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Forecasting the Variability of Stock Index Returns with the Multifractal Random Walk Model for Realized Volatilities

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Forecasting the Variability of Stock Index Returns with the Multifractal Random Walk Model for Realized Volatilities

by Cristina Sattarhoff and Thomas Lux

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Forecasting the Variability of Stock Index Returns with the Multifractal Random Walk Model for Realized Volatilities

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October 28, 2021

Abstract

We adapt the multifractal random walk model by Bacry et al. (2001) to realized volatilities (denoted RV-MRW) and take stock of recent theoretical insights on this model in Duchon et al. (2012) to derive forecasts of financial volatility. Moreover, we propose a new extension of the binomial Markov-switching multifractal (BMSM) model by Calvet and Fisher (2001) to the RV framework. We compare the predictive ability of the two against seven classical and multifractal volatility models. Forecasting performance is evaluated out-of-sample based on the empirical MSE and MAE as well as using model confidence sets following the methodology of Hansen et al. (2011). Overall, our empirical study for 14 international stock market indices has a clear message: The RV-MRW is *throughout* the best model for all forecast horizons under the MAE criterium as well as for large forecast horizons $h = 50$ and 100 days under the MSE criterion. Moreover, the RV-MRW provides most accurate 20-day ahead forecasts in terms of MSE for the great majority of indices, followed by RV-ARFIMA, the latter dominating the competition at the 5-day-horizon. These results are very promising if we consider that this is the first empirical application of the RV-MRW. Moreover, whereas RV-ARFIMA forecasts are often a time consuming task, the RV-MRW stands out due to its fast execution and straightforward implementation. The new RV-BMSM appears to be specialized in short term forecasting, the model providing most accurate one-day ahead forecasts in terms of MSE for the same number of cases as RV-ARFIMA.

JEL Classification: C20, G12

Keywords: Realized volatility, multiplicative volatility models, multifractal random walk, long memory, international volatility forecasting.

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1 Introduction

There exists a large body of literature demonstrating that volatility models based on realized volatilities (RV) produce far more accurate volatility forecasts than models using asset returns only.[1] This paper confirms this message based on an extensive empirical study covering fourteen international stock market indices and a collection of nine competing models, including four models based on RV.

We also consider the two representative models of the relatively new class of multifractal volatility models: the Markov-switching multifractal model by Calvet and Fisher (2001) and the multifractal random walk model by Bacry et al. (2001), this being the first study to compare the two. The multifractal (or multi-scaling) property is a nonlinear behavior of returns over different time horizons (minute, daily, monthly returns, etc.) which manifests itself in different degrees of long-term dependence of different absolute power of returns (i.e., different measures of volatility), a salient empirical regularity that has first been discovered by Ding et al. (1993). It has been argued that this ubiquitous feature of financial data arises from the interplay of investors with heterogenous investment horizons (cf. Ghashghaie et al., 1996). The models that have appeared under the heading 'multifractal' in recent literature have all generating principles that (unlike the classical GARCH-type and stochastic volatility models) intrinsically capture this new stylized fact.

Notably, we also consider the little-known multifractal random walk model which we adapt here to realized volatility, short: RV-MRW, the present paper also being the first empirical application of the approach by Duchon et al. (2012) for the derivation of volatility forecasts for the MRW model. As it turns out, based on the evaluation of model confidence sets calculated from mean absolute errors (MAE) of forecasts, RV-MRW proves to be the best model throughout, i.e., for all fourteen indices used in our study and for all forecast horizons between 1 day and 100 days. Second, our results using mean squared forecast errors (MSE) show that the RV-MRW clearly also provides the most accurate forecasts under this criterion for the overwhelming majority of all indices under scrutiny at large horizons ≥ 20 days, the model also outperforming the more traditional RV-ARFIMA (autoregressive fractionally integrated moving average specification for RV).

