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Working Paper

Fighting collusion: An implementation theory approach

Cardiff Economics Working Papers, No. E2021/19

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Suggested Citation: Azacis, Helmuts; Vida, Péter (2021) : Fighting collusion: An implementation theory approach, Cardiff Economics Working Papers, No. E2021/19, Cardiff University, Cardiff Business School, Cardiff

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Cardiff Economics Working Papers

Working Paper No. E2021/19

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August 2021

ISSN 1749-6010

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Fighting Collusion: An Implementation Theory Approach*

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August 16, 2021

Abstract

A competition authority has an objective, which specifies what output profile firms need to produce as a function of production costs. These costs change over time and are only known by the firms. The objective is implementable if in equilibrium, the firms cannot collude on their reports to the competition authority. Assuming that the firms can only report prices and quantities, we characterize what objectives are one-shot and repeatedly implementable. We use this characterization to identify conditions when the competitive output is implementable. We extend the analysis to the cases when a buyer also knows the private information of firms and when the firms can supply hard evidence about their costs.

Keywords: Collusion, Antitrust, (Repeated) Implementation, Monotonicity, Price-Quantity Mechanism, Hard Evidence

JEL Classification Numbers: C72; C73; D71; D82; L41

*This research was undertaken when the first author was an invited senior research fellow at the Corvinus Institute for Advanced Studies. He gratefully acknowledges the Institute's hospitality and financial support. The second author acknowledges financial support from Labex MME-DII. We would also like to thank Martin Peitz and Renault Régis for their helpful comments and suggestions.

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1 Introduction

It is well understood that the success of antitrust enforcement crucially depends on the information that a competition authority (CA) can gather about firms and markets. To study information acquisition by CA, we adopt an implementation theory approach in this paper. CA has a certain objective about firms' output. For example, it wants the firms to produce the competitive equilibrium output. This objective depends on the firms' production costs, which are unknown to CA, but commonly known by the firms. Therefore, CA must rely on the firms' reports, which need not be truthful. The question that we address, is whether CA can design a game such that the firms cannot collude on their reports in equilibrium in order to mislead CA, and CA can implement its objective. We study this question both in a one-shot and in an infinitely repeated setup with discounting.

Since we focus on the information acquisition, we assume that CA has perfect control over the production of each firm, while firms can only influence their outputs indirectly via their reports. Our model especially applies to regulated companies such as utilities. Over the past decades, regulators across the world have been adopting performance benchmarking of the regulated firms in order to improve their efficiency and service quality.[1] That is, firms are rewarded for being efficient, while their efficiency is measured against that of other firms. However, such benchmarking can also create incentives for the firms to coordinate information that they provide to the regulators. Therefore, it is important to study possible collusion by the regulated firms.

Motivated by the standard oligopoly models, we impose a restriction on the type of reports that the firms can make: each firm only announces a quantity it wants to produce and a price at which it wants to sell. Though, unlike oligopoly models, what firms produce can differ from what they announce. Therefore, we further require that the game possesses a "truthful" equilibrium, in which the firms announce and produce exactly those quantities that are specified by the objective, and all firms announce the price that equates the announced quantities with the exogenously given demand. We refer to this property of game as forthrightness.

We identify necessary and sufficient conditions on CA's objective for one-shot and repeated implementation. These conditions can be thought as a strengthening of Maskin monotonicity (Maskin, 1999) due to forthrightness and repeated interaction. Following the implementation literature, we now refer to CA's objective as a social choice function (SCF) and to the (stage) game that tells how much the firms must produce given their reports, as a mechanism. In the one-shot

[1] See, for example, the international comparison of benchmarking methods in the electricity sector by Jamasb and Pollitt (2000) or the assessment of benchmarking in the telecommunications, water, and energy sectors in the UK by Dassler, Parker, and Saal (2006), among many others.

setup, we start with the quantity mechanisms where each firm only announces the quantity that it wants to produce and we show that a condition, which we call q-monotonicity, is necessary and sufficient for implementation (Proposition 1). q-monotonicity is a rather demanding condition. For example, the SCF that requires the firms to produce the competitive equilibrium output and which we will call the competitive SCF, does not satisfy q-monotonicity in general (Proposition 2 and Example 1). This motivates us to study what can be implemented by price-quantity mechanisms where each firm, in addition to quantity, also announces a price. We find that a slightly modified condition, which we call pq-monotonicity, is now necessary and sufficient for implementation (Proposition 3). We also show that the competitive SCF satisfies pq-monotonicity under some fairly mild assumptions (Proposition 4). Therefore, we also adopt the price-quantity mechanisms for the rest of the paper.

In the repeated setup when the firms' costs change randomly from one period to another, we identify two conditions – pq-monotonicity and pq-stationary monotonicity – as necessary for implementation (Propositions 5 and 6). The first condition allows us to eliminate a situation when the firms only misreport their costs for one period and then behave truthfully thereafter, as the equilibrium outcome. The second condition allows us to eliminate a situation when the firms use the same misreporting strategy in every period, as the equilibrium outcome. We show that in the symmetric setup when the firms are identical and the SCF treats them equally, these two monotonicity conditions are also sufficient for repeated implementation (Proposition 7).

Repeated implementation of competitive SCF can fail because of high discount factor (Example 2) and forthrightness (Example 3). However, when the cost functions exhibit strictly increasing differences, we show that the competitive SCF is implementable for all values of the discount factor less than one, if the number of firms exceeds some number. For a given number of firms, we show that the competitive SCF is implementable if the discount factor is low enough, and we identify a condition when it is also implementable for the discount factors close to one (Proposition 8). Interestingly, an SCF can be implementable for low and high values of the discount factor, but not be implementable for intermediate values (Example 4).

Finally, we consider two extensions in the repeated setup. In the first extension, we allow a representative buyer also to be a participant in the game because antitrust investigations are often started after a complaint by a customer (see, e.g., Hay and Kelley, 1974). While the competitive SCF is not efficient from the firms' perspective (they prefer a collusive outcome), it becomes efficient once the interests of the buyer are also taken into account. We show that any efficient SCF is implementable if and only if an augmented SCF satisfies pq-monotonicity (Proposition 9). In our second extension, we incorporate hard evidence into the model. CA can request the firms to back any claims about the implied costs with supporting evidence. Clearly, more SCFs can be implemented with evidence be-

cause it can help when either of the two necessary monotonicity conditions fails. We show that (given a simplifying assumption) so-called evidence monotonicity is necessary and sufficient for repeated implementation in the symmetric setup (Propositions 10 and 11). If the firms can hide their profits but cannot exaggerate them, then the competitive SCF satisfies evidence monotonicity and, hence, is repeatedly implementable (Example 5).

The survey by Rey (2003) emphasizes the importance of implementation in the context of competition policy. Specifically, leniency programs where a firm receives a reduction in fines if it helps to uncover and prosecute a cartel, are designed to implement competitive behaviour of firms. Starting with Motta and Polo (2003); Spagnolo (2004); Aubert, Rey, and Kovacic (2006); Harrington (2008), there is large and growing literature that analyses the impact of leniency programs on collusion. There are some substantial differences between that literature and our approach. First, we mostly work with non-verifiable information, while that literature assumes verifiable information; namely, CA can find evidence of collusion with some probability. Second, in our model, CA is free to choose any price-quantity mechanism, while in that literature, CA treats the (often implicit) price-quantity mechanism as given; for example, it could be a Cournot or Bertrand model. Collusion by regulated firms that are subject to benchmarking, is studied by Tangerås (2002) and Dijkstra, Haan, and Mulder (2017). In particular, Tangerås (2002) considers information revelation in a static duopoly model when transfers from the regulator and side-payments between the firms are possible, and finds the implementable SCF that maximizes the welfare.

Our analysis builds on several strands of implementation literature. Repeated implementation has been studied in a general setup by Lee and Sabourian (2011); Mezzetti and Renou (2017); Āzacis and Vida (2019). Mezzetti and Renou (2017) show that a condition, called dynamic monotonicity, is necessary for repeated implementation. On one hand, pq-stationary monotonicity is a restriction of dynamic monotonicity to a special type of misreports, but on the other hand, it is a strengthening of dynamic monotonicity due to forthrightness. Further, Lee and Sabourian (2011) show that if an SCF is not efficient, then it is not repeatedly implementable for high enough discount factors. In Remark 2, we comment on the differences in assumptions that allow us to implement inefficient SCFs even when the discount factor is high.

Implementation with quantity and price-quantity mechanisms has previously been studied in static exchange economies. The seminal work in this area is by Dutta, Sen, and Vohra (1994); Sjöström (1994); Saijo, Tatamitani, and Yamato (1996). q- and pq-monotonicity resemble the necessary conditions that were identified in these papers. Bull and Watson (2004); Ben-Porath and Lipman (2012); Kartik and Tercieux (2012); Banerjee, Chen, and Sun (2021) study one-shot implementation with evidence. In particular, Kartik and Tercieux (2012) introduce the notion of evidence monotonicity. Compared to the literature, we are the first to study implementation with price-quantity mechanisms and with evidence in

the repeated setup.

The rest of the paper is organized as follows. We describe the basic elements of the model in Section 2. Then, we study one-shot implementation in Section 3 and repeated implementation in Section 4. We investigate repeated implementation of the competitive SCF in Section 5. In Section 6, we consider the two extensions to the model of Section 4. Finally, the appendix contains most of the proofs.

2 The Setup

There are $n \geq 2$ firms in a market and they produce a homogenous product. Let $I = \{1, \ldots, n\}$ denote the set of firms. The inverse demand function is $p = p(Q)$, where $Q = \sum_{i=1}^{n} q_i$ is the aggregate output and $q_i \in \mathbb{R}_+$ is the output of firm $i \in I$. It is assumed that $p(Q)$ is continuous and strictly decreasing for $p > 0$. Production costs depend on the state of the world. Let Θ be the set of possible states of the world, which is assumed to be finite. Each period a new state $\theta \in \Theta$ is drawn independently and identically according to a distribution function l. It is assumed that each state $\theta \in \Theta$ is realised with a strictly positive probability, $l(\theta) > 0$. The cost function of firm i is $c_i(q_i, \theta)$, which is assumed to be continuous and strictly increasing in q_i with $c_i(0, \theta) = 0$. Let $c_i'(q_i, \theta)$ denote the marginal cost function of firm i when $c_i(q_i, \theta)$ is differentiable in q_i. The profit of firm i in any period with state θ is given by

$$\pi_i(q, \theta) = p(Q)q_i - c_i(q_i, \theta),$$

where $q = (q_1, \ldots, q_n)$. Firms want to maximize their present discounted profits. Let δ be a discount factor, common to all firms.

The objective of competition authority (CA) is captured by a social choice function (SCF), which specifies the desired output profile for every state of the world, $f : \Theta \to \mathbb{R}_+^n$. It is assumed that f does not change over time and that CA never observes the state of the world of any period.

Except for Section 3.1 where we consider quantity mechanisms, we maintain the following assumption in the rest of the paper:

Assumption A1 *f is such that $p\left(\sum_{i \in I} f_i(\theta)\right) > 0$ for all θ.*

This assumption together with the assumption that $p(Q)$ is strictly decreasing for $p > 0$, ensures that there is a one-to-one relationship between the price and the aggregate output in the range of f. We could dispense of Assumption A1 if in the mechanisms that we consider below, CA required the firms to report the aggregate output instead of price.

We will specifically be interested in an SCF that selects a competitive output in every state of the world. To define this SCF (and later to study its implementability), we make the following assumption:

Assumption A2 *$c_i(q_i, \theta)$ is continuously differentiable and convex in q_i for all i and θ. $p(Q)$ is everywhere differentiable, except possibly at Q s.t. $p(Q) = 0$. For each θ, the solution q^* to the system of equations:*

$$p(Q) \leq c_i'(q_i, \theta), q_i \geq 0, (p(Q) - c_i'(q_i, \theta))q_i = 0 \text{ for } i = 1, \ldots, n, \tag{1}$$

is unique with $q_i^* > 0$ for all i.

Thus, given Assumption A2, the competitive SCF, which we denote by f^c, is such that for every θ, $f^c(\theta)$ is a solution to (1).

3 One-shot Implementation

We start by exploring implementation in a static setup. To elicit the state of the world, CA uses a mechanism (M, g) where M is a message space, which is the same for all firms, and $g : M^n \to \mathbb{R}_+^n$ is an outcome function that specifies an output profile for each profile of messages. Since below we will fix the message space M, we denote the mechanism (M, g) with its output function g. Let m_i be a generic message of firm i and let $m = (m_1, \ldots, m_n)$ be a generic message profile. m_{-i} is obtained by omitting the message of firm i from m. It is assumed that the firms send their messages simultaneously. The message profile m is a Nash equilibrium in state θ if $\pi_i(g(m), \theta) \geq \pi_i(g(m_i', m_{-i}), \theta)$ for all $i \in I$ and all $m_i' \in M$. Let $NE(g, \theta)$ denote the set of Nash equilibrium message profiles in state θ. The mechanism g implements an SCF f in Nash equilibrium if for every $\theta \in \Theta$, we have that $NE(g, \theta)$ is nonempty and $g(m) = f(\theta)$ for every $m \in NE(g, \theta)$. If there exists such a mechanism, then we say that f is implementable in Nash equilibrium.

In the implementation literature, the mechanisms that are used to implement SCFs, are often quite unnatural. The canonical mechanism for implementation in Nash equilibrium would require each firm to report a state, an output profile, and an integer.[2] To address this unattractive feature of the mechanisms, we impose a restriction on the mechanisms that CA can use. Following Dutta, Sen, and Vohra (1994); Sjöström (1994); Saijo, Tatamitani, and Yamato (1996), the messages of the firms are announcements about quantities and possibly also about prices. We start with quantity mechanisms, and next we consider price-quantity mechanisms.

3.1 Quantity Mechanisms

In quantity mechanisms, the message space of every firm is $M = \mathbb{R}_+$. We interpret the announcement of firm i as the output it should produce. However,

[2]For the description of the canonical mechanism, see the proof of Theorem 3 in Maskin (1999).

without further restrictions on the mechanisms, what firms announce can differ from what they actually produce. This, in turn, means that their announcements can potentially encode additional information besides their own desired output. Therefore, following Saijo, Tatamitani, and Yamato (1996), we impose the forthrightness requirement on the mechanisms, which says that it is an equilibrium to announce the socially desired output profile in every state and that in this equilibrium, the firms produce exactly what they have announced.

Definition 1 *A quantity mechanism g satisfies* forthrightness *w.r.t. f if for all $\theta \in \Theta$, $f(\theta) \in NE(g, \theta)$ and $g(f(\theta)) = f(\theta)$.*

We can view the Cournot model as a quantity mechanism where the outcome function is an identity map, $g(m) = m$ for all m. Forthrightness only requires that $g(m) = m$ for $m \in \{f(\theta)|\theta \in \Theta\}$.

Let $L_i(q, \theta) := \{q'|\pi_i(q, \theta) \geq \pi_i(q', \theta)\}$ denote the lower contour set of firm i in state θ at output q. $L_i(q, \theta)$ consists of all those output profiles that give to firm i weakly lower profits than q in state θ. Given any output profile q, q_{-i} is obtained by omitting the output of firm i from q. Let $f^{-1}(q_{-i}) := \{\theta \in \Theta|q_{-i} = f_{-i}(\theta)\}$ be the set of states that are consistent with q_{-i} given f, and let $\Lambda_i^f(q) := \cap_{\theta \in f^{-1}(q_{-i})} L_i(f(\theta), \theta)$. (Note, though, that $\Lambda_i^f(q)$ is independent of q_i.)

We now define a condition, which we call q-monotonicity, and show that it is necessary and sufficient for implementation with a quantity mechanism that satisfies forthrightness. q-monotonicity is similar to Condition W^* in Saijo, Tatamitani, and Yamato (1996), which is necessary for implementation with quantity mechanisms in exchange economies.[3]

Definition 2 *f satisfies* q-monotonicity *if for all $(\theta, \theta') \in \Theta^2$, $\Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$ implies $f(\theta') = f(\theta)$.*

Proposition 1 *f is implementable in Nash equilibrium with a quantity mechanism that satisfies forthrightness w.r.t. f if and only if f satisfies q-monotonicity.*

Although the definition of forthrightness allows for the possibility that there are other Nash equilibria besides $m = f(\theta)$ in state θ, it is, in fact, the unique equilibrium of the mechanism, which is used to prove the sufficiency part of Proposition 1.

The following condition, which is easier to check, is implied by q-monotonicity.

Definition 3 *f is* incentive compatible *if $\pi_i(f(\theta), \theta) \geq \pi_i(f(\theta'), \theta)$ for all i, θ, and $\theta' \in f^{-1}(f_{-i}(\theta))$.*

[3]Because Condition W^* is defined for exchange economies, it is not directly applicable to the present setup. However, if applied to the present setup, f^{-1} in Saijo, Tatamitani, and Yamato (1996) would be defined as $f^{-1}(q) := \{\theta \in \Theta|q = f(\theta)\}$. Therefore, Condition W^* more corresponds to pq-monotonicity, which we define later.

Proposition 2 *If f is implementable in Nash equilibrium with a quantity mechanism that satisfies forthrightness w.r.t. f, then f is incentive compatible.*

The following example shows that the competitive SCF is, in general, not implementable with a quantity mechanism that satisfies forthrightness.

Example 1. Let $n = 2$, $\Theta = \{(\theta_1, \theta_2), (\theta_1', \theta_2')\}$ with $\theta_1'/\theta_2' = \theta_1/\theta_2$ and $\theta_1 > \theta_1' > 0$, $p = \max\{0, a - Q\}$ with $a > \theta_1 + \theta_1/\theta_2$, $c_1(q_1, (\theta_1, \theta_2)) = \theta_1 q_1$ and $c_2(q_2, (\theta_1, \theta_2)) = \frac{1}{2}\theta_2 q_2^2$. The cost functions in the other state are the same, except θ_1 and θ_2 are replaced with θ_1' and θ_2', respectively. (Assumption A2 is satisfied.) The competitive SCF is $f^c(\theta_1, \theta_2) = (a - \theta_1 - \theta_1/\theta_2, \theta_1/\theta_2)$ and $f^c(\theta_1', \theta_2') = (a - \theta_1' - \theta_1/\theta_2, \theta_1/\theta_2)$. This SCF is not incentive compatible since $(\theta_1, \theta_2) \in f_2^{c^{-1}}(f_2^c(\theta_1', \theta_2'))$ and firm 1 strictly prefers outcome $f^c(\theta_1, \theta_2)$ in both states since $\theta_1 > \theta_1'$. ■

3.2 Price-Quantity Mechanisms

The failure to implement the competitive SCF with quantity mechanisms motivates us to study price-quantity mechanisms next. The message space of every firm in such mechanisms is $M = \mathbb{R}_{++} \times \mathbb{R}_+$. A typical message $m_i = (p_i, q_i)$ will have the following interpretation: p_i is firm i's announcement of the market price and q_i is its output. Note that we do not allow the firms to announce zero price, which we can do because of Assumption A1. If with quantity mechanisms, each firm announces how much it should produce, then with price-quantity mechanisms, it effectively also announces how much the other firms should jointly produce.

