

Glauben, Thomas; Grecksch, Wilfried; Burggraf, Christine

Conference Paper — Published Version

Stochastic control of individual's health investments

Suggested Citation: Glauben, Thomas; Grecksch, Wilfried; Burggraf, Christine (2015) : Stochastic control of individual's health investments, In: de León, Manuel Feng, Wei Feng, Zhaosheng Lopez Gomez, Julian Lu, Xin Martell, J.M. Parcet, Javier Peralta-Salas, Daniel Ruan, Weihua (Ed.): Dynamical Systems and Differential Equations, AIMS Proceedings 2015, 10th AIMS International Conference (Madrid, Spain), American Institute of Mathematical Sciences, Springfield, MO, pp. 159-168, <https://doi.org/10.3934/proc.2015.0159>

This Version is available at:

<https://hdl.handle.net/10419/249588>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by-nc-nd/4.0/>

STOCHASTIC CONTROL OF INDIVIDUAL'S HEALTH INVESTMENTS

CHRISTINE BURGGRAF

Leibniz Institute of Agricultural Development in Transition Economies
Theodor-Lieser-Street 2
Halle (Saale), 06120, Germany

WILFRIED GRECKSCH

Martin-Luther-University Halle-Wittenberg, Institute of Mathematics
Theodor-Lieser-Street 5
Halle (Saale), 06120, Germany

THOMAS GLAUBEN

Leibniz Institute of Agricultural Development in Transition Economies
Theodor-Lieser-Street 2
Halle (Saale), 06120, Germany

ABSTRACT. Grossman's health investment model has been one of the most important developments in health economics. However, the model's derived demand function for medical care predicts the demand for medical care to increase if the individual's health status increases. Yet, empirical studies indicate the opposite relationship. Therefore, this study improves the informative value of the health investment model by introducing a reworked Grossman model, which assumes a more realistic Cobb-Douglas health investment function with decreasing returns to scale. Because we introduced uncertainty surrounding individual's health status the resulting dynamic utility maximization problem is tackled by optimal stochastic control theory.

1. Introduction. Since 1972, Michael Grossman's health investment model [1, 2] has been one of the most important developments in the theory of the demand for medical care. In Grossman's approach, direct outlays on medical services and opportunity costs of the time invested in health are inputs to produce investments in a better health status. Despite its importance, Grossman's model is not without criticism [3]. Indeed, it is often argued that the model fails to account for the uncertainty of one's future health status and the uncertainty of the health investment efficiency [4]. Beyond that, a central criticism lies with the implications of the model's demand function for medical care. According to this demand function, the demand for medical care increases with an increasing health status. Yet empirical studies that have tested the implications of Grossman's model (see e.g., [5, 6]) indicate that people tend to demand more medical services if their health decreases, thus implying a negative relationship between health and health care. Even if a variety of different econometric methodologies and datasets is employed, this inconsistency between Grossman's theoretical implications and the empirical results seems to persist [9].

From this severe criticism, Zweifel [3] deduces only limited relevance of Grossman's health investment model to the practical work of health economists. Based on the current discussion of Zweifel [3], Kaestner [7], and Laporte [10], we aim to underscore the practical relevance of Grossman's health investment model. Especially promising is reworking the health investment production function [7, 8], which is usually assumed to be of constant returns to

2010 *Mathematics Subject Classification.* Primary: 93E20; Secondary: 92C50.

Key words and phrases. Health investment model, stochastic control, Bismut's approach, decreasing returns to scale, demand for medical care.

scale. Thereby, in contrast to other studies (e.g.; [9, 10]) we keep Grossman's standard model assumptions but simply rework the functional specification of the health investment production function. Further, we introduce uncertainty surrounding the health status in order to come closer to a real-world individual health investment problem.

2. Grossman's standard deterministic model assumptions. In this section we briefly present the standard problem setting of Grossman's deterministic health investment model to introduce the notation contained in this paper. The basic assumption of Grossman's model is that utility is generated by the amount of healthy time $h(t)$, as well as the consumption of household commodities $Z(t)$. For analytical convenience, an individual's lifetime utility function $U(T)$ is specified as being separable over time, together with the assumption of additive separability of preferences. Thus, under conditions of certainty, the individual's utility function can be expressed as

$$U(T) = \int_0^T U[Z(t), h(t)] e^{-\rho t} dt$$

where $h(t) = \phi(H(t))$ and $\frac{\partial U}{\partial Z} > 0$, $\frac{\partial^2 U}{\partial Z^2} < 0$, $\frac{\partial U}{\partial h} > 0$, $\frac{\partial^2 U}{\partial h^2} < 0$. The functions $U(T)$ and $h(t)$ are assumed to be increasing, strictly concave, and continuously differentiable in their arguments. The parameter ρ denotes the subjective discount rate. This utility function is maximized subject to the restrictions set by health and wealth time paths, as well as the production technology for the production of household commodities and health investments. These restrictions are detailed below.

