

A Service of

ZBW

Leibniz-Informationszentrum Wirtschaft Leibniz Information Centre for Economics

Bentahar, Imen

Working Paper Tail Conditional Expectation for vector-valued risks

SFB 649 Discussion Paper, No. 2006-029

Provided in Cooperation with: Collaborative Research Center 649: Economic Risk, Humboldt University Berlin

Suggested Citation: Bentahar, Imen (2006) : Tail Conditional Expectation for vector-valued risks, SFB 649 Discussion Paper, No. 2006-029, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin

This Version is available at: https://hdl.handle.net/10419/25112

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



WWW.ECONSTOR.EU

SFB 649 Discussion Paper 2006-029

Tail Conditional Expectation for vector-valued Risks

Imen Bentahar*



* Technische Universität Berlin, Germany

This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

http://sfb649.wiwi.hu-berlin.de ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin Spandauer Straße 1, D-10178 Berlin



Tail Conditional Expectation for vector valued risks

Imen Bentahar *

TU Berlin, Germany

bentahar@math.tu-berlin.de

PRELIMINARY VERSION

Abstract

In his paper we introduce a quantile-based risk measure for multivariate financial positions: the vector-valued Tail-conditional-expectation ($\mathcal{T}CE$). We adopt the framework proposed by Jouini, Meddeb, and Touzi [9] to deal with multi-assets portfolios when one accounts for frictions in the financial market. In this framework, the space of risks formed by essentially bounded random vectors, is endowed with some partial vector preorder \succeq accounting for market frictions. In a first step we provide a definition for quantiles of vector-valued risks which is compatible with the preorder \succeq . The $\mathcal{T}CE$ is then introduced as a natural extension of the "classical" real-valued tail-conditional-expectation. Our main result states that for continuous distributions $\mathcal{T}CE$ is equal to a

^{*}This work was partly supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk" $\,$

coherent vector-valued risk measure. We also provide a numerical algorithm for computing vector-valued quantiles and $\mathcal{T}CE$.

Key words : Risk measures, vector-valued risk measures, coherent risk-measures, quantiles, tail-conditional-expectation.

MSC Classification (2000): 91B28, 49L25, 35B05.

1 Introduction

In their seminal paper [2], Artzner et al. adopt an axiomatic approach to characterize economically *coherent* risk measures. The authors consider the resulting net worth of a financial position, at the end of a given investment period, and describe it by a realvalued random variable X on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Their definition of a risk measure relies on the following basic approach. An investor (or regulator) defines among the set of all possible financial positions, a subset \mathcal{A} of acceptable positions regarded as risk-free. Then, the risk measure $\rho(X)$ of a position $X \in \mathbb{R}^{\Omega}$, corresponds to the "extra" capital requirement that has to be invested at the beginning of the period in some "secure" instrument so that the resulting position is acceptable, i.e. X + $\rho(X) \in \mathcal{A}$. A set of axioms, namely: (i) subadditivity, (ii) monotonicity, (iii) positive homogeneity, and (iv) translation invariance, guarantees the economic coherence of a risk measure ρ . The notion of coherent risk measure has been extended to convex risk measure [6], and has been generalized to more complex spaces of risk, which allows to take into consideration financial positions with different types of cash streams structures. For instance, Delbaen [5] considers a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and identifies the space of risks with the space of essentially bounded random variables $L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$. Cheridito, Delbaen and Kupper [4] extend the definition and the dual representation of coherent and convex risk measures to the space of càdlàg processes. Jaschke and Küchler [8] consider coherent risk measures on abstract spaces of risk including deterministic, stochastic, single or multi-periodic cash-streams structures. The above mentioned generalizations, however, do not take into account the problem of portfolio aggregation. Indeed, if we consider realistic situations where investors have access to different markets and form multi-assets portfolios, in the presence of frictions such as : transaction costs, liquidity problems, irreversible transfers,... etc., a position cannot be merely described by a real-valued random net worth, or a realvalued process of cash-streams. Focusing on this problem, Jouini, Meddeb, and Touzi [9] describe financial positions through \mathbb{R}^d -valued random variables $X = (X^1, ..., X^d)$, where each component X^i corresponds to net worth at the end of the investment period of an investor's position on the i^{th} market. They restrict the space of risks to essentially bounded \mathbb{R}^d -valued random portfolios $X, X \in L^{\infty}_d$, and assume that the portfolios in L^{∞}_d are ordered according to a partial ordering relation \preceq which accounts for frictions on the financial market. By analogy to [2], the authors define a vector-valued risk measure as set-valued function R which associates to each risky portfolio X, deterministic portfolios $\bar{x} = (\bar{x}^1, ..., \bar{x}^d)$ where \bar{x}^i corresponds to some "extra" capital invested in a secure instrument from market i, and such that the position $X + \bar{x}$ is acceptable. Jouini et al. extend the axiomatic characterization of real-valued coherent risk measures to the multi-dimensional case, and they provide a dual representation result for coherent vector-valued risk measures.

Our main concern in this paper is to define a distribution-based vector-valued risk measure, to verify its coherency, and to propose a procedure to compute it.

In the one-dimensional context, risk measures of investment strategies are often defined in terms of a quantile of a given distribution. This is a natral procedure since the worst realizations of a financial position are concentrated on the left tail of its disribution. A typical example is the Value-at-Risk (VaR) which is a distribution tail related measure identifying the loss that is likely to be exceeded by a specified probability, over a given time horizon. A second important example is the *Tail Conditional Expectation* (TCE), which has been suggested as an alternative to VaR [2]. Indeed, while VaR fails to be a coherent risk measure (VaR is not subadditive), the risk mesure TCE results to coincide, under some conditions on the distribution of the risks, with a coherent risk measure, see [2], [3]. The main difficulty regarding the generalization of quantile-based measures to this framework is the fact that vector preorders are, in general, partial preorders. Then, what can be considered in a context of multidimensional portfolios as the analogous of "worst cases" or "tail distributions"? This is the first question we shall address by suggesting a suitable definition of quantiles for multi-dimensional portfolios. We shall then introduce Vector-valued Tail conditional Expectation as a natural extension to the "classical" real-valued TCE.

This paper is organized as follows. In Section 2 we recall the notion of vector-valued risk measures and acceptance sets, then we present our definition of the α -quantile of the distribution of an \mathbb{R}^d -valued portfolio and the corresponding definition of vector valued Tail Conditional Expectation. Our main result, provided in Section 3, states that for continuous distributions, the vector-valued Tail Conditional Expectation is equal the vector valued Generalized Worst Conditional Expectation (\mathcal{GWEC}) which is a coherent risk measure . Finally, we detail in Section 4 a numerical procedure to compute quantiles and Tail Conditional Expectations for vector-valued risks.

NOTATIONS : We first introduce the main notations of the paper.

Given an element x of a finite-dimensional vector space, we shall denote by x^i its i - th component.

For $i \leq d \in \mathbb{N}$, we shall denote by $\mathbf{1}_i$ the element of the canonical basis of \mathbb{R}^d defined by : $\mathbf{1}_i^j = 0$ if $j \neq i$, and $\mathbf{1}_i^i = 1$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall denote by L^d_{∞} the set of essentially bounded \mathbb{R}^d -valued random variables, and $||X||_{\infty}$ denotes the essential supremum of $X \in L^{\infty}_d$

Finally, for a set A, $\mathbf{1}_A$ states for its indicator function.

2 Definitions

We consider investors having access to $d \ge 1$ different markets. A financial position is then described by a *d*-dimensional random vector $X := (X^1, ..., X^d)'$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each component X^i represents the resulting wealth in the market *i* of the investor's positions or strategies. Following Jouini, Meddeb, and Touzi [9], we restrict our attention to financial positions in L_d^{∞} , and we assume that they are ordered according to the following rule :

$$X \succeq Y$$
 if and only if $X - Y \in K \mathbb{P}$ - a.s.

where K is a closed convex cone of \mathbb{R}^d satisfying the following conditions

$$\mathbb{R}^d_+ \subset K \text{ and } K \neq \mathbb{R}^d.$$
 (2.1)

Observe that the condition $\mathbb{R}^d_+ \subset K$ implies that portfolios with nonnegative entries are nonnegative w.r.t. the partial order relation \succeq .

2.1 Vector-valued risk measures

For the convenience of the reader, we give hereafter a short remainder of the notions of vector-valued risk measures, and acceptance sets. We refer the interested reader to [9] for a thourough treatment.

A basic approach for describing the risk incurred by some agent consists of defining, within the positions he can take, a subset of desirable or *acceptable* positions. Thus a portfolio is considered to be risky or not, whether it lies or not in the acceptance set.

