, (27-29) - *formed* the annual *wage distributions* : **Calendar** year *Age-wage profiles* in **Tables 5a-5b** - and implied the **Cohort Life time** wages in **Table 7**.

It must be emphasized that the calculated results in **Tables (5a-5b, 7)** are **not** dealing with a *pure* Labor Economy, using only distinct Labor inputs. As stressed by Smith above, *Output* (Y) by 'Division of Labor' and *Labor productivity*,  $\frac{Y}{L} = y$ , involved 'machinery' [K] in *Production* functions,  $Y = F(L, K)$ ,  $y = f(k)$ ; let,  $p \cdot \frac{\partial Y}{\partial L}(k) = W_A(k)$ :

$$w = p \cdot \frac{\partial Y}{\partial L}; r = p \cdot \frac{\partial Y}{\partial K}; w = p \cdot \frac{\partial Y}{\partial L} = p \cdot y - r \cdot k \equiv W_A; k = 2.72 : W_A(2.72) = 582 - 0.053 \cdot 2.72 = 438 \quad (47)$$

where actual numbers of **Table 3** are used in (47-48), RHS. Next, we have by (47), (29):

$$\begin{aligned} \lambda_i(t) &\equiv \frac{L_i(t)}{L(t)} = \frac{l_i \cdot n_i(t)}{l_{15-69}(t) \cdot n_{15-69}(t)}; l_i = l_i(2010), i = 1, 2, \dots, M; M = 11 \\ w_i(t) &= \widetilde{w}_4(t) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{\lambda_i(t)^{\rho_i - 1}}{\lambda_4(t)^{\rho_4 - 1}}; k = 2.72, p \cdot y = 581972, w = W_A = 437552 \text{ DKK} \quad (48) \end{aligned}$$

Thus,  $\mathbf{w}_i(\mathbf{t})$ ,  $i = 1, 2, \dots, M$ , (48), give all *pecuniary* (money) **wages** in **Tables (5a-5b)**, and/or as exhibited in any/every **Figure 2-6**. As seen in (47-48),  $\forall t : k(t), y(t), W_A(t)$ , are *unchanged* during projection period 2010-2090;  $W_A(t)$  is the *arithmetic mean* wage rate of the nation's year-round, full-time workers, (46) - **exogenous**,  $W_A(2010)$ , cf. (42).

Around such arithmetic mean, **Macro** wage rate,  $\mathbf{W}_A(2010)$ , however, the 'Division of Labor' with different **Ages** (maturity/experience) of **workers** amply generate at **Micro** level a *changing wage structure* over *time*, given by the *Age wage profiles*,  $\mathbf{w}_i(\mathbf{t})$ ,  $\mathbf{i} = 1, 2, \dots, \mathbf{M}$ , (48). The *explicit* form (48) shows that the **money age-wage determinants** are: 1. [ $\mathbf{n}_i(\mathbf{t}), \mathbf{l}_{15-69}(\mathbf{t}), \mathbf{n}_{15-69}(\mathbf{t})$ ], by affecting continuously *changing*,  $\lambda_i(\mathbf{t})$ , **Age distributions**<sup>26</sup> of demographic induced *Labor supplies*,  $L(t) = \sum_{i=1}^M L_i(t)$ , (39), due to changing *Employment/Population* (Support) ratio,  $L(t)/N(t)$ , 2. the *endogenous* money **wage**,  $\widetilde{w}_4(\mathbf{t})$ , (29), 3. the CRESH **parameters**,  $(\alpha_i, \rho_i)$  in **Table 4**.

The *Age-wage solutions*,  $\mathbf{w}_i(\mathbf{t})$ ,  $i = 1, 2, \dots, M$ , (48) in **Tables (5a-5b)**, **Fig. 2-6**, with changing *Support* ratio,  $L(t)/N(t)$ , may be considered as **Micro Age-wage scenarios** evolving under **Macro 'steady-state'** conditions [ 'steady-state' *sizes* of aggregate capital-labor ratio ( $\mathbf{k}$ ), aggregate labor productivity ( $\mathbf{y}$ ), aggregate wage,  $\mathbf{w} = \mathbf{W}_A$  ].

<sup>26</sup>Edin and Holmlund, (1995, p.328-29) show how marked fluctuations ('shocks') in *calendar year sizes* of Swedish (Totals,  $N_{15-19}$ ) translate into substantial changes in *Age* distribution (*ratios*),  $n_i(t)$ ,  $i = 1, \dots, M$  - coinciding with rising/falling Youth relative wages. Adjusting  $n_i(t)$  changes  $\lambda_i(t)$  in entire *Age* distribution of  $L(t)$  - affecting *wage structure/calendar year Age wage profiles*,  $w_i(t)$ ,  $i = 1, \dots, M$ , (48).



The **shape** (*qualitative* properties) of the quantitative Age-wage *profiles* (48) will be *robust* and carry over to 'non-steady-state' conditions with *increasing* aggregate Labor productivity ( $Y/L = y$ ) and *increasing* per capita National Income, (46) - as the result of *Capital Accumulation* beyond 'capital widening' to 'capital deepening' [*increasing* capital-labor ratios,  $k(t)$ ] in well-known macro-, two sector<sup>27</sup>-, and multisector growth models. Total (Aggregate) Labor *supply*,  $L(t)$ , in such growth models could still be the demographic induced *Labor* supplies,  $L(t) = \sum_{i=1}^M L_i(t)$ , (39), that could allow for also generating *Micro* age-wage profiles,  $w_i(t)$ ,  $i = 1, 2, \dots, M$ , from quantitative *growth* models.

## 7 Final Comments and Conclusion

This paper generalized models of imperfect labor substitution/complementarity by simultaneously: (i) specifying the CRESH Labor Aggregator function - relaxing the assumption of single-level, Arrow et al. (1961, p.230), CES elasticity of substitution between labor age groups - dually CES complementarity elasticities of wages to age-group supplies, (ii) allowing for a much larger number of age groups than is common in the literature, (iii) CRESH modelling the evolution and consequences of several demographic variants over longer transition periods rather than having a constant age distribution of the population, the labor force, and within and between the cohorts.

We have quantitatively demonstrated the micro-macro economic impacts of the assumptions - alternative fertility scenarios in the demographic projections (2020-2090) - on calendar year (t) wage patterns (Age-wage profiles) in the 'short-run', coming years (decade), and in the 'long-run' upon the lifetime wage incomes for selected (Generations), Cohorts (T), within the period (2010-2035).

The CRESH Labor Aggregator functional form can easily be analytically extended (specified) to include CRESH Subaggregator functions for any relevant Disaggregated Labor categories. Furthermore, the CRESH Labor aggregate (or Labor subaggregates) can next be combined with other Production factors (Capital inputs) in proper specified CRESH Multi-factor Production functions to be applied in single-sector (Macro) or multi-

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<sup>27</sup>Equipment investment are among prime determinants to national growth performance (productivity, per capita growth), Jensen (2003, p.82). Machinery becomes "cheap as well as good," Mokyr (1990, p.87).

sector GE models. In such interaction, the National Aggregate wage, ( $W_A$ ), become endogenously generated and can provide unified macro equilibrium feedback in calendar years to forming the Age-wage profiles of Labor age-groups and to selected Labor cohorts.

We have come a long way and reached a higher vantage point, which offer a better outlook and apprehension of the roads passed. In closing, we look forward to see Demography and in particular Labor Economics promoting coherent quantifications and projections of real-world (calendar) Annual wages and full-time Employment (Labor years), based on relevant Demographic Register data and consistent with National Income Accounts.

## 8 App.A : Labor Substitution and Complementarity

### 8.1 Substitution elasticities and complementarity elasticities

Allen-Uzawa *partial substitution elasticities*,  $\sigma_{ij}$  of any factor pair,  $(L_i, L_j)$ , for CRESH, (7-8) - Hanoch (1971, p.699), Hanoch (1979, p.296), Guest & Jensen (2016, p.29) - are :

$$\sigma_{ij} = \frac{1}{(1 - \rho_i)(1 - \rho_j)\bar{\rho}} = \sigma_{ji} > 0, i \neq j; \bar{\rho} = \sum_{i=1}^M \frac{\varepsilon_i}{1 - \rho_i} \quad (49)$$

$$\sigma_{ii} = \frac{1}{(1 - \rho_i)} \left[ \frac{1}{(1 - \rho_i)\bar{\rho}} - \frac{1}{\varepsilon_i} \right] < 0, i = 1, \dots, M \quad (50)$$

where  $(\sigma_{ii})$  are the “total substitution elasticity” terms; the *variable* ( $\bar{\rho}$ ) is a weighted average of the parameters,  $1/(1 - \rho_i)$ , with the respective wage (cost) *shares* ( $\varepsilon_i$ ) as variable weights. Clearly, especially *larger* values of  $\rho_i$  and  $\rho_j$  give a *larger*  $\sigma_{ij}$ . The *restrictions* (8) imply that  $\sigma_{ij} > 0$ : *all* CRESH labor *inputs*  $L_i, i=1, \dots, M$ , are *substitutes*. If all  $\rho_i > 0$ , then **all**  $\sigma_{ij} > 1$ , (49). Note also that **any**  $\sigma_{ij}$  given by (49) via *shares*  $\varepsilon_i$ , (14), depends on *all* the *parameters*,  $\rho_i, \alpha_i$ , and *all* the *Labor inputs*  $L_i, i = 1, \dots, M$ .