We now define forthrightness for price-quantity mechanisms. For every $\theta \in \Theta$, let $m(\theta)$ denote the message profile such that $m_i(\theta) = (p, q_i)$ for all $i \in I$ where $(q_1, \ldots, q_n) = f(\theta)$ and $p = p\left(\sum_{i \in I} q_i\right)$.

Definition 4 *A price-quantity mechanism g satisfies* forthrightness *w.r.t. f if for all $\theta \in \Theta$, $m(\theta) \in NE(g, \theta)$ and $g(m(\theta)) = f(\theta)$.*

With some abuse of notation, let $f^{-1}(q) := \{\theta \in \Theta | q = f(\theta)\}$ and $\Lambda_i^f(q) := \cap_{\theta \in f^{-1}(q)} L_i(q, \theta)$. Note that $m(\theta) \in NE(g, \theta')$ for all $\theta' \in f^{-1}(f(\theta))$ if g satisfies forthrightness w.r.t. f.

Definition 5 *f satisfies pq-*monotonicity *if for all $(\theta, \theta') \in \Theta^2$, $\Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$ implies $f(\theta') = f(\theta)$.*

The definition of pq-monotonicity differs from that of q-monotonicity only in the definition of $\Lambda_i^f(q)$. From the definitions of $\Lambda_i^f(q)$ in the case of quantity and price-quantity mechanisms, it follows that if f does not satisfy pq-monotonicity, then it will also not satisfy q-monotonicity. Further, there is no difference between q-monotonicity and pq-monotonicity if in each state, f assigns identical outputs

to all firms, which we would normally demand from f if the firms themselves are identical.

We have a similar result to that of Proposition 1 for the case when CA uses a price-quantity mechanism.

Proposition 3 *f is implementable in Nash equilibrium with a price-quantity mechanism that satisfies forthrightness w.r.t. f if and only if f satisfies pq-monotonicity.*

When a price-quantity mechanism is employed, incentive compatibility of f is not anymore a necessary condition for implementation. To see it, suppose that $f_{-i}(\theta) = f_{-i}(\theta')$, but $f_i(\theta) \neq f_i(\theta')$ for some θ, θ'. With a quantity mechanism, both announcements of $f_i(\theta)$ and $f_i(\theta')$ by firm i are consistent with $f_{-i}(\theta)$ and the incentive compatibility ensures that firm i announces the right quantity. With a price-quantity mechanism, the announcement of firm i can be crosschecked against $p^{-1}(\hat{p}) - \sum_{j \neq i} f_j(\theta)$ where $\hat{p}$ is the common price announced by the other firms. Therefore, at most one of two announcements $f_i(\theta)$ and $f_i(\theta')$ can be consistent with the messages of the other firms, and we do not need anymore to require that f satisfies the incentive compatibility. This allows to implement the competitive SCF (that satisfies Assumption A2) with a price-quantity mechanism.

Proposition 4 *If Assumption A2 holds, then f^c satisfies pq-monotonicity.*

Proof: To prove that f^c satisfies pq-monotonicity, we will show that $f^c(\theta') \neq f^c(\theta)$ implies that there exists firm i, for which $\Lambda_i^{f^c}(f^c(\theta)) \not\subseteq L_i(f^c(\theta), \theta')$ holds. Consider Figure 1, which shows iso-profit lines of firm i as a function of its own output, q_i, and the aggregate output of all the other firms, $Q_{-i} := \sum_{j \neq i} q_j$. One can think that in the competitive equilibrium of state θ, firm i is choosing q_i and Q_{-i} to maximize its profit $\hat{\pi}_i(q_i, Q_{-i}, \theta) := \pi_i(q, \theta)$ subject to the constraint $q_i + Q_{-i} = \sum_j f_j^c(\theta)$, meaning that the aggregate output and, hence, price remain constant. The optimum is at $(f_i^c(\theta), \sum_{j \neq i} f_j^c(\theta))$. The slope of iso-profit line is

$$\frac{dQ_{-i}}{dq_i} = -\frac{p'(Q)q_i + p(Q) - c_i'(q_i, \theta)}{p'(Q)q_i},$$

which is equal to -1, the slope of the constraint, in the competitive equilibrium where $p(\sum_j f_j^c(\theta)) = c_i'(f_i^c(\theta), \theta)$ holds. The same is true in all other states of the world, for which $f^c(\theta)$ remains the competitive equilibrium outcome. It follows that the points on the constraint $q_i + Q_{-i} = \sum_j f_j^c(\theta)$ belong to $\Lambda_i^f(f(\theta))$.

Now, suppose that $f^c(\theta)$ is not a competitive equilibrium outcome in state θ'. Then, there exists a firm, say, firm i, for which $p(\sum_j f_j^c(\theta)) \neq c_i'(f_i^c(\theta), \theta')$.[4]

[4]This follows from Assumption A2 that the solution to (1) is unique.

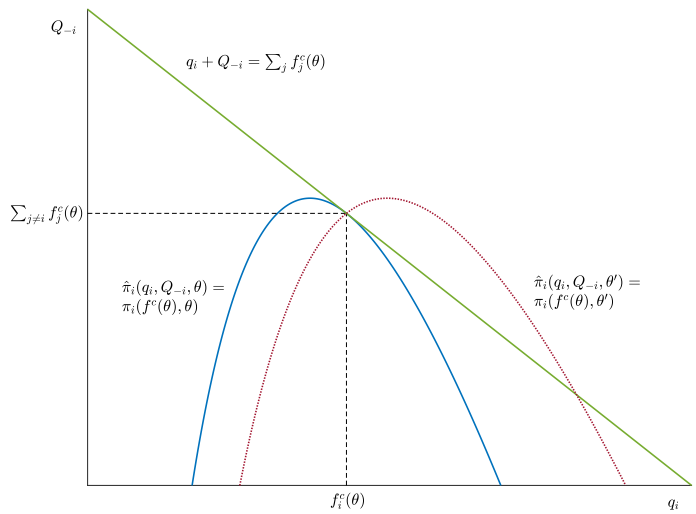

Figure 1: pq-monotonicity of f^c

It means its iso-profit line through point $(f_i^c(\theta), \sum_{j\neq i} f_j^c(\theta))$ is crossing the constraint in state θ'. Therefore, given the assumption that $f_j(\theta) > 0$ for all j, we can find a point on the constraint that firm i strictly prefers to $(f_i^c(\theta), \sum_{j\neq i} f_j^c(\theta))$.[5] That is, $\Lambda_i^{f^c}(f^c(\theta)) \not\subseteq L_i(f^c(\theta), \theta')$ holds. We can conclude that f^c satisfies pq-monotonicity. ■

4 Repeated Implementation

In the rest of the paper, we assume that the firms and CA interact for an infinite number of periods. The periods are indexed as $0, 1, 2, \ldots$, with period 0 being the initial period. A superscript to any variable will indicate the period. Thus, θ^t and m^t indicate period t state of the world and period t profile of messages, respectively. In period $t > 0$, a history of states is $\zeta^t = (\theta^0, \ldots, \theta^t)$, a history of messages is $\mu^t = (m^0, \ldots, m^{t-1})$, and a history is $h^t = (\mu^t, \zeta^{t-1})$. Let $\zeta^0 = \theta^0$ and $\zeta^{-1} = \mu^0 = h^0 = \emptyset$. For any t, let H^t denote the set of all possible period t histories.

We assume that CA employs a price-quantity mechanism in every period, but it can differ from one period to another. A pq-regime, r, describes which price-quantity mechanism is selected after every possible message history: $g^t =$

[5] In the absence of the assumption that $f_j(\theta) > 0$ for all j, it could be that the point $(f_i^c(\theta), \sum_{j\neq i} f_j^c(\theta))$ was located on the horizontal axis and any point that firm i strictly prefers to this point, would be below the horizontal axis and, hence, would not be feasible.

$r(\mu^t)$. Occasionally, it will be more convenient to condition r on h^t, with the understanding that $r(\mu^t, \zeta^{t-1}) = r(\mu^t, \hat{\zeta}^{t-1})$ for any $\zeta^{t-1}, \hat{\zeta}^{t-1}$. We assume that CA commits to a pq-regime at the start of period 0 and that the firms know which regime CA employs.

The firms learn period t state of the world before sending period t messages. At the end of period t, they also learn the period t messages of other firms. A (pure) strategy s_i of firm i specifies a message $m_i^t = s_i(h^t, \theta^t)$ for every t, h^t, and θ^t. We will condition strategies interchangeably on (h^t, θ^t) and (μ^t, ζ^t). Let $s = (s_1, \ldots, s_n)$ denote a strategy profile. Given t, h^t, and s, let $\rho(h^t|h^t, s) = 1$ and for any $\tau > t$, let

$$\rho(h^\tau|h^t, s) = \begin{cases} \rho(h^{\tau-1}|h^t, s) l(\theta) & \text{if } \mu^\tau = (\mu^{\tau-1}, s(h^{\tau-1}, \theta)) \text{ and } \zeta^{\tau-1} = (\zeta^{\tau-2}, \theta), \\ 0 & \text{otherwise.} \end{cases}$$

For any t and h^t, the continuation value of firm i before it knows period t state and given that the firms follow strategies s, is

$$v_i(s|h^t) = (1-\delta) \sum_{\tau=t}^{\infty} \sum_{h^\tau \in H^\tau} \delta^{\tau-t} \rho(h^\tau|h^t, s) l(\theta^\tau) \pi_i(g^\tau(s(h^\tau, \theta^\tau)), \theta^\tau),$$

where $g^\tau = r(h^\tau)$.[6]

A strategy profile s is a subgame perfect equilibrium (SPE) of r if for all i, t, h^t, and s_i', it is true that $v_i(s|h^t) \geq v_i((s_i', s_{-i})|h^t)$. (Note that this formulation also implies that firm i will not want to deviate from s_i once it learns θ^t.) A pq-regime r *repeatedly implements* f in SPE if the set of SPE is non-empty and for every SPE s, we have that $r(h^t)(s(h^t, \theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t such that $\rho(h^t|h^0, s) > 0$. f is *repeatedly implementable* in SPE if there exists a pq-regime that repeatedly implements it in SPE.

We now define forthrightness in the repeated setup. It requires that there exists an SPE such that on the equilibrium path, in every period, the firms announce and produce exactly those quantities that CA wants the firms to produce in that period according to f, and they all announce the price that equates demand with the announced quantities. Recall that for every $\theta \in \Theta$, $m(\theta)$ is such that $m_i(\theta) = (p, q_i)$ for all $i \in I$ where $(q_1, \ldots, q_n) = f(\theta)$ and $p = p\left(\sum_{i \in I} q_i\right)$.

Definition 6 *A pq-regime r satisfies* forthrightness *w.r.t. f if there exists a subgame perfect equilibrium s s.t. $s(h^t, \theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Let $\alpha : \Theta \to \Theta$ be a static deception where $\alpha(\theta) = \theta'$ says that the firms act as if the state was θ', although the true state is θ. Let $v_i(f \circ \alpha) := \sum_{\theta \in \Theta} l(\theta) \pi_i(f(\alpha(\theta)), \theta)$ denote the expected profit of firm i when the firms use the deception α. Let $v_i(f) := v_i(f \circ \alpha)$ when α is the identity map.

[6] v's also depend on r, but we do not make this dependence explicit.

Let v_i be a generic continuation value of firm i and let $v = (v_1, \ldots, v_n)$ be a generic profile of continuation values. For v_i to be feasible, it must be that $v_i \leq \overline{v}_i$ where $\overline{v}_i := \max_{\phi:\Theta\to\mathbb{R}^n_+} \sum_{\theta\in\Theta} l(\theta)\pi_i(\phi(\theta),\theta)$ denotes the highest continuation value that firm i can attain. Clearly, to attain $\overline{v}_i$, firm i must be a monopolist, that is, $\phi_{-i}(\theta) = 0_{-i}$ for all θ must hold for ϕ that maximizes the ex ante profit of firm i. If in some period with state θ, the firms produce an output profile q and expect v in continuation, then the present discounted profit of firm i is given by $(1-\delta)\pi_i(q,\theta) + \delta v_i$. Let $L_i(q, v, \theta) := \{(q', v')|(1-\delta)\pi_i(q,\theta) + \delta v_i \geq (1-\delta)\pi_i(q',\theta) + \delta v'_i\}$ denote the lower contour set of firm i in state θ at quantity-value pair (q, v), and let $\Lambda^f_i(q) := \cap_{\theta\in f^{-1}(q)} L_i(q, v(f), \theta)$.

Mezzetti and Renou (2017, Theorem 1) have proven that for an arbitrary message space M, an SCF must satisfy the so-called dynamic monotonicity in order to be repeatedly implementable. Dynamic monotonicity says that for every dynamic deception that results in an undesirable output profile after some history of states, there exists a firm that has incentives to deviate from this deception after some (possibly different) history of states.[7] One can also define a version of dynamic monotonicity when $M = \mathbb{R}_{++} \times \mathbb{R}_+$ and the regime satisfies forthrightness, and show that this dynamic monotonicity is necessary for repeated implementation.

Checking whether an SCF satisfies dynamic monotonicity, however, can be a daunting task because there is an infinity of possible dynamic deceptions.[8] For this reason, we will assume below that all the firms are identical and that an SCF assigns them identical outputs. With this assumption, we will show that we only need to consider two types of dynamic deceptions. The first type are stationary deceptions according to which the same static deception is used in every period. The second type of dynamic deception is the one where the firms deceive only for one period but otherwise they behave truthfully. For each of these two types of dynamic deceptions, we will define a corresponding notion of monotonicity and show that both notions of monotonicity are necessary and together they are sufficient for an SCF to be repeatedly implementable. Finally, note that any dynamic deception that belongs to the aforementioned two types, can be described with a single static deception. Therefore, the two notions of monotonicity are stated using only static deceptions.

Definition 7 *f satisfies pq-*stationary monotonicity *if for any $\alpha : \Theta \to \Theta$, (a) implies (b):*

a. $\Lambda^f_i(f(\alpha(\theta))) \subseteq L_i(f(\alpha(\theta)), v(f \circ \alpha), \theta)$ *for all* $i \in I$ *and* $\theta \in \Theta$,

[7] A dynamic deception specifies which static deception to use after each possible history of states. It is formally defined in Appendix B right before the proof of Proposition 7.

[8] Though, in Āzacis and Vida (2019, Theorem 1), we provide an alternative characterization of dynamic monotonicity, which allows to test dynamic monotonicity of SCF using numerical methods.

b. $f(\alpha(\theta)) = f(\theta)$ *for all* $\theta \in \Theta$.

Proposition 5 *If f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f, then f satisfies pq-stationary monotonicity.*

We can think of $(f, v(f))$ as an SCF that besides an output profile, also specifies the continuation values of firms (which are independent of the state). The following definition is identical to Definition 5, except that the SCF is given by $(f, v(f))$ instead of f and the profit of firm i in state θ is given by $(1 - \delta)\pi_i(q, \theta) + \delta v_i$ instead of $\pi_i(q, \theta)$.

Definition 8 $(f, v(f))$ *satisfies pq*-monotonicity *if for any $\alpha : \Theta \to \Theta$, (a) implies (b):*

a. $\Lambda_i^f(f(\alpha(\theta))) \subseteq L_i(f(\alpha(\theta)), v(f), \theta)$ *for all $i \in I$ and $\theta \in \Theta$,*

b. $f(\alpha(\theta)) = f(\theta)$ *for all $\theta \in \Theta$.*

Remark 1 *pq-monotonicity of f implies pq-monotonicity of $(f, v(f))$, but the converse is not true. From Proposition 4, it follows that $(f^c, v(f^c))$ satisfies pq-monotonicity if Assumption A2 holds.*

Proposition 6 *If f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f, then $(f, v(f))$ satisfies pq-monotonicity.*

From now on we assume that all firms are identical and that they are treated identically:

Assumption A3 *For every θ, $c_i(\cdot, \theta) = c_j(\cdot, \theta)$ and $f_i(\theta) = f_j(\theta)$ for all $i, j \in I$.*

Even though the firms are identical, their interests are not fully aligned because each firm prefers to be the only firm in the market.

Proposition 7 *Suppose Assumption A3 holds. If f satisfies pq-stationary monotonicity and $(f, v(f))$ satisfies pq-monotonicity, then f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f.*

We apply the result of Proposition 7 to study repeated implementation of the competitive SCF in the next section. Because of Remark 1, we only need to verify whether f^c satisfies *pq*-stationary monotonicity to conclude that it is repeatedly implementable.

5 Repeated Implementation of the Competitive SCF

We start with two examples to show why f^c might fail to be pq-stationary monotonic. Both examples satisfy Assumptions A2 and A3. The message of the first example is that it is harder to satisfy pq-stationary monotonicity when δ is large because firms put more weight on the continuation value in their discounted profits.

Example 2. Let $n = 2$, $\Theta = \{\theta_1, \theta_2\}$, $l(\theta_1) = 0.5$, $p = \max\{0, 2700 - 60Q\}$, $c_i(q_i, \theta_1) = 90q_i^2$ and $c_i(q_i, \theta_2) = 60q_i^2 + q_i^3$ for $i = 1, 2$. The competitive SCF is $f_i^c(\theta_1) = 9$ and $f_i^c(\theta_1) = 10$ for $i = 1, 2$. The firms are better off if they adopt the following deception in every period: $\alpha(\theta_1) = \alpha(\theta_2) = \theta_1$ because $\pi_i(f^c(\theta_1), \theta_2) = 8,991 > \pi_i(f^c(\theta_2), \theta_2) = 8,000$.