The stock of health capital $H(t)$ depreciates on a progressive depreciation rate and can be revalued upwards by investments in health capital [2]. Therefore, the equation of motion in health is expressed as follows

$$\dot{H}(t) = I(t) - \delta(t)H(t)$$

with $\delta(t) > 0$, $\dot{\delta}(t) > 0 \forall t \in [0, T]$, and the deterministic initial condition $H(0) = H_0$. Moreover, $H(t) > H_{\min} \forall t \neq T$. The variable $I(t)$ indicates investments in health capital and $\delta(t)$ is the depreciation rate of health capital. Investments in health are produced by medical services $M(t)$ and time invested in health $m(t)$ subject to one's level of knowledge $E(t)$. Under the condition of non-joint production functions, i.e. market goods and time inputs can be additively split between separate production processes for health investments and household commodities, the general form of the health investment function can be expressed as

$$I(t) = f_I(M(t), m(t); E(t))$$

with $I(t) \geq 0 \forall t \in [0, T]$. The consumer produces non-medical household commodities $Z(t)$ that are produced by non-medical market goods $Q(t)$ and consumption time $k(t)$ subject to one's level of knowledge $E(t)$. These non-medical household commodities are produced by

$$Z(t) = f_Z(Q(t), k(t); E(t)).$$

Over one's entire lifetime, expenditures on medical care and other market goods are restricted by the period's initial wealth plus the periodical wealth surplus. Wealth $A(t)$ develops over the whole lifetime according to the following equation of motion

$$\dot{A}(t) = w(t)l(t) + r(t)A(t) + y(t) - p_Q(t)Q(t) - p_M(t)M(t) \quad (1)$$

with $A(0) = A_0$, $A_0 > 0$, and $A(T) \geq 0$. The labor time $l(t)$ is valued by the wage rate $w(t)$, market inputs $Q(t)$ are valued by market prices p_Q , and medical services $M(t)$ are valued with the market prices for medical care p_M . Further, the consumer receives other income $y(t)$ and interest revenues given the interest rate $r(t)$. Besides the constraint on wealth, an individual's time is also constrained. This time constraint is given by

$$\Omega(t) = l(t) + m(t) + k(t) + s(t) \quad (2)$$

with $\Omega(t) - s(t) = h(t)$. Hence, total time $\Omega(t)$ available in each period t has to be fully divided into labor time $l(t)$, time invested in gross health investments $m(t)$, consumption time $k(t)$, and sick time $s(t)$.

3. Minimum short-run costs of producing health investments. Since the customer's utility function is assumed to be inter-temporally separable, the customer solves two separate optimization problems: an inter-temporal utility maximization problem and an intra-temporal cost minimization problem. Beginning with the intra-temporal cost minimization problem, the customer determines those input bundles that minimize the short run costs of attaining each unit of health investment $I(t)$ and household commodities $Z(t)$ subject to the constraints imposed by the respective production functions and the given level of knowledge.

Regarding the functional form of the health investment production function, there is still little empirical evidence to support the selection process [9, 14]. However, productions in the human capital-dependent field of health investments provide sufficient justification for decreasing returns to scale because of reasonably less automation and rationalization. Despite this fact, the majority of theoretical studies that utilize Grossman's pure investment model to derive a demand function for health care apply a health investment production function with constant returns to scale (e.g., [2]). However, inspired by the basic theoretical remarks of Ehrlich and Chuma [7], in this analysis health investments are assumed to be subjected to decreasing returns to scale.

Proposition 3.1. *Minimizing a linear short-term cost function of health investments subject to a Cobb-Douglas health investment production function with decreasing returns to scale yields a dual health investment cost function with increasing marginal costs.*

Proof of Proposition 3.1. Let the short-term cost function $C_I(t) = M(t)p_M(t) + m(t)w(t)$ with $C_I(t) \geq 0 \forall t \in [0, T]$ be minimized subject to a Cobb-Douglas health investment production function $I(t) = E(t)(M(t))^k(m(t))^\mu$ with some $k + \mu < 1$ and $\alpha = \frac{1}{k+\mu}$. The constants k and μ are the output elasticities of market inputs and time for health investments, respectively. The parameter α stands for the inverse scale elasticity. Since $\theta I(t) > E(t)(\theta M(t))^k(\theta m(t))^\mu$ with a scaling factor $\theta > 0$, our health investment production function is of decreasing returns to scale. Then by Lagrange multiplier method there exists a local extreme with the resulting cost-minimizing factor inputs of the forms

$$\begin{aligned} M^*(t) &= \left(\frac{k}{\mu}\right)^{\frac{\mu}{k+\mu}} p_M^{-\frac{\mu}{k+\mu}} w(t)^{\frac{\mu}{k+\mu}} E(t)^{-\frac{1}{k+\mu}} I(t)^{\frac{1}{k+\mu}}, \\ m^*(t) &= \left(\frac{k}{\mu}\right)^{-\frac{k}{k+\mu}} p_M^{\frac{k}{k+\mu}} w(t)^{-\frac{k}{k+\mu}} E(t)^{-\frac{1}{k+\mu}} I(t)^{\frac{1}{k+\mu}}, \end{aligned} \quad (3)$$

and the dual cost function for health investments of the form

$$\begin{aligned} C_I^*(t) &= \pi_H(t)I(t)^\alpha, \\ \pi_H(t) &= \left[\left(\frac{k}{\mu}\right)^{\frac{\mu}{k+\mu}} + \left(\frac{k}{\mu}\right)^{-\frac{k}{k+\mu}} \right] p_M^{\frac{k}{k+\mu}} w(t)^{\frac{\mu}{k+\mu}} E(t)^{-\frac{1}{k+\mu}}. \end{aligned} \quad (4)$$

Due to $\frac{\partial C_I^*(t)}{\partial I(t)} > 0$ and $\frac{\partial^2 C_I^*(t)}{\partial I(t)^2} > 0$, we have positive increasing marginal costs of health investments. \square

Remark 1. For simplicity and because we are not interested in the household commodity production, we assume the household commodity production to be of constant returns to scale.