Following the example of Lux et al. (2014) for the application of the lognormal Markov-switching multifractal model to realised volatility (RV-LMSM), we also extend the binomial

[1] Just to give a few examples, Koopman et al. (2005) evaluate one-step ahead forecasts from GARCH (generalised autoregressive conditional heteroscedasticity), SV (stochastic volatility), RV-UC (unobserved components model for RV) and RV-ARFIMA (autoregressive fractionally integrated moving average model for RV). They report best forecast performance for the RV-ARFIMA, closely followed by RV-UC. Koopman and Scharth (2013) propose an original SV model which they fit using both RV and returns data. The authors also consider two relatively new models: the heterogeneous autoregressive (HAR) model of Corsi (2009) and the high-frequency-based volatility (HEAVY) model of Shephard and Sheppard (2010). The RV-SV model delivers most accurate forecasting results, with higher performance gains for the longer horizon. Lux et al. (2014) distinguish between the turbulent period of mid 2007 to 2009 and the rather tranquil period from mid 2005 to mid 2007. Whereas their multifractal specification for the RV (RV-LMSM) proves superior forecasting performance during the crisis period over a wide variety of alternative models, overall, RV-ARFIMA clearly dominates the forecasting competition.

Markov-switching multifractal (BMSM) model by Calvet and Fisher (2001) to the RV framework. This new model proves to have superior forecasting capabilities in the short term, competing with the RV-ARFIMA for both one-day and five-day forecasts. As a matter of fact, both RV-BMSM and RV-ARFIMA provide most accurate forecasts in 50% of the cases in our sample for one-day forecasts in terms of MSE, respectively.

Overall, the best models based on the evaluation of mean squared forecast errors are RV-MRW, RV-ARFIMA and RV-BMSM, where we can observe a transition from RV-MRW dominating long term forecasts to RV-ARFIMA dominating medium- and short-term forecasts and RV-BMSM specialized in one-day forecasts.

The paper is organized as follows: The next section reviews some basic concepts of volatility modeling and shortly describes the volatility models considered. Section 3 gives a detailed exposition of the RV-MRW, section 4 of the RV-BMSM. Our empirical study in section 5 encompasses information on the dataset, the estimation results and the forecasting design, and reports our main findings for the forecast performance of the various models. We conclude in the sixth section.

2 Volatility models

Volatility models consider risk adjusted (zero mean) financial returns $x_t = (p_t - p_{t-1}) - \mu_t$, with $p_t = \ln(P_t)$ the logarithmic asset price at time t and $\mu_t = E_{t-1}[\ln(p_t) - \ln(p_{t-1})]$, the conditional mean of the return series given the public information available at time $t-1$. The focus is on the modelling of financial volatility σ_t according to various specifications within the following general framework:

$$x_t = \sigma_t u_t, \tag{1}$$

where u_t is Gaussian white noise $u_t \sim \mathcal{N}(0,1)$ (Andersen et al., 2006). This construction reflects the economic ideas behind the efficient market hypothesis: the return fraction μ_t constitutes the fair payment expected in t, whereas x_t is the excess return due to arrival of new information, which market participants are unable to predict (cf. Fama, 1965).

The seminal GARCH[1,1] (generalised autoregressive conditional heteroscedasticity) model of Bollerslev (1986) assumes that the volatility dynamics is governed by

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{2}$$

where the restrictions on the parameters are $\omega > 0$, $\alpha, \beta \geq 0$ and $\alpha+\beta < 1$. A well known stylized fact of financial time series is the so called *leverage effect*, which describes the negative correlation between volatility and returns. This is based on the empirical finding that return fluctuations intensify after negative financial news and are less pronounced after positive financial news (Black, 1976). To account for this stylized fact we consider the GJR-GARCH model, which was

named after the authors who introduced it (Glosten et al., 1993):

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \varsigma x_{t-1}^2 \mathcal{I}(x_{t-1} < 0) + \beta \sigma_{t-1}^2, \tag{3}$$

where $\mathcal{I}(\bullet)$ is the indicator function taking the value of 1 if $x_{t-1} < 0$ and 0 otherwise.

While the models of the GARCH family conceive the volatility dynamics as a linear process, multifractal volatility models, in contrast, are characterized by a multiplicative structure with a hierarchy of heterogeneous volatility components. The systematic arrangement of these volatility components provides for a very parsimonious framework with only a small number of parameters to estimate. In the Markov-switching multifractal (MSM) model, instantaneous volatility is determined by the product of k volatility components $M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}$ and a scale factor σ^2:

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)}. \tag{4}$$

Following the basic hierarchical principle of the multifractal approach, each volatility component $M_t^{(i)}$ will be renewed at time t with a probability γ_i depending on its rank within the hierarchy of multipliers, and will remain unchanged with probability $1 - \gamma_i$. Convergence of the discrete-time MSM of eq. (4) to a limiting Poisson process in continuous time requires to formalize transition probabilities as follows (cf. Calvet and Fisher, 2001):

$$\gamma_i = 1 - (1 - \gamma_k)^{(b^{i-k})}. \tag{5}$$

In this paper we set the number of multipliers $M_t^{(i)}$ to $k = 8$ and use the pre-specified parameter values $\gamma_k = 0.5$ and $b = 2$ (cf. Lux, 2008).[2] Following Calvet and Fisher (2004), the Binomial MSM (BMSM) is characterized by binomial random draws, each component $M_t^{(i)}$ taking the values m_0 or $2 - m_0$ $(1 \leq m_0 < 2)$ with equal probability. This configuration guarantees an expectation of unity for all $M_t^{(i)}$. Due to the finite set of states and the Markov property of the BMSM, this model can be estimated using exact maximum likelihood (ML).

In the Lognormal MSM (LMSM) model, multipliers are determined by random draws from the lognormal distribution with parameters λ and ν, i.e. $M_t^{(i)} \sim LN(-\lambda, \nu^2)$. Normalisation via $E[M_t^{(i)}] = 1$ leads to the restriction $\nu = \sqrt{2\lambda}$. The estimation of the LMSM parameters λ and σ^2 can be accomplished with the generalized method of moments (GMM) estimation technique based on the pertinent moment conditions for the log differences of absolute returns $\xi_{t,T} = \ln|x_t| - \ln|x_{t-T}|$ for various lags T as outlined in Lux (2008).

Here we evaluate volatility forecasts based on both model specifications. We compute BMSM forecasts based on the conditional probabilities of the current states of the volatility components and the transition matrix of the model according to Bayes' rule. Forecasting of the LMSM model is performed by means of best linear forecasts (cf. Brockwell and Davis, 1991, chap. 5) together

[2] An in-depth analysis of this model can be found in Calvet and Fisher (2004) and Lux (2008).

with the generalized Levinson-Durbin algorithm developed by Brockwell and Dahlhaus (2004) (see Lux, 2008, for further details).

The multifractal random walk (MRW) is a different multifractal specification obtained as the limit process over a continuum of volatility components in the limit when the sampling interval reaches zero. Section 3 introduces the MRW in detail.

The above volatility models are estimated on the basis of daily returns x_t. It is, however, also possible to model the volatility σ_t in (1) directly and fit it using daily *realized volatilities* v_t calculated from intraday returns:

$$v_t = \sum_{j=1}^{1/\Delta} \left(p(t-1+j\cdot\Delta) - p(t-1+j\cdot\Delta-\Delta)\right)^2 \tag{6}$$

for a sampling interval $0 < \Delta < 1$, $1/\Delta$ integer (e.g. $\Delta = 102^{-1}$ when using 5 min intervals and assuming that a trading day has 8.5 hours). In this study we consider both types of implementations and make the distinction between model types using the prefix *RV* for the latter.

We adopt a forecasting method for the MRW based on the history of RV introduced by Duchon et al. (2012) which is the subject of the next section. Lux et al. (2014) already adapted the LMSM model to the forecasting of RV using parameter estimates obtained via GMM together with best linear forecasts.[3] Additionally, in this paper we propose a new RV model extension for the BMSM, described in more detail in section 4.

The MRW, LMSM and RV-LMSM implement the idea of a *lognormal-normal mixture distribution* of returns governed by the integrated volatility. This idea goes back to Clark (1973) and is in accordance with more recent empirical evidence by Andersen et al. (2000, 2001a,b, 2003) and Bandorff-Nielsen and Shephard (2002). According to Andersen et al. (2003) standardized daily returns $\overline{r}_t = (p_t - p_{t-1})/\sqrt{v_t}$ are normally distributed whereas realized standard deviations $\sqrt{v_t}$ are lognormally distributed. Based on these considerations we also employ the popular ARFIMA (autoregressive fractionally integrated moving average) specification to the logarithmic RV with mean μ:

$$(1-\phi L)\,(1-L)^d\left(\ln\sigma_t^2 - \mu\right) = (1-\theta L)\,\eta_t \tag{7}$$

where ϕ and θ are the first order AR and MA parameters, respectively, d the parameter of fractional differentiation and η_t white noise.