We show that f^c does not satisfy pq-stationary monotonicity for large enough δ. To incentivize firm i to deviate when the firms follow deception α, but not to deviate when they behave truthfully, we need to find (q, v_i) s.t.

$$(1-\delta)\pi_i(f^c(\theta_1), \theta_1) + \delta v_i(f^c) \geq (1-\delta)\pi_i(q, \theta_1) + \delta v_i, \tag{2}$$

$$(1-\delta)\pi_i(f^c(\theta_1), \theta_2) + \delta v_i(f^c \circ \alpha) < (1-\delta)\pi_i(q, \theta_2) + \delta v_i. \tag{3}$$

After combining both inequalities and simplifying, one arrives at the following necessary condition:

$$\begin{aligned}(1-\delta)(c_i(q_i, \theta_1) - c_i(q_i, \theta_2)) > {} & (1-\delta)(c_i(f_i^c(\theta_1), \theta_1) - c_i(f_i^c(\theta_1), \theta_2)) \\ & +\delta l(\theta_2)(\pi_i(f^c(\theta_1), \theta_2) - \pi_i(f^c(\theta_2), \theta_2)). \end{aligned} \tag{4}$$

$c_i(q_i, \theta_1) - c_i(q_i, \theta_2)$ is maximized for $q_i = 20$. Thus, setting $q_i = 20$ and evaluating costs and profits at the specified values, we find that the above inequality will not be satisfied for $\delta \geq \frac{4598}{5589} \approx 0.8227$.

Let us write the r.h.s. of (2) and (3) as $w_i - (1-\delta)c_i(q_i, \theta)$ for $\theta \in \Theta$ where $w_i := (1-\delta)p(Q)q_i + \delta v_i$. Also, let us denote the l.h.s. of (2) and (3) as $v_i(\theta_1, \theta_1)$ and $v_i(\theta_2, \theta_1)$, respectively. Then, we can conveniently illustrate (2) and (3) when they hold with equalities, in the (q_i, w_i) space, which is done in Figure 2 for two values of δ. When $\delta = 0$, the iso-profit lines intersect and we can find (q_i, w_i) that gives the right incentives to firm i. (We can pick any point above the solid red line, but below the dotted blue line.) But when $\delta = 0.9$, the lines do not cross and we cannot satisfy (3) without violating (2). ■

The next example shows that pq-stationary monotonicity of f^c might fail because of the requirement that a pq-regime satisfies forthrightness.

Example 3. Let $n = 5$, $\Theta = \{\theta_1, \theta_2, \theta_3\}$, $l(\theta_1) = l(\theta_2) = \frac{1}{3}$, $p = \max\{0, 22 - Q\}$, and for all $i \in I$, the cost functions are as follows:

$$c_i(q_i, \theta_1) = \begin{cases} 2q_i & \text{if} \quad q_i \leq 6, \\ \frac{2q_i^3 + 4 \cdot 6^3}{3 \cdot 6^2} & \text{if} \quad q_i > 6, \end{cases}$$

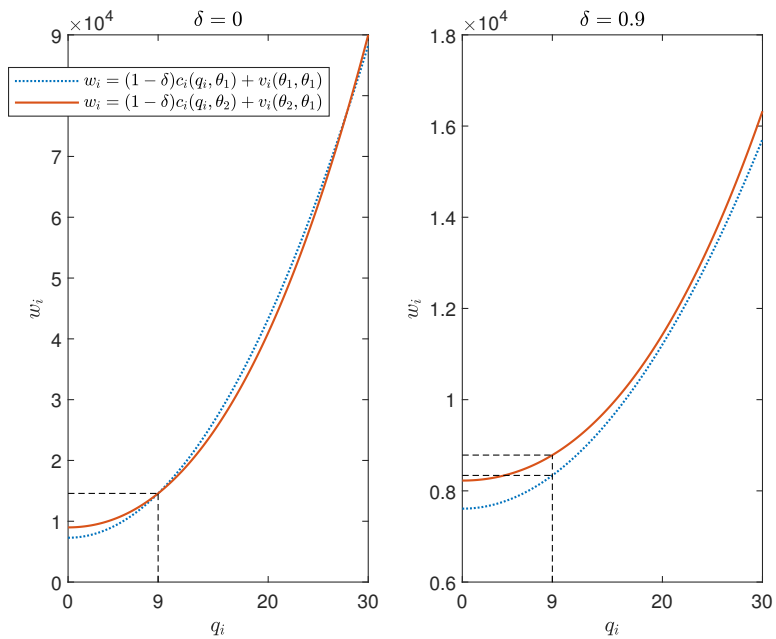

Figure 2: Illustration of Example 2

$$c_i\left(q_i,\theta_2\right) = \begin{cases} 4q_i & \text{if} \quad q_i \leq 5, \\ \frac{2q_i^2+2\cdot 5^2}{5} & \text{if} \quad q_i > 5, \end{cases}$$

$$c_i\left(q_i,\theta_3\right) = \begin{cases} 4q_i & \text{if} \quad q_i \leq 11, \\ \frac{q_i^4+3\cdot 11^4}{11^3} & \text{if} \quad q_i > 11. \end{cases}$$

All these cost functions are continuously differentiable (to be consistent with Assumption A2). The competitive SCF is $f_i^c(\theta_1) = 4$ and $f_i^c(\theta_2) = f_i^c(\theta_3) = 3.6$ for all $i \in I$. The firms are better off if they adopt the following deception: $\alpha(\theta_1) = \alpha(\theta_2) = \alpha(\theta_3) = \theta_2$ because $\pi_i(f^c(\theta_2),\theta_1) = 7.2 > \pi_i(f^c(\theta_1),\theta_1) = 0$. Also, $v_i(f^c) = 0$ and $v_i(f^c \circ \alpha) = 2.4$.

To eliminate deception α, we need to find (q,v) s.t. for some i, $(q,v) \in \Lambda_i^{f^c}(f^c(\alpha(\theta_1)))$ but $(q,v) \notin L_i(f^c(\alpha(\theta_1)), v(f^c \circ \alpha), \theta_1)$ or

$$\begin{aligned} 0 &\geq (1-\delta)\pi_i(q,\theta_2) + \delta v_i, \\ 0 &\geq (1-\delta)\pi_i(q,\theta_3) + \delta v_i, \\ 7.2(1-\delta) + 2.4\delta &< (1-\delta)\pi_i(q,\theta_1) + \delta v_i. \end{aligned}$$

Figure 3 plots the above expressions for $\delta = 0.85$ when they hold with equalities, in the (q_i, w_i) space where $w_i = (1-\delta)p(Q)q_i + \delta v_i$. The figure shows that it is impossible to find such a pair (q,v) that would satisfy all three inequalities: (q,v) must lie below both the solid blue line and the dashed red line, but above

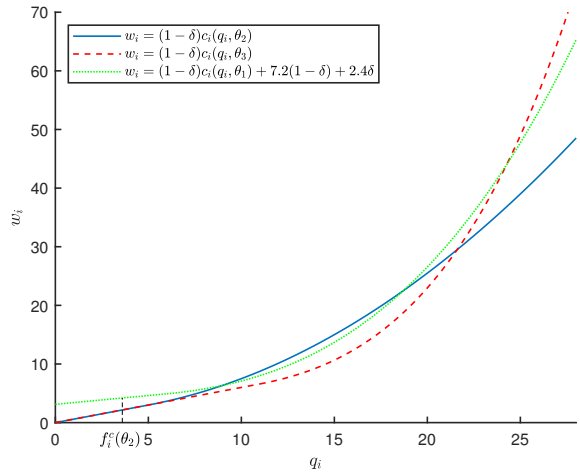

Figure 3: Failure of pq-stationary monotonicity in Example 3 when $\delta = 0.85$

the dotted green line. At the same time, if we did not impose forthrightness and allowed the firms to send different messages (on the equilibrium path) in states θ_2 and θ_3, then we can eliminate both deception $\alpha(\theta_1) = \theta_2$ and $\alpha(\theta) = \theta$ for $\theta = \theta_2, \theta_3$ (because the solid blue line intersects the dotted green line) and deception $\alpha'(\theta_1) = \theta_3$ and $\alpha'(\theta) = \theta$ for $\theta = \theta_2, \theta_3$ (because the dashed red line intersects the dotted green line).[9] ■

We now introduce an assumption that rules out situations like in Examples 2 and 3 when the isoprofit lines never intersect.

Assumption A4 *$\Theta \subset \mathbb{R}_{++}$, $c_i(q_i, \theta) = \theta c(q_i)$ for all $\theta \in \Theta$ and $i \in I$, $c(0) = 0$, $p(0) > \max_{\theta \in \Theta} \theta c'(0)$, $c'(q_i) > 0$, and $c''(q_i) \geq 0$ for all $q_i \geq 0$. $p(Q)$ is twice differentiable and $p''Q + 2p' \leq 0$ for all Q, except possibly at Q s.t. $p(Q) = 0$.*

Assumption A4 implies that the cost function, which is the same for all firms, exhibits strictly increasing differences. The assumption is a special case of Assumption A2, except that it only guarantees that there is a unique symmetric solution to (1).[10] Therefore, whenever Assumption A4 is invoked, we require that f^c is such that for every θ, $f^c(\theta)$ is a symmetric solution to (1), that is, $p(nf_i^c(\theta)) = \theta c'(f_i^c(\theta))$ for all i and θ.

Let f^m denote an SCF that assigns the symmetric cartel output to the firms in every state. That is, given Assumption A4, $f_i^m(\theta)$ is a solution to $p(nf_i^m(\theta)) +$

[9] v_i must also be feasible, that is, $v_i \leq \overline{v}_i$ must hold where $\overline{v}_i = 83.8775$. According to Figure 3, it is enough to consider $w_i < 70$ or $v_i < 70/\delta \approx 82.3529$. Thus, the feasibility of v_i is satisfied. In fact, this is the reason why we set $n = 5$.

[10] That is, Assumption A4 does not rule out the existence of additional asymmetric solutions to (1). For example, there is a continuum of solutions to (1) in Example 4.

$p'(nf_i^m(\theta))nf_i^m(\theta) = \theta c'(f_i^m(\theta))$ for all i and θ. ($c'' \geq 0$ and $p''Q + 2p' \leq 0$ ensure that the second order condition is satisfied.)

Proposition 8 *Suppose Assumption A4 holds.*

1. *There exists $\underline{\delta} > 0$ s.t. f^c satisfies pq-stationary monotonicity for all $\delta \in [0, \underline{\delta})$.*
2. *Suppose $\overline{v}_1 > v_1(f^c) + \frac{\max_\theta \theta(v_1(f^m) - v_1(f^c))}{\min_{\theta,\theta'|\theta>\theta'} \theta - \theta'}$ holds. Then, there exists $\overline{\delta} < 1$ s.t. f^c satisfies pq-stationary monotonicity for all $\delta \in (\overline{\delta}, 1)$.*
3. *There exists $\overline{n}$ s.t. f^c satisfies pq-stationary monotonicity for all $\delta \in [0, 1)$ if $n > \overline{n}$.*

Remark 2 *Lee and Sabourian (2011) have shown in their Theorem 1 that an SCF f is not repeatedly implementable for large enough discount factors if f is not efficient in the range, meaning, there exists a deception α such that $v(f \circ \alpha) > v(f)$.*[11] *In general, f^c is not efficient from the firms' perspective because $v(f^m) > v(f^c)$. The result of Proposition 8 that f^c can even be implemented when δ is arbitrary close to 1, however, does not contradict Theorem 1 of Lee and Sabourian (2011) because they assume bounded payoffs: there exists some finite K s.t. $\max_{i\in I,\theta\in\Theta,q\in\mathbb{R}^n_+} |\pi_i(q,\theta)| < K$. If the firms faced capacity constraints, then Theorem 1 of Lee and Sabourian (2011) would apply and the implementation of f^c would fail for large δ's.*

In the final example of the section, we show that there can be $0 < \underline{\delta} < \overline{\delta} < 1$ such that f^c is not *pq*-stationary monotonic when $\delta \in [\underline{\delta}, \overline{\delta}]$, but it is *pq*-stationary monotonic when $\delta \in [0, \underline{\delta}) \cup (\overline{\delta}, 1)$. Thus, in general, it is not true that if an SCF is *pq*-stationary monotonic for some δ, then it is also *pq*-stationary monotonic for any smaller δ. The example also shows that even if δ is close to 1, the number of firms that is needed for f^c to satisfy *pq*-stationary monotonicity does not need to be large.

Example 4. Let $\Theta = \{\theta_1, \theta_2\}$ with $a > \theta_1 > \theta_2 > 0$, $p = \max\{0, a - Q\}$, $c_i(q_i, \theta) = \theta q_i$ for all $i \in I$ and $\theta \in \Theta$. Thus, Assumption A4 is satisfied. The competitive SCF is $f_i^c(\theta) = \frac{a-\theta}{n}$ for all $i \in I$ and $\theta \in \Theta$. The firms are better off if they adopt the following deception: $\alpha(\theta_1) = \alpha(\theta_2) = \theta_1$ because $\pi_i(f^c(\theta_1), \theta_2) = (\theta_1 - \theta_2)\frac{a-\theta_1}{n} > \pi_i(f^c(\theta_2), \theta_2) = 0$. Note that $v_i(f^c) = 0$, $v_i(f^c \circ \alpha) = l(\theta_2)(\theta_1 - \theta_2)\frac{a-\theta_1}{n}$, and $\overline{v}_i = l(\theta_1)\left(\frac{a-\theta_1}{2}\right)^2 + l(\theta_2)\left(\frac{a-\theta_2}{2}\right)^2$.

From the proof of Proposition 8, f^c is *pq*-stationary monotonic if $v_1^* < \overline{v}_1$ where v_1^* is given by (10). In the example, $v_1^* < \overline{v}_1$ takes the following form:

$$l(\theta_1)\left(\frac{a-\theta_1}{2}\right)^2 + l(\theta_2)\left(\frac{a-\theta_2}{2}\right)^2 - l(\theta_2)(a-\theta_1)\frac{(\theta_1 - \max\{0, a - q_1^*\})q_1^*}{nq_1^* - a + \theta} > 0.$$

[11] We study the implementation of SCFs, which are efficient in the range, in Section 6.1.

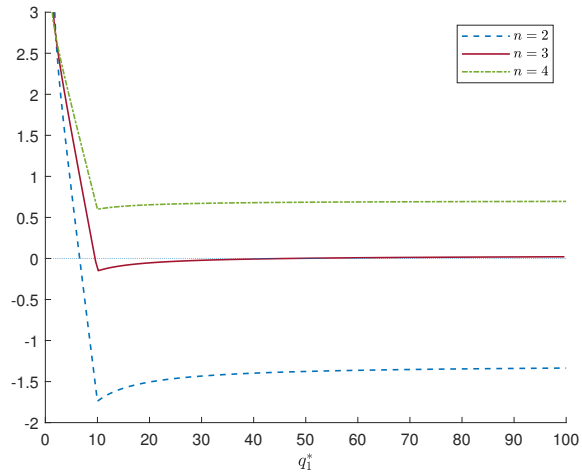

Figure 4: Feasibility constraint in Example 4

We plot the l.h.s. of the above inequality in Figure 4 for $a = 10$, $\theta_1 = 8$, $\theta_2 = 5.8$, $l(\theta_1) = 0.5$, and several values of n. Also, given q_1^*, we can find the corresponding δ from (9) in the proof of Proposition 8 when it holds with equality:

$$\delta = \frac{nq_1^* - a + \theta_1}{nq_1^* - (1 - l(\theta_2))(a - \theta_1)}.$$

Thus, when $n = 2$, f^c is pq-stationary monotonic for all $q_1^* < \underline{q}_1 \approx 6.5889$ or, equivalently, for all $\delta < \underline{\delta} \approx 0.9179$. When $n = 3$, f^c is pq-stationary monotonic for all $q_1^* < \underline{q}_1 \approx 9.5484$ and all $q_1^* > \overline{q}_1 \approx 47.0435$ or, equivalently, for all $\delta < \underline{\delta} \approx 0.9638$ and all δ s.t. $1 > \delta > \overline{\delta} \approx 0.9929$. When $n \geq 4$, f^c is pq-stationary monotonic for all $\delta < 1$. ■

Finally, from the proof of Proposition 8, it follows that under Assumption A4, f^c can only fail to be pq-stationary monotonic because of the constraint on feasible continuation values. However, if we allow for the monetary transfers from CA to firms, this constraint will not apply. Hence, we have the following corollary.

Corollary 1 *Suppose Assumption A4 holds. If monetary transfers from the CA to firms are feasible, then f^c satisfies pq-stationary monotonicity for all $\delta \in [0, 1)$.*

6 Extensions

6.1 Buyer as Another Participant

In a well-known survey of price fixing conspiracies in the US, Hay and Kelley (1974, Table 1) report that in 14% of cases (7 out of 49 cases), the conspiracy

was uncovered after a complaint by a customer. Carree, Günster, and Schinkel (2010) have surveyed all formal decisions by the European Commission on antitrust cases during 1957-2004 and have found that almost 100 of these cases started after a complaint as opposed to 29 cases that started with a leniency application (see their Table 1). Thus, complaints play an important role in detecting anticompetitive behaviour. While a detailed breakdown of complaints is not available, some of these complaints are filed by the customers who are the victims of the anticompetitive practices. This motivates us to study an extension to the model where a representative buyer also participates in the regime by sending messages to CA.

We denote the buyer as participant 0 in the regime and, hence, use subscript 0 to denote the variables corresponding to the buyer. The total number of participants in the regime now is $n + 1 \geq 3$. Let $I_0 := I \cup \{0\}$. When the buyer with quasilinear utility is a price taker, the inverse demand function stands for the marginal utility of the buyer from consuming the good. Therefore, the gross utility of the buyer from consuming Q units is given by $\int_0^Q p(x)dx$. And since the firms jointly receive $p(Q)Q$ in revenue, we can think that the buyer faces a nonlinear payment tariff given by $p(Q)Q$ and his net utility is $\pi_0(q, \theta) = \int_0^Q p(x)dx - p(Q)Q = -\int_0^Q p'(x)xdx$.

We assume that CA still employs a pq-regime. For every $\theta \in \Theta$, let now $m(\theta)$ denote the message profile such that $m_i(\theta) = (p, q_i)$ for all $i \in I_0$ where $(q_1, \ldots, q_n) = f(\theta)$, $q_0 = \sum_{i=1}^n q_i$, and $p = p(q_0)$. Since the implemented outcome must always satisfy $q_0 = \sum_{i=1}^n q_i$, we will write q or (q_i, q_{-i}) for some $i \in I$ to denote a vector of outputs $(q_1, \ldots, q_n)$, that is, this vector does not contain q_0. Definition 6 of forthright pq-regime stays the same. Definition 7 of pq-stationary monotonicity of f and Definition 8 of pq-monotonicity of $(f, v(f))$ also remain the same, except we replace I with I_0 and any vector of continuation values v also includes the continuation value of the buyer. Note though that $L_0(q, v, \theta) = L_0(q, v, \theta')$ for all q, v, θ, θ' because the buyer's utility is independent of the realized state. Consequently, $\Lambda_0^f(q) = L_0(q, v(f), \theta)$ for all q and θ. Propositions 5 and 6 about the necessity of pq-stationary monotonicity of f and pq-monotonicity of $(f, v(f))$ also remain valid since nothing in the proofs depends on the exact form of π_0. However, we cannot directly invoke Proposition 7 because the regime in the proof only allows messages from the firms. Rather than reproving Proposition 7 with the buyer added, we now prove that any f, which is efficient in the range, is repeatedly implementable provided that $(f, v(f))$ satisfies pq-monotonicity.[12]

Definition 9 *f is* efficient in the range *if there does not exist $\alpha : \Theta \to \Theta$ such that $v_i(f \circ \alpha) \geq v_i(f)$ for all $i \in I_0$ and $v_i(f \circ \alpha) > v_i(f)$ for some $i \in I_0$.*

[12] A similar result is proven in Proposition 1 of Āzacis and Vida (2019), but there M is allowed to be arbitrary and, consequently, r is not required to satisfy forthrightness.