It follows from (49) that, although *all*  $(\sigma_{ij})$  are *variable* elasticities of substitution (**VES**), they have nevertheless an **invariant** (constant) **CRESH** pattern:

$$\frac{\sigma_{ik}}{\sigma_{jk}} = \frac{(1 - \rho_j)}{(1 - \rho_i)}; \rho_i > \rho_j : \sigma_{ik} > \sigma_{jk}; \forall k \neq i, j : \frac{\sigma_{ij}}{\sigma_{kl}} = \frac{(1 - \rho_k)(1 - \rho_l)}{(1 - \rho_i)(1 - \rho_j)} \quad (51)$$

The *restrictions* (8) and *expressions* (49-51) were obtained by Hanoch (1971, p.698) via Lagrangian cost minimizing *factor demand* functions that correspond to a unique CRESH *minimum Cost* function,  $\mathbf{C}(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_M, \mathbf{L}_A)$ , or *unit cost* functions,  $\mathbf{c}(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_M)$ ,

$$C(w_1, w_2, \dots, w_M, L_A) = c(w_1, w_2, \dots, w_M) L_A = \sum_{i=1}^M w_i L_i; L_i = \frac{\partial C}{\partial w_i}; E(C, w_i) = \varepsilon_i \quad (52)$$

*dual* to the *implicit* CRESH *production* (aggregator) function, (7-8), (10),  $\varepsilon_i = \frac{w_i L_i}{C}$ , (14).

The own-price/cross-price *factor demand* elasticities corresponding to (49-50), (52), are:

$$E(L_i, w_i) = \varepsilon_i \sigma_{ii}; E(L_i, w_j) = \varepsilon_j \sigma_{ij}; E(L_j, w_i) = \varepsilon_i \sigma_{ji}; i = 1, \dots, M. \quad (53)$$

$E(L_i, w_i), E(L_i, w_j)$  are *conditional* (compensated, *fixed*:  $L_A$ ) Labor demand elasticities.

Like two-factor production/cost functions, the *changes* in factor *shares* ( $\varepsilon_i$ ) are ruled by,

$$\frac{\partial \varepsilon_i}{\partial w_j} \begin{matrix} \geq \\ \leq \end{matrix} 0 \Leftrightarrow \sigma_{ij} \begin{matrix} \geq \\ \leq \end{matrix} 1 \Leftrightarrow E(\varepsilon_i, w_j) = \varepsilon_j (\sigma_{ij} - 1); i \neq j, i = 1, \dots, M \quad (54)$$

The CRESH elasticities, (53) and (49-50), satisfy the standard summation properties :

$$\sum_{j=1}^M E(L_i, w_j) = \sum_{j=1}^M \varepsilon_j \sigma_{ji} = 0 ; \quad \sum_{j=1}^M \varepsilon_j E(L_j, w_i) = 0 \quad (55)$$

Actual *parametric* CRESH substitution elasticities,  $\sigma_{ij}$ ,  $\sigma_{ii}$ , (49-50), and factor demand elasticities,  $E(L_i, w_i)$ ,  $E(L_i, w_j)$ , (53), (55), are shown in **Table 8A-8B**, using the validated CRESH *parameter* values :  $(\rho_i, \alpha_i)$  in **Table 4**, (column 3-4), and the *Labor inputs*  $L_i$ ,  $i=1, \dots, M$ ,  $L_A = L$ , (2010), in **Table 2** (column 5).

The  $\sigma_{ij}$  formulas (56) are the Uzawa (1962, p.293) *duality forms* of Allen (1938, p.504)

$$\sigma_{ij} = \frac{C \frac{\partial^2 C}{\partial w_i \partial w_j}}{\frac{\partial C}{\partial w_i} \frac{\partial C}{\partial w_j}} , \quad \sigma_{ii} = \frac{C \frac{\partial^2 C}{\partial w_i^2}}{[\frac{\partial C}{\partial w_i}]^2} ; \quad c_{ij} = \frac{f \frac{\partial^2 f}{\partial L_i \partial L_j}}{\frac{\partial f}{\partial L_i} \frac{\partial f}{\partial L_j}} , \quad c_{ii} = \frac{f \frac{\partial^2 f}{\partial L_i^2}}{[\frac{\partial f}{\partial L_i}]^2} \quad (56)$$

*partial elasticity of substitution* ( $\sigma_{ij}$ ). But it is *impossible* to *apply* the beautiful and *simple*  $\sigma_{ij}$  formulas (56) to get the CRESH results (49-50), as the relevant **dual** CRESH *cost function*,  $C(w_1, w_2, \dots, w_M, L_A)$ , (52), has *no closed* form. However, such existing unknown *dual* CRESH *Cost function* (52) would by  $\sigma_{ij}$  (56) give the same CRESH *parametric substitution elasticities* (49-50) - as were successfully derived from the first and second order conditions for CRESH Lagrangian cost minimization by Hanoch (1971, p.697-98).

The Hicks *partial complementarity elasticity* ( $c_{ij}$ ), (56), for any factor pair  $(L_i, L_j)$  of CRESH function,  $L_A = f(L_1, L_2, \dots, L_M)$ , (10), are defined exactly in *analogy* with  $\sigma_{ij}$  of C, (52), see Sato and Koizumi (1973, p.47)<sup>28</sup> ; cf. Hicks (1970).<sup>29</sup> Note that the *size* of  $L_A$  [level of output,  $Y$  (note 35)] is *not held constant* in complementarity elasticities,  $c_{ij}$ . In fact, *positive*  $c_{ij}$  measures exactly the degree to which two factor inputs *jointly* contribute to a *change* in  $L_A$  [ $Y$ ] - as the cross-partial derivative  $\frac{\partial^2 f}{\partial L_i \partial L_j}$  shows in (56). Thus in contrast to  $\sigma_{ij}$ , (49), *larger* values of  $\rho_i$  and  $\rho_j$  give *smaller* numbers for  $c_{ij}$ , (57).

<sup>28</sup> Sato & Koizumi (1973, p.46) considered an *explicit* production function as,  $Y = F(X_1, X_2, \dots, X_M)$ ,  $Y =$  output,  $X_i =$   $i$ -th input, with derivatives,  $\forall X_i > 0 : \frac{\partial F}{\partial X_i} > 0$ ,  $\frac{\partial^2 F}{\partial X_i^2} < 0$ ,  $Y = \sum_{i=1}^M \frac{\partial F}{\partial X_i} X_i$ .

The *complementarity elasticities*,  $c_{ij}$ ,  $c_{ii}$ , are defined as :  $c_{ij} = \frac{F \frac{\partial^2 F}{\partial X_i \partial X_j}}{\frac{\partial F}{\partial X_i} \frac{\partial F}{\partial X_j}}$ ,  $c_{ii} = \frac{F \frac{\partial^2 F}{\partial X_i^2}}{[\frac{\partial F}{\partial X_i}]^2}$

The problem with applying these  $c_{ij}$ ,  $c_{ii}$ , definitions to CRESH function,  $L_A = f(L_1, L_2, \dots, L_M)$ , above in (56) is that  $f$ , (10), is *not* - as here  $F(X_1, X_2, \dots, X_M)$  - an *explicit* function. However, we *know* (can calculate, as explained in section 8.2 and **Appendix B**) the *derivatives* of  $f$ , (10), to use in (56), and  $L_A = f$  drops out of (56) - as seen from CRESH formulas (72-73), (78-79), and finally stated in (57-58).

<sup>29</sup>On related issues of Derived Factor Demand, see Sato & Koizumi (1970, p.109), Hicks (1970, p.294).

Table 8A. Partial substitution elasticities ( $\sigma_{ij}, \sigma_{ii}$ ) - Denmark 2010 - by (49-50) and Table 4.

	$L_1$	$L_2$	$L_3$	$L_4$	$L_5$	$L_6$	$L_7$	$L_8$	$L_9$	$L_{10}$	$L_{11}$
$L_1$	-809.675	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	8.679
$L_2$	8.679	-144.081	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	8.679
$L_3$	5.786	5.786	-47.513	3.858	2.893	2.315	2.315	2.893	3.858	5.786	5.786
$L_4$	5.786	5.786	3.858	-26.883	2.893	2.315	2.315	2.893	3.858	5.786	5.786
$L_5$	4.340	4.340	2.893	2.893	-15.561	1.736	1.736	2.170	2.893	4.340	4.340
$L_6$	3.472	3.472	2.315	2.315	1.736	-11.409	1.389	1.736	2.315	3.472	3.472
$L_7$	3.472	3.472	2.315	2.315	1.736	1.389	-11.530	1.736	2.315	3.472	3.472
$L_8$	4.340	4.340	2.893	2.893	2.170	1.736	1.736	-16.036	2.893	4.340	4.340
$L_9$	5.786	5.786	3.858	3.858	2.893	2.315	2.315	2.893	-23.6932	5.786	5.786
$L_{10}$	8.679	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	-67.354	8.679
$L_{11}$	8.679	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	-419.695
$\sum_{j=1}^M \varepsilon_j \sigma_{j\mu}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 8B. Conditional factor demand elasticities,  $E(L_i, w_i), E(L_i, w_j)$  - by (53) and Table 4.