We estimate the RV-ARFIMA(1,d,1) model above via exact ML and maximize the (concentrated) log-likelihood function numerically.[4] We also looked at alternative model orders. After prior evaluation of significance of the parameters we chose model order based on information

[3]The authors employ the corresponding moment conditions for the log differences of volatilities $\zeta_{t,T} = \ln\sigma_t - \ln\sigma_{t-T}$ for various lags T (Lux et al., 2014).

[4]This is implemented in the package 'arfima' by Veenstra and McLeod (2018) for R (R Core Team, 2020).

criteria and in accordance with the Box-Jenkins methodology.[5]

Table 1 provides an overview of the models considered:

Models for returns	Models for RV
LMSM	RV-LMSM
BMSM	RV-BMSM
MRW	RV-MRW
GARCH[1,1]	RV-ARFIMA
GJR-GARCH[1,1,1]	

Table 1: The models.

3 The MRW model

The MRW was first proposed in Bacry et al. (2001). The model was generalized by Muzy and Bacry (2002) and Bacry and Muzy (2003), who introduced a continuous random cascade model on the upper half-plane by means of an independently scattered infinitely divisible two-dimensional random measure:

$$M_\ell(dt) = e^{2\omega_\ell(t)}dt. \tag{8}$$

This cascade model intends to capture the information cascade from long-term to short-term traders and generates the so called *multifractal random measure* in the limit of small scales $\ell \to 0^+$.

At the core of this construction is the *magnitude process* $\omega_\ell(t)$ with a similar multiplicative structure like eq. (4) which is generating the multifractal scaling. Its autocovariances are a function of the intermittency coefficient λ^2, $0 \leq \lambda^2 < \frac{1}{2}$, measuring the degree of multifractal scaling and of the correlation length (or integral scale) T:

$$\mathrm{Cov}\left[\omega_\ell(t), \omega_\ell(t+h)\right] = \begin{cases} \lambda^2\left(\ln\left(\frac{T}{\ell}\right) + 1 - \frac{h}{\ell}\right), & 0 \leq h < \ell \\ \lambda^2 \ln\left(\frac{T}{h}\right), & \ell \leq h < T \\ 0, & h \geq T \end{cases}. \tag{9}$$

The (log-normal) MRW is obtained by compounding a self-similar stochastic process, here

[5]Since the distribution of RV can be well approximated by the log-normal distribution, it seems natural to adopt the ARFIMA framework for the logarithmic RV. However, we also tested the ARFIMA specification for the levels σ_t^2. This proved unsatisfactory due to a large number of significant autocorrelations in the residuals.

Brownian motion, with the multifractal random measure above:

$$X(t) = B\left(\lim_{\ell \to 0^+} \int_0^t e^{2\omega_\ell(u)} du\right). \tag{10}$$

An equivalent representation is given by the stochastic integral:

$$X(t) = \lim_{\ell \to 0^+} \int_0^t e^{\omega_\ell(u)} dB(u). \tag{11}$$

Here $dB(u)$ is Gaussian white noise with mean 0 and variance σ^2, independent of $\omega_\ell(u)$, whereas the multifractal random measure plays the part of a stochastic variance. The MRW increments over unit time intervals model the risk adjusted returns x_t introduced in section 2:[6]

$$x_t = X(t) - X(t-1) = \lim_{\ell \to 0^+} \int_{t-1}^t e^{\omega_\ell(u)} dB(u). \tag{12}$$

The parameters to estimate are λ^2, T and the scale factor σ^2. We estimate this model via the generalized method of moments (GMM). In the GMM framework the unknown parameter vector φ is obtained by minimizing the distance of empirical moments from their theoretical counterparts, i.e.

$$\widehat{\varphi}_T = \arg\min_{\varphi \in \Phi} f_T(\varphi)' A_T f_T(\varphi), \tag{13}$$

with Φ the parameter space, $f_T(\varphi)$ the vector of differences between sample moments and analytical moments, and A_T a positive definite and possibly random weighting matrix. Under some regularity conditions the GMM estimator $\widehat{\varphi}_T$ is consistent and asymptotically normally distributed (cf. Harris and Mátyás, 1999).[7]

Bacry et al. (2008) derived a discretized version of the MRW increments and an approximate distribution in the limit of small intermittency $\lambda^2 \ll 1$ given by the rescaled product of a Gaussian white noise and a log-normal process. This facilitated the formulation of approximate expressions for the mean and the autocovariance function of the log absolute returns:

$$\mathrm{E}\left[\ln|x_t|\right] = \ln(\sigma) - \frac{\gamma + \ln(2)}{2} - \lambda^2 (1.5 + \ln(T)) + o\left(\lambda^2\right), \tag{14}$$

[6] An in-depth analysis of this model can be found in Bacry and Muzy (2003) and Bacry et al. (2013).