Definition 2.1 A coherent acceptance set is a subset \mathcal{A} of L^{∞}_d satisfying

$$\begin{array}{lll} C0- & \mathcal{A} \text{ is a closed and convex cone of } L_d^{\infty} \text{ containing } 0 \ . \\ C1- & \text{For all } X \in \mathcal{A} \text{, for all } Y \in L_d^{\infty} \text{, } Y \succeq X \implies Y \in \mathcal{A} \text{.} \\ C2- & \mathbb{R}^d \not\subset \mathcal{A} \text{.} \end{array}$$

A vector valued risk measure associates with each risk X a set, R(X), of deterministic portfolios $\bar{x} \in \mathbb{R}^d$ such that: the modified position $X + \bar{x}$ is "accetable".

Definition 2.2 A coherent risk measure is a set-valued map $R : L_d^{\infty} \to \mathbb{R}^d$ satisfying the following axioms

$$\begin{array}{lll} A0- & For \ all \ X \in L_d^{\infty}, R(X) \ is \ closed, \ and \ \ 0 \in R(0) \neq \mathbb{R}^d \ . \\ A1- & For \ all \ X, Y \in L_d^{\infty}, \ X \succeq Y \Rightarrow R(Y) \subset R(X) \ . \\ A2- & For \ all \ X, Y \in L_d^{\infty}, \ R(X) + R(Y) \subset R(X+Y) \ . \\ A3- & For \ all \ X \in L_d^{\infty}, \ for \ all \ t > 0, \ R(tX) = tR(X) \ . \\ A4- & For \ all \ X \in L_d^{\infty}, \ for \ all \ x \in \mathbb{R}^d \ , \ R(X+\bar{x}) = \ \{-\bar{x}\} + R(X) \end{array}$$

The following Proposition states the relation between coherent acceptance sets and coherent risk measures.

Proposition 2.1 Let \mathcal{A} be some subset of L_d^{∞} and define the set-valued map $R_{\mathcal{A}}$: $L_d^{\infty} \to \mathbb{R}^n$ as follows

$$R_{\mathcal{A}}(X) := \left\{ \bar{x} \in \mathbb{R}^d : X + \bar{x} \in \mathcal{A} \right\}$$

Then \mathcal{A} is a coherent acceptance set if and only if $R_{\mathcal{A}}$ is a coherent risk measure.

Example 2.1 Fix some $\alpha \in (0, 1)$. The (vector-valued) Worst Conditional Expectation at level α is the set-valued map defined on L_d^{∞} by

$$\mathcal{W}CE_{\alpha}(X) := \left\{ \bar{x} \in \mathbb{R}^d : \mathbb{E}\left[X + \bar{x} | B \right] \succeq 0 \text{ for all } B \in \mathcal{F} \text{ with } \mathbb{P}(B) > \alpha \right\}$$

The vector-valued worst conditional expectation, WCE_{α} , has been introduced by Jouini et al. [9] as a natural extension of the coherent real-valued worst conditional expectation, WCE_{α} defined, for a real-valued random variable \mathcal{X} , by

$$WCE_{\alpha}(\mathcal{X}) := -\inf \{ \mathbb{E}[\mathcal{X} | B] , B \in \mathcal{F}, \mathbb{P}(B) \geq \alpha \}$$

It can be easily checked, that WCE_{α} is a coherent vector-valued risk measure.

Remark 2.1 Actually, in [9], the authors introduce the notion of (d, n)-coherent risk measure: by assuming that the convex cone K satisfies the additional property of *substitutability*

for all
$$i = n + 1, ..., d$$
: $-\mathbf{1}_i + \delta \mathbf{1}_1$ and $\mathbf{1}_i - \gamma \mathbf{1}_1 \in K$ for some $\delta, \gamma > 0$.

it possible to regulate risks in L_d^{∞} by using deterministic portfolios of the form $\bar{x} = (x, 0, ..., 0)$, where $x \in \mathbb{R}^n$. In this paper we restrict our analysis to (d, d)-coherent risk measures in order to simplify our presentation. Our work adapts without any difficulty to more general (d, n)-coherent risk measure.

We end this subsection by introducing a new vector-valued coherent risk measure based on the WCE_{α} . We call this risk measure the *Generalized Worst Conditional Expectation* at level α , we denote it by \mathcal{GWCE}_{α} , and we defined by

$$\mathcal{GWCE}_{\alpha}(X) = \bigcup_{\tilde{X}} \mathcal{WCE}_{\alpha}(\tilde{X})$$
(2.2)

where the union is taken over all random variables \tilde{X} on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ having the same distribution as X.

Proposition 2.2 $\mathcal{G}WCE_{\alpha}$ is a coherent distribution-based risk measure.

Proof. We only prove that \mathcal{GWCE}_{α} satisfies the subadditivity axiom A2, the other properties (A1, A3 - 4) being easy to check.

Let X, Y be in L^{∞}_d , and \bar{x}, \bar{y} respectively in $\mathcal{GWCE}_{\alpha}(X)$ and $\mathcal{GWCE}_{\alpha}(Y)$. This means that $\bar{x} \in \mathcal{WCE}_{\alpha}(\tilde{X}')$ (reps. $\bar{y} \in \mathcal{WCE}_{\alpha}(\tilde{Y}')$), where the random vector \tilde{X}' (resp. \tilde{Y}') defined on $(\tilde{\Omega}^x, \tilde{\mathcal{F}}^x, \tilde{\mathbb{P}}^x)$ (resp. $(\tilde{\Omega}^y, \tilde{\mathcal{F}}^y, \tilde{\mathbb{P}}^y)$) has the same distribution as X (resp. Y). We have to show that $\bar{x} + \bar{y} \in \mathcal{GWCE}_{\alpha}(X + Y)$.

Define the product probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) := (\tilde{\Omega}^x \times \tilde{\Omega}^y, \tilde{\mathcal{F}}^x \otimes \tilde{\mathcal{F}}^y, \tilde{\mathbb{P}}^x \otimes \tilde{\mathbb{P}}^y)$, and the random variables

$$\tilde{X}(\omega,\omega') = \tilde{X}'(\omega)$$
 and $\tilde{Y}(\omega,\omega') = \tilde{Y}'(\omega')$ for $(\omega,\omega') \in \tilde{\Omega}$.

For all $\tilde{B} = B^x \times B^y \in \tilde{\mathcal{F}}^x \otimes \tilde{\mathcal{F}}^y$ such that $\tilde{\mathbb{P}}(\tilde{B}) \geq \alpha$ we have necessarily that $\min\left\{\tilde{\mathbb{P}}^x(B^x); \tilde{\mathbb{P}}^x(B^x)\right\} \geq \alpha$, and we easily verify, using the definition of $\mathcal{W}CE_\alpha$ $\tilde{\mathbb{E}}\left[\bar{x} + \tilde{X}|\tilde{B}\right] = \tilde{\mathbb{E}}^x\left[\bar{x} + \tilde{X}'|B^x\right] \in K$ and $\tilde{\mathbb{E}}\left[\bar{y} + \tilde{Y}|\tilde{B}\right] = \tilde{\mathbb{E}}^y\left[\bar{y} + \tilde{Y}'|B^y\right] \in K$, hence $\tilde{\mathbb{E}}\left[(\bar{x} + \bar{y})\tilde{X} + \tilde{Y}|\tilde{B}\right] \in K$, and by arbitrariness of \tilde{B} we deduce that $\bar{x} + \bar{y} \in \mathcal{W}CE_\alpha(\tilde{X} + \tilde{Y}) \subset \mathcal{G}XCE_\alpha(X + Y)$.

2.2 Quantiles of vector-valued risks

Given a <u>real-valued</u> random variable \mathcal{X} , and a confidence level $\alpha \in (0, 1)$, the α % worst realisations of \mathcal{X} , situated at the left tail of its distribution, are described by the lower and upper α quantiles defined by

$$q_{\alpha}(\mathcal{X}) := \inf\{\xi \in \mathbb{R} : \mathbb{P}(\mathcal{X} \le \xi) \ge \alpha\} \text{ and } q^{\alpha}(X) := \inf\{\xi \in \mathbb{R} : \mathbb{P}(X \le \xi) > \alpha\}$$

The main difficulty in the multidimensional framework is that the vector ordering \succeq is a partial ordering. Hence, the possible realizations of a *d*-dimensional portfolio $X \in L_d^{\infty}$ are not comparable and speaking about "worst realizations" does not make sense. In order to find an analogous, in this context, to tails of distribution, we consider subsets A of \mathbb{R}^d , which may contain non-comparable possible values of X, but satisfy the following requirement :

$$X(\omega) \in A \Rightarrow X(\omega) - K \subset A$$
,

i.e., if a realization X(w) is among the set A, then all positions which are more risky then X(w) are also contained in A.