	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$	$w_7$	$w_8$	$w_9$	$w_{10}$	$w_{11}$	$\sum_{j=1}^M E(L_i, w_j)$
$L_1$	-4.947	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	0.101	0.000
$L_2$	0.053	-4.716	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	0.101	0.000
$L_3$	0.035	0.189	-3.083	0.418	0.408	0.362	0.358	0.397	0.467	0.381	0.068	0.000
$L_4$	0.035	0.189	0.250	-2.915	0.408	0.362	0.358	0.397	0.467	0.381	0.068	0.000
$L_5$	0.027	0.142	0.188	0.314	-2.194	0.271	0.269	0.298	0.350	0.285	0.051	0.000
$L_6$	0.021	0.114	0.150	0.251	0.245	-1.783	0.215	0.238	0.280	0.228	0.041	0.000
$L_7$	0.021	0.114	0.150	0.251	0.245	0.217	-1.785	0.238	0.280	0.228	0.041	0.000
$L_8$	0.027	0.142	0.188	0.314	0.306	0.271	0.269	-2.202	0.350	0.285	0.051	0.000
$L_9$	0.035	0.189	0.250	0.418	0.408	0.362	0.358	0.397	-2.867	0.381	0.068	0.000
$L_{10}$	0.053	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	-4.429	0.101	0.000
$L_{11}$	0.053	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	-4.899	0.000
$\sum_{j=1}^M \varepsilon_j E(L_j, w_j)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

## 8.2 Labor Complementarity elasticities

The elasticity  $c_{ij}$  formulas (56) are simple ; but the CRESH,  $\mathbf{L}_A = \mathbf{f}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , (10), did *not* exist in *closed* form. For CRESH, (7-8), the *complementarity elasticities*  $c_{ij}$ , (56), are *explicitly* derived in **Appendix B**, (78-79), as *parametrically* given by :

$$c_{ij} = 1 - \rho_i - \rho_j + \tilde{\rho} = c_{ji}, \quad i \neq j; \quad \tilde{\rho} = \sum_{i=1}^M \varepsilon_i \rho_i \quad (57)$$

$$c_{ii} = 1 - 2\rho_i - (1 - \rho_i)/\varepsilon_i + \tilde{\rho} < 0, \quad i = 1, \dots, M \quad (58)$$

where  $(c_{ii})$  are the “total complementarity elasticity” terms ; *variable*  $(\tilde{\rho})$  is a weighted average of parameters  $(\rho_i)$ , with the respective wage (cost) *shares*  $(\varepsilon_i)$  as variable weights. **CRESH**  $(c_{ij})$  are all *variable complementarity* elasticities, (57), but they have an **invariant** (constant) **CDEC** (*constant difference of elasticity of complementarity*) pattern:

$$c_{ik} - c_{jk} = \rho_i - \rho_j; \quad c_{ij} - c_{kl} = (\rho_i + \rho_j) - (\rho_k + \rho_l) \quad (59)$$

Note that *unlike* substitution elasticities,  $\sigma_{ij}$ , (49), the *restrictions* (8) do **not** impose a particular *sign* upon **all** the **complementarity** elasticities,  $c_{ij}$ , (57).

**Wage Income** function,  $\mathbf{W} = \mathbf{W}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M, \mathbf{W}_A)$  - as a **dual** to *Wage Cost* function,  $\mathbf{C}(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_M, \mathbf{L}_A)$ , (52) - is an *alternative Wage Sum* formulation with *important applications* in sections 4-5 for the *elasticities*  $c_{ij}$ , (57), see **Appendix B**, (81-88):

$$W(L_1, L_2, \dots, L_M, W_A) = W_A f(L_1, L_2, \dots, L_M) = W_A L_A; \quad w_i = \frac{\partial W}{\partial L_i}, \quad E(W, L_i) = \varepsilon_i \quad (60)$$

$$W(L_1, L_2, \dots, L_M, W_A) = c(w_1, w_2, \dots, w_M) L_A = W_A L_A = \sum_{i=1}^M w_i L_i; \quad W_A = c(w_1, w_2, \dots, w_M) \quad (61)$$

$$c_{ij} = c_{ij}(L_1, L_2, \dots, L_M) = \frac{W \frac{\partial^2 W}{\partial L_i \partial L_j}}{\frac{\partial W}{\partial L_i} \frac{\partial W}{\partial L_j}}; \quad i = 1, \dots, M, \quad j = 1, \dots, M; \quad c_{ii} = c_{ii}(L_1, L_2, \dots, L_M) = \frac{W \frac{\partial^2 W}{\partial L_i^2}}{[\frac{\partial W}{\partial L_i}]^2} \quad (62)$$

*Factor price (wage) elasticities w.r.t own-, cross supply increases* are, cf. (53), (57-58),

$$E(w_i, L_i) = \varepsilon_i c_{ii}, \quad E(w_i, L_j) = \varepsilon_j c_{ij}, \quad E(w_j, L_i) = \varepsilon_i c_{ji}; \quad i = 1, \dots, M. \quad (63)$$

$E(w_i, L_i), E(w_i, L_j)$ , are *conditional* (fixed  $\mathbf{W}_A$ ) **partial wage** elasticities of *group*  $(i)$ .

Like two-factor production/cost functions, *wage shares*  $(\varepsilon_i)$ , (14), (60), follow the rules:

$$\frac{\partial \varepsilon_i}{\partial L_j} \begin{matrix} \geq \\ \leq \end{matrix} 0 \Leftrightarrow c_{ij} \begin{matrix} \geq \\ \leq \end{matrix} 1 \Leftrightarrow E(\varepsilon_i, L_j) = \varepsilon_j (c_{ij} - 1); \quad i \neq j, \quad i = 1, \dots, M \quad (64)$$

Note that  $\mathbf{c}_{ij}$  in (57) depend on *all parameters*,  $\rho_i$ ,  $\alpha_i$ , and also via :  $\varepsilon_i$ ,  $(\tilde{\rho})$ , on *all the Labor inputs*,  $L_i, i = 1, \dots, M$ , cf. (14), (49). Thereby is  $\mathbf{c}_{ij}$  the relevant and adequate *tool* (summary measure) - with our CRESH forms, (57) - to answer *distributional* (absolute *wage share*) issues with formula (64) ; cf. (54). See Sato and Koizumi (1973, p.486).

CRESH elasticities, (62-63), (57-58), (89), have standard *summation* properties, cf. (55):

$$\sum_{j=1}^M E(w_i, L_j) = \sum_{j=1}^M \varepsilon_j c_{ji} = 0 ; \quad \sum_{j=1}^M \varepsilon_j E(w_j, L_i) = 0 \quad (65)$$

Actual *parametric CRESH complementarity* elasticities,  $c_{ij}$ ,  $c_{ii}$ , (57-58), and the *wage effect* elasticities,  $E(w_i, L_i)$ ,  $E(w_i, L_j)$ , (63), (65), are shown in **Tables 8C-8D**, cf. CRESH *parameter* values :  $(\rho_i, \alpha_i)$  in **Table 4**, (column 3-4), and the *Labor inputs*  $L_i, i=1,\dots,M, L_A = L$ , (2010), in **Table 2** (column 5).

Note in **Table 8D** that the numerically *highest wage* elasticities,  $E(w_i, L_i)$ , are :  $E(w_5, L_5)$ ,  $E(w_6, L_6)$ ,  $E(w_7, L_7)$ ,  $E(w_8, L_8)$ , i.e., being most *sensitive to own supply* increases. These same *middle age* groups *gain* most by *larger cross supplies* from other age-groups, i.e., have the *highest wage* elasticities,  $E(w_i, L_j)$ ,  $i = 5, 6, 7, 8, i \neq j$ , in **Table 8D** - they have also the *largest cross complementarity* elasticities in **Table 8C**.

Finally, let us note from (49) and (57), that **if**  $\forall \rho_i = \rho$ , cf. CES, (9), then we have,

$$\forall \rho_i = \rho : \quad \sigma_{ij} = \frac{1}{(1 - \rho_i)(1 - \rho_j)\tilde{\rho}} = \frac{1}{1 - \rho} ; \quad c_{ij} = 1 - \rho_i - \rho_j + \tilde{\rho} = 1 - \rho \quad (66)$$

i.e., substitution elasticities ( $\sigma_{ij}$ ) and *dual complementarity* elasticities ( $c_{ij}$ ) are simply *reciprocals* of each other, and there would also be simple "reciprocal" relations between *factor demand* elasticities (53) and the so-called "*inverse factor demand*" [conditional *partial wage*] elasticities, (63). But with the much *richer parametric* class of **CRESH** production/aggregator functions and their duality relations, the *simple reciprocals* in (66) evidently **no longer apply** - and clearly **Table 8C** is *neither* the *reciprocal* of **Table 8A**.

With *demographic Age* groups and *exogenous Labor supplies* ( $L_i$ ), (39), it is  $\mathbf{c}_{ij}$ , (57-58), and  $E(w_i, L_i)$ ,  $E(w_i, L_j)$ ,  $E(w_j, L_i)$ ,  $E(w_i, L_j)$ , (63-65), **Tables 8C-8D** that are the *relevant* elasticities - which are behind all the *Age-wage group* results in **Tables 5-7**.

Table 8C. Partial complementarity elasticities ( $c_{ij}, c_{ii}$ ) - Denmark 2010 - by (57-58) and Table 4.