Proposition 9 *If f is efficient in the range and $(f, v(f))$ satisfies pq-monotonicity, then f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f.*

Note that we do not invoke Assumption A3 in Proposition 9.

If $(f, v(f))$ satisfies pq-monotonicity when the set of agents is I, then it still satisfies pq-monotonicity when the set of agents is I_0. Hence, we know from Remark 1 that $(f^c, v(f^c))$ satisfies pq-monotonicity if Assumption A2 holds. f^c is also efficient in the range since $f^c(\theta)$ maximizes the total surplus,

$$\sum_{i=0}^{n} \pi_i(q, \theta) = \int_0^{\sum_{i=1}^{n} q_i} p(x)dx - \sum_{i=1}^{n} c_i(q_i, \theta)$$

in state θ. This leads to the following result.

Corollary 2 *Suppose Assumption A2 holds and the buyer is also one of the participants in the regime. Then, f^c is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f^c.*

Note, however, f^c is not the only SCF that is efficient in the range. For example, f^m that maximizes the joint profits of the firms, is also efficient in the range. Even though, in every state θ, the total surplus is higher when $f^c(\theta)$ instead of $f^m(\theta)$ is implemented, it is impossible to transfer any of additional surplus to the firms because of the restriction that the payment from the buyer to the firms takes the form $p(Q)Q$.

6.2 Hard Evidence

The literature that studies the impact of leniency programs on collusion, explicitly or implicitly assumes the existence of hard evidence, which proves collusion by firms (see, for example, Motta and Polo, 2003; Spagnolo, 2004; Aubert, Rey, and Kovacic, 2006). Therefore, we now extend the model of Section 4 by assuming that the firms possess hard evidence, which varies with the state of the world. Intuitively, the existence of hard evidence can help with implementation exactly when either pq-stationary monotonicity of f or pq-monotonicity of $(f, v(f))$ fails. We build on Kartik and Tercieux (2012) who studied one-shot implementation with evidence, and we show that a condition, called evidence monotonicity, is necessary and sufficient for repeated implementation of SCFs (under certain assumptions). It says that for any deception that violates either pq-stationary monotonicity of f or pq-monotonicity of $(f, v(f))$, it must be that either the deception cannot be supported with evidence or a firm can supply evidence that would not be available if the firms were not deceiving.

Thus, we now assume that in each period t, firm $i \in I$ possesses a set of evidence $E_i(\theta^t) \neq \emptyset$, which only depends on the state of the world of that period,

$\theta^t \in \Theta$. A generic piece of evidence of firm i is denoted as e_i. In state θ, firm i can only present evidence that it has, that is, $e_i \in E_i(\theta)$. We refer to $\{E_i(\theta)\}_{i \in I, \theta \in \Theta}$ as an evidence structure. For simplicity, the evidence structure does not change over time. Let $E(\theta) = E_1(\theta) \times \ldots \times E_n(\theta)$ for every θ and $E = \cup_{\theta \in \Theta} E(\theta)$, with a generic element $e = (e_1, \ldots, e_n)$. Besides the firms' messages, the outcome function of a mechanism now also depends on the evidence supplied by the firms, $g : M^n \times E \to \mathbb{R}^n_+$.[13] We also slightly modify μ^t by including the evidence that is provided by the firms, in the history of messages. That is, let now $\mu^t = (\mu^{t-1}, (m^{t-1}, e^{t-1}))$ for all $t > 0$. No other changes in notation are required. In particular, the definitions of regime and strategies remain the same (given the modified history of messages).

We refer to $e : \Theta \to E$ s.t. $e(\theta) \in E(\theta)$ for all θ as an evidence function.

Definition 10 *A pq-regime r satisfies* forthrightness *w.r.t. f if there exists a subgame perfect equilibrium s s.t. $s(h^t, \theta^t) = (m(\theta^t), e(\theta^t))$ and $r(h^t)(m(\theta^t), e(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$ and e is some evidence function.*

Let $\mathcal{A}$ be the set of all deceptions which violate either *pq*-stationary monotonicity of f or *pq*-monotonicity of $(f, v(f))$, that is, for any $\alpha \in \mathcal{A}$, part (a) in either Definition 7 or Definition 8 holds, but part (b) does not.

Definition 11 *f satisfies* evidence monotonicity *if there is an evidence function e s.t. for every $\alpha \in \mathcal{A}$, there exist $i \in I$ and $\theta \in \Theta$ s.t. either $e_i(\alpha(\theta)) \notin E_i(\theta)$ or $E_i(\theta) \not\subseteq E_i(\alpha(\theta))$.*

To keep matters simple, we make the following assumption:

Assumption A5 *f is s.t. $f(\theta) \neq f(\theta')$ for all $\theta, \theta' \in \Theta$.*

Given this assumption, we can replace $\Lambda_i^f(f(\alpha(\theta)))$ with $L_i(f(\alpha(\theta)), v(f), \alpha(\theta))$ in Definitions 7 and 8.

Proposition 10 *Suppose f satisfies Assumption A5. If f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f, then f satisfies evidence monotonicity.*

Proposition 11 *Suppose Assumptions A3 and A5 hold. If f satisfies evidence monotonicity, then f is repeatedly implementable in subgame perfect equilibrium with a pq-regime that satisfies forthrightness w.r.t. f.*

[13] This formulation assumes that firm i can only submit a single piece of evidence. To allow it to submit more than one piece of evidence, we can assume that if $e_i, e_i' \in E_i(\theta)$ for some θ, then there is also $e_i'' \in E_i(\theta)$ such that $e_i'' = \{e_i\} \cup \{e_i'\}$.

Even though we assume the symmetry of firms and the SCF, we do not require that the evidence structure is also symmetric in Proposition 11.

We finish this section by providing an example of natural evidence structure, which guarantees that f^c is repeatedly implementable.

Example 5. Suppose that f^c is such that Assumption A4 holds. (Hence, Assumption A5 also automatically holds.) Assume that the evidence structure is such that for any $\theta, \theta' \in \Theta$, if $\pi_i(f^c(\theta'), \theta) > \pi_i(f^c(\theta'), \theta')$ for all $i \in I$, then $E_j(\theta) \not\subseteq E_j(\theta')$ for some $j \in I$. The motivation for this assumption is that firm j can prove in state θ that it can earn higher profits with the output profile $f^c(\theta')$ than in state θ'. (It seems reasonable to assume that it is easier to under-report profits than to over-report them.) We argue that in this case, f^c satisfies evidence monotonicity. We know that $(f^c, v(f^c))$ satisfies pq-monotonicity. Therefore, $\mathcal{A}$ only contains those deceptions that violate pq-stationary monotonicity of f^c.

Take any deception $\alpha \in \mathcal{A}$ such that $v_i(f^c \circ \alpha) > v_i(f^c)$ for all i. (Note that we are in a symmetric setup.) There must exist a state θ such that $\alpha(\theta) > \theta$, that is, the firms exaggerate their costs in order to produce less. Therefore, $\pi_i(f^c(\alpha(\theta)), \theta) > \pi_i(f^c(\alpha(\theta)), \alpha(\theta))$ holds for all $i \in I$. Given the assumed evidence structure, $E_j(\theta) \not\subseteq E_j(\alpha(\theta))$ for some $j \in I$. From the definition of evidence monotonicity, any such α cannot violate evidence monotonicity of f^c.

Take now any deception $\alpha \in \mathcal{A}$ such that $v_i(f^c) \geq v_i(f^c \circ \alpha)$ for all i. In this case, $L_i(f^c(\alpha(\theta)), v(f^c \circ \alpha), \theta) \subseteq L_i(f^c(\alpha(\theta)), v(f^c), \theta)$ holds for all i and θ. Therefore, if pq-stationary monotonicity of f^c fails for this α, then pq-monotonicity of $(f^c, v(f^c))$ must also fail for the same α, which is a contradiction.

We conclude that f^c satisfies evidence monotonicity given the assumed evidence structure and, according to Proposition 11, f^c is repeatedly implementable in SPE with a pq-regime that satisfies forthrightness w.r.t. f^c. Finally, note that in this example, the evidence function in the definition of evidence monotonicity can be anything. ■

7 Concluding Remarks

To our knowledge, we are the first to study repeated collusion by firms as an implementation problem. This research can be extended in several directions. For example, one could allow the evidence structure in Section 6.2 to change over time depending on the past messages or outputs. We mention two more possible extensions. First, there are no restriction on the regime r in Proposition 7 once a deviation from the equilibrium path has occurred. However, in a more realistic setup, CA would face constraints on rewards and punishments that it can apply. That is, besides forthrightness, we may want to impose further restrictions on r off the equilibrium path. Second, in the model, CA controls the output, while the firms can only influence it indirectly via their messages. In a more complex model with several profit-relevant variables, the firms could directly decide on

some of these variables, while the other variables would be controlled by CA.

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Appendix

A Proofs of Section 3

We first introduce some additional notation and define a "modulo" game. Let $\overline{q}_i$ be such that $\pi_i((\overline{q}_i, 0_{-i}), \theta) < \pi_i(f(\theta'), \theta)$ for all (θ, θ'). $\overline{q}_i$ represents a bad outcome for firm i. Let $\partial\Lambda_i^f(q)$ denote the boundary of the set $\Lambda_i^f(q)$. That is, if $q' \in \partial\Lambda_i^f(q)$, then there does not exist $q''_{-i} < q'_{-i}$ such that $(q'_i, q''_{-i}) \in \Lambda_i^f(q)$. Note that $\Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta')$ if and only if $\partial\Lambda_i^f(f(\theta)) \subset L_i(f(\theta), \theta')$. Finally, let $|X|$ denote the cardinality of set X.

The modulo game. Suppose firms have announced outputs q. For all $i \in I$, let z_i be defined as follows: if q_i is an irrational number, then $z_i = 0$. If q_i

is a rational number, then it can be written as a ratio of two integer numbers whose only common divisor is number 1. Then, z_i is the number given by the last $\lfloor \log_{10} n \rfloor + 1$ digits of the integer in the numerator. (If the numerator has less than $\lfloor \log_{10} n \rfloor + 1$ digits, then z_i is simply equal to the numerator.) Let $i := \left(\sum_{j \in I} z_j\right) (\mod n) + 1$. We say that i is the winner of the modulo game given q.

Note that for any q and $\epsilon > 0$, we can always find a rational number q_i' such that $|q_i - q_i'| < \epsilon$ and i is the winner of the modulo game if i announces q_i', while the others announce q_{-i}.

Proof of Proposition 1: *Necessity.* Suppose g implements f in Nash equilibrium and satisfies forthrightness w.r.t. f. Since m_i is an announcement about a quantity, we will use q_i instead of m_i to denote the message. Let $G_i(q_{-i}) := \{g(q_i, q_{-i}) | q_i \in M\}$. This is the set of output profiles that firm i can induce given that the other firms announce q_{-i}. Take any i and θ. Because $f(\theta'') = (f_i(\theta''), f_{-i}(\theta)) \in NE(g, \theta'')$ for all $\theta'' \in f^{-1}(f_{-i}(\theta))$, $G_i(f_{-i}(\theta)) \subseteq L_i(f(\theta''), \theta'')$ for all $\theta'' \in f^{-1}(f_{-i}(\theta))$. Hence, $G_i(f_{-i}(\theta)) \subseteq \Lambda_i^f(f(\theta))$.

Suppose there is a pair (θ, θ') such that $\Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$. Then, $G_i(f_{-i}(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$ and $f(\theta) \in NE(g, \theta')$. Because g implements f, it must be that $f(\theta') = g(f(\theta)) = f(\theta)$.

Sufficiency. We define a quantity mechanism and show that it satisfies forthrightness w.r.t. f and implements f in Nash equilibrium. Let $D(q) := \{i \in I | f^{-1}(q_{-i}) \neq \emptyset\}$ denote the set of potential deviators when the firms announce q.

The mechanism is as follows. Given $m = q$,

i. If $q = f(\theta)$ for some $\theta \in \Theta$, then $g(q) = q$.

ii. If $q \neq f(\theta)$ for any $\theta \in \Theta$ and $D(q) = \{i\}$, then $g(q) = q' \in \partial\Lambda_i^f(q)$ s.t. $q_i' = q_i$ and $q_j' = q_k'$ for all $j, k \neq i$. If no such q' exists, then let $q_i' = \overline{q}_i$ and $q_j' = 0$ for all $j \neq i$.[14]

iii. If $q \neq f(\theta)$ for any $\theta \in \Theta$ and $|D(q)| > 1$, then $g(q) = q'$ where $q_i' = \overline{q}_i$ if $i \in D(q)$ and $q_i' = 0$ if $i \notin D(q)$.

iv. If $q \neq f(\theta)$ for any $\theta \in \Theta$ and $D(q) = \emptyset$, then $g(q) = (q_i, 0_{-i})$ where i is the winner of the modulo game given q.

We claim that there is no Nash equilibrium that falls under parts (ii)-(iv) of the mechanism. Clearly, there is no Nash equilibrium corresponding to part (iv): every firm has incentives to be the only firm in the market. If q falls under part (ii), then every firm $j \notin D(q)$ expects strictly less than its monopoly profit. (To

[14]Given q_i, there exists at most one $q' \in \partial\Lambda_i^f(q)$ s.t. $q_i' = q_i$ and $q_j' = q_k'$ for all $j, k \neq i$. Thus, by announcing q_i, firm i specifies not just its own output, but also the output of the other firms. However, it can be that there does not exist $q' \in \partial\Lambda_i^f(q)$ s.t. $q_i' = q_i$. This, for example, happens if $q_i = 0$, while the profit of firm i is strictly negative for all $q' \in \Lambda_i^f(q)$.

see that this is also true for $n = 2$, note that if $D(q) = \{i\}$ and $q' \in \partial\Lambda_i^f(q)$ with $q'_i = q_i = 0$, then from the definition of $\partial\Lambda_i^f(q)$, also $q_j = 0$ for $j \neq i$. Thus, firm j cannot be earning the monopoly profit when part (ii) applies.) Any firm $j \notin D(q)$ can trigger part (iv) of the mechanism and earn arbitrarily close to its monopoly profit. Suppose q falls under part (iii). Then, every firm $i \in D(q)$ can trigger part (i) of the mechanism since by the definition of $D(q)$, there exists q'_i such that $(q'_i, q_{-i}) = f(\theta)$ for some θ. From the definition of $\overline{q}_i$, such a deviation is profitable.

Now, suppose the true state is θ. Suppose the firms announce $q = f(\theta)$, in which case part (i) applies and the outcome is also $f(\theta)$. Any firm can deviate and trigger part (ii) and possibly part (iii). Any deviation that triggers part (iii), is clearly suboptimal. By triggering part (ii), firm i can obtain an outcome in $\partial\Lambda_i^f(f(\theta)) \subset \Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta)$. Again, such a deviation is unprofitable. Thus, announcing $f(\theta)$ when the state is θ, is a Nash equilibrium and the mechanism satisfies forthrightness w.r.t. f.

Suppose that in state θ, the firms announce $q = f(\theta')$ for some $\theta' \neq \theta$. We claim that if no firm has incentives to deviate, it must be that $\partial\Lambda_i^f(f(\theta')) \subset L_i(f(\theta'), \theta)$ holds for all i. Note that firm i cannot obtain every $q' \in \partial\Lambda_i^f(f(\theta'))$ when it triggers part (ii). However, if there exists q' s.t. $q' \in \partial\Lambda_i^f(f(\theta'))$, but $q' \notin L_i(f(\theta'), \theta)$, then it is also true that $q'' \in \partial\Lambda_i^f(f(\theta'))$ and $q'' \notin L_i(f(\theta'), \theta)$ for q'' s.t. $q''_i = q'_i$, $q''_j = \left(\sum_{k \in I \setminus \{i\}} q'_k\right)/n$ for all $j \neq i$ because firm i is indifferent how other firms share their output among themselves. Thus, if no firm wants to deviate, $\partial\Lambda_i^f(f(\theta')) \subset L_i(f(\theta'), \theta)$ holds for all i and q-monotonicity implies that $f(\theta) = f(\theta')$. Hence, in the (unique) Nash equilibrium, the desired outcome is implemented. ■

Proof of Proposition 2: Suppose g implements f in Nash equilibrium and satisfies forthrightness w.r.t. f. We use q_i to denote the message of firm i. Let $G_i(q_{-i}) := \{g(q_i, q_{-i}) | q_i \in M\}$. Take any i and θ. Because of forthrightness, $f(\theta') \in G_i(f_{-i}(\theta))$ for all $\theta' \in f^{-1}(f_{-i}(\theta))$, that is, firm i can induce the output profile $f(\theta')$ when the competitors announce $f_{-i}(\theta)$. Also, because $f(\theta) \in NE(g, \theta)$, $\pi_i(f(\theta), \theta) \geq \pi_i(f(\theta'), \theta)$ holds for all $\theta' \in f^{-1}(f_{-i}(\theta))$. ■

Proof of Proposition 3: *Necessity.* Suppose g implements f in Nash equilibrium and satisfies forthrightness w.r.t. f. Let $G(m_{-i}) := \{g(m_i, m_{-i}) | m_i \in M_i\}$. Take any θ. Because $m(\theta) \in NE(g, \theta'')$ for all $\theta'' \in f^{-1}(f(\theta))$, $G(m_{-i}(\theta)) \subseteq L_i(f(\theta), \theta'')$ for all $i \in I$ and $\theta'' \in f^{-1}(f(\theta))$. Hence, $G(m_{-i}(\theta)) \subseteq \Lambda_i^f(f(\theta))$ for all $i \in I$. Suppose there is θ' such that $\Lambda_i^f(f(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$. Then, $G(m_{-i}(\theta)) \subseteq L_i(f(\theta), \theta')$ for all $i \in I$ and $m(\theta) \in NE(g, \theta')$. Because g implements f, it must be that $f(\theta') = g(m(\theta)) = f(\theta)$.