Proposition 3.2. *Minimizing a linear short-term cost function of commodities subject to a Cobb-Douglas commodity production function with constant returns to scale yields a dual commodity cost function with constant marginal costs.*

Proof of Proposition 3.2. Let the short-term cost function $C_Z(t) = Q(t)p_Q(t) + k(t)w(t)$ with $C_Z(t) \geq 0 \forall t \in [0, T]$ be minimized subject to a Cobb-Douglas commodity production function $Z(t) = E(t)(Q(t))^\zeta ((k(t))^\vartheta)$ with some $\zeta + \vartheta = 1$. The constants ζ and ϑ are the output elasticities of market inputs and time for commodity production, respectively. Since $\theta Z(t) = E(t)(\theta Q(t))^\zeta (\theta k(t))^\vartheta$ with a scaling factor $\theta > 0$, our commodity production function is of constant returns to scale. Then by Lagrange multiplier method there exists a local extreme with the dual cost function for commodity production of the form

$$\begin{aligned} C_Z^*(t) &= \pi_z(t)Z(t), \\ \pi_z(t) &= \left[\left(\frac{\zeta}{\vartheta} \right)^\vartheta + \left(\frac{\zeta}{\vartheta} \right)^{-\zeta} \right] p_Q^\zeta w(t)^\vartheta E(t)^{-1}. \end{aligned} \quad (5)$$

Due to $\frac{\partial C_Z^*(t)}{\partial Z(t)} > 0$ and $\frac{\partial^2 C_Z^*(t)}{\partial Z(t)^2} = 0$, we have positive constant marginal costs of commodity production. \square

Integrating the time constraint (2), the dual cost function of health investments (4), and the dual cost function of household commodities (5) into the wealth constraint (1), the full-wealth constraint of Grossman's problem setting can be written as

$$\dot{A}(t) = r(t)A(t) + w(t)h(t) + y(t) - \pi_H(t)I(t)^\alpha - \pi_z(t)Z(t).$$

4. Optimal stochastic control of health investments. Now, the inter-temporal utility maximization problem can be solved. In this regard, the individual chooses the trajectories of health investments and household commodities that maximize the present value of utility subject to the restrictions imposed by the model. However, individual's health is generally subject to sudden health shocks, which have an increasing likelihood of occurrence with age. This means that health develops with some kind of uncertainty.¹ Therefore, we model health capital as a linear generalized Brownian motion with drift (or Ito stochastic differential equation). Hence, in this model illness affects health capital in the sense of an extraordinary depreciation of the health stock. We now consider the probability space $(\Theta, \mathcal{F}, \mathcal{P})$, where Θ is the non-empty space of health outcomes, \mathcal{F} denotes the σ -algebra (or σ -field) of subsets of Θ , and \mathcal{P} is the probability measure defined on \mathcal{F} , i.e. $\mathcal{P}: \mathcal{F} \rightarrow [0, 1]$, which fulfills the axioms of Kolmogorov [15]. Health capital $H(t)$ develops by a stochastic process that shows a stochastic differential equation in the sense of Ito. The stochastic noise process W is a Wiener process caused by random shocks to the health capital. A Wiener process $\{W(t), t \in [0, T]\}$ is a continuous-time dependent stochastic process on the probability space $(\Theta, \mathcal{F}, \mathcal{P})$ with following properties²

- (i) $W(0) = 0$,
- (ii) for $0 \leq t_1 \leq \dots \leq t_n$, the increments $W(t_i) - W(t_{i-1})$ with $i = 1, \dots, n$ are independent random variables,
- (iii) for $0 \leq s < t$, the increment $W(t) - W(s)$ has a normal distribution $N(0, t - s)$,
- (iv) W is continuous with respect to the time $t \geq 0$, and
- (v) the path $W(t)$ for $t \geq 0$ is nowhere differentiable.

It follows from (iii) that the variance of $W(t) - W(s)$ increases linearly with the length of the time interval $[s, t]$. Further, with the Wiener process $\{W(t), t \geq 0\}$ defined on probability space $(\Theta, \mathcal{F}, \mathcal{P})$, the random variable $\{W(s), 0 \leq s \leq t\}$ produces the σ -algebra \mathcal{F}_t , where $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$. \mathcal{F}_t contains all past realizations of the Wiener process. Hence,

¹The authors do not consider the terminology differences between risk and uncertainty.