Such subsets are eligible for defining the analogous of quantiles for random vectors. We introduce the set :

$$\mathcal{Q} := \left\{ A \in \mathcal{B}(\mathbb{R}^d) : A - K = A \right\}$$
(2.3)

Definition 2.3 Let X be in L_d^{∞} . The lower α -quantile of X is the set

$$\mathcal{Q}_{\alpha}(X) := \{ A \in \mathcal{Q} : \mathbb{P}(X \in A) \ge \alpha \}$$

and the upper α -quantile of X is the set

$$\mathcal{Q}^{\alpha}(X) := \{ A \in \mathcal{Q} : \mathbb{P}(X \in A) > \alpha \}$$

2.3 Tail Conditional Expectation

Let X be in L_d^{∞} , and A be some set in $\mathcal{Q}_{\alpha}(X)$, the set :

$$\left\{ \eta \in \mathbb{R}^d, \ \mathbb{E}[\eta + X \mid X \in A] \succeq 0 \right\} = \left\{ -\mathbb{E}[X \mid X \in A] \right\} + K$$

represents the set of deterministic portfolios η such that the conditional expected value of the modified position $X + \eta$, when X falls in the set A of "bad" scenarii, is admissible. An immediate proposal for the generalization to the vector-valued framework of TCE_{α} and TCE^{α} , which are defined for a real-valued random variable \mathcal{X} by

$$TCE_{\alpha}(\mathcal{X}) := \mathbb{E}[-\mathcal{X}|\mathcal{X} \le q_{\alpha}(\mathcal{X})] \text{ and } TCE^{\alpha}(\mathcal{X}) := \mathbb{E}[-\mathcal{X}|\mathcal{X} \le q^{\alpha}(\mathcal{X})]$$

is the following.

Definition 2.4 Let X be in L^{∞}_d and $\alpha \in (0,1)$.

The lower vector valued tail conditional expectation of X at level α is defined by

$$\mathcal{T}CE_{\alpha}(X) := \left\{ \eta \in \mathbb{R}^{d} : \forall A \in \mathcal{Q}_{\alpha}(X), \mathbb{E}\left[\eta + X \mid X \in A\right] \succeq 0 \right\} \quad (2.4)$$
$$= \bigcap_{A \in \mathcal{Q}_{\alpha}(X)} \left(\left\{ -\mathbb{E}\left[X \mid X \in A\right] \right\} + K \right) .$$

The upper vector valued tail conditional expectation of X at level α is defined by

$$\mathcal{T}CE^{\alpha}(X) := \left\{ \eta \in \mathbb{R}^{d} : \forall A \in \mathcal{Q}^{\alpha}(X), \mathbb{E}\left[\eta + X \mid X \in A\right] \succeq 0 \right\} \quad (2.5)$$
$$= \bigcap_{A \in \mathcal{Q}^{\alpha}(X)} \left(\left\{ -\mathbb{E}\left[X \mid X \in A\right] \right\} + K \right) .$$

Proposition 2.3 Let α be in (0,1). $\mathcal{T}CE_{\alpha}$ and $\mathcal{T}CE^{\alpha}$ define on L_d^{∞} set valued maps satisfying axioms A0, A1, A3 and A4.

In general $\mathcal{T}CE_{\alpha}$, resp. $\mathcal{T}CE^{\alpha}$, does not satisfy axiom A2, i.e. it is not subadditive, and hence is not a coherent risk measure.

2.4 Consistency with the one-dimensional framework

We end this Section by verifying that the definitions given above are consistent with the usual definitions of quantiles and Tail conditional expectation of real-valued risk. Clearly, in the one-dimensional context

$$\mathcal{Q} = \{(-\infty, a], a \in \mathbb{R}\} \cup \{(-\infty, a), a \in \mathbb{R}\}$$

Proposition 2.4 For all \mathcal{X} in L^{∞} , and α in (0, 1),

$$q_{\alpha}(\mathcal{X}) = \sup \bigcap_{A \in \mathcal{Q}_{\alpha}(\mathcal{X})} A \text{ and } q^{\alpha}(\mathcal{X}) = \sup \bigcap_{A \in \mathcal{Q}^{\alpha}(\mathcal{X})} A$$

Proof. By the definitions of $q_{\alpha}(\mathcal{X})$, and $\mathcal{Q}_{\alpha}(\mathcal{X})$, for all $\varepsilon > 0$, and $A \in \mathcal{Q}_{\alpha}(\mathcal{X})$

 $(-\infty, q_{\alpha}(\mathcal{X}) - \varepsilon] \subset A$, and $(-\infty, q_{\alpha}(\mathcal{X}) + \varepsilon] \in \mathcal{Q}_{\alpha}(\mathcal{X})$.

It follows that

$$(-\infty, q_{\alpha}(\mathcal{X})) = \bigcap_{\varepsilon > 0} (-\infty, q_{\alpha}(\mathcal{X}) - \varepsilon] \subset \bigcap_{A \in \mathcal{Q}_{\alpha}(\mathcal{X})} A ,$$

and
$$\bigcap_{A \in \mathcal{Q}_{\alpha}(\mathcal{X})} A \subset \bigcap_{\varepsilon > 0} (-\infty, q_{\alpha}(\mathcal{X}) + \varepsilon] = (-\infty, q_{\alpha}(\mathcal{X})] ,$$

hence : $q_{\alpha}(\mathcal{X}) = \sup \bigcap_{A \in \mathcal{Q}_{\alpha}(\mathcal{X})} A$. Similarly, we have that $(-\infty, q^{\alpha}(\mathcal{X})) \subset \bigcap_{A \in \mathcal{Q}^{\alpha}(\mathcal{X})} A \subset (-\infty, q^{\alpha}(\mathcal{X})]$, hence : $q^{\alpha}(\mathcal{X})$ $= \sup \bigcap_{A \in \mathcal{Q}^{\alpha}(\mathcal{X})} A$.

The following Proposition states the relation between the (real-valued) Tail conditional expectation, and the vector-valued Tail conditional expectation. **Proposition 2.5** For all \mathcal{X} in L^{∞} , and for all α in (0,1)

$$TCE_{\alpha}(\mathcal{X}) = \min \mathcal{T}CE_{\alpha}(\mathcal{X}) \text{ and } TCE^{\alpha}(\mathcal{X}) = \min \mathcal{T}CE^{\alpha}(\mathcal{X})$$

Proof. $\mathcal{T}CE_{\alpha}(\mathcal{X})$ is given by

$$\mathcal{T}CE_{\alpha}(\mathcal{X}) = \bigcap_{A \in \mathcal{Q}_{\alpha}(\mathcal{X})} [-E[\mathcal{X} | \mathcal{X} \in A], \infty) ,$$

1. Notice that for all $n \ge 1$, $\left(-\infty, q_{\alpha}(\mathcal{X}) + \frac{1}{n}\right] \in \mathcal{Q}_{\alpha}(\mathcal{X})$, we deduce that for all $n \ge 1$

$$-\mathbb{E}\left[\mathcal{X} \mid \mathcal{X} \le q_{\alpha}(\mathcal{X}) + \frac{1}{n}\right] \le \min \mathcal{T}CE_{\alpha}(\mathcal{X})$$

By a dominated convergence argument the left hand side of the last inequality converges to $TCE_{\alpha}(\mathcal{X})$ as n goes to ∞ , then $TCE_{\alpha}(\mathcal{X}) \leq \min TCE_{\alpha}(\mathcal{X})$.

2. Notice that $\mathcal{Q}_{\alpha}(X) \subset \overline{\mathcal{Q}}_{\alpha}(\mathcal{X})$ where the set is given by

$$\bar{\mathcal{Q}}_{\alpha}(\mathcal{X}) := \{(-\infty, a], a \ge q_{\alpha}(X)\} \cup \{(-\infty, a), a > q_{\alpha}(X)\}.$$

Consequently, to verify that $TCE_{\alpha}(\mathcal{X}) \geq \min TCE_{\alpha}(\mathcal{X})$, it is sufficient to check that for all $A \in \overline{\mathcal{Q}}_{\alpha}(\mathcal{X}), TCE_{\alpha}(X) \geq E[-X|X \in A].$

2.1. Let $a \ge q_{\alpha}(\mathcal{X})$, set $p_{\alpha} := \mathbb{P}(\mathcal{X} \le q_{\alpha}(\mathcal{X}))$, and $p_a := \mathbb{P}(\mathcal{X} \le a)$. We have $p_a \ge p_{\alpha}$, and

$$TCE_{\alpha}(\mathcal{X}) - (-E\left[\mathcal{X} \mid \mathcal{X} \leq a\right]) = E\left[\mathcal{X} \mid \mathcal{X} \leq a\right] - E\left[\mathcal{X} \mid \mathcal{X} \leq q_{\alpha}(\mathcal{X})\right]$$
$$= \frac{1}{p_{\alpha}p_{a}} \left\{ p_{\alpha}\mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq a}\right] - p_{a}\mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_{\alpha}(\mathcal{X})}\right] \right\}$$
$$= \frac{1}{p_{\alpha}p_{a}} \left\{ p_{\alpha}\mathbb{E}\left[\mathcal{X}\mathbf{1}_{q_{\alpha}(\mathcal{X}) < \mathcal{X} \leq a}\right] - (p_{a} - p_{\alpha})\mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_{\alpha}(\mathcal{X})}\right] \right\}$$
$$\geq \frac{1}{p_{a}p_{\alpha}} \left(p_{a} - p_{\alpha} \right) \left\{ p_{\alpha}q_{\alpha}(\mathcal{X}) - \mathbb{E}\left[\mathcal{X}\mathbf{1}_{\mathcal{X} \leq q_{\alpha}(\mathcal{X})}\right] \right\}$$
$$\geq 0.$$

2.2 A similar computation shows that if $a > q_{\alpha}(X)$, then

$$TCE_{\alpha}(\mathcal{X}) - (-E[\mathcal{X} | \mathcal{X} \leq a]) \geq 0.$$

We prove similarly that $TCE^{\alpha}(\mathcal{X}) = \min TCE^{\alpha}(\mathcal{X})$.