[7] GMM was implemented with MATLAB (MathWorks, 2018) using a revised and reprogrammed version of the GMM toolbox by Kyriakoulis (2005). We computed HAC (heteroskedasticity and autocorrelation consistent) standard errors with the fast algorithm by Heberle and Sattarhoff (2017) mainly based on the fast Fourier transform.

$$\operatorname{Cov}\left[\ln|x_t|, \ln|x_{t+h}|\right] = \begin{cases} \lambda^2 g(h,T) + o\left(\lambda^2\right), & 1 \le h \le T-1 \\ o\left(\lambda^2\right), & h \ge T+1 \end{cases} \tag{15}$$

where γ is the Euler constant, $o\left(\lambda^2\right)$ satisfies $\lim_{\lambda^2 \to 0^+} o\left(\lambda^2\right)/\lambda^2 = 0$ and

$$g(h,T) = \begin{cases} \ln(T) + 1.5 - 2\ln(2), & h = 1, T \ge 2 \\ \ln\left(\frac{T}{h}\right) - \frac{(h+1)^2}{2}\ln\left(1+\frac{1}{h}\right) - \frac{(h-1)^2}{2}\ln\left(1-\frac{1}{h}\right) + 1.5, & 2 \le h \le T-1 \end{cases}. \tag{16}$$

The authors estimate the MRW parameters via GMM using various lags of the autocovariance function above together with the variance condition $E\left[x_t^2\right] = \sigma^2$ (Bacry et al., 2008, 2013). Bacry et al. (2013) show within the framework of a Monte Carlo simulation study that the estimation of the intermittency coefficient λ^2 has very small bias and MSE, being also reliable in finite samples. However, the asymptotic normality of this estimator is compromised even for fairly large datasets. Instead, the authors recommend computation of confidence intervals using Monte Carlo simulations.

In this paper, we deviate from Bacry et al. (2008) by fitting the mean explicitly to the sample average (Sattarhoff, 2011). In contrast, following Bacry et al. (2008, 2013) the mean condition would enter the estimation only through the computation of the sample autocovariances. Based on simulations, Sattarhoff (2011) shows that the explicit use of the mean condition is preferable as it achieves a better convergence to the asymptotic distribution, with normally distributed estimates for λ^2 and $\ln\sigma$ starting at approximately 2,000 data points, i.e. about 8 years of daily financial data.

Out-of-sample h-step ahead forecasting of the MRW model is performed for the zero-mean quantity x_t^2 by means of best linear forecasts (cf. Brockwell and Davis, 1991) together with the generalized Levinson-Durbin algorithm developed by Brockwell and Dahlhaus (2004).

One major problem in the applications of the MRW framework is the estimation of the correlation length T. Previous empirical applications report very large error bars for this parameter, e.g. Bacry et al. (2008). In a sense, T is a nuisance parameter that only serves to bring the multifractal apparatus into a manageable time series format with convenient asymptotic properties. Empirically, tests for long memory in financial volatility have always found long-range correlations without any indication of a finite correlation length (cf. Lux and Ausloos, 2002). One might thus, prefer to rather get rid of this parameter when applying the MRW framework to practical tasks like volatility forecasting.

Duchon et al. (2012) have developed the pertinent theory of the limit $T \to \infty$. Among other results, they establish the following approximate formula for conditional expectations of second moments:

$$E[\sigma_{t+h}|\sigma_t, \sigma_{t-1}, \ldots, \sigma_{t-N}] \approx e^{\frac{\lambda^2 C}{2}} h^{\frac{\lambda^2}{2}} \prod_{\tau=0}^{N} (\sigma_{t-\tau})^{\alpha^*_{h,\tau+1}} \tag{17}$$

with $C \approx 1.33$ and

$$\alpha^*_{h,\tau} = \frac{2}{\pi}\left(Arctan\sqrt{\frac{\tau}{h}} - Arctan\sqrt{\frac{\tau-1}{h}}\right),$$

with the vector $\sigma_{t-\tau}$ being implemented using either absolute returns or realized volatility as a proxy of past realizations of the standard deviation. With correlation length $T \to \infty$, all past observations should be used just as in the linear forecasts for a process with long-term memory.