²For more information on these properties, see for example Malliaris and Brock [16].

it is assumed that the consumer knows all the available past information generated by the Wiener process. As time goes on, consumer information increases because the consumer observes additional realizations of the random variable. In the stochastic case, health capital is assumed to develop as a Brownian motion with drift given by

$$dH(t) = [I(t) - \delta(t)H(t)] dt + \sigma(t, H(t), I(t))dW(t)$$

and wealth develops over time according to

$$dA(t) = [rA(t) + y(t) + wh(t) - \pi_H(t)I(t)^\alpha - \pi_Z(t)Z(t)] dt.$$

These stochastic differential equations are defined by the corresponding integral equations

$$\begin{aligned} H(t) &= H(0) + \int_0^t [I(\tau) - \delta(\tau)H(\tau)] d\tau + \int_0^t \sigma(\tau, H(\tau), I(\tau)) dW(\tau), \\ A(t) &= A(0) + \int_0^t [rA(\tau) + y(\tau) + wh(\tau) - \pi_H(\tau)I(\tau)^\alpha - \pi_Z(\tau)Z(\tau)] d\tau, \end{aligned}$$

for all t with a probability of 1, where the admissible controls are adapted processes so that the above integrals are defined. The behavior of the continuous time stochastic process $H(t)$ is characterized by the sum of a Lebesgue integral and an Ito integral. Under the assumption that health capital develops in an Ito stochastic process, the expected value and variance of the health increment consecutive to any decision $I(t)$ are known. The expected value of $H(t)$ is given by

$$\mathbb{E}[H(t)] = \mathbb{E}[H(0)] + \mathbb{E}\left[\int_0^t [I(\tau) - \delta(\tau)H(\tau)] d\tau\right]$$

since $\mathbb{E}[dW] = 0$. The variance of $H(t)$ is given by $\text{Var}[H] = \sigma^2 dt$. Hence, the probability of downward shocks in the sense of unexpected illness and upward shocks in the sense of unexpected recovery from different kinds of illness increases with age t . For this application, $\mathbb{E}[I(t) - \delta(t)H(t)]$ is the expected instantaneous drift rate of the Ito process, and $\sigma(t, H(t), I(t))$ is the instantaneous diffusion rate. Here, the linear structure of the deterministic drift rate is mirrored by a linear diffusion rate of the form $\sigma(t, H(t), I(t)) = \beta + \sigma_H H(t) + \sigma_I I(t)$.

Theorem 4.1. (*Bismut's approach*) Suppose that $I^*(t), Z^*(t), H(t)$, and $A(t)$ solve

$$\max_{I, Z} \mathbb{E} \int_0^T U[Z(t), h(t)] e^{-\rho t} dt \quad (6)$$

subject to

$$dH(t) = [I(t) - \delta(t)H(t)] dt + \sigma(t, H(t), I(t))dW(t), \quad (7)$$

$$dA(t) = [rA(t) + y(t) + wh(t) - \pi_H(t)I(t)^\alpha - \pi_Z(t)Z(t)] dt, \quad (8)$$

$$I(t) = f_I(M(t), m(t), E(t)), \quad (9)$$

$$Z(t) = f_Z(Q(t), k(t), E(t)), \quad (10)$$

$$H(0) = H_0, \quad T \text{ fix}, \quad (11)$$

$$A(0) > 0, A(T) \geq 0, \quad (12)$$

$$I \in [0, \infty]. \quad (13)$$

Then the resulting Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= U[Z(t), h(t)] e^{-\rho t} \\ &+ \varphi_A(t) [rA(t) + y(t) + wh(t) - \pi_H I(t)^\alpha - \pi_Z(t)Z(t)] \\ &+ \varphi_H(t) [I(t) - \delta(t)H(t)] + B(t)\sigma(t, H(t), I(t)). \end{aligned} \quad (14)$$

Given $\varphi_{HH}(t) = \frac{\partial \varphi_{HH}(t)}{\partial H(t)}$, the following relations hold for optimal values of $I(t)$ and $Z(t)$:

$$dH(t) = [I(t) - \delta(t)H(t)] dt + \sigma(t, H(t), I(t))dW(t), \quad (15)$$

$$dA(t) = [rA(t) + y(t) + wh(t) - \pi_H(t)I(t)^\alpha - \pi_Z(t)Z(t)] dt, \quad (16)$$

$$\frac{\partial \mathcal{H}}{\partial Z(t)} = \frac{\partial U}{\partial Z} e^{-\rho t} - \varphi_A(t)\pi_Z(t) = 0, \quad (17)$$

$$\frac{\partial \mathcal{H}}{\partial I(t)} = -\varphi_A(t)\alpha\pi_H I(t)^{\alpha-1} + \varphi_H(t) + B(t)\sigma_I = 0, \quad (18)$$

$$d\varphi_A(t) = [-\varphi_A(t)r] dt, \quad (19)$$

$$d\varphi_H(t) = \left[-e^{-\rho t} \frac{\partial U}{\partial h} \frac{\partial h}{\partial H} - \varphi_A(t)w \frac{\partial h}{\partial H} + \varphi_H(t)\delta(t) - B(t)\sigma_H \right] dt + B(t)dW(t) \quad (20)$$

with the transversality conditions $\varphi_H(H(T), T) = 0$, $\varphi_{HH}(H(T), T) = 0$, $\varphi_A(A(T), T) = 0$ with $\frac{\partial \mathcal{H}}{\partial T} < 0$.

Remark 2. All price data is assumed to be constant.