3 Coherency of Tail Conditional Expectation

In general, the (real valued) tail conditional expectation is not subadditive, hence does not define a coherent risk measure. In the previous literature equality between the (real-valued) tail conditional expectation and the coherent (real-valued) worst conditional expectation has been established under two sets of assumptions.

Proposition 3.1 [2] Assume that Ω is finite and that the probability on Ω is uniform. If X is a risk such that no two values of X in different states are ever equal, then $TCE_{\alpha}(X) = WCE_{\alpha}(X)$

We refer the reader to [2] for the proof of this result.

Proposition 3.2 [3] Let $\alpha \in (0,1)$ and X a real valued random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X^-] < \infty$. Then :

$$\begin{cases} \mathbb{P}(X < q_{\alpha}(X)) > 0 \quad and \quad \mathbb{P}(X \le q^{\alpha}(X)) = \alpha \\ or \\ \mathbb{P}(X < q_{\alpha}(X)) = 0 \quad and \quad \mathbb{P}(X = q^{\alpha}(X)) = 0 \\ iff \quad WCE_{\alpha}(X) = TCE_{\alpha}(X) = TCE^{\alpha}(X) . \end{cases}$$
(3.1)

In particular, (3.1) holds if the distribution of X is continuous.

We refer the reader to [3].

The main results of this paper extends Proposition 3.1 to the multidimensional framework and state that, in the case of a general probability space, if the distribution of X is continuous, and the condition $\mathbb{P}[X \in X(B)] = \mathbb{P}[B]$, for all B in \mathcal{F} holds, then

$$\mathcal{W}CE_{lpha}(X) = \mathcal{T}CE_{lpha}(X)$$
 .

The inclusion $WCE_{\alpha}(X) \subset TCE_{\alpha}(X)$ is straightforward. The following paragraphs are dedicated to the proof of the second inclusion.

3.1 Finite probability space

Theorem 3.1 Assume that Ω is finite and that the probability on Ω is uniform. If X is a risk such that no two values of X in different states are ever equal, then $WCE_{\alpha}(X) = TCE_{\alpha}(X).$

The proof of this Proposition is based on the following Proposition .

Proposition 3.3 Assume that Ω is finite and that the probability on Ω is uniform. Let X be a risk such that no two values of X in different states are ever equal. Then, for all B in \mathcal{F} , there exists $A \subset \mathbb{R}^d$ such that:

(i)
$$A \in \mathcal{Q}_{\alpha}(X)$$

(ii) $\mathbb{P}[X \in A] = \mathbb{P}[B]$
(iii) $\mathbb{E}[X | B] \succeq \mathbb{E}[X | X \in A]$

In the proof of Proposition 3.3, we are going to use the notation :

$$\mathcal{E}(D) := (D - K) \setminus D, \quad D \subset \mathbb{R}^d.$$
(3.2)

The set $\mathcal{E}(D)$ is such that :

1. if $d \in D$, $e \notin D$, and $d \succeq e$, then $e \in \mathcal{E}(D)$, 2. if $e \in \mathcal{E}(D)$ then there exists some $d \in D$ such that $d \succeq e$. We can qualify the subset $\mathcal{E}(D)$ as the "bad positions induced by D" and not contained in D.

Proof of Proposition 3.3 Let *B* be in \mathcal{F} . Notice that, since $X(\omega) \neq X(\omega')$ whenever $\omega \neq \omega'$,

$$\mathbb{P}(B) = \mathbb{P}(X \in X(B))$$
, and $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_{X \in X(B)}]$.

If $\mathbb{P}[X \in \mathcal{E}(X(B))] = 0$ then we verify immediately that the set $A := X(B) \cup \mathcal{E}(X(B))$ satisfies (i), (ii), and (iii).

Assume that $\mathbb{P}[X \in \mathcal{E}(X(B))] \neq 0$. The idea is to replace elements from X(B) by elements from $\mathcal{E}(X(B))$ in a convenient way, in order to obtain a set A with the desired properties.

Set $N := Card(\Omega)$. Then

$$\mathbf{p} := \mathbb{P}[X = X(\omega)] = \frac{1}{N}$$
, for all $\omega \in \Omega$.

Consider the (finite) sequence $(A_n)_{0 \le n \le N}$ of subsets of \mathbb{R}^d defined as follows:

1. $A_0 := X(B)$, 2. for $n, 0 \le n < N$: $- \text{ if } \mathbb{P}[X \in \mathcal{E}(A_n)] = 0$: we set $A_{n+1} := A_n$, $- \text{ if } \mathbb{P}[X \in \mathcal{E}(A_n)] > 0$: $A_{n+1} := (A_n \setminus \{a_n\}) \cup \{e_n\}$, where (a_n, e_n) can be chosen such that

 $(a_n, e_n) \in A_n \times \mathcal{E}(A_n) \cap X(\Omega)^2$, $a_n \succeq e_n$, and $a_n \notin \mathcal{E}(A_n \setminus \{a_n\})$.

Indeed, since $\mathbb{P}[X \in \mathcal{E}(A_n)] > 0$, we can chose some $e_n \in \mathcal{E}(A_n) \cap X(\Omega)$. By definition of $\mathcal{E}(A_n)$, there exists some $a^0 \in A_n$ such that $e_n \in \{a^0\} - K$. Then, consider the sequence $(a^k)_{k\geq 0}$ defined by induction as follows

$$\begin{aligned} a^{k+1} &:= a \in \{a \in A_n \cap X(\Omega) : a \neq a^k, a \succeq a^k\} \text{ if this set is non-empty} \\ a^{k+1} &:= a^k \text{ if } \{a \in A_n \cap X(\Omega) : a \neq a^k, a \succeq a^k\} = \emptyset. \end{aligned}$$

Since $X(\Omega)$ is finite, it is easy to see that for some $k^* \ge 0$, $a^k = a^{k^*}$ for all $k \ge k^*$, and that $a_n := a^{k^*}$ satisfies $: a_n \succeq e_n$, and $a_n \notin \mathcal{E}(A_n \setminus \{a_n\})$.

In this procedure, while $\mathcal{E}(A_n)$ is nonempty, A_{n+1} is obtained by replacing an element of A_n by a worst position contained in $\mathcal{E}(A_n)$. Notice that the property $a_n \notin \mathcal{E}(A_n \setminus \{a_n\})$ implies that $a_n \notin A_{n+k}$ for all $k \geq 1$.

Since the probability \mathbb{P} is uniform, the sequence (A_n) is such that $\mathbb{P}[X \in A_n] = \mathbb{P}[X \in X(B)]$ for n = 0, ..., N, in particular

$$\mathbb{P}(B) = P[X \in A_N] .$$

We also verify for $0 \le n < N$ that $\,$:

$$\mathbb{E}\left[X\mathbf{1}_{X\in A_{n+1}}\right] = \mathbb{E}\left[X\mathbf{1}_{X\in A_{n}}\right] \text{ or}$$
$$\mathbb{E}\left[X\mathbf{1}_{X\in A_{n+1}}\right] = \mathbb{E}\left[X\mathbf{1}_{X\in A_{n}}\right] - \mathbf{p}\left(a_{n} - e_{n}\right) \text{ where } a_{n} \succeq e_{n}.$$

Hence $\mathbb{E}[X\mathbf{1}_{X\in A_n}] \succeq \mathbb{E}[X\mathbf{1}_{X\in A_{n+1}}]$. In particular

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_{X \in X(B)}] \succeq \mathbb{E}[X\mathbf{1}_{X \in A_N}].$$

It remains to verify that $\mathbb{P}[X \in \mathcal{E}(A_N)] = 0$ to conclude that $\tilde{A}_N := A_N \cup \mathcal{E}(A_N)$ satisfies (*i*), (*ii*), and (*iii*).

Assume to the contrary that $\mathbb{P}[X \in \mathcal{E}(A_N)] \neq 0$. Recall that, by construction, the sequence (A_n) , for each $n \geq 0$, $a_n \notin A_{n+k}$, for all $k \geq 1$. Hence $\{a_0, ..., a_N\}$ is a set of N+1 distinct elements from $X(\Omega)$ which contradicts the fact that $Card(\Omega) = N$.