Sufficiency. Let $m = ((p_1, q_1), \ldots, (p_n, q_n))$. Let $D(m) := \{i \in I | \exists m'_i \text{ s.t. } (m'_i, m_{-i}) = m(\theta) \text{ for some } \theta\}$ denote the set of potential deviators when the firms announce m. Note that if $i \in D(m)$, it must be that the rest of the firms have announced the same price, say, p. Also, because of Assumption A1, m'_i in the definition of

$D(m)$ is unique and given by $(p, \hat{q}_i)$ such that $p\left(\sum_{j\in I\setminus\{i\}} q_j + \hat{q}_i\right) = p$. In the following mechanism, $\hat{q}_i$ should be understood as having been constructed in this way, given m and given that $i \in D(m)$.

The mechanism is as follows:

i. If $m = m(\theta)$ for some $\theta \in \Theta$, then $g(m) = f(\theta)$.

ii. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $D(m) = \{i\}$, then $g(m) = q' \in \partial\Lambda_i^f(\hat{q}_i, q_{-i})$ s.t. $q'_i = q_i$ and $q'_j = q'_k$ for all $j, k \neq i$. If no such q' exists, then let $q'_i = \overline{q}_i$ and $q'_j = 0$ for all $j \neq i$.

iii. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $|D(m)| > 1$, then $g(m) = q'$ where $q'_i = \overline{q}_i$ if $i \in D(m)$ and $q'_i = 0$ if $i \notin D(m)$.

iv. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $D(m) = \emptyset$, then $g(m) = (q_i, 0_{-i})$ where i is the winner of the modulo game given q.

The proof that this mechanism implements f if it satisfies pq-monotonicity, is analogous to the one of Proposition 1. ■

B Proofs of Section 4

Proof of Proposition 5: Suppose f is repeatedly implementable in SPE with a pq-regime that satisfies forthrightness w.r.t. f. Thus, there exists an SPE s s.t. $s(h^t, \theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$. Fix this s. Given s, we define another SPE $\hat{s}$, which we will then use to prove that f must satisfy pq-stationary monotonicity. We first explain $\hat{s}$ in words. $\hat{s}$ only differs from s when a unilateral deviation from the equilibrium play occurs. Specifically, suppose that $\hat{\theta}, \tilde{\theta} \in f^{-1}(f(\theta))$ for some θ. Suppose that firm i deviates from s in period t. Then, the continuation values that the firms expect starting period $t+1$, can differ depending if period t state is $\hat{\theta}$ or $\tilde{\theta}$ because the firms can condition their future play on period t state. $\hat{s}$ will have a property that the continuation values are the same whether period t state is $\hat{\theta}$ or $\tilde{\theta}$. The reason why such an equilibrium exists is that because of forthrightness, CA cannot infer from the messages whether the state is $\hat{\theta}$ or $\tilde{\theta}$ and, hence, cannot offer different mechanisms in the continuation.

We now proceed with defining $\hat{s}$ formally. First, let $\hat{s}(h^t, \theta^t) = s(h^t, \theta^t) = m(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$. Next, take any t, θ^t, $h^t = (\mu^t, \zeta^{t-1})$ s.t. $\rho(h^t|h^0, s) > 0$. Suppose that period t state is some $\hat{\theta}^t \in f^{-1}(f(\theta^t))$. Note that $s(h^t, \hat{\theta}^t) = m(\theta^t)$. Take any firm i. We now define the output-value pair that firm i induces by announcing some $m_i \neq m_i(\theta^t)$:

$$(q, v(\hat{\theta}^t)) := (r(\mu^t)(m_i, m_{-i}(\theta^t)), v(s|((\mu^t, (m_i, m_{-i}(\theta^t))), (\zeta^{t-1}, \hat{\theta}^t)))).$$

While, given $(m_i, m_{-i}(\theta^t))$, q does not depend on $\hat{\theta}^t$, $v(\hat{\theta}^t)$ can depend on it since firms can condition their future messages on the state in period t. Firm i will not have incentives to deviate from $m_i(\theta^t)$ in state $\hat{\theta}^t$ if $(q, v(\hat{\theta}^t)) \in L_i(f(\theta^t), v(f), \hat{\theta}^t)$. While it is not guaranteed that $(q, v(\tilde{\theta}^t)) \in \Lambda_i^f(f(\theta^t))$ for all $\tilde{\theta}^t \in f^{-1}(f(\theta^t))$, this must be true for the state in which $v_i(\tilde{\theta}^t)$ is minimal among $\tilde{\theta}^t \in f^{-1}(f(\theta^t))$. Suppose that this state is $\hat{\theta}^t$, that is, $(q, v(\hat{\theta}^t)) \in \Lambda_i^f(f(\theta^t))$ holds.

$\hat{s}$ will be defined in such a way that for every $\tilde{\theta}^t \in f^{-1}(f(\theta^t))$, the continuation value after the history $((\mu^t, (m_i, m_{-i}(\theta^t))), (\zeta^{t-1}, \tilde{\theta}^t))$ is $v(\hat{\theta}^t)$ rather than $v(\tilde{\theta}^t)$. Thus, let $\zeta^t = (\zeta^{t-1}, \tilde{\theta}^t)$ for any $\tilde{\theta}^t \in f^{-1}(f(\theta^t))$ and let $\hat{\zeta}^t = (\zeta^{t-1}, \hat{\theta}^t)$. For all $\tau > t$ and all θ^τ, let $\zeta^\tau = (\zeta^{\tau-1}, \theta^\tau)$ and $\hat{\zeta}^\tau = (\hat{\zeta}^{\tau-1}, \theta^\tau)$. Let $\mu^{t+1} = (\mu^t, (m_i, m_{-i}(\theta^t))$. For all $\tau > t+1$ and all m^τ, let $\mu^{\tau+1} = (\mu^\tau, m^\tau)$. Let $\hat{s}$ be such that $\hat{s}(\mu^\tau, \zeta^\tau) = s(\mu^\tau, \hat{\zeta}^\tau)$ for all $\tau > t$. Intuitively, $\hat{s}$ does not depend on which exact state in $f^{-1}(f(\theta^t))$ is realized in period t (given h^t and $(m_i, m_{-i}(\theta^t))$). By construction, it follows that $v(\hat{s}|(\mu^{t+1}, \zeta^t)) = v(s|(\mu^{t+1}, \hat{\zeta}^t))$ and

$$(r(\mu^t)(m_i, m_{-i}(\theta^t)), v(\hat{s}|(\mu^{t+1}, \zeta^t))) \in \Lambda_i^f(f(\theta^t)). \tag{5}$$

Since s are SPE strategies of the subgame starting after history $(\mu^{t+1}, \hat{\zeta}^t)$ and because profits from period $t+1$ onwards do not depend on period t history of states, it must be that $\hat{s}$ forms SPE strategies of the subgame starting after history (μ^{t+1}, ζ^t).

Now, repeat the above process for all t, θ^t, h^t s.t. $\rho(h^t|h^0, s) > 0$, and all i and $m_i \neq m_i(\theta^t)$. The only remaining histories are the ones that occur after there has been a multilateral deviation from the equilibrium path of s. For any such history h^t in any period t and any θ, let $\hat{s}(h^t, \theta) = s(h^t, \theta)$.

The constructed $\hat{s}$ is also SPE of the game induced by r. By construction, (5) holds for all t, θ^t, $h^t = (\mu^t, \zeta^{t-1})$ s.t. $\rho(h^t|h^0, s) > 0$, and all i and $m_i \neq m_i(\theta^t)$, which means that no unilateral deviation from the equilibrium path is profitable. Also, from the definition of $\hat{s}$, it follows that $\hat{s}$ forms NE strategies in the subgames that are reached after a (unilateral or multilateral) deviation from the equilibrium path.

We now use $\hat{s}$ to prove that f must satisfy pq-stationary monotonicity. Thus, suppose it does not satisfy pq-stationary monotonicity. That is, there exists α such that in Definition 7, part (a) holds, but part (b) does not. Fix this α. Given $\hat{s}$ and α, we now define another strategy profile $\tilde{s}$. We first explain $\tilde{s}$ in words. Under $\tilde{s}$, the firms follow strategies $\hat{s}$, but they deceive according to α as long as no deviation has occurred. If a deviation ever occurs, the firms start conditioning $\tilde{s}$ on the true states, but they still behave as if the states that were drawn up to and including the period of first deviation, were the ones given by the deception.

Formally, $\tilde{s}$ is defined recursively with the help of a couple of auxiliary variables. Let $\tilde{\zeta}^0 = \alpha(\theta^0)$ and $d(\mu^0, \zeta^0) = 1$. The period 0 strategies are $\tilde{s}(\mu^0, \zeta^0) = \hat{s}(\mu^0, \tilde{\zeta}^0)$. Also, let $d(\mu^1, \zeta^1) = d(\mu^0, \zeta^0) \cdot \mathbf{1}_{\{(m^0, \zeta^0) = (\hat{s}(\mu^0, \tilde{\zeta}^0), \zeta^0)\}}$, where $\mathbf{1}_{\{X\}}$ is an indicator function taking value 1 if X is true and 0 otherwise. In period 1, if

$d(\mu^1, \zeta^1) = 1$, then $\tilde{\zeta}^1 = (\tilde{\zeta}^0, \alpha(\theta^1))$; otherwise, $\tilde{\zeta}^1 = (\tilde{\zeta}^0, \theta^1)$. Let $\tilde{s}(\mu^1, \zeta^1) = \hat{s}(\mu^1, \tilde{\zeta}^1)$ and $d(\mu^2, \zeta^2) = d(\mu^1, \zeta^1) \cdot \mathbf{1}_{\{((\mu^1, m^1), \zeta^1) = ((\mu^1, \hat{s}(\mu^1, \tilde{\zeta}^1)), \zeta^1)\}}$. Suppose we have defined the variables up to period $t-1$. In period t, if $d(\mu^t, \zeta^t) = 1$, then $\tilde{\zeta}^t = (\tilde{\zeta}^{t-1}, \alpha(\theta^t))$; otherwise, $\tilde{\zeta}^t = (\tilde{\zeta}^{t-1}, \theta^t)$. Let $\tilde{s}(\mu^t, \zeta^t) = \hat{s}(\mu^t, \tilde{\zeta}^t)$ and $d(\mu^{t+1}, \zeta^{t+1}) = d(\mu^t, \zeta^t) \cdot \mathbf{1}_{\{(\mu^t, m^t), \zeta^t) = ((\mu^t, \hat{s}(\mu^t, \tilde{\zeta}^t)), \zeta^t)\}}$.

By assumption, there exists θ such that $f(\alpha(\theta)) \neq f(\theta)$. Therefore, the constructed strategy profile $\tilde{s}$ selects undesirable outcomes on its path. Since it is assumed that r implements f, $\tilde{s}$ cannot be an SPE. Because the future profits do not depend on the history of states, by construction, $\tilde{s}$ implies NE play in the subgames that follow after a deviation from the path of $\tilde{s}$ has occurred. That is, if $\tilde{s}$ are not SPE strategies, it must be because there is a profitable deviation on the path. Thus, there exist t, θ^t, $h^t = (\mu^t, \zeta^{t-1})$ s.t. $\rho(h^t|h^0, s) > 0$, i, and m_i such that[15]

$$\begin{aligned}(q, v) &= (r(\mu^t)(m_i, m_{-i}(\alpha(\theta^t))), v(\tilde{s}|((\mu^t, (m_i, m_{-i}(\alpha(\theta^t)))), (\zeta^{t-1}, \theta^t)))) \\ &\notin L_i(f(\alpha(\theta^t)), v(f \circ \alpha), \theta^t).\end{aligned}$$

By part (a) in Definition 7, $(q, v) \notin \Lambda_i^f(f(\alpha(\theta^t)))$. Also, from the definition of $\tilde{s}$, there exists $\tilde{\zeta}^{t-1}$ such that

$$(q, v) = (r(\mu^t)(m_i, m_{-i}(\alpha(\theta^t))), v(\hat{s}|((\mu^t, (m_i, m_{-i}(\alpha(\theta^t)))), (\tilde{\zeta}^{t-1}, \alpha(\theta^t))))).$$

However, this contradicts (5), which must hold for all t, $\tilde{\theta}^t$, $\tilde{h}^t$ s.t. $\rho(\tilde{h}^t|h^0, s) > 0$, and all i and $m_i \neq m_i(\tilde{\theta}^t)$, including $\tilde{\theta}^t = \alpha(\theta^t)$ and $\tilde{h}^t = (\mu^t, \tilde{\zeta}^{t-1})$. It follows that for any α, if part (a) in Definition 7 holds, then part (b) must also hold. That is, f must satisfy pq-stationary monotonicity. ■

Proof of Proposition 6: Suppose f is repeatedly implementable in SPE with a pq-regime that satisfies forthrightness w.r.t. f. Consider again SPE $\hat{s}$ that we constructed in the proof of Proposition 5. We will use this strategy profile to prove that $(f, v(f))$ must satisfy pq-monotonicity. Thus, suppose there exists α such that part (a) in Definition 8 holds. Given this α and $\hat{s}$, we define another strategy profile $\tilde{s}$. Let $\zeta^0 = \theta^0$, $\hat{\zeta}^0 = \alpha(\theta^0)$, and $\tilde{s}(\theta^0) = \hat{s}(\alpha(\theta^0))$ for all θ^0. For all $t > 0$ and all θ^t, let $\zeta^t = (\zeta^{t-1}, \theta^t)$ and $\hat{\zeta}^t = (\hat{\zeta}^{t-1}, \theta^t)$. For all $t > 0$ and all m^t, let $\mu^t = (\mu^{t-1}, m^{t-1})$. For all $t > 0$, let $\tilde{s}$ be such that $\tilde{s}(\mu^t, \zeta^t) = \hat{s}(\mu^t, \hat{\zeta}^t)$.

Since the future profits do not depend on period 0 state, by construction, $\tilde{s}$ implies subgame perfect play from period 1 onwards for all h^1. To show that $\tilde{s}$ are SPE strategies, it remains to check that there is no profitable deviation in period 0. From the definition $\tilde{s}$ and from (5), we have that for all i, m_i, and θ^0,

$$\begin{aligned}&(r(\mu^0)(m_i, m_{-i}(\theta^0)), v(\tilde{s}|(\mu^1, \zeta^0))) \\ =\ &(r(\mu^0)(m_i, m_{-i}(\alpha(\theta^0))), v(\hat{s}|(\mu^1, \hat{\zeta}^0))) \in \Lambda_i^f(f(\alpha(\theta^0))),\end{aligned}$$

[15] Even though the firms use a stationary deception, the regime itself does not need to be stationary. Therefore, we cannot restrict attention to $t = 0$.

where $\mu^1 = (m_i, m_{-i}(\theta^0))$. That is, any output-value pair that firm i can obtain by deviating in any state θ^0, will lie in $\Lambda_i^f(f(\alpha(\theta^0)))$. Because, according to part (a) in Definition 8, $\Lambda_i^f(f(\alpha(\theta^0))) \subseteq L_i(f(\alpha(\theta^0)), v(f), \theta^0)$ for all i and θ^0, no firm has a profitable deviation in period 0. Thus, $\tilde{s}$ is an SPE. Because it is assumed that f is repeatedly implementable, it must be that part (b) in Definition 8 also holds, or else an undesirable output profile would be produced in some state θ^0. ■

Before proving Proposition 7, we introduce some further notation and define a dynamic deception. For each i, let $q_i(Q_{-i}, \theta) := \arg\max_{q_i} \pi_i(q_i, Q_{-i}, \theta)$ and $\overline{v}_i(Q_{-i}) := \sum_\theta l(\theta)\pi_i(q_i(Q_{-i}, \theta), Q_{-i}, \theta)$. (Note $\overline{v}_i(0) = \overline{v}_i$.) Also, let $\underline{v}_i(q_i) := \sum_\theta l(\theta)\pi_i((q_i, 0_{-i}), \theta)$. Let $\overline{\overline{q}}_i$ be such that $(1-\delta)\pi_i((\overline{\overline{q}}_i, 0_{-i}), \theta) < (1-\delta)\pi_i(f(\theta'), \theta) + \delta \min\{0, v_i(f)\}$ for all (θ, θ'). $\overline{\overline{q}}_i$ represents an even worse outcome than $\overline{q}_i$ for firm i. Also, an outcome-value pair (q', v) belongs to the boundary of $\Lambda_i^f(q)$, that is, $(q', v) \in \partial\Lambda_i^f(q)$ if there does not exist v' s.t. $(q', v') \in \Lambda_i^f(q)$ and $v_i < v_i' \leq \overline{v}_i$.

A dynamic deception allows the firms to deceive differently in different periods. Thus, a dynamic deception β specifies a state $\tilde{\theta}^t \in \Theta$ for every t and $\zeta^t = (\zeta^{t-1}, \theta^t)$: $\beta(\zeta^{t-1}, \theta^t) = \tilde{\theta}^t$.[16] Given t and ζ^t, let $l(\zeta^t|\zeta^t) = 1$ and for any $\tau > t$, let

$$l(\zeta^{\tau+1}|\zeta^t) = \begin{cases} l(\zeta^\tau|\zeta^t)l(\theta) & \text{if } \zeta^{\tau+1} = (\zeta^\tau, \theta), \\ 0 & \text{otherwise.} \end{cases}$$

For any t and ζ^t, the discounted value of future profits of firm i if the firms follow the dynamic deception β is

$$v_i(f \circ \beta(\zeta^t)) = (1-\delta) \sum_{\tau > t} \sum_{\zeta^{\tau-1}} \sum_{\theta^\tau} \delta^{\tau-t-1} l(\zeta^{\tau-1}|\zeta^t) l(\theta^\tau) \pi_i(f(\beta(\zeta^{\tau-1}, \theta^\tau)), \theta^\tau).$$

Proof of Proposition 7: We start by defining a pq-regime. After any history, the regime will call one of two mechanisms and this will be done with the help of an auxiliary mapping d.

Regime r. Let $d(\mu^0) = (0, 0)$. For any $t \geq 0$ and μ^t, if $d(\mu^t) = (0, 0)$, then $r(\mu^t) = \hat{g}$; otherwise, $r(\mu^t) = \check{g}$. How d is determined for $t > 0$ and μ^t is given in the description of mechanisms below. ■

Mechanism $\hat{g}$. Suppose the mechanism is called after the message history μ^t. Given $m = ((p_1, q_1), \ldots, (p_n, q_n))$, let $D(m) := \{i \in I | \exists m_i' \text{ s.t. } (m_i', m_{-i}) = m(\theta) \text{ for some } \theta\}$. Because of Assumption A3, $m_i(\theta) = m_j(\theta)$ for all $i, j \in I$ and $|D(m)| \in \{0, 1, n\}$. $|D(m)| = n$ and $m \neq m(\theta)$ for any θ can only occur when $n = 2$. Otherwise, when $n > 2$, $|D(m)| = n$ only if $m = m(\theta)$ for some θ.