	$L_1$	$L_2$	$L_3$	$L_4$	$L_5$	$L_6$	$L_7$	$L_8$	$L_9$	$L_{10}$	$L_{11}$
$L_1$	-32.713	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	0.022
$L_2$	0.022	-6.089	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	0.022
$L_3$	0.122	0.122	-4.402	0.222	0.322	0.422	0.422	0.322	0.222	0.122	0.122
$L_4$	0.122	0.122	0.222	-2.545	0.322	0.422	0.422	0.322	0.222	0.122	0.122
$L_5$	0.222	0.222	0.322	0.322	-2.415	0.522	0.522	0.422	0.322	0.222	0.222
$L_6$	0.322	0.322	0.422	0.422	0.522	-2.578	0.622	0.522	0.422	0.322	0.322
$L_7$	0.322	0.322	0.422	0.422	0.522	0.622	-2.608	0.522	0.422	0.322	0.322
$L_8$	0.222	0.222	0.322	0.322	0.422	0.522	0.522	-2.491	0.322	0.222	0.222
$L_9$	0.122	0.122	0.222	0.222	0.322	0.422	0.422	0.322	-2.258	0.122	0.122
$L_{10}$	0.022	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	-3.020	0.022
$L_{11}$	0.022	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	-17.113
$\sum_{i=1}^M \varepsilon_j c_{ji}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 8D. Partial factor price (wage) elasticities,  $E(w_i, L_i)$ ,  $E(w_i, L_j)$  by (63) and Table 4.

	$L_1$	$L_2$	$L_3$	$L_4$	$L_5$	$L_6$	$L_7$	$L_8$	$L_9$	$L_{10}$	$L_{11}$	$\sum_{j=1}^M E(w_i, L_j)$
$w_1$	-0.200	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	0.000	0.000
$w_2$	0.000	-0.199	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	0.000	0.000
$w_3$	0.001	0.004	-0.286	0.024	0.045	0.066	0.065	0.044	0.027	0.008	0.001	0.000
$w_4$	0.001	0.004	0.014	-0.276	0.045	0.066	0.065	0.044	0.027	0.008	0.001	0.000
$w_5$	0.001	0.007	0.021	0.035	-0.341	0.082	0.081	0.058	0.039	0.015	0.003	0.000
$w_6$	0.002	0.011	0.027	0.046	0.074	-0.403	0.096	0.072	0.051	0.021	0.004	0.000
$w_7$	0.002	0.011	0.027	0.046	0.074	0.097	-0.404	0.072	0.051	0.021	0.004	0.000
$w_8$	0.001	0.007	0.021	0.035	0.059	0.082	0.081	-0.342	0.039	0.015	0.003	0.000
$w_9$	0.001	0.004	0.014	0.024	0.045	0.066	0.065	0.044	-0.273	0.008	0.001	0.000
$w_{10}$	0.000	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	-0.199	0.000	0.000
$w_{11}$	0.000	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	-0.200	0.000
$\sum_{j=1}^M \varepsilon_j E(w_j, L_i)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000



## 9 Appendix B : CRESH complementarity elasticities

**A1.** The *partial complementarity elasticity*,  $(c_{ij})$ , between any factor pair  $(L_i, L_j)$  within in the *implicit* CRESH function,  $\mathbf{L}_A = \mathbf{f}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M)$ , (10), was defined in (56) as :

$$c_{ij} \equiv c_{ij}(L_1, L_2, \dots, L_M) = \frac{f \frac{\partial^2 f}{\partial L_i \partial L_j}}{\frac{\partial f}{\partial L_i} \frac{\partial f}{\partial L_j}} ; \quad i = 1, \dots, M, \quad j = 1, \dots, M ; \quad c_{ii} \equiv c_{ii}(L_1, L_2, \dots, L_M) = \frac{f \frac{\partial^2 f}{\partial L_i^2}}{[\frac{\partial f}{\partial L_i}]^2} \quad (67)$$

The *first-order* derivatives in (67) were already given in (12) as,

$$\forall L_i > 0 : \quad \frac{\partial f}{\partial L_i} = - \frac{\partial F / \partial L_i}{\partial F / \partial L_A} = \frac{\alpha_i \rho_i (L_i / L_A)^{\rho_i - 1}}{\sum_{i=1}^M \alpha_i \rho_i (L_i / L_A)^{\rho_i}} > 0, \quad i = 1, \dots, M \quad (68)$$

The *second-order* derivatives in (67) are derived from the second term (ratio) in (68) as,

$$\frac{\partial^2 f}{\partial L_i \partial L_j} = \frac{-1}{[\frac{\partial F}{\partial L_A}]^3} \left[ \frac{\partial^2 F}{\partial L_i \partial L_j} \left[ \frac{\partial F}{\partial L_A} \right]^2 - \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_j} \frac{\partial F}{\partial L_A} - \frac{\partial^2 F}{\partial L_j \partial L_A} \frac{\partial F}{\partial L_i} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_A^2} \frac{\partial F}{\partial L_i} \frac{\partial F}{\partial L_j} \right] \quad (69)$$

$$\frac{\partial^2 f}{\partial L_i^2} = \frac{-1}{[\frac{\partial F}{\partial L_A}]^3} \left[ \frac{\partial^2 F}{\partial L_i^2} \left[ \frac{\partial F}{\partial L_A} \right]^2 - 2 \frac{\partial^2 F}{\partial L_i \partial L_A} \frac{\partial F}{\partial L_i} \frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_A^2} \left[ \frac{\partial F}{\partial L_i} \right]^2 \right] \quad (70)$$

Insert first-order and second-order derivatives (68-70) of  $L_A = f(L_1, L_2, \dots, L_M)$  into (67):

$$c_{ij} = -L_A \left[ \frac{\frac{\partial^2 F}{\partial L_i \partial L_j} \frac{\partial F}{\partial L_A}}{\frac{\partial F}{\partial L_i} \frac{\partial F}{\partial L_j}} - \frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} - \frac{\frac{\partial^2 F}{\partial L_j \partial L_A}}{\frac{\partial F}{\partial L_j}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}} \right] \quad (71)$$

In CRESH cases, we have,  $\partial^2 F / \partial L_i \partial L_j = 0$ , if  $i \neq j$ , cf. (74-75). Hence (71) becomes :

$$c_{ij} = -L_A \left[ - \frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} - \frac{\frac{\partial^2 F}{\partial L_j \partial L_A}}{\frac{\partial F}{\partial L_j}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}} \right], \quad i \neq j \quad (72)$$

$$c_{ii} = -L_A \left[ \frac{\frac{\partial^2 F}{\partial L_i^2} \frac{\partial F}{\partial L_A}}{[\frac{\partial F}{\partial L_i}]^2} - 2 \frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}} \right], \quad i = j \quad (73)$$

To obtain *explicit* CRESH formulas from (72-73), the *parametric* expressions of the first-order and second-order derivatives of the CRESH function,  $F(L_A, L_1, L_2, \dots, L_M)$ , (7) are now needed. We already have the first-order derivatives of  $F$  as, cf. (11),

$$\frac{\partial F}{\partial L_i} = \frac{\gamma \alpha_i \rho_i (L_i / L_A)^{\rho_i - 1}}{L_A} \equiv \frac{\gamma \varepsilon_i \beta}{L_i}; \quad \frac{\partial F}{\partial L_A} = - \frac{\gamma \sum_{i=1}^M \alpha_i \rho_i (L_i / L_A)^{\rho_i}}{L_A} \equiv - \frac{\gamma \beta}{L_A} \quad (74)$$

where,  $\beta \equiv \sum_{i=1}^M \alpha_i \rho_i (\mathbf{L}_i / \mathbf{L}_A)^{\rho_i}$  ;  $\varepsilon_i = \alpha_i \rho_i (\mathbf{L}_i / \mathbf{L}_A)^{\rho_i} / \beta$ ,  $i=1, \dots, M$ , cf. (14).

The second-order derivatives of  $F$  are derived from the second terms (ratios) in (74) as,

$$\frac{\partial^2 F}{\partial L_i \partial L_j} = 0; \quad \frac{\partial^2 F}{\partial L_i \partial L_A} = -\gamma \alpha_i \rho_i^2 L_i^{\rho_i - 1} L_A^{-1 - \rho_i} = -\frac{\gamma \rho_i \varepsilon_i \beta}{L_i L_A}; \quad \frac{\partial^2 F}{\partial L_j \partial L_A} = -\frac{\gamma \rho_j \varepsilon_j \beta}{L_j L_A} \quad (75)$$

$$\frac{\partial^2 F}{\partial L_i^2} = \frac{1}{L_A} \gamma \alpha_i \rho_i (\rho_i - 1) (L_i/L_A)^{\rho_i - 2} \frac{1}{L_A} = \frac{\gamma (\rho_i - 1) \varepsilon_i \beta}{L_i^2} \quad (76)$$

$$\frac{\partial^2 F}{\partial L_A^2} = \gamma \sum_{i=1}^M \alpha_i \rho_i (1 + \rho_i) L_i^{\rho_i} L_A^{-2 - \rho_i} = \frac{\gamma \beta}{L_A^2} \left[ \sum_{i=1}^M (1 + \rho_i) \varepsilon_i \right] = \frac{\gamma \beta}{L_A^2} (1 + \tilde{\rho}) \quad (77)$$

where,  $\tilde{\rho} = \sum_{i=1}^M \varepsilon_i \rho_i$ . Finally, inserting (74-77) into (72-73) give,

$$c_{ij} = -L_A \left[ -\frac{\frac{-\gamma \rho_i \varepsilon_i \beta}{L_i L_A}}{\frac{\gamma \varepsilon_i \beta}{L_i}} - \frac{\frac{-\gamma \rho_j \varepsilon_j \beta}{L_j L_A}}{\frac{\gamma \varepsilon_j \beta}{L_j}} + \frac{\frac{\gamma \beta}{L_A^2} (1 + \tilde{\rho})}{-\frac{\gamma \beta}{L_A}} \right] = -\rho_i - \rho_j + 1 + \tilde{\rho} ; \tilde{\rho} = \sum_{i=1}^M \varepsilon_i \rho_i \quad (78)$$

$$c_{ii} = -L_A \left[ \frac{\frac{\gamma (\rho_i - 1) \varepsilon_i \beta}{L_i^2} \left[ -\frac{\gamma \beta}{L_A} \right]}{\left[ \frac{\gamma \varepsilon_i \beta}{L_i} \right]^2} - 2 \frac{\frac{-\gamma \rho_i \varepsilon_i \beta}{L_i L_A}}{\frac{\gamma \varepsilon_i \beta}{L_i}} + \frac{\frac{\gamma \beta}{L_A^2} (1 + \tilde{\rho})}{-\frac{\gamma \beta}{L_A}} \right] = \frac{\rho_i - 1}{\varepsilon_i} - 2\rho_i + 1 + \tilde{\rho} \quad (79)$$