Proof of Theorem ?? 1. Since $\{X^{-1}(A), A \in \mathcal{Q}_{\alpha}(X)\} \subset \{B \in \mathcal{F}, \mathbb{P}(B) \geq \alpha\}$, we have that $\mathcal{W}CE_{\alpha}(X) \subset \mathcal{T}CE_{\alpha}(X)$.

2. To get the reverse inclusion, we have to show that for all x in $\mathcal{T}CE_{\alpha}(X)$, and for all B in \mathcal{F} with $\mathbb{P}(B) \geq \alpha$, $\mathbb{E}[x + X | B] \succeq 0$.

Let x be in $\mathcal{T}CE_{\alpha}(X)$, and B in \mathcal{F} with $\mathbb{P}(B) \geq \alpha$. From Proposition 3.3, there exists a subset A of \mathbb{R}^d such that A - K = A, A closed, $\mathbb{P}(A) = \mathbb{P}(B)$, and $\mathbb{E}[X\mathbf{1}_B]$ $\succeq \mathbb{E}[X\mathbf{1}_{X\in A}]$. In particular :

$$\mathbb{E}[X | B] - \mathbb{E}[X | X \in A] = \frac{1}{\mathbb{P}(B)} (\mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X \in A}]) \succeq 0,$$

$$A \in \mathcal{Q}_{\alpha}(X) \text{ hence } \mathbb{E}[x + X | X \in A] \succeq 0.$$

Consequently

$$\mathbb{E}[x+X \mid B] = \mathbb{E}[x+X \mid X \in A] + (\mathbb{E}[X \mid B] - \mathbb{E}[X \mid X \in A])$$
$$= \mathbb{E}[x+X \mid X \in A] + \frac{1}{\mathbb{P}(B)} (\mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X \in A}]) \succeq 0.$$

3.2 General probability space

Theorem 3.2 Let X be a risk in L^{∞}_d having a continuous probability density f. Then

$$\mathcal{GWCE}_{\alpha}(X) = \mathcal{T}CE_{\alpha}(X)$$

The proof of this Theorem relies on the following Proposition.

Proposition 3.4 Let X be in L^{∞}_d . Assume that X has a continuous probability density f. Then, for all B in $\mathcal{F}^x := \sigma(X)$, there exists $A \subset \mathbb{R}^d$ such that:

(i)
$$A \in \mathcal{Q}_{\alpha}(X)$$

(ii) $\mathbb{P}[X \in A] = \mathbb{P}[B]$
(iii) $\mathbb{E}[X | B] \succeq \mathbb{E}[X | X \in A]$

We defer the proof of the Proposition 3.4, and start by proving Theorem 3.2

Proof of Theorem 3.2 1. Since $\{X^{-1}(A), A \in \mathcal{Q}_{\alpha}(X)\} \subset \{B \in \mathcal{F}, P(B) \geq \alpha\}$, it is easy to check that $\mathcal{GWCE}_{\alpha}(X) \subset \mathcal{TCE}_{\alpha}(X)$.

2. To get the reverse inclusion, we are going to show that for all x in $\mathcal{T}CE_{\alpha}(X)$, and for all B in $\mathcal{F}^x = \sigma(X)$ with $\mathbb{P}(B) \ge \alpha$, $\mathbb{E}[x + X | B] \succeq 0$. Indeed this implies that

$$\mathcal{T}CE_{\alpha}(X) \subset \mathcal{W}CE_{(\Omega,\mathcal{F}^x,\mathbb{P}),\alpha}(\tilde{X}) \subset \mathcal{G}WCE_{\alpha}(X)$$

Let x be in $\mathcal{T}CE_{\alpha}(X)$, and B in \mathcal{F}^x with $\mathbb{P}(B) \geq \alpha$. Proposition 3.4 shows that there exists a subset A of \mathbb{R}^d such that A - K = A, A closed, $\mathbb{P}(A) = \mathbb{P}(B)$, and $\mathbb{E}[X\mathbf{1}_B]$ $\succeq \mathbb{E}[X\mathbf{1}_{X \in A}]$. In particular :

$$\mathbb{E}[X | B] - \mathbb{E}[X | X \in A] = \frac{1}{\mathbb{P}(B)} \left(\mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X \in A}]\right) \succeq 0,$$

$$A \in \mathcal{Q}_{\alpha}(X) \text{ hence } \mathbb{E}[x + X | X \in A] \succeq 0.$$

Consequently

$$\mathbb{E}[x+X \mid B] = \mathbb{E}[x+X \mid X \in A] + (\mathbb{E}[X \mid B] - \mathbb{E}[X \mid X \in A])$$
$$= \mathbb{E}[x+X \mid X \in A] + \frac{1}{\mathbb{P}(B)} (\mathbb{E}[X\mathbf{1}_B] - \mathbb{E}[X\mathbf{1}_{X \in A}]) \succeq 0.$$

We now turn to the proof of Proposition 3.4. The proof consists in two steps. The first one is

Lemma 3.1 We introduce the set S of pairs of subsets (δ, β) such that:

- 1. β and δ are open subsets of \mathbb{R}^d ,
- 2. $\delta \cap \beta = \emptyset$,
- 3. $\delta \cap [(D \setminus \delta) K] = \emptyset$,

- $4. \ \beta K = \beta,$
- 5. $d \succeq b$ for all (d, b) in $\delta \times \beta$,
- 6. $\mathbb{P}_X(\beta \cap \mathcal{E}(D)) = \mathbb{P}_X(\delta \cap D)$.

Elements of S are ordered as follows :

$$(\delta,\beta) \propto (\bar{\delta},\bar{\beta}) \quad iff \ \delta \subset \bar{\delta} \ and \ \beta \subset \bar{\beta} \ .$$
 (3.3)

Then S admits a maximal element w.r.t the order relation ∞ .

Proof. Notice that each totally ordered finite sequence $\{(\delta_{i_0}, \beta_{i_0}), ..., (\delta_{i_N}, \beta_{i_N})\}$ of elements of S, admits a maximum. We shall denote by $m(i_0, ..., i_N)$ the index of the maximum of $\{(\delta_{i_0}, \beta_{i_0}), ..., (\delta_{i_N}, \beta_{i_N})\}$.

Let $(\delta_i, \beta_i)_{i \in I}$ be a totally ordered family of elements of S. we claim that $(\delta_i, \beta_i)_{i \in I}$ has an upperbound.

Set
$$\delta_{\infty} := \bigcup_{i \in I} \delta_i$$
 and $\beta_{\infty} := \bigcup_{i \in I} \beta_i$.

It is clear that $(\delta_{\infty}, \beta_{\infty})$ is an upperbound for $(\delta_i, \beta_i)_{i \in I}$, provided it is contained in \mathcal{S} . This is what we proof through the Steps 1-6.

1. δ_{∞} and β_{∞} are open subsets of \mathbb{R}^d .

2. By definition of δ_{∞} , β_{∞} , the order relation \propto , and using property (2) of the elements of S:

$$\delta_{\infty} \cap \beta_{\infty} \subset \bigcup_{i,j \in I} \delta_i \cap \beta_j \subset \bigcup_{i,j \in I} \delta_{m(i,j)} \cap \beta_{m(i,j)} = \bigcup_{i,j \in I} \emptyset = \emptyset.$$

3. For all *i* in I, $(D \setminus \delta_{\infty}) \subset (D \setminus \delta_i)$, then

$$\delta_{\infty} \cap [(D \setminus \delta_{\infty}) - K] = \bigcup_{i \in I} \delta_{i} \cap [(D \setminus \delta_{\infty}) - K]$$
$$\subset \bigcup_{i \in I} \delta_{i} \cap [(D \setminus \delta_{i}) - K] = \bigcup_{i \in I} \emptyset = \emptyset.$$

4. $\beta_{\infty} - K = \bigcup_{i \in I} \beta_i - K = \bigcup_{i \in I} \beta_i = \beta_{\infty}.$

5. Let d be in δ_{∞} , and b be in β_{∞} . Let i, j be in I such that $d \in \delta_i$, and $b \in \beta_j$. Then $(d, b) \in \delta_{m(i,j)} \times \beta_{m(i,j)}$, and it follows from property (5) of elements of S that $d \succeq b$.

Consequently, for all (d, b) in $\delta_{\infty} \times \beta_{\infty}$, $d \succeq b$.

6. It remains to show that $\mathbb{P}[X \in \beta_{\infty} \cap \mathcal{E}(D)] = \mathbb{P}[X \in \delta_{\infty} \cap D]$ to conclude that $(\delta_{\infty}, \beta_{\infty})$ is an element of \mathcal{S} . In order to establish this property we use the following Lemma which proof is reported later on.

Lemma 3.2 Let G and H be two subsets of \mathbb{R}^d , then

$$\mathbb{P}(X \in G \cap H) = \sup_{C \in \mathcal{C}(G)} \mathbb{P}(X \in C \cap H) ,$$

where $\mathcal{C}(G)$ denotes the set of all compact subsets of G.