Bismut's approach [17] is based on Pontryagin's Maximum Principle [13, 21]. The adjoint variables $\varphi_H(t)$ and $\varphi_A(t)$ are in the nature of Lagrange multipliers of the states $H(t)$ and $A(t)$, respectively. As such, they measure the shadow prices of the associated state variables at a particular point in time. In the stochastic case the marginal value of health capital at time t is given by $\varphi_H(t) = \frac{\partial \varphi_H(t)}{\partial H(t)} \mathbb{E} \left\{ \int_t^T U[Z(\tau), h(\tau)] e^{-\rho\tau} d\tau | \mathcal{F}_t \right\}$, which is the partial derivative of the conditional expectation of the utility function from time t to T with respect to $H(t)$ and with $I(t)$ being the optimal policy. Bismut's random variable $B(t)$ corresponds to $\frac{\partial \varphi_H(t)}{\partial H(t)} \sigma(t, H(t), I(t))$ and provides one's instantaneous attitude towards risk. This variable is positive if the individual is risk taking and negative if the individual is risk averse. From equation (7) it follows $B(t)dW(t) = \varphi_{HH} [dH - (I(t) - \delta(t)H(t)) dt]$. Therefore, in line with the general definitions of Malliaris and Brock [16], $B(t)dW(t)$ is a correction term in the evolution of the marginal value of health capital, which evaluates in terms of $\varphi_H(t)$ the difference between dH and $\mathbb{E}[dH]$, where $\mathbb{E}[dH] = [I(t) - \delta(t)H(t)] dt$.

From (18) and $\eta(t) = \frac{\varphi_H(t)}{\varphi_A(t)}$, the flow equilibrium condition for health investments can be derived

$$\eta(t) = \alpha\pi_H I(t)^{\alpha-1} + \frac{1}{\varphi_A(0)} B(t)\sigma_I, \quad (21)$$

with optimal health investment in period t

$$I^*(t) = \left(\frac{\eta(t)}{\alpha\pi_H} - \frac{B(t)\sigma_I}{\alpha\pi_H \varphi_A(0)} \right)^{\frac{1}{\alpha-1}}.$$

Suppose that $\eta(t)$ is defined as the relative shadow price of health capital. Then, if $-B(t) > 0$, i.e. assuming that the consumer is risk-averse, condition (21) implies that the consumer will invest in his health up to the point where the expected relative shadow price of health capital equals the marginal investment costs in health, minus the marginal risk of health investment valued at its costs. Because the risk-averse consumer fears health capital losses, he will tend to invest more in his health with the same $\eta(t)$ as he would, if no risk is involved. This result complies with the results of the static model setting of Dardanoni and Wagstaff [11] and the simplified retirement version of Picone *et al.* [12]. In Figure 1, the optimal health investment in the stochastic case is achieved at $I^*(t)^s$, which is higher than the optimal investment in the deterministic case $I^*(t)^d$ at the same $\eta(t)$, i.e. $I^*(t)^s > I^*(t)^d$, which provides a cushion against the effects of a health shock.

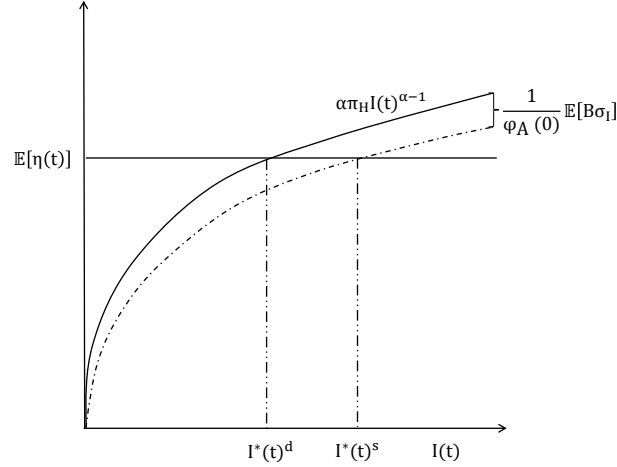


FIGURE 1. Optimal investment of a risk-averse consumer

From $\eta = \frac{\varphi_H(t)}{\varphi_A(t)}$ it follows that

$$d\varphi_H(t) = d\eta(t)\varphi_A(t) + \eta(t)\dot{\varphi}_A(t)dt, \quad (22)$$

$$\mathbb{E}[d\varphi_H(t)] = \mathbb{E}[d\eta(t)\varphi_A(t) + \eta(t)\dot{\varphi}_A(t)dt]. \quad (23)$$

Substituting (19),(22)-(23) in (20) and taking the expected value, a continuous stock equilibrium condition for $H(t)$ can be derived as follows:

$$\begin{aligned} \mathbb{E}[\eta(t)] \left[\delta(t) + r - \frac{\mathbb{E}[d\eta(t)]}{\mathbb{E}[\eta(t)]dt} \right] &= \frac{1}{\varphi_A(0)} e^{(r-\rho)t} \mathbb{E} \left[\frac{\partial U}{\partial h} \frac{\partial h}{\partial H} \right] + w \mathbb{E} \left[\frac{\partial h}{\partial H} \right] \\ &+ \frac{1}{\varphi_A(0)} e^{rt} \mathbb{E}[B(t)\sigma_H]. \end{aligned} \quad (24)$$

According to equation (24) the stock of health capital is optimal in each t if the expected marginal cost of health capital is equal to the expected marginal efficiency of health capital minus the marginal risk of health, which is discounted and normalized with the shadow price of tangible assets at $t = 0$. Thereby, the expected marginal cost of holding an additional unit of health capital in period t considers interest earnings forgone by holding an additional unit of health capital, the health capital depreciation costs from holding an additional unit of health capital, and the expected offsetting capital gain from buying the investment good at time t instead of waiting until time $t + dt$. The expected marginal efficiency of health capital consists of two parts: the expected additional labor income from an infinitesimal increase of health capital and the expected direct marginal utility of health capital, discounted and normalized with the shadow price of wealth at $t = 0$.