Labor *complementarity elasticities*  $c_{ij}$  (78-79) satisfy *regularity* (summation) property :

$$\begin{aligned} \sum_{j=1}^M \varepsilon_j c_{ji} &= \sum_{j=1, j \neq i}^M \varepsilon_j c_{ji} + \varepsilon_i c_{ii} = \sum_{j=1, j \neq i}^M \varepsilon_j (1 - \rho_i - \rho_j + \tilde{\rho}) + \varepsilon_i (1 - 2\rho_i + \frac{\rho_i - 1}{\varepsilon_i} + \tilde{\rho}) \\ &= -\sum_{j \neq i}^M \varepsilon_j \rho_j - \rho_i \sum_{j \neq i}^M \varepsilon_j - 2\rho_i \varepsilon_i + \rho_i + \tilde{\rho} = -\sum_{j=1}^M \varepsilon_j \rho_j + \tilde{\rho} = -\tilde{\rho} + \tilde{\rho} = 0 \quad (80) \end{aligned}$$

CRESH complementarity elasticities (78-80) were seen in **Table 8C** for Denmark (2010).

**A2. Wage Income function,  $\mathbf{W}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_M, \mathbf{W}_A)$**  - Wage Sum,  $W_A L_A$ , defined as,

$$W(L_1, L_2, \dots, L_M, W_A) = W_A f(L_1, L_2, \dots, L_M) = c(w_1, w_2, \dots, w_M) L_A = \sum_{i=1}^M w_i L_i \equiv W_A L_A \quad (81)$$

From *Wage Income* function (81), we get the *basic dual* expressions, cf. (14), (52), (67),

$$E(W, L_i) = \frac{\partial W}{\partial L_i} \frac{L_i}{W} = \frac{w_i L_i}{W} = \frac{\partial f}{\partial L_i} \frac{L_i}{f} = E(W, L_i) = \varepsilon_i = E(L_A, L_i) ; E(W, W_A) = 1 \quad (82)$$

$$\frac{W \frac{\partial^2 W}{\partial L_i \partial L_j}}{\frac{\partial W}{\partial L_i} \frac{\partial W}{\partial L_j}} = \frac{W_A f}{W_A} \frac{W_A \frac{\partial^2 f}{\partial L_i \partial L_j}}{W_A \frac{\partial f}{\partial L_i} W_A \frac{\partial f}{\partial L_j}} = \frac{f \frac{\partial^2 f}{\partial L_i \partial L_j}}{\frac{\partial f}{\partial L_i} \frac{\partial f}{\partial L_j}} = c_{ij} = c_{ji} ; i = 1, \dots, M, j = 1, \dots, M \quad (83)$$

$$w_i = w_i(L_1, L_2, \dots, L_M, W_A) = \frac{\partial W(L_1, L_2, \dots, L_M, W_A)}{\partial L_i} = W_A \cdot \frac{\partial f(L_1, L_2, \dots, L_M)}{\partial L_i} \quad (84)$$

where  $w_i$  (84) is the "shadow value" (*marginal value-added*:  $W_A \cdot \frac{\partial f}{\partial L_i}$ ) of *one unit* increase of specific Labor *inputs* from age-group (i),  $\mathbf{L}_i$ , i.e.,  $\mathbf{w}_i$  is the *nominal factor price* (money annual wage) of  $\mathbf{L}_i$  - being obtained as **Inverse factor demand** price or named '*partial market equilibrium*' **wage** for the *Labor supply* of Age (i),  $\mathbf{L}_i$  - with *fixed* Labor *supplies* of all the other Age groups - and with a *fixed aggregate* (average) annual wage,  $\mathbf{W}_A$ , to

the *Total* (Aggregate) *Labor market equilibrium* [complying with full general equilibrium of competitive product and factor markets].

Finally, using (82-83), we shall derive the annual *partial wage elasticities* of the optimal (Pareto efficient) **annual wages** ( $w_i$ ), [*Inverse factor demands*], (84), with respect to *partial* variation of **own labor supply** ( $L_i$ ) and any **cross labor supply** ( $L_j$ ), i.e.,

$$\frac{\partial w_i(L_1, L_2, \dots, L_M, W_A)}{\partial L_j} = \frac{\partial^2 W(L_1, L_2, \dots, L_M, W_A)}{\partial L_i \partial L_j} ; \quad i = 1, \dots, M, \quad j = 1, \dots, M \quad (85)$$

$$E(w_i, L_j) = \frac{\partial w_i}{\partial L_j} \frac{L_j}{w_i} = \frac{\partial^2 W}{\partial L_i \partial L_j} \frac{L_j}{w_i} = \frac{\frac{\partial^2 W}{\partial L_i \partial L_j} L_j}{\frac{\partial W}{\partial L_i}} = \frac{W \frac{\partial^2 W}{\partial L_i \partial L_j} \cdot \frac{w_j L_j}{W}}{\frac{\partial W}{\partial L_i} \frac{\partial W}{\partial L_j}} = \varepsilon_j c_{ij} \quad (86)$$

$$E(w_j, L_i) = \frac{\partial w_j}{\partial L_i} \frac{L_i}{w_j} = \frac{\partial^2 W}{\partial L_j \partial L_i} \frac{L_i}{w_j} = \frac{\frac{\partial^2 W}{\partial L_j \partial L_i} L_i}{\frac{\partial W}{\partial L_j}} = \frac{W \frac{\partial^2 W}{\partial L_j \partial L_i} \cdot \frac{w_i L_i}{W}}{\frac{\partial W}{\partial L_j} \frac{\partial W}{\partial L_i}} = \varepsilon_i c_{ij} \quad (87)$$

$$E(w_i, L_i) = \frac{\partial w_i}{\partial L_i} \frac{L_i}{w_i} = \frac{\partial^2 W}{\partial L_i^2} \frac{L_i}{w_i} = \frac{\frac{\partial^2 W}{\partial L_i^2} L_i}{\frac{\partial W}{\partial L_i}} = \frac{W \frac{\partial^2 W}{\partial L_i^2} \cdot \frac{w_i L_i}{W}}{[\frac{\partial W}{\partial L_i}]^2} = \varepsilon_i c_{ii} \quad (88)$$

where *complementarity elasticities*,  $c_{ij}$ , (78-79), are the relevant numbers for obtaining the **basic partial wage elasticities**, (63), (86-88), that are involved in our demographic *population* (cohort) *impact* analyses (calculations) over the projection period, 2020-2090. *Annual wage elasticities* of *Labor supply* (63), (86-88) have *summation* properties, as (65):

$$(i) \quad \sum_{j=1}^M E(w_i, L_j) = \sum_{j=1}^M \varepsilon_j c_{ji} = 0 ; \quad (ii) \quad \sum_{j=1}^M \varepsilon_j E(w_j, L_i) = 0 \quad (89)$$

Annual wage elasticities (63), (86-89), were illustrated in **Table 8D** for *Denmark* (2010).

By the way, the "adding-up", summing-property (89, *i*) is easily understood to hold from the "shadow-value" (wage) functions (84) being homogeneous of degree *zero* in increasing *all* labor supplies - by *derivatives* of the Wage Income function and Aggregator function, (81), (being homogeneous of degree *one* in labor supplies). Increasing proportionally *all* labour supplies does *not* change the *relative wages* - hence economically (89, *i*). Actual checking (89, *ii*) for CRESH (80) was more cumbersome ; but this was of course *necessary* for the CRESH formula **demographic-labor applications** in sections 4-5.

## 9.1 Price functions, Inverse demands - Hotelling-Wold identity

9.1.1. *Existence of consumer good price functions.* With regular (monotone, quasiconcave, smooth) **Utility** functions,  $u = U(q_1, \dots, q_n)$ , and *Budget constraint*,  $P_1 q_1 + \dots + P_n q_n = C$ ,

there is one and *only one set* of consumer good prices,  $P_i$ ,  $i = 1, 2, \dots, n$ , for which exogenously *fixed quantities*,  $(q_1, \dots, q_n)$ , are *optimal* (max.U) ; this price set is given by :

$$\frac{P_i}{C} = \varphi_i(q_1, \dots, q_n) \equiv \frac{\frac{\partial U}{\partial q_i}(q_1, \dots, q_n)}{\frac{\partial U}{\partial q_1} \cdot q_1 + \frac{\partial U}{\partial q_2} \cdot q_2 + \dots + \frac{\partial U}{\partial q_n} \cdot q_n} ; P_i = \varphi_i C, \quad i = 1, 2, \dots, n \quad (90)$$

$P_i$  - *price functions*, named as *Hotelling-Wold identity* - shown in Hotelling (1935, p.71)<sup>30</sup>, Wold (1944, p.70), Wold & Juren (1953, p.92,p.145) - or **Inverse uncompensated consumer good** ('Marshall') *demand functions*, cf. Diewert (1974, p.131), Cornes (1992, p.37).