By Lemma (3.2) : $\mathbb{P}[X \in \delta_{\infty} \cap D] = \sup_{C \in \mathcal{C}(\delta_{\infty})} \mathbb{P}[X \in C \cap D].$

Let C be a compact subset of $\bigcup_{i \in I} \delta_i \ C(\delta_\infty)$. Since $\delta_i, i \in I$ are open subsets of \mathbb{R}^d , there exists a finite sequence $\{\delta_{i_1}, ..., \delta_{i_N}\}$ such that $C \subset \bigcup_{k=1}^N \delta_{i_k}$.

The finite sequence $\{(\delta_{i_1}, \beta_{i_1}), ..., (\delta_{i_N}, \beta_{i_N})\}$ is totally ordered in \mathcal{S} , hence it admits a maximum. Let $i_0 := m(i_1, ..., i_N)$ be the index of its maximum. Then by definition of the order relation ∞ , and by property (6) of the elements of \mathcal{S} :

$$\bigcup_{k=1}^{N} \delta_{i_k} \subset \delta_{i_0} \text{ and } \mathbb{P}\left[X \in \delta_{i_0} \cap D\right] = \mathbb{P}\left[X \in \beta_{i_0} \cap \mathcal{E}(D)\right]$$

it follows:

$$\mathbb{P}[X \in C \cap D] \leq \mathbb{P}[X \in \delta_{i_0} \cap D] = \mathbb{P}[X \in \beta_{i_0} \cap \mathcal{E}(D)] \leq \mathbb{P}[X \in \beta_{\infty} \cap \mathcal{E}(D)] .$$

Taking the supremum over $\mathcal{C}(\delta_{\infty})$, we get $: \mathbb{P}[X \in \delta_{\infty} \cap D] \leq \mathbb{P}[X \in \beta_{\infty} \cap \mathcal{E}(D)].$

A similar argument shows that the reverse inequality holds. Then

$$\mathbb{P}\left[X \in \delta_{\infty} \cap D\right] = \mathbb{P}\left[X \in \beta_{\infty} \cap \mathcal{E}(D)\right]$$

In S, each totally ordered family admits an upperbound, then, by Zorn's Lemma, S admits a maximal element.

Proof of Lemma 3.2 Since the distribution of X is absolutely continuous w.r.t to the Lebesgue measure, for each set S in \mathbb{R}^d

$$\mathbb{P}\left(X \in S\right) = \sup_{C \in \mathcal{C}(S)} \mathbb{P}\left(X \in C\right) \,,$$

where $\mathcal{C}(S)$ denotes the set of all compact subsets of S (see [?]). Let C be in $\mathcal{C}(G \cap H)$, then C is in $\mathcal{C}(G)$, it follows that

$$\mathbb{P}\left(X \in G \cap H\right) = \sup_{C \in \mathcal{C}(G \cap H)} \mathbb{P}\left(X \in C \cap H\right) \leq \sup_{C \in \mathcal{C}(G)} \mathbb{P}\left(X \in C \cap H\right) \,.$$

The reverse inequality is straightforward, since for all C in $\mathcal{C}(G)$, $C \cap H \subset G \cap H$, hence $\mathbb{P}(X \in C \cap H) \leq \mathbb{P}(X \in G \cap H)$.

Now we are ready for the proof of the second step of Proposition 3.4.

Proof of Proposition 3.4

Let B be in \mathcal{F} , and set D := X(B). X is such that

$$\mathbb{P}(B) = \mathbb{P}(X \in D)$$
 and $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[X\mathbf{1}_{X \in D}]$.

Assume that $\mathbb{P}(X \in \mathcal{E}(D)) = 0$. Then, clearly the set $A := D \cup \mathcal{E}(D) = D - K$ satisfies each of the requirement (i), (ii), and (iii).

Now we concentrate on the case where $\mathbb{P}(X \in Ec(D)) > 0$. We are going to prove that it is possible to obtain a subset A with the required properties by substituting elements from D by "worst" elements w.r.t to the preoerder \succeq , i.e. elements taken from $\mathcal{E}(D)$. Lemma 3.1 states that the set S admits a maximal element. Let (δ^*, β^*) be such an element.

1. If $\mathbb{P}[X \in D \setminus \delta^*] = 0$, then the set $A := \beta^* \cap \mathcal{E}(D)$ satisfies (i), (ii), and (iii).

Indeed, by definition of $\mathcal{E}(D)$, $\mathcal{E}(D) - K = \mathcal{E}(D)$. The property (4) of elements of \mathcal{S} implies $\beta^* - K = \beta^*$. Hence

$$A - K = [\beta^* \cap \mathcal{E}(D)] - K = \beta^* \cap \mathcal{E}(D) = A$$

Using $\mathbb{P}[X \in D \setminus \delta^{\star}] = 0$, and the property (6) of elements on \mathcal{S}

$$\mathbb{P}[X \in A] = \mathbb{P}[X \in \beta^* \cap \mathcal{E}(D)] = \mathbb{P}[X \in D \cap \delta^*] = \mathbb{P}[X \in D] .$$

Since $P[X \in D] = \mathbb{P}(B) \ge \alpha$, it follows that

$$A \in \mathcal{Q}_{\alpha}(X)$$
 and $\mathbb{P}(X \in A) = P(B)$

Using $\mathbb{P}[D \setminus \delta^{\star}] = 0$, and $p := \mathbb{P}[X \in \delta^{\star} \cap D] = \mathbb{P}[X \in \beta^{\star} \cap \mathcal{E}(D)]$

$$\mathbb{E} [X\mathbf{1}_{B}] - \mathbb{E} [X\mathbf{1}_{X\in A}]$$

$$= \mathbb{E} [X\mathbf{1}_{B}] - \mathbb{E} [X\mathbf{1}_{X\in A}]$$

$$= \mathbb{E} [X\mathbf{1}_{X\in D\cap\delta^{\star}}] - \mathbb{E} [X\mathbf{1}_{X\in\beta^{\star}\cap\mathcal{E}(D)}]$$

$$= \frac{1}{p} \int \int (X(\omega) - X(\omega')) \mathbf{1}_{X\in D\cap\delta^{\star}}(\omega) \mathbf{1}_{X\in\beta^{\star}\cap\mathcal{E}(D)}(\omega') dP(\omega) dP(\omega')$$

$$\succeq 0$$

where the last inequality follows from property (5) of elements of \mathcal{S} .

2. Now we consider the case where $\mathbb{P}[X \in D \setminus \delta^*] > 0$, and we set

$$D^{\star} := (D \setminus \delta^{\star}) \cup (\mathcal{E}(D) \cap \beta^{\star})$$

We shall prove that in this case $\mathbb{P}[X \in \mathcal{E}(D^*)] = 0$, it then follows that the set $A := D^* \cup \mathcal{E}(D^*)$ satisfies (i), (ii) and (iii).

Assume to the contrary that $\mathbb{P}[X \in \mathcal{E}(D^*)] > 0$. We are going to show that, then, it is possible to find $(\bar{\delta}, \bar{\beta})$ in \mathcal{S} such that $(\delta^*, \beta^*) \propto (\bar{\delta}, \bar{\beta})$, which is in contradiction with the fact that (δ^*, β^*) is a maximal element.

2.1. Notice that since f is continuous, and the cone K contains \mathbb{R}^d_+ , then $\mathbb{P}[X \in D \setminus \delta^*]$ implies that there exists some $a \in \mathcal{E}(D^*) \cap int[D \setminus \delta^* - K]$, with f(a) > 0. Let r > 0 such that $B(a, r) \subset int[D \setminus \delta^* - K]$, and $z \in B(a, r) \cap int[a + K]$, b in $\in D \setminus \delta^*$ such that z = b - k for some k in K. Then u := b - a is in int[K]. 2.2. Now consider the function Ψ defined on \mathbb{R} by :

$$\Psi(t) = \mathbb{P}\left(X \in \left[(a+t.u)+K\right] \cap \left[D \setminus \delta^{\star}\right]\right) - \mathbb{P}\left[X \in \left[(a+t.u)-K\right] \cap \mathcal{E}(D \setminus \delta^{\star})\right]$$

We verify that

$$\lim_{t \to +\infty} \Psi(t) = -\mathbb{P}\left(X \in \mathcal{E}(D \setminus \delta^*)\right) < 0$$
$$\lim_{t \to -\infty} \Psi(t) = \mathbb{P}\left(X \in D \setminus \delta^*\right) > 0$$

In fact, since a is in b+int(K), there exists some $\eta > 0$ such that $B(a,\eta) \subset b+int(K)$. Let $\lambda := \frac{3\|X\|_{\infty}}{\eta}$, then

$$B(a', 3||X||_{\infty}) \subset b + int(K)$$
, with $a' := b + \lambda(a - b)$.