5. Sufficient condition for a global maximum.

Theorem 5.1. *The necessary conditions (15)-(20) are sufficient for the existence of the global maximum.*

Proof. The Hamiltonian (14) is maximized with respect to the admissible controls for any t from $[0, T]$ and regarding any trajectory of the introduced stochastic process, i.e. for every t the given trajectories evolve into a deterministic optimization problem with probability 1.

With $\frac{\partial^2 \mathcal{H}}{\partial I^2} < 0$, $\frac{\partial^2 \mathcal{H}}{\partial Z^2} < 0$ and using Arrow's theorem, the conditions of the maximum principle are also sufficient for the global maximum. To check the Arrow sufficiency condition, some definitions are needed. Let the maximized Hamiltonian function \mathcal{H}^0 be the value of the Hamiltonian when evaluated at the maximizing controls

$$\begin{aligned} \mathcal{H}^0(t, H, A, \varphi_A, \varphi_H) &= F(t, H, A, I^*, Z^*) \\ &+ \varphi_A f_A(t, H, A, I^*, Z^*) + \varphi_H f_H(t, H, A, I^*, Z^*). \end{aligned}$$

If $\mathcal{H}^0(t, H, A, \varphi_A, \varphi_H)$ is a concave function of H and A , then I^* , Z^* , H^* , and A^* will maximize (6) subject to (7)-(13). Since $\frac{\partial C_I^*(t)}{\partial I(t)} > 0$ and $\frac{\partial^2 C_I^*(t)}{\partial I(t)^2} > 0$, it follows that the dual cost function of health investments $C_I^*(t) = \pi_H I(t)^\alpha$ with $\alpha > 1$ is a monotonic increasing function of health investments. Therefore, we are able to derive $I(t)^* = \left[\frac{1}{\alpha \pi_H} \frac{\varphi_H(t)}{\varphi_A(t)} \right]^{\frac{1}{\alpha-1}}$. Thus, since marginal utilities $\varphi_H(t) \geq 0$, $\varphi_A(t) \geq 0$, the production parameter $\alpha > 0$, and $\pi_H > 0$, it follows that $I^*(t) \geq 0$. Because of equation (17) $\frac{\partial U}{\partial Z} e^{-\rho t} - \varphi_A(t) \pi_Z(t) = 0$ and the assumption that $\frac{\partial U}{\partial Z} > 0$, $\frac{\partial^2 U}{\partial Z^2} < 0$ an inverse function of $U_Z(t)$ exists of the form $Z^*(t) = U_Z(t)^{\prime-1}(e^{\rho t}, \varphi_A(t), \pi_Z(t))$. Hence, the maximized deterministic Hamiltonian function according to Arrow's Theorem can be written as

$$\begin{aligned} \mathcal{H}^0 &= U \left[U_Z(t)^{\prime-1}(e^{\rho t}, \varphi_A(t), \pi_Z(t)), h(t) \right] e^{-\rho t} \\ &+ \varphi_A(t) \left[rA(t) + y(t) + wh(t) - \pi_H \left(\left[\frac{1}{\alpha \pi_H} \frac{\varphi_H(t)}{\varphi_A(t)} \right]^{\frac{1}{\alpha-1}} \right)^\alpha \right. \\ &\quad \left. - \pi_Z U_Z(t)^{\prime-1}(e^{\rho t}, \varphi_A(t), \pi_Z(t)) \right] + \varphi_H \left[\left[\frac{1}{\alpha \pi_H} \frac{\varphi_H(t)}{\varphi_A(t)} \right]^{\frac{1}{\alpha-1}} - \delta(t)H(t) \right], \end{aligned}$$

with $|D| = \begin{bmatrix} \mathcal{H}_{AA}^0 & \mathcal{H}_{AH}^0 \\ \mathcal{H}_{HA}^0 & \mathcal{H}_{HH}^0 \end{bmatrix}$ and $|D^0| = \begin{bmatrix} \mathcal{H}_{HH}^0 & \mathcal{H}_{HA}^0 \\ \mathcal{H}_{AH}^0 & \mathcal{H}_{AA}^0 \end{bmatrix}$ whose principal minors are $|D_1| = 0$, $|D_2| = 0$, $|D_1^0| < 0$ and $|D_2^0| = 0$. If $|D_1|$ and $|D_1^0|$ are referred to $|\check{D}_1|$ whereas $|D_2|$ and $|D_2^0|$ are referred to $|\check{D}_2|$, then the test for semi-definiteness is as follows: $|\check{D}_1| \leq 0$ and $|\check{D}_2| = 0$. In conclusion, the quadratic form involved is negative semi-definite, meaning that the maximized Hamiltonian \mathcal{H} is concave for every t and with probability 1. Hence, $I^*(t)$, $Z^*(t)$, $H^*(t)$ and $A^*(t)$ maximize (6) subject to (7)-(13). \square