Hence, the product on the right-hand side extends over the entire sample up to time t. This also renders the estimation of the scale parameter σ unnecessary, as it is 'learned' in the out-of-sample forecasting from the previous realizations that appear in the iterative formula of eq. (17).

Curiously enough, the paper by Duchon et al. (2012) does not include any empirical application of their theoretical results, and to the best of our knowledge, it has also never been applied by other authors since.

Additionally, in the present paper we adapt the GMM framework for the use of realized volatilities:

$$\sigma_t^2 = \lim_{\ell \to 0^+} \int_{t-1}^{t} e^{2\omega_\ell(u)} du. \tag{18}$$

The parameters λ^2 and T can be estimated based on the pertinent moment conditions for the logarithmic volatility provided in Bacry et al. (2013):

$$\mathrm{E}\left[\ln \sigma_t^2\right] = -2\lambda^2 \left(\ln T + 1.5\right) + o\left(\lambda^2\right) \tag{19}$$

$$\mathrm{Cov}\left[\ln \sigma_t^2, \ln \sigma_{t+h}^2\right] = \begin{cases} 4\lambda^2 g\left(h, T\right) + o\left(\lambda^2\right), & 1 \le h \le T-1 \\ o\left(\lambda^2\right), & h \ge T+1 \end{cases} \tag{20}$$

Hence, we employ the MRW twice:

- First, we estimate model parameters via GMM on the basis of daily returns x_t and calculate forecasts for the quantity x_{t+h}^2 based on the returns history using best linear forecasts as outlined above.
- Second, we calculate forecasts for the quantity σ_{t+h} based on the prediction formula by Duchon et al. (2012). For this purpose, we estimate the volatilities $\sigma_{t-\tau}^2$ using realized volatilities v_t from the in-sample period. This is coupled with the new estimation procedure based on realized volatilities outlined above. We call this second approach the RV-MRW model.

The parameter estimates for both model variants are available in an online appendix.

4 The RV-BMSM model

We also adapt the BMSM model by Calvet and Fisher (2004) to the RV framework. This is an appealing idea since in this new model we can combine optimal forecasts according to Bayes' rule with the RV measures, which hopefully provides for a better identification of the unobserved volatility process. Inspired by the proximity of realized standard deviations $\sqrt{v_t}$ to the lognormal distribution we use a lognormal random variable $\eta_t \sim \mathcal{LN}\left(\phi, \xi^2\right)$ for the innovations of the volatility dynamics which is independent of the volatility components $M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}$:

$$\ln \sigma_t^2 = \sum_{i=1}^{k} \ln M_t^{(i)} + 2 \ln \eta_t. \tag{21}$$

The other model characteristics remain unchanged: The volatility components are binomially distributed taking the values m_0 or $2 - m_0$ $(1 \leq m_0 < 2)$ with equal probability. Each volatility component $M_t^{(i)}$ will be renewed at time t with probability $\gamma_i = 1 - (1 - 0.5)^{(2^{i-k})}$ depending on its rank within the hierarchy of multipliers, and will remain unchanged with probability $1 - \gamma_i$. The model is, therefore, a Markov-switching process with $d = 2^k$ states with the $d \times d$ transition matrix A with elements $a_{ij} = P\left(M_{t+1} = s_j | M_t = s_i\right)$, $1 \leq i, j \leq d$. Here $M_t = \left(M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}\right)$ denotes the vector of volatility components and s_i, s_j are state vectors. The number of components k was set again equal to eight.