Given explicit,  $u = U(q_1, \dots, q_n)$ , the functions,  $\varphi_i(q_1, \dots, q_n)$ , (90), are easily obtained.

9.1.2. *Existence of factor price functions.* Given a regular (monotone, concave, smooth, homogeneous of degree one) **Production** function,  $Y = g(x_1, \dots, x_m)$ , generating *Total revenue* (Factor income, Value-added),  $V \equiv PY = w_1x_1 + \dots + w_mx_m$ , there is one and *only one set* of factor prices,  $w_i$ ,  $i = 1, 2, \dots, n$ , for which the factor *quantities*,  $(x_1, \dots, x_n)$ , are *optimal* (maximizing profit) ; this factor price set is given by :

$$\frac{w_i}{PY} = \psi_i(x_1, \dots, x_m) \equiv \frac{\frac{\partial g}{\partial x_i}(x_1, \dots, x_m)}{\frac{\partial g}{\partial x_1} \cdot x_1 + \frac{\partial g}{\partial x_2} \cdot x_2 + \dots + \frac{\partial g}{\partial x_m} \cdot x_m} ; w_i = P \frac{\partial g}{\partial x_i}(x_1, \dots, x_m), \quad i = 1, 2, \dots, m \quad (91)$$

$w_i$  - *factor price functions*, RHS, (91), are the money *value marginal* factor productivity equations - or the **Inverse factor demand** functions [for competitive general equilibrium in both product (*fixed P*) and factor markets (*fixed supply of other factors,  $x_j, j \neq i$* ) ]

9.1.3. *Existence of Age annual wage functions.* Given a regular (monotone, concave, smooth, homogeneous of degree one) **Labor Aggregator** function,  $L_A = f(L_1, L_2, \dots, L_M)$ , giving *Total wage income* (Labor earnings),  $W \equiv W_A L_A = w_1L_1 + \dots + w_M L_M$ , there is *only one set* of annual wages,  $w_i$ ,  $i = 1, 2, \dots, n$ , for which Labor *supplies* of Age groups,  $(L_1, L_2, \dots, L_M)$ , are used *efficiently* (maximizing Total wages); this wage set is given by:

$$\frac{w_i}{W_A L_A} = \Psi_i(L_1, \dots, L_M) \equiv \frac{\frac{\partial f}{\partial L_i}(L_1, \dots, L_M)}{\frac{\partial f}{\partial L_1} \cdot L_1 + \frac{\partial f}{\partial L_2} \cdot L_2 + \dots + \frac{\partial f}{\partial L_M} \cdot L_M} ; w_i = W_A \frac{\partial f}{\partial L_i}(L_1, \dots, L_M), \quad i = 1, 2, \dots, M \quad (92)$$

$$\frac{w_i L_i}{W_A L_A} = \Psi_i(L_1, \dots, L_M) \cdot L_i \equiv \frac{\frac{\partial f}{\partial L_i}(L_1, \dots, L_M) \cdot L_i}{\frac{\partial f}{\partial L_1} \cdot L_1 + \frac{\partial f}{\partial L_2} \cdot L_2 + \dots + \frac{\partial f}{\partial L_M} \cdot L_M} = \varepsilon_i, \quad i = 1, 2, \dots, M ; \sum_{i=1}^M \varepsilon_i = 1 \quad (93)$$

$w_i$  - *annual wage functions*, RHS, (92), are the money *value marginal* Labor *contributions* of  $L_i$  to Wage sum,  $W$  - or **Inverse Labor demand** functions [for competitive equilibrium of the Aggregate Labor market (*fixed  $W_A$* ) and *fixed* Labor supply of other *ages,  $L_j, j \neq i$* ]. RHS, (93), shows that wage functions  $w_i$  (92) meet Total Wage ( $W$ ) *accounting identity*.

**CRESH** implementations of (92-93) are seen as **Age-wage profiles**,  $w_i$ ,  $i=1, \dots, M$ , in (25-27), (84),  $w_{iJ}$ , (30), (33), and to **partial wage elasticities** in (85-89). Using RHS (91) gives Macro **wages**,  $W_{AJ}$ ,  $J=I, II$ , (37-38), by **CRESH production** function (34-36).

<sup>30</sup>Hotelling calls :  $P_i = \varphi_i(q_1, \dots, q_n)C$ ,  $i = 1, \dots, n$ , *demand functions* ; see also Hotelling (1932, p.590).

## 10 App.C : Canonical Wage Structure Model - CRESH

Card and Lemieux (2001, pp.709) used two **CES Subaggregators**,  $L_{AJ}$  : College Labor,  $C = I$ , and High-school Labor,  $H = II$  - stated in notation, (9), (30), and  $\rho_J = \rho$ ,

$$L_{AJ} : L_{AI} = \left[ \sum_{i=1}^M \alpha_{iI} L_{iI}^\rho \right]^{\frac{1}{\rho}}, \quad L_{AII} = \left[ \sum_{i=1}^M \alpha_{iII} L_{iII}^\rho \right]^{\frac{1}{\rho}}; \quad -\infty < \rho \leq 1, \quad \sigma = \frac{1}{1-\rho} \quad (94)$$

As in existing literature, **Aggregate output** ( $Y$ ) comes with **CES** function of  $L_{AI}$ ,  $L_{AII}$  :

$$Y = \left[ \sum_{J=I}^{II} a_J(t) L_{AJ}^{\rho_y} \right]^{\frac{1}{\rho_y}}; \quad \frac{\partial Y}{\partial L_{iI}} = \frac{\partial Y}{\partial L_{AI}} \cdot \frac{\partial L_{AI}}{\partial L_{iI}}; \quad \frac{\partial Y}{\partial L_{iII}} = \frac{\partial Y}{\partial L_{AII}} \cdot \frac{\partial L_{AII}}{\partial L_{iII}}; \quad \frac{w_{iI}}{w_{iII}} = \frac{\frac{\partial Y}{\partial L_{iI}}}{\frac{\partial Y}{\partial L_{iII}}} \quad (95)$$

The *marginal product* (output) of workers in *age group* ( $i$ ) - with College (I) or High School education (II) - are seen (by chain rule) in (95). Pareto efficient utilization of different labor qualities (I, II) requires that *relative wages*,  $\frac{w_{iI}}{w_{iII}}$ , are equated to relative marginal products, RHS, (95). The partial derivatives of the CES functions in (94) and (95) imply that *relative wages* in *same age group* ( $i$ ),  $\frac{w_{iI}}{w_{iII}}$ , satisfy equation (96)<sup>31</sup>, LHS:

$$\begin{aligned} \frac{w_{iI}(t)}{w_{iII}(t)} &= \frac{a_I(t) \alpha_{iI}}{a_{II}(t) \alpha_{iII}} \left[ \frac{L_{AI}(t)}{L_{AII}(t)} \right]^{\rho_y - \rho} \left[ \frac{L_{iI}(t)}{L_{iII}(t)} \right]^{\rho - 1}; \quad \rho_y - \rho = -\frac{1}{\sigma_y} + \frac{1}{\sigma}, \quad \rho - 1 = -\frac{1}{\sigma} \quad (96) \\ &\equiv \frac{a_I(t) \alpha_{iI}}{a_{II}(t) \alpha_{iII}} \left[ \frac{L_{AI}(t)}{L_{AII}(t)} \right]^{-\frac{1}{\sigma_y}} \left[ \frac{\lambda_{iI}(t)}{\lambda_{iII}(t)} \right]^{-\frac{1}{\sigma}}, \quad \lambda_{iI} = \frac{L_{iI}}{L_{AI}}, \quad \lambda_{iII} = \frac{L_{iII}}{L_{AII}}, \quad \sum_{i=1}^M \lambda_{iJ} = 1 \quad (97) \end{aligned}$$

which is equivalent to expression (97) - with *Employment (Supply)*<sup>32</sup> *ratios* (Labor proportion of College/High school workers), and using *Age* composition (distribution) *within* College/High school workers ( $\lambda_{iJ}$ ), (97). Evidently from (97), *larger substitution elasticities* ( $\sigma_y$  and  $\sigma$ ) imply *smaller* changes in the relative wages,  $\frac{w_{iI}}{w_{iII}}$  [or log changes  $r_i(t)$ , (98)], coming from variation in *Aggregate Supply ratios* and *Age* compositions ( $\lambda_{iJ}$ ).

$$r_i(t) \equiv \log \frac{w_{iI}(t)}{w_{iII}(t)}, \quad i = 1 = 26 - 30, 31 - 35, \dots, M = 7 = 56 - 60; \quad I, II \quad (98)$$

<sup>31</sup>Card and Lemieux (2001, p.710, equation 7) presents (96), LHS, in logarithmic form, which is more convenient for parameter estimation purposes. We do not enter estimation - will only discuss the results.

<sup>32</sup>In relative wages (97), the aggregate supply *ratio* (relative supplies) is also seen in *share* form,  $\lambda_{AJ}$  :

$$\frac{w_{iI}(t)}{w_{iII}(t)} = \frac{a_I(t) \alpha_{iI}}{a_{II}(t) \alpha_{iII}} \left[ \frac{\lambda_{AI}(t)}{\lambda_{AII}(t)} \right]^{-\frac{1}{\sigma_y}} \left[ \frac{\lambda_{iI}(t)}{\lambda_{iII}(t)} \right]^{-\frac{1}{\sigma}}, \quad \lambda_{AI} = \frac{L_{AI}}{L_{AI} + L_{AII}}, \quad \lambda_{AII} = \frac{L_{AII}}{L_{AI} + L_{AII}}, \quad \sum_{J=I}^{II} \lambda_{AJ} = 1$$

For changes in log shares -  $\frac{d\lambda_{AJ}}{\lambda_{AJ}} \cdot 100$  - of aggregate labor input groups and their relative wages changes, see Katz and Murphy (1992, p.39-40,49,67-68) - where the aggregate labor supplies,  $L_{AJ}(t)$ , are measured in so-called '*efficiency units*'. No *Age-specific* full-time equivalents  $L_{iJ}$  appear in Katz & Murphy (1992).