6. Optimal demand for medical care. Given the optimality conditions (21) and (24), the structural demand function for medical care can be derived to constitute the model's theoretical predictions. However, since the majority of influencing effects on the demand for medical care remain ambiguous in sign [18], it is preferable to deal with submodels. Further, estimating sub-models avoids using non-linear estimation methods [1, 5]. Grossman himself stresses the pure investment model, i.e. the marginal utility of healthy days $\frac{\partial U}{\partial h}$ is set to zero [2]. Applying the pure investment model, the equilibrium condition for health capital (24) can be reduced to the logarithmic form

$$\log \mathbb{E}[\eta_i(t)] + \log \delta_i(t) - \log \mathbb{E}[\Psi_{1i}(t)] = \log w_i(t) + \log \mathbb{E} \left[\frac{\partial h_i(t)}{\partial H_i(t)} \right] + \log \mathbb{E}[\Psi_2(t)], \quad (25)$$

with $\Psi_1(t) = \frac{\delta(t)}{\delta + r - \frac{\mathbb{E}[\dot{\eta}(t)]}{\mathbb{E}[\eta(t)]dt}}$ and $\Psi_2(t) = 1 + \frac{\frac{1}{\varphi_A(0)} e^{\rho t} \mathbb{E}[B\sigma_H]}{w \mathbb{E} \left[\frac{\partial h(t)}{\partial H(t)} \right]}$. The subscript i denotes reference to the i -th individual. The constructed variable $\Psi_1(t)$ indicates the share of the depreciation rate in the adjustment factor of the marginal health capital costs. The constructed variable $\Psi_2(t)$ is employed as a health risk indicator on expected wage income. According

to Grossman [2] and Wagstaff [5] we assume the following specifications

$$h_i(t) = \Omega - \beta_1 H_i(t)^{-\beta_2}, \quad (26)$$

$$\log \delta_i(t) = \log \delta_0 + \beta_3 t_i + \beta_4 X_{1i}, \quad (27)$$

$$\log \Psi_{1i}(t) = \beta_9 t_i, \quad (28)$$

$$E(t) = e^{\beta_6 \check{E}}, \quad (29)$$

with $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$, $\beta_6 > 0$, $\beta_9 > 0$ and $h_i(H_{\min}(T)) = 0$. Variable $X(t)$ indicates some individual's characteristic with $\beta_4 > 0$ if this characteristic is health damaging. Further, we assume the health risk indicator to be of the form

$$\Psi_2(t) = \beta_7 \check{B}_i(t) \sigma_H + \beta_8 t, \quad (30)$$

with $\beta_7 > 0$, $\beta_8 > 0$ and assuming some $\check{B}_i(t) < 0$ indicating a risk-averse consumer.

Given the functional specifications (4) and (26)-(30) and the stock equilibrium condition (24), the structural demand equations of the pure investment model can be derived. Hence, the optimal expected demand for health investments is given by

$$\begin{aligned} \log \mathbb{E}[I_i^*(t)] &= \beta_{01} - \frac{\beta_5 \alpha}{\alpha - 1} \log p_M + \frac{\beta_5 \alpha}{\alpha - 1} \log w_i(t) + \frac{\beta_6 \alpha}{\alpha - 1} \check{E}_i + \frac{\beta_9 + \beta_8 - \beta_3}{\alpha - 1} t_i \\ &\quad - \frac{\beta_4}{\alpha - 1} X_{1t} - \frac{1 + \beta_2}{\alpha - 1} \log \mathbb{E}[H_i(t)] - \beta_7 \log \mathbb{E}[\check{B}_i(t) \sigma_H] + u_{1i}, \end{aligned} \quad (31)$$

with the constant $\beta_{01} = \left(-\log \left[\left(\frac{k}{\mu} \right)^{\frac{\mu}{k+\mu}} + \left(\frac{k}{\mu} \right)^{-\frac{k}{k+\mu}} \right] + \log \beta_1 \beta_2 - \log \alpha \right) / (\alpha - 1)$, coefficient $\beta_5 = k$ and error term $u_{1i} = -\frac{1}{\alpha-1} \log \delta_0$. Substituting (31) in (3) yields the structural demand function for medical care with

$$\begin{aligned} \log \mathbb{E}[M_i^*(t)] &= \beta_{02} - \left(1 + \frac{\beta_5 \alpha}{\alpha - 1} \right) \log p_M(t) + \left(1 + \frac{\beta_5 \alpha}{\alpha - 1} \right) \log w_i(t) \\ &\quad + \frac{\beta_6 \alpha}{\alpha - 1} \check{E}_i(t) + (\beta_9 + \beta_8 - \beta_3) \frac{\alpha}{\alpha - 1} t_i - \frac{\beta_4 \alpha}{\alpha - 1} X_{1i} \\ &\quad - (1 + \beta_2) \frac{\alpha}{\alpha - 1} \log \mathbb{E}[H_i(t)] - \beta_7 \alpha \log \mathbb{E}[\check{B}_i(t) \sigma_H] + u_{2i}, \end{aligned} \quad (32)$$

with the constant $\beta_{02} = (1 - \beta_5 \alpha) \ln \left(\frac{\beta_5}{1 - \beta_5} \right) + \alpha \beta_{01}$ and the error term $u_{2i} = \alpha u_{1i}$.