By translation: b+(a-a')-K, which is equal to $a+\lambda(b-a)-K$ contains $B(a',3||X||_{\infty})$. We deduce that

$$\forall \lambda' > \lambda, B(a', 3 \|X\|_{\infty}) \subset a + \lambda(b - a) - K .$$

It follows that

$$\forall \lambda' > \lambda , \ \mathbb{P}\left[X \in \left[(a + \lambda'.u) + K\right] \cap \left[D \setminus \delta^{\star}\right]\right] = 0$$

and $\mathbb{P}\left[X \in \left[(a + \lambda'.u) - K\right] \cap \mathcal{E}(D \setminus \delta^{\star})\right] = \mathbb{P}\left[X \in \mathcal{E}(D \setminus \delta^{\star})\right]$.

Then

$$\lim_{t \to \infty} \Psi(t) = \lim_{t \to \infty} \mathbb{P} \left[X \in \left[(a + \lambda' . u) + K \right] \cap \left[D \setminus \delta^* \right] \right]$$
$$-\mathbb{P} \left[X \in \left[(a + \lambda' . u) - K \right] \cap \mathcal{E}(D \setminus \delta^*) \right]$$
$$= -\mathbb{P} \left[X \in \mathcal{E}(D \setminus \delta^*) \right]$$

Similar arguments provide the limit at $-\infty$.

We state the following Lemma which is proved later on.

Lemma 3.3 The function Ψ is continuous.

Since Ψ is continuous, there exists t^* such that $\Psi(t^*) = 0$.

Consider the subsets $[(a + t^*.u) - int(K)]$ and $[(a + t^*.u) + int(K)]$, they are open, disjoint and their union is nonempty as it contains either a or b. We obtain by setting

$$\overline{\delta} := \delta^* \cup [(a + t^*.u) + int(K)]$$
 and $\overline{\beta} := \beta^* \cup [(a + t^*.u) - int(K)]$

an element $(\bar{\delta}, \bar{\beta})$ of S which majorates strictly (δ^*, β^*) . This is in contradiction with the fact that (δ^*, β^*) is maximal.

We then conclude that if $\mathbb{P}[X \in D \setminus \delta^*] > 0$ then $\mathcal{E}(D^*)$ is \mathbb{P}_X -null, which ends the proof. \Box

Proof of Lemma 3.3. It is sufficient to prove that for any subset A and any closed convex cone K with nonempty interior, the mapping $\Psi_1 : t \mapsto \mathbb{P}_X(A \cap [(a+t.u)+K])$ is continuous; then Ψ is continuous as the sum of two continuous applications. Notice that

$$K(t') := (a + t') + K \subset (a + t.u) - K =: K(t) \text{ for } t \le t' \in \mathbb{R}$$

Then:

$$\begin{aligned} |\Psi(t) + \Psi(t')| &= \Psi(t) - \Psi(t') \\ &= \mathbb{P} \left[X \in A \cap (K(t) \setminus K(t')) \right] \\ &\leq \|f\|_{B(0, \|X\|_{\infty})} L(B(0, \|X\|_{\infty}) \cap (K(t) \setminus K(t'))) \end{aligned}$$

where f is the density of X and L denotes the Lebesgue measure on \mathbb{R}^d The result then follows from the continuity of the Lebesgue measure on the space of bounded convex bodies equipped with the Pompeiu-Hausdorff-Blaschke metric (see [10]).

Remark 3.1 Let X be a risk in L_d^{∞} having a continuous probability density f. Then

$$\mathcal{T}CE_{\alpha}(X) = \bigcap_{A \in \mathcal{Q}_{\alpha}(X), A \text{ closed}} -\{\mathbb{E}[X|X \in A]\} + K$$

Indeed, since f is continuous for each borel set A contained in $\mathcal{Q}_{\alpha}(X)$, for each $n \geq 1$ there exists some closed set A_n such that $A_n \subset A$ and $\mathbb{P}(X \in A_n) \geq \mathbb{P}(X \in A) - 1/n$. Then the sequence $(\bar{A}_n := A_n - K)$ is a sequence of closed subsets contained in $Q_{\alpha}(X)$ and satisfying $\mathbb{E}\left[X|X \in \bar{A}_n\right] \xrightarrow[n \to \infty]{} \mathbb{E}\left[X|X \in A\right]$, and the required result follows from the closedness of $\mathcal{T}CE_{\alpha}(X)$.

4 Numerical computation of TCE

4.1 The discrete distribution case

In the case of a discrete distribution, the $\mathcal{T}CE$ can be computed through a rather simple algorithm since the Quantile set is a finite collection of finite subsets of \mathbb{R}^d . The main idea of the algorithm is to compute recursively the quantiles of the distribution :

• assume that the random vector X has N realisations denoted by $x_i, i = 1, ..., N$,

• and denote by $\mathbb{Q}_a lpha$ the list of α quantiles, inizialized to an empty list

end

where

```
Algorithm : Quantile(A, i, c)
```

Begin

var integer j; $A := A + \{x_i\}$; if (c > 0)thenfor j = i + 1 to N : Quantile(A, j, c - 1);

else

```
if (\mathbb{P}(X \in A) \ge \alpha) then Q := Q + \{A\};

A := A - (\text{last element in } A);
```

 $end \ if$

A := A - (last element in A);

end

Example 4.1 We consider the example of an investor having access to two financial markets indexed by i, i = 1, 2. Each maket i consists of a non risky asset B^i , and a risky asset S^i , and the currency in markets i is denoted by m_i . For each i, transfers between the B^{i} 's account and the S^{i} 's account are free of charges, hence the position of the investor in the market i can be described by a single account X^i . However transfers between the two markts are subject to proportional transaction costs. Taking the currency m^1 as reference, when the investor transfers 1 m_1 from account i, he receives $1 - \lambda^{ij} m_1$ in the account j.

Let $\Omega := \{\omega_k, 1 \le k \le 10\}$, and $\mathbb{P}(\omega_k) = \frac{1}{10}$ for k = 1, ..., 10. Denote by $X = (X^1, X^2)$ the random vector representing the Profit&Loss (P&L) (expressed in m^1) at a future date T of some investor's position. The realisations of X are given in the following table.

	X^1	X^2		X^1	X^2
ω_1	35	10	ω_6	5	0
ω_2	20	5	ω_7	8	-2
ω_3	25	30	ω_8	-8	8
ω_4	10	20	ω_9	-10	6
ω_5	15	-10	ω_{10}	-15	-10

Figures 4.1, 4.1, 1 and 1, report $\mathcal{T}CE_{\alpha}(X)$ for different values of α . On each of these figures, $\mathcal{T}CE_{\alpha}(X)$ corresponds to the hatched rectangle at the top right corner. It is obtained as the intersection of the sets $\{-\mathbb{E}[X|X \in A]\} + K, A \in \mathcal{Q}_{\alpha}(X)$. We also reported on these figures the realizations of $X : \circ$, and the points $\{-\mathbb{E}[X|X \in A]\}, A \in \mathcal{Q}_{\alpha}(X) : \circ$.

Application : In order to secure his position the investor can proceed as follows

- choose $\bar{x} = (\bar{x}^1, \bar{x}^2)$ in $\mathcal{T}CE_{\alpha}(X)$,

- place $(x^1 + x^2 * K)e^{-r^1T} m_1$ in B^1 ,

- buy x^2 unities of the exchange options paying 1 m_2 for $K m_1$.

The resulting P&L of modified position at the end of the investment period is equal to $X + \bar{x}$, and satisfies, according to the defition of $\mathcal{T}CE_{\alpha}$: $\mathbb{E}[X + \bar{x}|X \in A] \succeq 0$ for all $A \in \mathcal{Q}_{\alpha}(X)$.







Figure 2: $\alpha = 4\%$





Figure 4: $\alpha = 6\%$

4.2 The continuous case : a numerical approximation

Let X be in L_d^{∞} , and L > 0 such that $\mathbb{P}\left(X \in \prod_{k=1}^d [-L, L]\right) = 1$. For each $n \ge 1$ we consider the grid

$$G_n := \prod_{k=1}^d \left\{ -L + i \frac{2L}{n} , \ 0 \le i \le n \right\} ,$$

then Ω can be partitioned into $\Omega = \bigcup_{\ell=1}^{n^d} A_{\ell}$, where

$$A_{\ell} = \left\{ X \in \prod_{k=1}^{d} \left[-L + (j_k - 1) \frac{2L}{n}, -L + j_k \frac{2L}{n} \right] \right\}$$

each $\ell = 1 + \sum_{k=1}^{d} n^{k-1} (j_k - 1)$ with $1 \le j_1, ..., j_d \le n$

For each $n \ge 1$, we define the random vector X_n by

for

$$X_n = \left(-L * (j_k - 1)\frac{2L}{n}, 1 \le k \le d\right)' \text{ on } A_\ell$$

for each $1 \le j_1, ..., j_d \le n$, and $\ell = 1 + \sum_{k=1}^d n^{k-1}(j_k - 1)$.