The outcome function of the mechanism is as follows:

[16] Firms could also condition their deception on the history of messages that they send and on the history of mechanisms that are selected by the regime. However, because the firms select their messages deterministically and the regime also selects the mechanisms deterministically, it is sufficient to condition the deception only on the history of states.

i. If $m = m(\theta)$ for some $\theta \in \Theta$, then $g(m) = f(\theta)$. Set $d^* = (0, 0)$.

ii. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $D(m) = \{i\}$, then $g(m) = q'$ s.t. $q'_i = q_i$ and $q'_j = \max\{0, (p^{-1}(p_i) - q_i)/(n-1)\}$ for all $j \neq i$ provided that there exists v s.t. $(q', v) \in \partial\Lambda_i^f(\hat{q}_i, q_{-i})$ where $\hat{q}_i = q_j$ for some $j \neq i$. Set $d^* = (i, v_i)$.

If no such v exists, then $g(m) = (\overline{q}_i, 0_{-i})$. Set $d^* = (i, \underline{v}_i(\overline{q}_i))$.

iii. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $|D(m)| = n = 2$, then $g(m) = (\overline{q}_1, \overline{\overline{q}}_2)$. Set $d^* = (1, \underline{v}_1(\overline{q}_1))$.

iv. If $m \neq m(\theta)$ for any $\theta \in \Theta$ and $D(m) = \emptyset$, then $g(m) = (q_i, 0_{-i})$ where i is the winner of the modulo game given q. Set $d^* = (i, \overline{v}_i)$.

Let $\mu^{t+1} = (\mu^t, m)$ and $d(\mu^{t+1}) = d^*$. ■

Mechanism $\check{g}$. Suppose the mechanism is called after the message history μ^t and $d(\mu^t) = (i, v_i)$. Given $m = ((p_1, q_1), \ldots, (p_n, q_n))$, if $\max\{0, \underline{v}_i(\overline{q}_i)\} < v_i \leq \overline{v}_i$, then $g(m) = (q_i, q'_{-i})$ s.t. $\overline{v}_i(\sum_{j \neq i} q'_j) = v_i$. (It is irrelevant how $\sum_{j \neq i} q'_j$ is divided among $j \neq i$.) If $v_i \leq \max\{0, \underline{v}_i(\overline{q}_i)\}$, then $g(m) = (q'_i, 0_{-i})$ s.t. $\underline{v}_i(q'_i) = v_i$.

Let $\mu^{t+1} = (\mu^t, m)$ and $d(\mu^{t+1}) = d(\mu^t)$. ■

Lemma 1 *There exists a subgame perfect equilibrium s s.t. $s(h^t, \theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t | h^0, s) > 0$.*

Proof of Lemma 1: For all t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$, let s be defined as follows:

- If $d(\mu^t) = (0, 0)$, then $s_i(h^t, \theta^t) = m_i(\theta^t)$ for all i.
- If $d(\mu^t) = (i, v_i)$, then $s_i(h^t, \theta^t) = (\cdot, q_i(Q_{-i}, \theta^t))$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$. (The first coordinate in i's message can be any p_i.) $s_j(h^t, \theta^t)$ for $j \neq i$ can be anything.

If the firms follow the specified strategies, then the mechanism $\hat{g}$ is selected for every t, and the outcome is $f(\theta^t)$ for every θ^t, that is, the desired output is implemented in every period. Next, we verify that no firm has incentives to deviate from s. First we consider deviations in the subgames off the path and next we consider deviations in the subgames on the path.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some $i \in I$ and $v_i \leq \overline{v}_i$. Thus, the firms face the mechanism $\check{g}$. Since the firms are also facing exactly the same problem in all future periods irrespective of what their period t messages are, the best that firm i can do is to announce $m_i = (\cdot, q_i(Q_{-i}, \theta^t))$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$. Any messages by other firms are optimal because their messages do not affect the outcome. (If $v_i \leq \max\{0, \underline{v}_i(\overline{q}_i)\}$, then the message of firm i also does not affect the outcome.) Also note that given s, after history h^t, but before period t state is realized, firm i expects exactly the continuation value of v_i.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (0,0)$, in which case the firms face the mechanism $\hat{g}$. Given that other firms follow s, if firm i deviates, the messages will fall under part (ii) of the mechanism or possibly part (iii) when $n = 2$. By construction, any deviation will result in an output-value pair $(q', v) \in \Lambda_i^f(f(\theta^t))$ and, therefore, is not profitable. (From the definition of $\overline{q}_i$, note that $((\overline{q}_i, 0_{-i}), (\underline{v}_i(\overline{q}_i), 0_{-i})) \in \Lambda_i^f(f(\theta))$ for all θ.) We conclude that s is indeed an SPE. It also follows that r satisfies forthrightness w.r.t. f. ■

In the continuation, for any t and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some $i \in I$ and $v_i \leq \overline{v}_i$, it should be understood that firm i behaves as specified in the second bullet point in the proof of Lemma 1. This guarantees that it receives the continuation value of v_i, which is the best it can get.

Lemma 2 *There does not exist a subgame perfect equilibrium s s.t. $r(h^t) = \check{g}$ for some t and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Proof of Lemma 2: If $\check{g}$ is played on the equilibrium path, there must be some $\tau < t$, θ^τ, and $h^\tau = (\mu^\tau, \zeta^{\tau-1})$ s.t. $\rho(h^\tau|h^0, s) > 0$, $d(\mu^\tau) = (0,0)$, and $s(h^\tau, \theta^\tau) \neq m(\theta)$ for any θ. That is, period τ messages fall under parts (ii)-(iv) of mechanism $\hat{g}$. If $s(h^\tau, \theta^\tau)$ falls under part (ii), any firm $j \notin D(s(h^\tau, \theta^\tau))$ expects strictly less than $(1-\delta)\pi_j((q_j(0, \theta^\tau), 0_{-j}), \theta^\tau) + \delta\overline{v}_j(0)$, while it can obtain profits arbitrarily close to these by deviating to a message that triggers part (iv) of $\hat{g}$ and wins the modulo game. (If $d(\mu^\tau, s(h^\tau, \theta^\tau)) = (i, v_i)$ with $v_i \neq 0$, then firm i must be producing a positive output in every period $\tau' > \tau$ and firm $j \neq i$ cannot be a monopolist, while if $v_i = 0$, then from the description of $\check{g}$, it follows that $q_j^{\tau'} = 0$.) Similarly, if $s(h^\tau, \theta^\tau)$ falls under part (iv), each firm has incentives to win the modulo game. If $n = 2$ and $s(h^\tau, \theta^\tau)$ falls under part (iii), there exists θ s.t. $s_2(h^\tau, \theta^\tau) = m_2(\theta) = (p_2, q_2)$. Firm 1 can trigger part (ii) and obtain profits arbitrarily close to $(1-\delta)\pi_1(f(\theta), \theta^\tau) + \delta v_1(f)$ by announcing $m_1 = (p_2 + \epsilon, q_2)$ where ϵ is a small positive number. (If it announced $m_1 = (p_2, q_2)$, then because of Assumption A3, part (i) of $\hat{g}$ would apply.) By definition of $\overline{q}_1$, these profits are strictly higher than $(1-\delta)\pi_1((\overline{q}_1, \overline{\overline{q}}_2), \theta^\tau) + \delta\underline{v}_1(\overline{q}_1)$ that firm 1 obtains if part (iii) of $\hat{g}$ applies. We conclude that s s.t. $r(h^t) = \check{g}$ for some t and h^t s.t. $\rho(h^t|h^0, s) > 0$ cannot be an SPE. ■

Lemma 3 *In any subgame perfect equilibrium s, $s(h^t, \theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Proof of Lemma 3: From Lemma 2, we know that in any SPE s, the mechanism $\hat{g}$ is always selected on the equilibrium path. Therefore, for every t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$, it must be that $s(h^t, \theta^t) = m(\tilde{\theta}^t)$ for some $\tilde{\theta}^t$. Fix some SPE s. Given s, dynamic deception β is defined as follows. For every t and $\zeta^t = (\zeta^{t-1}, \theta^t)$, let $\beta(\zeta^t) = \tilde{\theta}^t$ where $\tilde{\theta}^t$ is s.t. $s(\mu^t, \zeta^t) = m(\tilde{\theta}^t)$ and μ^t is the history of messages that is induced by s and ζ^{t-1}. In fact, as long as no deviation

from s has occurred, we can think that firms' strategies are given by β. Also, given any t, θ^t, and ζ^{t-1}, firm i expects a profit of

$$(1-\delta)\pi_i(f(\beta(\zeta^{t-1},\theta^t)),\theta^t)+\delta v_i(f\circ\beta(\zeta^{t-1},\theta^t))$$

if the firms follow s or, equivalently, β.

We can think that for a given t and ζ^t, $\beta(\zeta^t,\cdot)$ specifies a static deception that the firms use in period $t+1$ and which we denote as α^{ζ^t}. (Thus, the static deception that the firms use in period 0, is denoted $\alpha^{\zeta^{-1}}(\cdot) := \beta(\zeta^{-1},\cdot)$.) Since Θ is assumed to be finite, the number of possible static deceptions is also finite. Among all static deceptions α^{ζ^t} for all $t \geq -1$ and ζ^t, find the one for which $v(f\circ\alpha^{\zeta^t})$ is maximal. (Recall that the firms are symmetric.) Suppose this deception is played in period $\tau+1$ when period τ history of states is ζ^τ and let us denote this deception simply as $\alpha(\cdot)$. Further, suppose that $\alpha(\theta)\notin f^{-1}(f(\theta))$ for some θ. Since f satisfies pq-stationary monotonicity, then for any i (since firms are symmetric) and some θ', there exists $(q,v)\in\partial\Lambda_i^f(f(\alpha(\theta')))$ s.t.

$$\begin{aligned}(1-\delta)\pi_i(q,\theta')+\delta v_i &> (1-\delta)\pi_i(f(\alpha(\theta')),\theta')+\delta v_i(f\circ\alpha)\\ &\geq (1-\delta)\pi_i(f(\beta(\zeta^\tau,\theta'),\theta')+\delta v_i(f\circ\beta(\zeta^\tau,\theta')).\end{aligned}$$

Because firm i only cares about the total output of its competitors, we can assume that q is s.t. $q_j=q_k$ for all $j,k\neq i$. Further, firm i can secure (q,v_i) after the history (ζ^τ,θ') by announcing $(p(\sum_{j\in I}q_j),q_i)$, which triggers part (ii) of $\hat{g}$.[17] Since (q,v_i) gives higher profit to firm i, it wants to deviate from the deception β, which contradicts that s is an SPE.

Thus, among the static deceptions that are selected by β, any deception α that results in the highest continuation value, is s.t. $\alpha(\theta)\in f^{-1}(f(\theta))$ for all θ and, hence, $v(f\circ\alpha)=v(f)$ holds. That is, $v(f)$ is the maximal continuation value that the firms can obtain by deceiving. (Therefore, if α is s.t. $v(f\circ\alpha)=v(f)$, then $\alpha(\theta)\in f^{-1}(f(\theta))$ for all θ.) It also follows that for all $t\geq -1$ and ζ^t, α^{ζ^t} is s.t. $v(f\circ\alpha^{\zeta^t})\leq v(f)$.

Now, if for some $t\geq 0$ and $\zeta^t=(\zeta^{t-1},\theta^t)$, α^{ζ^t} is s.t. $v(f\circ\alpha^{\zeta^t})<v(f)$, then $v(f\circ\beta(\zeta^t))<v(f)$ also holds. We claim that any firm i again has a profitable deviation: it can trigger part (ii) of $\hat{g}$ in period t and obtain profits arbitrarily close to $(1-\delta)\pi_i(f(\beta(\zeta^t)),\theta^t)+\delta v_i(f)$ by announcing $m_i^t=(p_j+\epsilon,q_j)$ for a small ϵ where $(p_j,q_j):=m_j(\beta(\zeta^t))$ for any $j\neq i$. Thus, it is true that for all $t\geq 0$ and ζ^t, $\alpha^{\zeta^t}(\theta)\in f^{-1}(f(\theta))$ for all θ.

The only remaining case is that $\alpha^{\zeta^{-1}}$ is s.t. $v(f\circ\alpha^{\zeta^{-1}})<v(f)$. (Here we cannot invoke the argument of the previous paragraph because a firm would need to deviate before the start of period 0, which we do not allow.) Because

[17] Given q, v_i is the same for all v s.t. $(q,v)\in\partial\Lambda_i^f(f(\alpha(\theta')))$. Thus, by announcing $(p(\sum_{j\in I}q_j),q_i)$, firm i not only specifies q, but also v_i. The exact v_{-i} is irrelevant because the present discounted profit of firm i does not depend on it.

$v(f \circ \alpha^{\zeta^{-1}}) < v(f)$, it must be that $\alpha^{\zeta^{-1}}(\theta) \notin f^{-1}(f(\theta))$ for some θ. Since $(f, v(f))$ satisfies pq-monotonicity, then for any i, there exists $(q, v) \in \partial\Lambda_i^f(f(\alpha^{\zeta^{-1}}(\theta)))$ s.t.[18]

$$\begin{aligned}(1-\delta)\pi_i(q,\theta) + \delta v_i &> (1-\delta)\pi_i(f(\alpha^{\zeta^{-1}}(\theta)),\theta) + \delta v_i(f) \\ &= (1-\delta)\pi_i(f(\beta(\zeta^{-1},\theta),\theta) + \delta v_i(f \circ \beta(\zeta^{-1},\theta)).\end{aligned}$$

Again, we can assume that q is s.t. $q_j = q_k$ for all $j, k \neq i$ because the profit of firm i only depends on $\sum_{j \in I \setminus \{i\}} q_j$. Firm i can secure (q, v_i) by triggering part (ii) of $\hat{g}$ in state θ of period 0 and, therefore, it will want to deviate from the deception β. Thus, $\alpha^{\zeta^{-1}}(\theta) \in f^{-1}(f(\theta))$ must hold for all θ.

We have shown that for s to be an SPE, it must be the case that for all $t \geq 0$ and $\zeta^t = (\zeta^{t-1}, \theta^t)$, $\alpha^{\zeta^{t-1}}(\theta^t) = \beta(\zeta^{t-1}, \theta^t) \in f^{-1}(f(\theta^t))$ holds. Since $m(\theta) = m(\theta')$ for all $\theta, \theta' \in f^{-1}(f(\theta^t))$, it follows that $s(h^t, \theta^t) = m(\beta(\zeta^{t-1}, \theta^t)) = m(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$. Also, the description of $\hat{g}$ implies that $r(h^t)(s(h^t, \theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$. Finally, since s was an arbitrary SPE, the same applies to all SPE and, hence, r implements f in SPE. ■

■

C Proofs of Section 5

Proof of Proposition 8: Since we are in a symmetric environment, we will consider the incentives of firm 1 to deviate from a deception. We start with a couple of observations. First, we know from Proposition 4 that f^c satisfies pq-stationary monotonicity for $\delta = 0$. Therefore, in the following, we assume that $\delta \in (0, 1)$. Second, because of Assumption A4, $0 < f_1^c(\theta) < f_1^c(\theta')$ if $\theta > \theta'$. One implication is that $f^{c-1}(f^c(\theta)) = \{\theta\}$ and $\Lambda_1^{f^c}(f^c(\theta)) = L_1(f^c(\theta), v(f^c), \theta)$ for all $\theta \in \Theta$.

Consider any stationary deception α s.t. $\alpha(\theta') \neq \theta'$ for some $\theta' \in \Theta$. This deception would result in an undesirable output profile in state θ'. To incentivize firm 1 to deviate when the firms deceive according to α, but not to deviate when they behave truthfully, we need to find (q, v_1) s.t.

$$(1-\delta)\pi_1(f^c(\theta),\theta) + \delta v_1(f^c) \geq (1-\delta)\pi_1(q,\theta) + \delta v_1, \tag{6}$$

$$(1-\delta)\pi_1(f^c(\theta),\theta') + \delta v_1(f^c \circ \alpha) < (1-\delta)\pi_1(q,\theta') + \delta v_1, \tag{7}$$

for some θ, θ' s.t. $\alpha(\theta') = \theta \neq \theta'$. Suppose that $v_1(f^c \circ \alpha) \leq v_1(f^c)$ holds. Then, we can choose $v_1 = v_1(f^c)$ and, according to Proposition 4, we can find q that satisfies (6) and (7). Therefore, in the continuation, we assume that $v_1(f^c \circ \alpha) > v_1(f^c)$

[18] It is enough to consider $\alpha^{\zeta^{-1}}$ s.t. $\alpha^{\zeta^{-1}}(\theta') = \theta'$ for all $\theta' \neq \theta$ because the firms do not deceive after period 0 anymore.

holds. It means that there must exist θ, θ' s.t. $\alpha(\theta') = \theta > \theta'$. That is, the firms will increase their profits if they produce less than the competitive output, which requires that they exaggerate their costs. Next, we ask when firm 1 will have incentives to deviate from α in this particular state θ'.

We can equivalently rewrite (6) and (7) as:

$$\begin{aligned}&(1-\delta)[\theta(c(q_1) - c(f_1^c(\theta)) + p(nf_1^c(\theta))f_1^c(\theta) - p(Q)q_1] + \delta v_1(f^c) \geq \delta v_1 \\ &> (1-\delta)[\theta'(c(q_1) - c(f_1^c(\theta)) + p(nf_1^c(\theta))f_1^c(\theta) - p(Q)q_1] + \delta v_1(f^c \circ \alpha). \qquad (8)\end{aligned}$$

(8) tells what values v_1 can take, given q. It can, however, be that any v_1 that satisfies the inequalities in (8), is not feasible because it exceeds the ex ante monopoly profit of firm 1, $\overline{v}_1$. It is easier to satisfy the feasibility constraint if we set $q_{-1} = 0_{-1}$, which we can always do. That is, if (q, v_1) satisfies (8), then so does $((q_1, 0_{-1}), v_1')$ where v_1' is given by $\delta(v_1' - v_1) = (1-\delta)(p(Q) - p(q_1))q_1 \leq 0$. Thus, in the continuation, we assume that $q_{-1} = 0_{-1}$ and $p(Q) = p(q_1)$ in (8).