According to the parameter settings for α and β 's, this inquiry provides a demand function for medical care in which the expected demand for medical care in t increases with an increasing wage rate and an increasing educational level, but it decreases with increasing prices of medical services. These results are in line with common results of general household production models and demand theory. Further, $-(1 + \beta_2) \frac{\alpha}{\alpha - 1} < 0$ and $-\beta_7 \alpha < 0$ imply that the expected demand for medical care in t increases with an expected worsening health status in t and an expected higher health associated risk for that t , respectively. These theoretically predicted relationships are substantially confirmed by empirical evidence.

7. Conclusions. The hitherto criticized empirical inconsistency of the standard health investment model is no longer valid if the health investment production function is specified to be of decreasing returns to scale. Hence, our stochastic modification offers reasonable predictions, provided that the functional forms are properly specified. For future research, it is suitable to assume spill-over effects on other person's survival as done by Kuhn *et al.* [19] for a simplified deterministic individual life-cycle model based on Yaari [20].

REFERENCES

- [1] M. Grossman, *The demand for health: a theoretical and empirical investigation*, National Bureau of Economic Research, Cambridge, 1972.
- [2] M. Grossman, The human capital model, in *Handbook of health economics* (eds. A. J. Culyer and J. P. Newhouse) Elsevier Science, Amsterdam, (2000), 347408.
- [3] P. Zweifel, The Grossman model after 40 years, *Eur. J. Health Econ.*, **13** (2012), 677–682.
- [4] U. G. Gerdtham, M. Johannesson, L. Lundberg and D. G. L. Isacson, The demand for health: results from new measures of health capital, *Eur. J. Health Econ.*, **15** (1999), 501–521.
- [5] A. Wagstaff, The demand for health: Some new empirical evidence, *J. Health Econ.*, **53** (1986), 195–233.
- [6] M. Erbsland, W. Ried and V. Ulrich, Health, health care, and the environment: Econometric evidence from German micro data, *Health Econ.*, **4** (1995), 169–182.
- [7] I. Ehrlich and H. Chuma, A Model of the Demand for Longevity and the Value of Life Extension, *J. Pol. Econ.*, **98** (1990), 761–782.
- [8] R. Kaestner, The Grossman model after 40 years: A reply to Peter Zweifel, *Eur. J. Health Econ.*, **14** (2013), 357–360.
- [9] T. J. Galama, P. Hulleger, E. Meijer and S. Outcault, Is there empirical evidence for decreasing returns to scale in a health capital model?, *Health Econ.*, **21** (2012), 1080–1100.
- [10] A. Laporte, *Should the Grossman model retain its iconic status in health economics?*, CCHE/CCES Working paper, 2014.
- [11] V. Dardanoni and A. Wagstaff, Uncertainty, inequalities in health and the demand for health, *J. Health Econ.*, **6** (1987), 283–290.
- [12] G. Picone, M. Uribe, R. M. Wilson, The effect of uncertainty on the demand for medical care, health capital and wealth, *J. Health Econ.*, **17** (1998), 171–185.
- [13] L. S. Pontryagin, W. Boltjanski, R. V. Gamkrelidze and E. F. Mishchenko, *Mathematische Theorie optimaler Prozesse, (German) [Mathematical Theory of Optimal Processes]*, 2nd edition, Oldenburg-Verlag, Muenchen, 1967.
- [14] A. A. Leibowitz, The demand for health and health concerns after 30 years, *J. Health Econ.*, **23** (2004), 663–671.
- [15] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung, (German) [Foundations of the theory of probability]*, 2nd edition, Oldenburg-Verlag, Muenchen, 1967.
- [16] A. G. Malliaris and W. A. Brock, *Stochastic methods in economics and finance*, Elsevier/North-Holland, New York, 1982.
- [17] J. M. Bismut, Conjugate convex functions in optimal stochastic control, *J. Math. Anal. Appl.*, **44** (1973), 384–404.
- [18] W. Ried, Comparative dynamic analysis of the full Grossman model, *J. Health Econ.*, **17** (1998), 383–425.
- [19] M. Kuhn, S. Wrzaczek, A. Prskawetz, G. Feichtinger, Externalities in a Life-Cycle Model with Endogenous Survival, *J. Math. Econ.*, **47** (2011), 627–641.
- [20] M. E. Yaari, Uncertain lifetime, life insurance, and the theory of the consumer, *Rev. Econ. Stud.*, **32** (1965), 137–150.
- [21] W. H. Fleming, R. W. Rishel, *Deterministic and stochastic optimal control*, Springer, New York, 1975.

Received September 2014; revised August 2015.

E-mail address: burgggraf@iameo.de

E-mail address: wilfried.grecksch@mathematik.uni-halle.de

E-mail address: glauben@iameo.de