In contrast to the BMSM model for financial returns, the volatility process σ_t^2 of RV-BMSM follows a lognormal distribution conditional on the volatility state $M_t = s_j$ with mean $\sum_{i=1}^{k} \ln s_j^{(i)} + 2\phi$ and variance $4\xi^2$. This process can again be implemented with realized volatility v_t. Since BMSM for realized volatility is a new addition to the zoo of multifractal models, we provide the details of our estimation algorithm for this variant which is based on an iterative formalization of the likelihood function. To derive the conditional likelihoods of all observations, we compute the conditional probability vector $\Pi_{t|t} = \left(\Pi_{t|t}^1, \Pi_{t|t}^2, \ldots, \Pi_{t|t}^d\right)$ with components $\Pi_{t|t}^j = P(M_t = s_j | \ln \sigma_t^2, \ln \sigma_{t-1}^2 \ldots, \ln \sigma_1^2)$ for all time points t in the sample recursively based on the history of volatilities

$$\Pi_{t|t} = \frac{\omega\left(\ln \sigma_t^2\right) * \left(\Pi_{t-1|t-1} A\right)}{\left[\omega\left(\ln \sigma_t^2\right) * \left(\Pi_{t-1|t-1} A\right)\right] \mathbf{1}_k} \tag{22}$$

where $\omega\left(\ln \sigma_t^2\right) = \left[f_{\ln \sigma_t^2}\left(x | M_t = s_1\right), \ldots, f_{\ln \sigma_t^2}\left(x | M_t = s_d\right)\right]$ contains the conditional density functions of $\ln \sigma_t^2$ with

$$f_{\ln \sigma_t^2}\left(x | M_t = s_j\right) = \frac{1}{2\xi\sqrt{2\pi}} e^{-\frac{1}{2} \cdot \frac{\left(x - E\left[\ln \sigma_t^2 | M_t = s_j\right]\right)^2}{4\xi^2}} \tag{23}$$

for each $1 \leq j \leq d$. $\mathbf{1}_d$ is the column vector of ones $\mathbf{1}_d = (1, \ldots, 1)^\top \in \mathbf{R}^d$ and $*$ denotes the

element-wise product of vectors.

We estimate the parameters m_0, ϕ and ξ via ML. We start the recursion with $m_0 = 1.4$ (Lux, 2008). We derive initial values for ϕ and ξ based on the conditional mean and conditional variance of the volatility process: We set ξ^2 to a quarter of the sample variance of log RV and

$$\phi = \frac{1}{2}\overline{\ln v_t} - 2\ln m_0 - 2\ln(2 - m_0)\,. \tag{24}$$

As a by-product of the estimation, the conditional probabilities of the unobserved volatility states can be exploited to compute one-step and multi-step optimal forecasts of future volatility on the basis of the transition matrix of the model according to Bayes' rule.

5 Empirical study

5.1 Estimation results

Our dataset comprizes daily returns and daily realized volatilities over the sample period 2000/01–2018/10 for a collection of 14 international stock market indices as provided by the Oxford-Man Institute's realised library (Heber et al., 2009). As an exception, the data for the Italian index FTSE MIB is available only starting with 2009/01, for OMX Stockholm – starting with 2005/10 and in the case of the Japanese Nikkei 225 starting with 2000/02. Table 2 displays the regional distribution of the markets, including 13 national indices and one regional index designed to capture the stock markets performance in the Eurozone, the EURO STOXX 50.

An online appendix reports some descriptive statistics of returns, absolute returns and squared returns. According to the Kolmogorov and Anderson-Darling normality tests (significance level of 5%) as well as the high kurtosis values, the distribution of our data deviates throughout from normality. We could also reject the null hypothesis of no serial correlation using the Ljung-Box-Pierce statistic (Q statistic) for each data series at very small significance levels. For absolute and squared returns, the usual result of an even stronger rejection applies with highly significant positive values observed over long lags of their empirical autocorrelation functions. However, in the case of raw returns, the autocorrelations, although significant, are much less pronounced.[8] This is in accordance with the Lagrange multiplier (LM) test for serial correlation, which, as opposed to the Ljung-Box-Pierce test, cannot reject the hypothesis of no serial correlations in the raw returns. The LM test is more adequate for financial data since it is robust against heteroskedasticity.[9]

We modeled this overall relatively weak autocorrelation structure present in the raw returns data using an AR(1) model and extracted the centered returns in (1) from the filtered series. We estimated the nine volatility models under scrutiny (see Table 1) using as our in-sample

[8] As an exception, we failed to reject the null hypothesis of no serial correlation in the case of raw returns for the indices FTSE MIB, HANG SENG and Nikkei 225 based on the Ljung-Box-Pierce statistic.