If we take the right-most and left-most expressions in (8) and simplify, we find that q_1 must satisfy the following inequality:

$$(1-\delta)(\theta - \theta')c(q_1) > (1-\delta)(\theta - \theta')c(f_1^c(\theta)) + \delta(v_1(f^c \circ \alpha) - v_1(f^c)). \qquad (9)$$

From (9), it follows that $q_1 > f_1^c(\theta)$ because $\theta > \theta'$ and $v_1(f^c \circ \alpha) > v_1(f^c)$.

To sum up, if we can select q_1 that satisfies (9) and v_1 that satisfies (8) (with $q_{-1} = 0_{-1}$), then we can incentivize firm 1 to deviate from deception α in state θ'. Now, for any $\delta \in (0, 1)$, we can always find q_1 that satisfies (9). Therefore, we only need to determine when we can find $v_1 \leq \overline{v}_1$ that satisfies (8).

Let q_1^* and v_1^* be the values of q_1 and v_1, respectively, such that both inequalities in (8) are in fact equalities. (Hence, (9) also holds with equality.) Using (9) when $q_1 = q_1^*$ and $v_1 = v_1^*$, we can eliminate δ in the first line of (8) to obtain that

$$v_1^* = v_1(f^c) + \frac{v_1(f^c \circ \alpha) - v_1(f^c)}{\theta - \theta'} \left\{ \theta - \frac{p(q_1^*)q_1^* - p(nf_1^c(\theta))f_1^c(\theta)}{c(q_1^*) - c(f_1^c(\theta))} \right\}. \qquad (10)$$

If $v_1^* < \overline{v}_1$, then we can always find $q_1 > q_1^*$ and $v_1 \leq \overline{v}_1$ that satisfy (8) and (9). Therefore, next we study when $v_1^* < \overline{v}_1$ holds.

Part 1: The term in the curly brackets of (10) can be written as

$$\frac{-(p(q_1^*) - p(nf_1^c(\theta)))q_1^* + \theta[c(q_1^*) - c(f_1^c(\theta)) - c'(f_1^c(\theta))(q_1^* - f_1^c(\theta))]}{c(q_1^*) - c(f_1^c(\theta))}$$

where we have used that $p(nf_1^c(\theta)) = c'(f_1^c(\theta))$. When q_1^* is sufficiently close to $f_1^c(\theta)$, the above expression is negative because the expression in the square brackets is close to zero and $p(q_1^*) > p(nf_1^c(\theta))$. Hence, $v_1^* < \overline{v}_1$ holds. From (9), there is an increasing one-to-one relationship between δ and q_1^*. Therefore, we can conclude that there exists $\underline{\delta} > 0$ such that $v_1^* < \overline{v}_1$ is satisfied for $\delta \in (0, \underline{\delta})$.

Since α and θ' were arbitrary, this completes the proof of the first part of the proposition.
Part 2: For any $\epsilon > 0$, we can pick $\tilde{q}_1$ such that the term in the curly brackets of (10) is less than $\theta + \epsilon$ for all $q_1^* > \tilde{q}_1$. If $\overline{v}_1 > v_1(f^c) + \frac{\max_\theta \theta(v_1(f^m) - v_1(f^c))}{\min_{\theta,\theta'|\theta>\theta'} \theta - \theta'}$ holds, then there exists $\tilde{q}_1$ or, equivalently, $\overline{\delta}$ such that $v_1^* < \overline{v}_1$ is satisfied for all $q_1^* > \tilde{q}_1$ or, equivalently, for all $\delta \in (\overline{\delta}, 1)$. This proves the second part of the proposition.
Part 3: Finally, the term in the curly brackets of (10) can be upper-bounded, for example, with $\theta + \frac{p(nf_1^c(\theta))f_1^c(\theta)}{c(nf_1^c(\theta)) - c(f_1^c(\theta))}$ for all $q_1^* > f_1^c(\theta)$ or, equivalently, all $\delta \in (0, 1)$. One can verify that as long as $f_1^c(\theta) > 0$, $\frac{df_1^c(\theta)}{dn} < 0$, while $\frac{dnf_1^c(\theta)}{dn} \geq 0$. Thus, the term in the curly brackets is decreasing in n. Likewise, $\frac{v_1(f^m)}{dn} < 0$ and $\frac{v_1(f^c)}{dn} \leq 0$; the latter with strict inequality if $v_1(f^c) > 0$. Since $v_1(f^m) \geq v_1(f^c \circ \alpha)$, $v_1(f^c \circ \alpha)$ must also tend to 0 as n increases. We conclude that there exists $\overline{n}$ such that $v_1^* < \overline{v}_1$ is satisfied for all $n > \overline{n}$. This proves the final part of the proposition. ■

D Proofs of Section 6 [Not for publication]

Note that some of the notation that is used in the proofs of this section, is defined in Sections A and B.
Proof of Proposition 9: In the proof, we will often refer to the firms and the buyer as agents. Let $D(m) := \{i \in I_0 | \exists m_i' \text{ s.t. } (m_i', m_{-i}) = m(\theta) \text{ for some } \theta\}$ denote the set of potential deviators when the agents announce $m = ((p_0, q_0), \ldots, (p_n, q_n))$. Note that if $i \in D(m)$, then there exists $p > 0$ s.t. $p_j = p$ for all $j \in I_0 \backslash \{i\}$. Also, $p = p(\sum_{j=1}^n q_j)$ must hold if $i = 0$ and $p = p(q_0)$ must hold if $i \neq 0$. Finally, m_i' in the definition of $D(m)$ is given by $m_i' = (p, \sum_{j=1}^n q_j)$ if $i = 0$ and $m_i' = (p, q_0 - \sum_{j \in I \backslash \{i\}} q_j)$ if $i \neq 0$.

We first prove the claim that if $m \neq m(\theta)$ for any θ, then the set of potential deviators either only consists of the buyer or a subset of firms, but not both together. It means that if we want to punish the potential deviators, we do not need to punish the buyer and firms at the same time.

Claim 1 *If for some m, $0 < |D(m)| < n + 1$, then either $D(m) = \{0\}$ or $D(m) \subseteq I$. If $|D(m)| = n + 1$, then $m = m(\theta)$ for some θ.*

Proof of Claim 1: If $0 \in D(m)$, then there exists $p > 0$ s.t. $p_i = p$ for all $i \in I$ and $p = p(\sum_{i=1}^n q_i)$. If $|D(m)| < n+1$, then it must be that $(p_0, q_0) \neq (p, \sum_{i=1}^n q_i)$. (Or, otherwise, $m = m(\theta)$ for some θ and, hence, $|D(m)| = n + 1$.) In this case, because $n \geq 2$, one cannot find m_i' for any $i \in I$ s.t. $(m_i', m_{-i}) = m(\theta)$ for some θ. Therefore, $D(m) = \{0\}$. This proves the first part of the claim.

If $|D(m)| = n + 1$, then there exists $p > 0$ s.t. $p_i = p$ for all $i \in I_0$ and $p = p(q_0) = p(\sum_{j=1}^n q_j)$ and, hence, $q_0 = \sum_{i=1}^n q_i$. Consider any $i \in I$. Because $i \in D(m)$, there exists m_i' s.t. $(m_i', m_{-i}) = m(\theta)$ for some θ. It must be that $m_i' = (p, q_0 - \sum_{j \in I \backslash \{i\}} q_j) = (p, q_i) = m_i$. Thus, $m = m(\theta)$ for some θ. ■

The pq-regime is similar to the one given in the proof of Proposition 7.

Regime r. Let $d(\mu^0) = (0,0)$. For any $t \geq 0$ and μ^t, if $d(\mu^t) = (0,0)$, then $r(\mu^t) = \dot{g}$; otherwise, $r(\mu^t) = \ddot{g}$. How d is determined for $t > 0$ and μ^t is given in the description of mechanisms below. ■

Mechanism $\dot{g}$. Suppose the mechanism is called after the message history μ^t.

Given $m = ((p_0, q_0), \ldots, (p_n, q_n))$, the outcome function of the mechanism is as follows:

i. If $D(m) = I_0$ and, hence, $m = m(\theta)$ for some $\theta \in \Theta$, then $g(m) = f(\theta)$. Set $d^* = (0,0)$.

ii. If $D(m) = \{0\}$, then $g(m) = (q_1, \ldots, q_n)$. Set $d^* = (0, v_0(f))$.

iii. If $D(m) = \{i\}$ for some $i \in I$, then $g(m) = q'$ s.t. $q'_i = q_i$ and $q'_j = \max\{0, (p^{-1}(p_i) - q_i)/(n-1)\}$ for all $j \in I \backslash \{i\}$, provided that there exists v s.t. $(q', v) \in \partial \Lambda_i^f(\hat{q}_i, q_{-i})$ where $\hat{q}_i = q_0 - \sum_{j \in I \backslash \{i\}} q_j$. Set $d^* = (i, v_i)$.

   If no such v exists, then $g(m) = q'$ s.t. $q'_i = \overline{q}_i$ and $q'_j = 0$ for all $j \in I \backslash \{i\}$. Set $d^* = (i, \underline{v}_i(\overline{q}_i))$.

iv. If $1 < |D(m)| < n+1$, then $g(m) = q'$ s.t. $q'_{i^*} = \overline{q}_{i^*}$ where $i^* := \min\{i | i \in D(m)\}$, $q'_j = \overline{\overline{q}}_j$ for all $j \in D(m) \backslash \{i^*\}$, and $q'_j = 0$ for all $j \in I \backslash D(m)$. Set $d^* = (i^*, \underline{v}_{i^*}(\overline{q}_{i^*}))$.

v. If $D(m) = \emptyset$, then $g(m) = q'$ s.t. $q'_i = q_i$ and $q'_j = 0$ for all $j \in I \backslash \{i\}$ where $i \in I$ is the winner of the modulo game given q. (The buyer does not participate in the modulo game.) Set $d^* = (i, \overline{v}_i)$.

Let $\mu^{t+1} = (\mu^t, m)$ and $d(\mu^{t+1}) = d^*$. ■

Mechanism $\ddot{g}$. Suppose the mechanism is called after the message history μ^t and $d(\mu^t) = (i, v_i)$. Given $m = ((p_0, q_0), \ldots, (p_n, q_n))$,

i. If $i = 0$, then $g(m) = q'$ where $q'_i = Q/n$ for all $i \in I$ and Q is s.t. $-\int_0^Q p'(x) x dx = v_0(f)$.

ii. If $i \neq 0$ and $\max\{0, \underline{v}_i(\overline{q}_i)\} < v_i \leq \overline{v}_i$, then $g(m) = q'$ s.t. $q'_i = q_i$ and q'_j for all $j \in I \backslash \{i\}$ satisfy $\overline{v}_i(\sum_{j \in I \backslash \{i\}} q'_j) = v_i$.

iii. If $i \neq 0$ and $v_i \leq \max\{0, \underline{v}_i(\overline{q}_i)\}$, then $g(m) = q'$ s.t. $q'_j = 0$ for all $j \in I \backslash \{i\}$ and q'_i satisfy $\underline{v}_i(q'_i) = v_i$.

Let $\mu^{t+1} = (\mu^t, m)$ and $d(\mu^{t+1}) = d(\mu^t)$. ■

Lemma 4 *There exists a subgame perfect equilibrium s s.t. $s(h^t, \theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t | h^0, s) > 0$.*

Proof of Lemma 4: For all t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$, let s be defined as follows:

- If $d(\mu^t) = (0, 0)$, then $s_i(h^t, \theta^t) = m_i(\theta^t)$ for all i.
- If $d(\mu^t) = (i, v_i)$ and $i \neq 0$, then $s_i(h^t, \theta^t) = (\cdot, q_i(Q_{-i}, \theta^t))$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$. (The first coordinate in i's message can be anything.) $s_j(h^t, \theta^t)$ for all $j \in I_0 \backslash \{i\}$ can be anything.
- If $d(\mu^t) = (0, v_0(f))$, then $s_j(h^t, \theta^t)$ for all $j \in I_0$ can be anything.

If the agents follow the specified strategies, then the mechanism $\dot{g}$ is selected for every t, and the outcome is $f(\theta^t)$ for every θ^t, that is, the desired output is implemented in every period. Next, we verify that no agent has incentives to deviate from s. First we consider deviations in the subgames off the path and next we consider deviations in the subgames on the path.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some $i \neq 0$ and $v_i \leq \overline{v}_i$. Thus, the agents face the mechanism $\ddot{g}$. Since the agents will also face exactly the same problem in all future periods irrespective of what their period t messages are, the best that firm i can do is to announce $m_i = (\cdot, q_i(Q_{-i}, \theta^t))$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$. Any messages by other agents are optimal because their messages do not affect either current or future outcomes. If $d(\mu^t) = (0, v_0(f))$, no message by any agent affects current or future outcomes and, hence, all messages are optimal.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (0, 0)$, in which case the agents face the mechanism $\dot{g}$. Given that the firms follow s, if the buyer deviates, the messages will fall under part (ii) of the mechanism. In this case, the period t outcome will still be $f(\theta^t)$ and the continuation value of the buyer will still be $v_0(f)$. Thus, it is not profitable for the buyer to deviate. If instead some firm i deviates, the messages will fall under either part (iii) or (iv) of the mechanism. By construction, any deviation will result in an output-value pair $(q', v) \in \Lambda_i^f(f(\theta^t))$ and, therefore, is not profitable. We conclude that s is indeed an SPE. It also follows that r satisfies forthrightness w.r.t. f. ■

In the continuation, for any t and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some $i \in I$ and $v_i \leq \overline{v}_i$, it should be understood that firm i behaves as specified in the second bullet point in the proof of Lemma 4. This guarantees that it receives the continuation value of v_i, which is the best it can get.

Lemma 5 *There does not exist a subgame perfect equilibrium s s.t. $r(h^t) = \ddot{g}$ for some t and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Proof of Lemma 5: If $\ddot{g}$ is played on the equilibrium path, there must be some $\tau < t$, θ^τ, and $h^\tau = (\mu^\tau, \zeta^{\tau-1})$ s.t. $\rho(h^\tau|h^0, s) > 0$, $d(\mu^\tau) = (0, 0)$, and $s(h^\tau, \theta^\tau) \neq m(\theta)$ for any θ. That is, period τ messages fall under parts (ii)-(v) of mechanism $\dot{g}$. If $s(h^\tau, \theta^\tau)$ falls under part (ii), there exists firm $j \in I$ that expects

strictly less than its discounted monopoly profits $(1-\delta)\pi_j((q_j(0,\theta^\tau),0_{-j}),\theta^\tau) + \delta\overline{v}_j(0)$, while it can obtain profits arbitrarily close to these by deviating to a message that triggers part (v) and wins the modulo game. Similarly, if $s(h^\tau,\theta^\tau)$ falls under part (iii), then any firm $j \notin D(s(h^\tau,\theta^\tau))$ expects strictly less than $(1-\delta)\pi_j((q_j(0,\theta^\tau),0_{-j}),\theta^\tau) + \delta\overline{v}_j(0)$. (To see that it is also true for $n=2$, note that for firm j to earn monopoly profit in any period $\tau' > \tau$, it must be that firm i does not produce anything in that period, which means that $v_i = 0$ must hold. But then, by part (iii) of $\ddot{g}$, firm j does not produce anything either.) Again, it can obtain profits arbitrarily close to these by deviating to a message that triggers part (v) and wins the modulo game. Suppose $s(h^\tau,\theta^\tau)$ falls under part (iv). Then, by the definition of $D(s(h^\tau,\theta^\tau))$, for every firm $i \in D(s(h^\tau,\theta^\tau))$, there exists $m_i' = (p_i', q_i')$ s.t. $(m_i', s_{-i}(h^\tau,\theta^\tau)) = m(\theta)$ for some θ. By announcing $m_i'' = (p_i' + \epsilon, q_i')$, firm i can trigger part (iii) and secure profits arbitrarily close to $(1-\delta)\pi_i(f(\theta),\theta^\tau) + \delta\overline{v}_i(f)$, which is strictly more than what the firm expects under part (iv). Finally, if $s(h^\tau,\theta^\tau)$ falls under part (v), each firm has incentives to win the modulo game. We conclude that s s.t. $r(h^t) = \ddot{g}$ for some t and h^t s.t. $\rho(h^t|h^0,s) > 0$ cannot be an SPE. ■

Lemma 6 *In any subgame perfect equilibrium s, $s(h^t,\theta^t) = m(\theta^t)$ and $r(h^t)(m(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0,s) > 0$.*

Proof of Lemma 6: From Lemma 5, we know that in any SPE s, the mechanism $\dot{g}$ is always selected on the equilibrium path. Therefore, for every t, θ^t, and h^t s.t. $\rho(h^t|h^0,s) > 0$, it must be that $s(h^t,\theta^t) = m(\tilde{\theta}^t)$ for some $\tilde{\theta}^t$. Fix some SPE s. Given s, β is defined as follows. For every t and $\zeta^t = (\zeta^{t-1},\theta^t)$, let $\beta(\zeta^t) = \tilde{\theta}^t$ where $\tilde{\theta}^t$ is s.t. $s(\mu^t,\zeta^t) = m(\tilde{\theta}^t)$ and μ^t is the history of messages that is induced by s and ζ^{t-1}. Given any t and $\zeta^t = (\zeta^{t-1},\theta^t)$, agent i expects a payoff of $(1-\delta)\pi_i(f(\beta(\zeta^t)),\theta^t) + \delta v_i(f\circ\beta(\zeta^t))$ if the agents follow s.

Suppose that $v(f\circ\beta(\zeta^t)) \neq v(f)$ for some $t \geq 0$ and $\zeta^t = (\zeta^{t-1},\theta^t)$. Because f is efficient in the range, there is an agent $i \in I_0$ s.t. $v_i(f\circ\beta(\zeta^t)) < v_i(f)$. We claim that agent i has a profitable deviation. If this agent is the buyer, that is, $i=0$, he can trigger part (ii) of $\dot{g}$ by announcing any $m_0^t = (p_0,q_0)$ s.t. $p_0 \neq p(q_0)$. This will secure him a strictly higher utility of $(1-\delta)\pi_0(f(\beta(\zeta^t)),\theta^t) + \delta v_0(f)$. If this agent is a firm, it can trigger part (iii) of $\dot{g}$ by announcing $m_i^t = (p_i+\epsilon, q_i)$ for a small ϵ where $(p_i,q_i) = m_i(\beta(\zeta^t))$. This gives firm i profits arbitrarily close to $(1-\delta)\pi_i(f(\beta(\zeta^t)),\theta^t) + \delta v_i(f)$. Thus, it must be that $v(f\circ\beta(\zeta^t)) = v(f)$ for all $t \geq 0$ and $\zeta^t = (\zeta^{t-1},\theta^t)$.