For year (t),  $r_i(t)$  is called a College-High school *premium* or *wage gap* for *age group* (i). Ratios  $r_i(t)$ ,  $i = 1, \dots, 7$  is an *Age profile* (98) of *premiums/wage gaps* for *calendar year* (t). We will briefly discuss the *parameters*, (96-97), of the *Age profiles* (98) that Card and Lemieux (2001, p.715, 718) has *calculated* in *years*, 1959,1970,1975,1980,1985,1990,1995, for the United States, and roughly *same* calendar years for United Kingdom and Canada.

For all three countries, CES parameter estimates, Card & Lemieux (2001, pp.725-27), of  $\rho$ , (94), (96), were in the **range** :  $\rho = 0.77$  to  $\rho = 0.83$ <sup>33</sup>, i.e.,  $\sigma = 4.34$  to  $\sigma = 5.88$ .

The *estimated* sizes of the *age specific efficiency* (intensity) parameters -  $\alpha_{iI}$  and  $\alpha_{iII}$  - in (94), (96-97), for the seven (98) age groups,  $i = 1, \dots, M=7$ , are *not* available<sup>34</sup> (reported); they give the two *age-wage profiles* of,  $w_{iI}(t)$ ,  $w_{iII}(t)$ ,  $i = 1, 2, \dots, M$ , by (94) - as seen below.

In contrast to  $\alpha_{iI}$ ,  $\alpha_{iII}$ , the relative *efficiency* (intensity) parameters,  $a_I$ ,  $a_{II}$ , (95-97), are *not* time-invariant for the aggregates  $L_{AI}$  and  $L_{AII}$  in the *Production* function (95). Card & Lemieux (p.725) give for the three countries *estimates* of *year effects*, reflecting changes (technology *shocks*) to the ratios,  $\frac{a_I(t)}{a_{II}(t)}$ , in the calendar years above. These year effects are rising, and next replaced by *linear trends* :  $\frac{a_I(t)}{a_{II}(t)} = \beta t$ ,  $\beta \in (0.017, 0.020)$  for US,  $\beta \in (0.021, 0.018)$  for UK,  $\beta \approx 0$  for Canada. These trend *estimates* ( $\beta$ ) are combined with the final estimation of  $\rho_y$ , (95). Card & Lemieux (p.727) present *two* estimates of  $\rho_y$  [depending on sizes of the Aggregate Supply indexes for College Labor and High-school Labor, 1. Katz-Murphy indexes, 2.  $L_{AJ}$  in (94)]. Thus we see the *estimates* for the US : 1.  $\rho_y = 0.59$ , 2.  $\rho_y = 0.52$ , i.e.,  $\sigma_y = 2.44$  or  $\sigma_y = 2.08$ . For UK: 1.  $\rho_y = 0.53$ , 2.  $\rho_y = 0.66$ , i.e.,  $\sigma_y = 2.13$  or  $\sigma_y = 2.94$ . For Canada: 1.  $\rho_y = 0.93$ , 2.  $\rho_y = 0.87$ , i.e.,  $\sigma_y = 14.29$  or  $\sigma_y = 7.69$  (Canadian  $\rho_y$  are imprecise estimates, also  $\rho_y = 0.82$ ,  $\sigma_y = 3.57$ ).

By using second estimate (2) of  $(\rho_y, \sigma_y)$ , second trend coefficient of ( $\beta$ ), common estimate of  $(\rho_I = \rho_{II} = \rho, \sigma)$ , together with *relative Aggregate Labor Supplies*,  $L_{AI}(t)/L_{AII}(t)$ , and *relative Age-group Supplies*,  $L_{iI}(t)/L_{iII}(t)$ , [*relative Age distributions*,  $\lambda_{iI}(t)/\lambda_{iII}(t)$ ] we obtain with (96-97) their *Age-relative wage* profile,  $\mathbf{w}_{iI}(\mathbf{t})/\mathbf{w}_{iII}(\mathbf{t})$ ,  $i = 1, \dots, M = 7$ , for

<sup>33</sup>In **Table 4** (Col. 3), the range of the CRESH age-specific  $\rho_i$  was :  $\rho_i = 0.5$  to  $\rho_i = 0.8$  ; but  $\rho_i = 0.5$  applied only to two *age groups*, 40-44, 45-49. CES parameter  $\rho = 0.8$  ( $\sigma = 5$ ) can fit any data of relative wages pretty well ; cf. Guest & Jensen (2016, p.31, Fig.4 - misprint p.32: interchange titles of Fig 5,6).

<sup>34</sup>Card & Lemieux (p.713, eq.12a-12b) show how,  $\alpha_{iI}$ ,  $\alpha_{iII}$ , ( $\alpha'_j s, \beta'_j s$ ),  $\rho(\eta)$ , are estimated - and used to construct estimates of Aggregate Labor supplies,  $L_{AI}(t)$ ,  $L_{AII}(t)$ , ( $C_t, H_t$ ), in (94) ; cf. footnote 26.

the calendar *year* (t) - or in log version, *Age-relative wage* profile (98),  $\mathbf{r}_i(\mathbf{t})$ ,  $i = 1, \dots, M = 7$ , called the (*College premiums - wage gaps*) for calendar *year* (t).

The *shape* of *Age-relative wage* profiles,  $\frac{\mathbf{w}_{iI}(\mathbf{t})}{\mathbf{w}_{iII}(\mathbf{t})}$ , [or  $\mathbf{r}_i(\mathbf{t})$ ],  $i = 1, \dots, M$ , have changed over years 1970,1975,1980,1985,1990,1995 - but alike for three countries, Card & Lemieux (p.718). It has shifted upwards before 1980 - and *rotated* (twisted) after 1980-85, with *younger* workers (31-35, 36-40) *rising* much more than the older workers (46-50, 51-55).

Apart from 'biased' *technology* trends,  $a_I(t)/a_{II}(t) = \beta t$ , Card & Lemieux (p.707) see a *deceleration* (slower increases) in *relative* College Labor *supplies* since 1980 as the driving force behind the increased<sup>35</sup> *relative* wages (96-98). But behind such relative wages,<sup>36</sup> (96-98), we also want economic-analytically to know exactly what occur - in a consistent way - to the corresponding *calendar year* (t) *Age-wage profiles* :  $w_{iI}(t)$ ,  $i = 1, 2, \dots, M$ , and,  $w_{iII}(t)$ ,  $i = 1, 2, \dots, M$ . We will match CES version of  $w_{iJ}(t)$ , (30), to (94-97).

<sup>35</sup>See Acemoglu & Autor (2011, p.1052); Goldin & Margo (1992, p.7), Murphy & Welch (1992, p.294).

<sup>36</sup>As background for CRESH production function, (34-36), the general CES version is discussed here.

The CES production functions are (normalizing,  $\sum_{J=I}^{VI} a_J = 1$ , may need  $\gamma$  for dimensional reasons) :

$$Y = \gamma \left[ \sum_{J=I}^{VI} a_J (b_J X_J)^\rho \right]^{\frac{1}{\rho}} ; \gamma > 0 ; \sum_{J=I}^{VI} a_J = 1 ; -\infty < \rho \leq 1, \sigma = \frac{1}{1-\rho}$$

where  $b_J$  are *factor-augmenting* technology terms - parameters (constants) or specified functions of time (trends). The expressions,  $\mathbf{b}_J \mathbf{X}_J$ , are referred to as the *factor supplies* shown (measured) in "efficiency units". Usually, the variables (quantities),  $X_J$ ,  $J = I, \dots, VI$ , have their own *units of measurements* (e.g., labor, capital, etc.), which are entirely different matters (and problems) than attaching *factor-augmenting* terms (parameters, functions,  $a_J$ ) to each variable  $X_J$  in discrete or continuous time. For our purposes, we did neither consider any factor augmenting terms ( $b_J$ ) involved in  $L_{AJ}$ , (95), nor ( $b_{iJ}$ ) in  $L_{iJ}$ , (94).

Autor et al. (1998, p.1176-79) use such CES two-factor labor augmenting ( $b_I, b_{II}$ ) version of (95), giving the relative wages (ratios of marginal products of two labor types) as follows [notation, (96-97)]:

$$\frac{w_I(t)}{w_{II}(t)} = \frac{a_I(t)}{1 - a_I(t)} \left[ \frac{b_I(t)}{b_{II}(t)} \right]^\rho \left[ \frac{L_I(t)}{L_{II}(t)} \right]^{-\frac{1}{\sigma}} \equiv d(t) \left[ \frac{L_I(t)}{L_{II}(t)} \right]^{-\frac{1}{\sigma}} ; \rho = \frac{\sigma - 1}{\sigma}, \sigma = \sigma_y, \rho = \rho_y$$

$w_I = \frac{\partial Y}{\partial L_I} = a_I(t) b_I^\rho(t) \left[ \frac{Y}{L_I} \right]^{\rho-1}$ ,  $w_{II} = \frac{\partial Y}{\partial L_{II}} = a_{II}(t) b_{II}^\rho(t) \left[ \frac{Y}{L_{II}} \right]^{\rho-1}$ . Such technology term  $\left[ \frac{b_I(t)}{b_{II}(t)} \right]^\rho$  may be included in (101). No *Age-specific* full-time equivalents,  $L_{iJ}$  appear in Autor et al. (1998)

The 'parameter elements' within the composite variable,  $d(t)$ , are used to reflect technological- and relative factor demand shifts that may favor college equivalents,  $L_I(t)$ , raising the college premium/wage gap :  $r(t) \equiv \log \frac{w_I(t)}{w_{II}(t)}$ . Using Katz & Murphy (1992, p.69) point estimate of  $\sigma_y = 1/0.709 = 1.41$ , and lower/upper limits of  $\sigma_y$  (1, 2), Autor et al. (1998), calculate - for  $\sigma_y = 1, 1.4, 2$  - and with data,  $\frac{L_I(t)}{L_{II}(t)}$ , the *college premium*,  $r(t)$ , and implied relative demand *shifts*,  $d(t)$ , for decades in period 1940-1996.