For each $n \geq 1$, $\mathcal{T}CE_{\alpha}(X_n)$ can be computed by the algorithm described in the previous subsection. The following Proposition shows that the sequence $(\mathcal{T}CE_{\alpha}(X_n))$ allows to approximates elements of $\mathcal{T}CE_{\alpha}(X)$.

Proposition 4.1 The sequence $(X_n)_{n\geq 1}$ converges to X in probability and satisfies

$$\liminf_{n} \mathcal{T}CE_{\alpha}(X_{n}) := \bigcup_{n} \bigcap_{k \ge n} \mathcal{T}CE_{\alpha}(X_{k}) \quad \subset \mathcal{T}CE_{\alpha}(X)$$

Proof. By definition of the sequence (X_n) , $\mathbb{P}\left(|X_n - X| > \frac{2}{n}\right) = 0$ for all $n \ge 1$, hence (X_n) converges to X in probability.

Let η be in $\liminf_n \mathcal{T}CE_{\alpha}(X_n)$. Observe that it is sufficient to prove that $\mathbb{E}[X|X \in A]$ is in K for all A in $\mathcal{Q}_{\alpha}(X)$ with A closed (see Remark 3.1). Let A be a closed subset of \mathbb{R}^d contained in $\mathcal{Q}_{\alpha}(X)$. By the definition of the sequence $(X_n)_{n\geq 1}, X_n(\omega) \preceq X(\omega)$ for all $n \geq 1$ and $\omega \in \Omega$. Consequently, $\mathbb{P}(X_n \in A) \geq P(X \in A)$, and A is in $\mathcal{Q}_{\alpha}(X_n)$ for all $n \geq 1$. Since $\eta \in \liminf_n \mathcal{T}CE_{\alpha}(X_n)$, we have for n sufficiently large

$$\eta + \mathbb{E}\left[X_n | X_n \in A\right] \in K$$

On the other hand

$$\mathbb{E}\left[X_n | X_n \in A\right] \xrightarrow[n \to \infty]{} \mathbb{E}\left[X | X \in A\right] .$$

Indeed, by the definition of the sequence (X_n) , and the closedness of the set A we have that $\liminf_n \{X_n \in A\} = \{X \in A\}$, then

$$\lim_{n} \mathbb{P}(X_n \in A) = \mathbb{P}(\liminf_{n} \{X_n \in A\}) = P(X \in A)$$

and

$$\begin{aligned} |\mathbb{E}\left[X_n\mathbf{1}_{X_n\in A}\right] - \mathbb{E}\left[X\mathbf{1}_{X\in A}\right]| &\leq \mathbb{E}\left[|X_n - X|\right] + ||X||_{\infty} \left|\mathbb{P}(X_n\in A) - \mathbb{P}(X\in A)\right| \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

It follows from the closedness of the cone K, that $\eta + \mathbb{E}[X|X \in A]$ is in K. By arbitrariness of A in $\{A' \in \mathcal{Q}_{\alpha}(X), A' \text{ closed}\}$ we onclude to the required result. \Box

References

- Artzner P., Delbaen F., Eber J.M. and Heath D. (1997), Thinking coherently, Risk 10, 68-71.
- [2] Artzner P., Delbaen F., Eber J.M. and Heath D. (1998), Coherent Measures of Risk, Mathematical Finance 9, 203-228.

- [3] Acerbi C. and Tasche D. (2002), On the coherence of expected shortfall, Journal of Banking and Finance 26, 1487-1503.
- [4] Cheridito P., Delbaen F., Kluppelberg C. (2004), Coherent and convex monetary risk measures for bounded càdlàg processes, *Stochastic Processes and their Applications*, **112**, 1-22.
- [5] Delbaen F. (2000), Coherent Risk Measures on General Probability Spaces, preprint.
- [6] Föllmer H. and Schied A. (2002), Convex measures of risk and trading constraints, Finance and Stochastics 6, 429-447.
- [7] Grothendieck A. (1973). Topological Vector spaces, Gordon and Breach, New York.
- [8] Jaschke S. and Küchler U. (2001), Coherent Risk Measures and Good-Deal Bounds, Finance and Stochastics 5, 181-200.
- [9] Jouini E., Meddeb M. and Touzi N. (2004), Vector Valued coherent Risk Measures, Finance and Stochastics 8, 531-552.
- [10] Sangwine-Yager J.R (1993), Mixed Volumes, Handbook of convex Geometry, P.M Gruber and J.M. Willis, USA.

SFB 649 Discussion Paper Series 2006

For a complete list of Discussion Papers published by the SFB 649, please visit http://sfb649.wiwi.hu-berlin.de.

- 001 "Calibration Risk for Exotic Options" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
- 002 "Calibration Design of Implied Volatility Surfaces" by Kai Detlefsen and Wolfgang K. Härdle, January 2006.
- 003 "On the Appropriateness of Inappropriate VaR Models" by Wolfgang Härdle, Zdeněk Hlávka and Gerhard Stahl, January 2006.
- 004 "Regional Labor Markets, Network Externalities and Migration: The Case of German Reunification" by Harald Uhlig, January/February 2006.
- 005 "British Interest Rate Convergence between the US and Europe: A Recursive Cointegration Analysis" by Enzo Weber, January 2006.
- 006 "A Combined Approach for Segment-Specific Analysis of Market Basket Data" by Yasemin Boztuğ and Thomas Reutterer, January 2006.
- 007 "Robust utility maximization in a stochastic factor model" by Daniel Hernández–Hernández and Alexander Schied, January 2006.
- 008 "Economic Growth of Agglomerations and Geographic Concentration of Industries - Evidence for Germany" by Kurt Geppert, Martin Gornig and Axel Werwatz, January 2006.
- 009 "Institutions, Bargaining Power and Labor Shares" by Benjamin Bental and Dominique Demougin, January 2006.
- 010 "Common Functional Principal Components" by Michal Benko, Wolfgang Härdle and Alois Kneip, Jauary 2006.
- 011 "VAR Modeling for Dynamic Semiparametric Factors of Volatility Strings" by Ralf Brüggemann, Wolfgang Härdle, Julius Mungo and Carsten Trenkler, February 2006.
- 012 "Bootstrapping Systems Cointegration Tests with a Prior Adjustment for Deterministic Terms" by Carsten Trenkler, February 2006.
- 013 "Penalties and Optimality in Financial Contracts: Taking Stock" by Michel A. Robe, Eva-Maria Steiger and Pierre-Armand Michel, February 2006.
- 014 "Core Labour Standards and FDI: Friends or Foes? The Case of Child Labour" by Sebastian Braun, February 2006.
- 015 "Graphical Data Representation in Bankruptcy Analysis" by Wolfgang Härdle, Rouslan Moro and Dorothea Schäfer, February 2006.
- 016 "Fiscal Policy Effects in the European Union" by Andreas Thams, February 2006.
- 017 "Estimation with the Nested Logit Model: Specifications and Software Particularities" by Nadja Silberhorn, Yasemin Boztuğ and Lutz Hildebrandt, March 2006.
- 018 "The Bologna Process: How student mobility affects multi-cultural skills and educational quality" by Lydia Mechtenberg and Roland Strausz, March 2006.
- 019 "Cheap Talk in the Classroom" by Lydia Mechtenberg, March 2006.
- 020 "Time Dependent Relative Risk Aversion" by Enzo Giacomini, Michael Handel and Wolfgang Härdle, March 2006.
- 021 "Finite Sample Properties of Impulse Response Intervals in SVECMs with Long-Run Identifying Restrictions" by Ralf Brüggemann, March 2006.
- 022 "Barrier Option Hedging under Constraints: A Viscosity Approach" by Imen Bentahar and Bruno Bouchard, March 2006.

SFB 649, Spandauer Straße 1, D-10178 Berlin http://sfb649.wiwi.hu-berlin.de



This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

- 023 "How Far Are We From The Slippery Slope? The Laffer Curve Revisited" by Mathias Trabandt and Harald Uhlig, April 2006.
- 024 "e-Learning Statistics A Selective Review" by Wolfgang Härdle, Sigbert Klinke and Uwe Ziegenhagen, April 2006.
- 025 "Macroeconomic Regime Switches and Speculative Attacks" by Bartosz Maćkowiak, April 2006.
- 026 "External Shocks, U.S. Monetary Policy and Macroeconomic Fluctuations in Emerging Markets" by Bartosz Maćkowiak, April 2006.
- 027 "Institutional Competition, Political Process and Holdup" by Bruno Deffains and Dominique Demougin, April 2006.
- 028 "Technological Choice under Organizational Diseconomies of Scale" by Dominique Demougin and Anja Schöttner, April 2006.
- 029 "Tail Conditional Expectation for vector-valued Risks" by Imen Bentahar, April 2006.



SFB 649, Spandauer Straße 1, D-10178 Berlin http://sfb649.wiwi.hu-berlin.de