Suppose that $\beta(\zeta^t) \notin f^{-1}(f(\theta^t))$ and, hence, $f(\beta(\zeta^t)) \neq f(\theta^t)$ for some $t \geq 0$ and $\zeta^t = (\zeta^{t-1},\theta^t)$. Since $(f,v(f))$ satisfies pq-monotonicity, then there exist a firm i and an output-value pair $(q,v) \in \partial\Lambda_i^f(f(\beta(\zeta^t)))$ s.t.

$$(1-\delta)\pi_i(q,\theta^t) + \delta v_i > (1-\delta)\pi_i(f(\beta(\zeta^t)),\theta^t) + \delta v_i(f).$$

We can assume that q is s.t. $q_j = q_k$ for all $j,k \in I\backslash\{i\}$ because the profit of firm i only depends on $\sum_{j\in I\backslash\{i\}} q_j$. Firm i can secure (q,v_i) by triggering part

(iii) of $\dot{g}$ and, therefore, it will want to deviate from the deception β in period t after the history of states $\zeta^t = (\zeta^{t-1}, \theta^t)$. Thus, $\beta(\zeta^t) \in f^{-1}(f(\theta^t))$ must hold for all $t \geq 0$ and $\zeta^t = (\zeta^{t-1}, \theta^t)$. Since $m(\theta) = m(\theta')$ for all $\theta, \theta' \in f^{-1}(f(\theta^t))$, it follows that $s(h^t, \theta^t) = m(\beta(\zeta^{t-1}, \theta^t)) = m(\theta^t)$ for all t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $\rho(h^t|h^0, s) > 0$. Also, the description of $\dot{g}$ implies that $r(h^t)(s(h^t, \theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$. Finally, since s was an arbitrary SPE, the same applies to all SPE and, hence, r implements f in SPE. ■

■

Proof of Proposition 10: Suppose f is repeatedly implementable in SPE with a pq-regime that satisfies forthrightness w.r.t. f. Fix an SPE $\hat{s}$ s.t. $\hat{s}(h^t, \theta^t) = (m(\theta^t), e(\theta^t))$ and $r(h^t)(m(\theta^t), e(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$ and e is some evidence function. Also, suppose f does not satisfy evidence monotonicity. That is, for every evidence function e', there exists $\alpha^{e'} \in \mathcal{A}$ s.t. for all $i \in I$ and $\theta \in \Theta$, $e'_i(\alpha^{e'}(\theta)) \in E_i(\theta)$ and $E_i(\theta) \subseteq E_i(\alpha^{e'}(\theta))$. In particular, this is also true for e that is implied by $\hat{s}$.

Given $\hat{s}$ and α^e, we define another strategy profile $\tilde{s}$. Let $\tilde{s}$ be defined exactly as in the proof of Proposition 5 if α^e violates pq-stationary monotonicity of f. Otherwise, let $\tilde{s}$ be defined exactly as in the proof of Proposition 6. Note that for any t, $\tilde{s}(h^t, \theta^t) = (m(\alpha^e(\theta^t)), e(\alpha^e(\theta^t)))$ if $\rho(h^t|h^0, \tilde{s}) > 0$. Because it is assumed that $e_i(\alpha^e(\theta)) \in E_i(\theta)$ for all i and θ, evidence $e(\alpha^e(\theta^t))$ is indeed feasible.

Since $\alpha^e \in \mathcal{A}$, there exists θ such that $f(\alpha^e(\theta)) \neq f(\theta)$. Therefore, the constructed strategy profile $\tilde{s}$ selects an undesirable outcome on its path. Since we assume that the regime implements f, $\tilde{s}$ cannot be an SPE. Because the future profits do not depend on the history of states, by construction, $\tilde{s}$ implies NE play in the subgames that follow after a deviation from the path of $\tilde{s}$ has occurred. Therefore, if there exists a profitable deviation, it must be on the path. Thus, suppose there exist t, θ^t, $h^t = (\mu^t, \zeta^{t-1})$ s.t. $\rho(h^t|h^0, s) > 0$, i, and (m_i, e_i) such that

$$(q, v) = (r(\mu^t)((m_i, e_i), \tilde{s}_{-i}(h^t, \theta^t)), v(\tilde{s}|h^{t+1})) \notin L_i(f(\alpha^e(\theta^t)), v(f \circ \alpha^e), \theta^t),$$

where $h^{t+1} = ((\mu^t, ((m_i, e_i), \tilde{s}_{-i}(h^t, \theta^t))), (\zeta^{t-1}, \theta^t))$. Because $\alpha^e \in \mathcal{A}$ (and because of Assumption A5), $(q, v) \notin L_i(f(\alpha^e(\theta^t)), v(f), \alpha^e(\theta^t))$.

Note that because $E_i(\theta) \subseteq E_i(\alpha^e(\theta))$ holds for all i and θ, it is feasible for firm i to provide evidence e_i if period t state is $\alpha^e(\theta^t)$. Also, from the definition of $\tilde{s}$, there exists $\tilde{h}^t = (\mu^t, \tilde{\zeta}^{t-1})$ s.t. $\rho(\tilde{h}^t|h^0, s) > 0$ and

$$(q, v) = (r(\mu^t)((m_i, e_i), \hat{s}_{-i}(\tilde{h}^t, \alpha^e(\theta^t)), v(\hat{s}|\tilde{h}^{t+1})),$$

where $\tilde{h}^{t+1} = ((\mu^t, ((m_i, e_i), \hat{s}_{-i}(\tilde{h}^t, \alpha^e(\theta^t)))), (\tilde{\zeta}^{t-1}, \alpha^e(\theta^t)))$. That is, firm i can also secure outcome (q, v) when the firms follow strategies $\hat{s}$. And because $(q, v) \notin L_i(f(\alpha^e(\theta^t)), v(f), \alpha^e(\theta^t))$, $\hat{s}$ is not an SPE, which is a contradiction. It follows that if f is repeatedly implementable in SPE with a pq-regime that satisfies

forthrightness w.r.t. f, then it cannot be the case that for every evidence function e, there exists $\alpha^e \in \mathcal{A}$ s.t. for all $i \in I$ and $\theta \in \Theta$, $e_i(\alpha^e(\theta)) \in E_i(\theta)$ and $E_i(\theta) \subseteq E_i(\alpha^e(\theta))$. That is, f must satisfy evidence monotonicity. ■

Proof of Proposition 11: Let $e(\cdot)$ denote the evidence function given in Definition 11. We still use the pq-regime given in the proof of Proposition 7, but we modify the mechanism $\hat{g}$. (In the case of mechanism $\check{g}$, any evidence submitted by the firms is simply ignored.)

Mechanism $\hat{g}$. Suppose the mechanism is called after the message history μ^t and the firms send messages $m = ((p_1, q_1), \ldots, (p_n, q_n))$ and evidence e.

The outcome function of the mechanism is as follows:

i. If $(m, e) = (m(\theta), e(\theta))$ for some $\theta \in \Theta$, then $g(m) = f(\theta)$. Set $d^* = (0, 0)$.

ii. If $m = m(\theta)$ for some $\theta \in \Theta$, but $e \neq e(\theta)$, then $g(m) = (\overline{q}_1, \overline{\overline{q}}_2, \ldots, \overline{\overline{q}}_n)$. Set $d^* = (1, \underline{v}_1(\overline{q}_1))$.

iii. If there exist i and θ s.t. $m_j = m_j(\theta)$ for all $j \neq i$, and $m_i \neq m_i(\theta)$ if $n \geq 3$ or $m_i \neq m_i(\theta')$ for any θ' if $n = 2$, then

   (a) If $e_i \notin E_i(\theta)$, then $g(m, e) = (q_i, 0_{-i})$. Set $d^* = (i, \overline{v}_i)$.

   (b) If $e_i \in E_i(\theta)$, then $g(m, e) = q'$ s.t. $q'_i = q_i$ and $q'_j = \max\{0, (p^{-1}(p_i) - q_i)/(n-1)\}$ for all $j \neq i$ provided that there exists v s.t. $(q', v) \in \partial\Lambda_i^f(m(\theta))$. Set $d^* = (i, v_i)$.

   If no such v exists, then $g(m, e) = (\overline{q}_i, 0_{-i})$. Set $d^* = (i, \underline{v}_i(\overline{q}_i))$.

iv. If $n = 2$ and there exist θ and θ' s.t. $\theta \neq \theta'$, $m_i = m(\theta)$, and $m_j = m(\theta')$, then $g(m) = (\overline{q}_1, \overline{\overline{q}}_2)$. Set $d^* = (1, \underline{v}_1(\overline{q}_1))$.

v. For all other (m, e), $g(m, e) = (q_i, 0_{-i})$ where i is the winner of the modulo game given q. Set $d^* = (i, \overline{v}_i)$.

Let $\mu^{t+1} = (\mu^t, m)$ and $d(\mu^{t+1}) = d^*$. ■

Lemma 7 *There exists a subgame perfect equilibrium s s.t. $s(h^t, \theta^t) = (m(\theta^t), e(\theta^t))$ and $r(h^t)(m(\theta^t), e(\theta^t)) = f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Proof of Lemma 7: For all t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$, let s be defined as follows:

- If $d(\mu^t) = (0, 0)$, then $s_i(h^t, \theta^t) = (m(\theta^t), e(\theta^t))$ for all i.

- If $d(\mu^t) = (i, v_i)$, then $s_i(h^t, \theta^t) = ((\cdot, q_i(Q_{-i}, \theta^t)), \cdot)$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$ and the first and last coordinates can be anything (feasible). $s_j(h^t, \theta^t)$ for $j \neq i$ can be anything.

If the firms follow the specified strategies, then the mechanism $\hat{g}$ is selected for every t, and the outcome is $f(\theta^t)$ for every θ^t, that is, the desired output is implemented in every period. Next, we verify that no firm has incentives to deviate from s. First we consider deviations in the subgames off the path and next we consider deviations in the subgames on the path.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some i and $v_i \leq \overline{v}_i$. Thus, the firms face the mechanism $\check{g}$. Since the firms are also facing exactly the same problem in all future periods irrespective of what their messages and evidence in period t are, the best that firm i can do is to announce $m_i = (\cdot, q_i(Q_{-i}, \theta^t))$ where Q_{-i} is s.t. $\overline{v}_i(Q_{-i}) = \max\{0, v_i\}$ and supply any feasible evidence. Any messages and evidence by the other firms are optimal because they do not affect either current or future outcomes.

Consider any t, θ^t, and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (0, 0)$, in which case the firms face the mechanism $\hat{g}$. Given that other firms follow s, if firm i deviates, the messages will fall either under part (ii), (iii.b) or possibly (iv) of $\hat{g}$. By construction, any deviation will result in an output-value pair $(q', v) \in \Lambda_i^f(f(\theta^t))$ and, therefore, is not profitable. We conclude that s is indeed an SPE. It also follows that r satisfies forthrightness w.r.t. f. ■

In the continuation, for any t and $h^t = (\mu^t, \zeta^{t-1})$ s.t. $d(\mu^t) = (i, v_i)$ for some $i \in I$ and $v_i \leq \overline{v}_i$, it should be understood that firm i behaves as specified in the second bullet point in the proof of Lemma 7. This guarantees that it receives the continuation value of v_i, which is the best it can get.

Lemma 8 *There does not exist a subgame perfect equilibrium s s.t. $r(h^t) = \check{g}$ for some t and h^t s.t. $\rho(h^t|h^0, s) > 0$.*

Proof of Lemma 8: If $\check{g}$ is played on the equilibrium path, there must be some $\tau < t$, θ^τ, and $h^\tau = (\mu^\tau, \zeta^{\tau-1})$ s.t. $\rho(h^\tau|h^0, s) > 0$, $d(\mu^\tau) = (0, 0)$, and $s(h^\tau, \theta^\tau) \neq (m(\theta), e(\theta))$ for any θ. That is, period τ messages fall under parts (ii)-(v) of mechanism $\hat{g}$. If $s(h^\tau, \theta^\tau)$ falls under part (iii), any firm $j \neq i$ expects strictly less than $(1-\delta)\pi_j((q_j(0, \theta^\tau), 0_{-j}), \theta^\tau) + \delta\overline{v}_j(0)$, while it can obtain profits arbitrarily close to these by deviating to a message that triggers part (v) of $\hat{g}$ and wins the modulo game.[19] Similarly, if $s(h^\tau, \theta^\tau)$ falls under part (v), each firm has incentives to win the modulo game. If $s(h^\tau, \theta^\tau) = (m(\theta), \cdot)$ falls under part (ii), then firm 1 can profitably deviate: either firm 1 can trigger part (iii.a), which is clearly profitable, or firm 1 can trigger part (iii.b) and obtain profits arbitrarily close to $(1-\delta)\pi_1(f(\theta), \theta^\tau) + \delta\overline{v}_1(f)$ by announcing $m_1 = (p_1 + \epsilon, q_1)$ where $(p_1, q_1) = m_1(\theta)$ and ϵ is a small positive number. These profits are strictly higher than $(1-\delta)\pi_1((\overline{q}_1, \overline{\overline{q}}_2, \ldots, \overline{\overline{q}}_n), \theta^\tau) + \delta\underline{v}_1(\overline{q}_1)$ that firm 1 obtains if part (ii) of $\hat{g}$ applies. The argument is similar if $n = 2$ and $s(h^\tau, \theta^\tau)$ falls under part (iv), in which case there exists θ s.t. $s_2(h^\tau, \theta^\tau) = ((p_2, q_2), e_2)$ and $(p_2, q_2) = m_2(\theta)$. If

[19] The argument why this is true when $n = 2$ and part (iii.b) applies, is the same as in the proof of Lemma 2.

firm 1 can trigger part (iii.a), then it is clearly profitable to do so. If firm 1 can trigger part (iii.b), it can obtain profits arbitrarily close to $(1-\delta)\pi_1(f(\theta),\theta^\tau)+\delta\overline{v}_1(f)$ by announcing $m_1=(p_2+\epsilon,q_2)$ where ϵ is a small positive number. We conclude that s s.t. $r(h^t)=\check{g}$ for some t and h^t s.t. $\rho(h^t|h^0,s)>0$ cannot be an SPE. ■

Lemma 9 *In any subgame perfect equilibrium s, $s(h^t,\theta^t)=(m(\theta^t),e(\theta^t))$ and $r(h^t)(m(\theta^t),e(\theta^t))=f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0,s)>0$.*

Proof of Lemma 9: From Lemma 8, we know that in any SPE s, the mechanism $\hat{g}$ is always selected on the equilibrium path. Therefore, for every t, θ^t, and h^t s.t. $\rho(h^t|h^0,s)>0$, it must be that $s(h^t,\theta^t)=(m(\tilde{\theta}^t),e(\tilde{\theta}^t))$ for some $\tilde{\theta}^t$. Fix some SPE s. Given s, let a dynamic deception β be defined as follows. For every t and $\zeta^t=(\zeta^{t-1},\theta^t)$, let $\beta(\zeta^t)=\tilde{\theta}^t$ where $\tilde{\theta}^t$ is s.t. $s(\mu^t,\zeta^t)=(m(\tilde{\theta}^t),e(\tilde{\theta}^t))$ and μ^t is the history of messages that is induced by s and ζ^{t-1}.

As in the proof of Lemma 3, given β, we can define static deceptions α^{ζ^t} for all $t\geq -1$ and ζ^t. Let α denote a static deception that gives the highest continuation value: $v(f\circ\alpha)\geq v(f\circ\alpha^{\zeta^t})$ for all $t\geq -1$ and ζ^t. Suppose that $\alpha(\theta)\neq\theta$ or, because of Assumption A5, $f(\alpha(\theta))\neq f(\theta)$ for some θ. If $\alpha\notin\mathcal{A}$, then as in the proof of Lemma 3, we can find a profitable deviation for any firm from β. If $\alpha\in\mathcal{A}$, then because f satisfies evidence monotonicity, there exist i and θ' s.t. either $e_i(\alpha(\theta'))\notin E_i(\theta')$ or $E_i(\theta')\not\subseteq E_i(\alpha(\theta'))$ holds. In the former case, β and, consequently, s are, in fact, not feasible because the firms cannot submit evidence $e_i(\alpha(\theta'))$ in state θ', and we arrive at a contradiction. In the latter case, firm i can profitably deviate by triggering part (iii.a) of mechanism $\hat{g}$ in the period when the firms deceive according to α and the state is θ'.

We conclude that if s is an SPE, then $\alpha(\theta)=\theta$ for all θ. Consequently, $v(f)\geq v(f\circ\alpha^{\zeta^t})$ for all $t\geq -1$ and ζ^t. Suppose that for some $t\geq -1$, ζ^t, and θ, $\alpha^{\zeta^t}(\theta)\neq\theta$. Hence, it must be that $v(f)>v(f\circ\alpha^{\zeta^t})$. As in the proof of Lemma 3, we can rule this case out for all $t\geq 0$. It remains to consider the case when $\alpha^{\zeta^{-1}}(\theta)\neq\theta$ for some θ. If $\alpha^{\zeta^{-1}}\notin\mathcal{A}$, then as in the proof of Lemma 3, we can find a profitable deviation for any firm from β. If $\alpha^{\zeta^{-1}}\in\mathcal{A}$, then because f satisfies evidence monotonicity, there exists firm i and θ' s.t. either $e_i(\alpha^{\zeta^{-1}}(\theta'))\notin E_i(\theta')$ or $E_i(\theta')\not\subseteq E_i(\alpha^{\zeta^{-1}}(\theta'))$ holds. In the former case, β and, consequently, s are, in fact, not feasible and we arrive at a contradiction. In the latter case, firm i can profitably deviate by triggering part (iii.a) of mechanism $\hat{g}$ in period 0 when the state is θ'.

We have shown that for s to be an SPE, it must be the case that for all $t\geq 0$ and $\zeta^t=(\zeta^{t-1},\theta^t)$, $\alpha^{\zeta^{t-1}}(\theta^t)=\beta(\zeta^{t-1},\theta^t)=\theta^t$ holds. It follows that $s(h^t,\theta^t)=(m(\beta(\zeta^{t-1},\theta^t)),e(\beta(\zeta^{t-1},\theta^t)))=(m(\theta^t),e(\theta^t))$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0,s)>0$. Also, the description of $\hat{g}$ implies that $r(h^t)(s(h^t,\theta^t))=f(\theta^t)$ for all t, θ^t, and h^t s.t. $\rho(h^t|h^0,s)>0$. Finally, since s was an arbitrary SPE, the same applies to all SPE and, hence, r implements f in SPE. ■

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