In (37-38), the *exogenous*  $\mathbf{W}_{AJ}$  referred to *Average wage* of particular Subaggregates of workers,  $L_{AJ}$ . With a single-sector (output), aggregate **CES** production function, (95), the efficiency in production implies that  $\mathbf{W}_{AJ}$  are *marginal value products* of **Labor**, i.e.,

$$\frac{W_{AI}}{P} = \frac{\partial Y}{\partial L_{AI}} = a_I(t) \left[ \frac{Y}{L_{AI}} \right]^{\rho_y - 1}; \quad \frac{W_{AII}}{P} = \frac{\partial Y}{\partial L_{AII}} = a_{II}(t) \left[ \frac{Y}{L_{AII}} \right]^{\rho_y - 1}; \quad Y = \left[ \sum_{J=I}^{II} a_J(t) L_{AJ}^{\rho_y} \right]^{\frac{1}{\rho_y}} \quad (99)$$

Thus with (99) and CES (94), **calendar wages** of Age group (i),  $w_{iJ}(t)$ , (30), becomes:

$$w_{iI}(t) = P \cdot a_I(t) \left[ \frac{Y}{L_{AI}} \right]^{\rho_y - 1} \alpha_{iI} \lambda_{iI}^{\rho - 1}(t); \quad w_{iII}(t) = P \cdot a_{II}(t) \left[ \frac{Y}{L_{AII}} \right]^{\rho_y - 1} \alpha_{iII} \lambda_{iII}^{\rho - 1}(t); \quad i = 1, \dots, M \quad (100)$$

Accordingly, (100) is consistent with *relative wages*, LHS (101) - formally similar to (97):

$$\frac{w_{iI}(t)}{w_{iII}(t)} = \frac{a_I(t)}{a_{II}(t)} \frac{\alpha_{iI}}{\alpha_{iII}} \left[ \frac{L_{AI}(t)}{L_{AII}(t)} \right]^{-\frac{1}{\sigma_y}} \left[ \frac{\lambda_{iI}(t)}{\lambda_{iII}(t)} \right]^{-\frac{1}{\sigma}}, \quad L_{AI} = L_I = \sum_{i=1}^M L_{iI}; \quad L_{AII} = L_{II} = \sum_{i=1}^M L_{iII} \quad (101)$$

*Note.* In (99-101), *Aggregate Labor supplies* ( $L_{AI}, L_{AII}$ ) are Age-group *sums* (RHS (101)).

We now need to scrutinize the concepts and the actual numbers of the *Aggregate Labor supply* variables,  $L_{AJ}$ ,  $J=I, II$ , that appear in the CES functions, (94-95). The CES *Subaggregator* formulas (94) are often called *Aggregate Supply indexes* of College/High school Labor resources. Clearly, larger values of CES parameter  $\rho$  ( $\sigma$ ) affect the isoquant maps of (94) analogously to the role of  $\rho_y$  ( $\sigma_y$ ) for the isoquant maps of (95), implying e.g., that *larger* supply index numbers  $L_{AI}$  are attained with *smaller* sizes of  $L_{iI}$  in (94); but such larger abstract-theoretical  $L_{AI}$  total supply numbers are neither directly observable nor satisfy simple labor accounting identities stated in RHS (101). With well-defined *measuring units* as 'full-time worker (College/High school) equivalents', the direct sum (accounting) of sub-groups in RHS (101) are important to satisfy - as in **Table 2** (Col.5). In short, Subaggregator formulas (94) are *not* used to obtain (predict) *Total quantities*,  $L_{AJ}$ , but (94) are used to *generate* sub-group *wages* to form Total Wage Income, (25-27).



Only *derivatives* (31-32) of *Aggregator* functions,  $L_{AJ} = f_J$ , are used in *imputing wages*  $w_{iJ}$ , (100), to the *age-groups* (i) of Total Labor supply,<sup>37</sup>  $L_{AJ} = \sum_{i=1}^M L_{iJ}$ , RHS (101).

The *size* (level) of the Average wages -  $\mathbf{W}_{AJ}$  were not explained *economically*. The CES Production functions (95) attempted such economic explanation of,  $W_{AJ}$ , through the *marginal products* of Labor, (99),  $L_{AJ}$ ,  $J = I, II$ , which implied the calendar *Age-wage* profiles,  $w_{iJ}(t)$ ,  $i = 1, \dots, M$ , (100), of College/High school Labor,  $J = I, II$ . *Shifting* (twisting, *rotating*) of the *two* calendar CES *Age-wage* profiles is seen by (100) to follow from: 1. *divergent* evolutions of the (sum) *Totals*,<sup>38</sup>  $L_{AI}(t)$ ,  $L_{AII}(t)$ , 2. *divergent* evolutions of the **Age distributions**,  $\lambda_{iJ}$ ,  $i = 1, \dots, M$ , *within* the two categories,  $J = I, II$ .

With CRESH, CES *calendar year Age-wage* profiles (100), are replaced by inserting - *marginal value products* of Labor, (99) - into CRESH **Age-wage profile** formula (33):

$$w_{iI}(t) = P \cdot a_I(t) \left[ \frac{Y}{L_{AI}} \right]^{\rho_y - 1} \frac{\alpha_{iI} \rho_{iI} \lambda_{iI}^{\rho_{iI} - 1}}{\sum_{i=1}^M \alpha_{iI} \rho_{iI} \lambda_{iI}^{\rho_{iI}}} ; w_{iII}(t) = P \cdot a_{II}(t) \left[ \frac{Y}{L_{AII}} \right]^{\rho_y - 1} \frac{\alpha_{iII} \rho_{iII} \lambda_{iII}^{\rho_{iII} - 1}}{\sum_{i=1}^M \alpha_{iII} \rho_{iII} \lambda_{iII}^{\rho_{iII}}} \quad (102)$$

Evidently, more elaborate **shifting**<sup>39</sup> by two CRESH **calendar year Age wage** profiles.<sup>40</sup> Replacing CES, (99) by (37-38) give in (102) Age wage profiles affected by  $(K_{III}, K_{IV}, K_V)$ .

<sup>37</sup>Although relative wages in LHS (101) formally look 'similar' to (97), there is an ambiguity about the numerical size of,  $L_{AI}$ ,  $L_{AII}$ , appearing in Sub-aggregators, (94), Production function (95), and relative wages, (96-97). As discussed above, the absolute size of,  $L_{AI}$ ,  $L_{AII}$ , are irrelevant for using Sub-aggregators, (94) to age-wage imputations, where only **age distributions**,  $\lambda_{iJ}$ ,  $i=1, \dots, M$ , mattered cf. (31-32). However, the total imputed wage sums are:  $W_{AJ} L_{AJ} = \sum_{i=1}^M w_{iJ} L_{iJ} \equiv w_J L_J \equiv W_J$ ,  $J = I, II$ , i.e.,  $L_{AJ}$  as in RHS (101) - and not as  $L_{AJ}$  determined in (94). Thus  $L_{AJ}$  in (95) is neither (94), but RHS (101) - no ages are involved in the factor substitutions by (95). The text above (p.42) mentioned two estimates of  $(\rho_y, \sigma_y)$  by : 1. Katz-Murphy indexes, 2.  $L_{AJ}$  in (94). Thus,  $(\rho_y, \sigma_y)$  estimates by the latter Labor supply numbers (94) are inadequate - instead we wanted estimates of  $\alpha_{iJ}$ , cf. footnote 23.

<sup>38</sup>The time path of the ratio,  $\frac{L_{AI}(t)}{L_{AII}(t)}$  is called "intercohort shifts in the relative supply of highly educated workers", "intercohort trend in educational attainment", and  $\lambda_{iI}(t), \lambda_{iII}(t)$ ,  $i = 1, \dots, M$ , are called "differences in *age distributions* of educational attainment", Card & Lemieux (p.707). Cohorts means *calendar year* (t) total *Labor supplies*,  $L_{AJ}(t)$ ,  $J = I, II$ . Cohorts in the sense of Labor *supplies*,  $L_i^*(T)$ , [life-cycle ages (i)], and life time supplies,  $L^*(T)$ , cf. (44), are not seen in Card & Lemieux (2001).

<sup>39</sup>Analysis and explanations of the 'relative demand shifts'/'trends' behind college premium/wage inequality (note 25) need of a variety of models, including trade, cf. Borjas & Ramey (1994, p.12), Borjas et al. (1997, p.11, 40-41), Topel (1997, p.68).

<sup>40</sup>A *pure* labor economy model by the production and substitution with only *two Labor* factors (categories),  $L_{AI}(t), L_{AII}(t)$ , (95), is of course an abstraction for explaining,  $W_{AJ}$ , (99).

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