[9] As an exception, we rejected the null of no serial correlations with the LM test for the raw returns of Shanghai Composite, S&P BSE Sensex and NIFTY 50 at the 5% significance level.

country	index	time span	country	index	time span
USA	DJIA	00/01–18/10	France	CAC 40	00/01–18/10
	Nasdaq 100	00/01–18/10	Germany	DAX	00/01–18/10
	S&P 500	00/01–18/10	Italy	FTSE MIB	09/01–18/10
			Sweden	OMX Stockholm	05/10–18/10
China	Shanghai Composite	00/01–18/10	UK	FTSE 100	00/01–18/10
Hongkong	HANG SENG	00/01–18/10	euro area	EURO STOXX 50	00/01–18/10
Japan	Nikkei 225	00/02–18/10			
India	NIFTY 50	00/01–18/10			
	S&P BSE Sensex	00/01–18/10			

Table 2: The dataset.

period all observations from the first observation available until the end of 2015/12. The online appendix contains the estimation results.

5.2 Forecast evaluation

Based on these estimations we calculate volatility forecasts using a rolling window for the out-of-sample period 2016/01–2018/10. Throughout, we evaluate forecast accuracy against RV as a proxy for the true volatility.[10] We employ daily RV data provided by Heber et al. (2009).[11]

We evaluate forecast accuracy based on the values for the relative empirical MSE and MAE criteria, i.e., we divide the values of the loss functions by the corresponding values when computing forecasts with the historical volatility. We further calculate model confidence sets for each forecast horizon using the methodology by Hansen et al. (2011). Overall, we find that RV models show superior performance compared with models based on returns.

[10] RV-ARFIMA and RV-BMSM produce forecasts of the quantity $\ln \sigma_t^2$. For reasons of comparison with the alternative models, we transformed these forecasts into forecasts of σ_t^2 and evaluated them consistently against RV v_t. Forecasts from package 'arfima' by Veenstra and McLeod (2018) come along with estimates of the forecast variances for each forecast horizon. In the case of RV-BMSM the ML estimate for ξ^2 can be utilized for this purpose.

[11] We use realized kernels (Barndorff-Nielsen et al., 2008). We also evaluated forecasts against alternative realized measures provided in the Oxford-Man Institute's realized library (Heber et al., 2009) without significant changes in our results.

MSE and MAE values

Whereas RV-MRW is the consistent winner under the MAE criterium for all financial markets, the forecast results in terms of MSE values vary as a function of the forecast horizon. Table 3 displays the models with the lowest relative MSE values. Throughout, the best models are RV-ARFIMA, RV-BMSM and RV-MRW with the following differences:

At short horizons of $h = 1$ and $h = 5$ days RV-ARFIMA and RV-BMSM perform best most of the time. Whereas for one-day forecasts these models appear to have similar forecasting performance, each of them delivering lowest MSE values in the case of nine indices and (even) identical MSE values in the case of four indices up to three decimals, RV-ARFIMA dominates in a larger number of cases than RV-BMSM for the five-day forecasts. Interestingly, as an exception, in the case of FTSE 100 the GJR-GARCH specification performs favorably at $h = 5$ and 10. The situation is much more definite at longer horizons $h > 10$ days where RV-MRW outperforms all models most of the time. $h = 10$ appears intermediate with RV-MRW and RV-ARFIMA scoring best in the same number of cases. The detailed results for the relative MSE and MAE can be found in Appendix A.

	forecast horizon					
index	1	5	10	20	50	100
DJIA	B	A	A	M	M	M
Nasdaq 100	B	A	A/M	M	M	M
S&P 500	B	A	M	M	M	M
Shanghai Composite	B	B	M	M	M	M
HANG SENG	A/B	A/B	A/M	M	M	M
Nikkei 225	A	A	A	A	A	M
NIFTY 50	A	B	A/B/M	M	M	M
BSE Sensex	A/B	A	A	M	M	M

	forecast horizon					
index	1	5	10	20	50	100
CAC 40	B	A/B	M	M	M	M
DAX	A/B	M	M	M	M	M
FTSE MIB	A	A	A/M	M	M	M
OMX Stockholm	A/B	B/M	M	M	M	M
FTSE 100	A	X	A/X	A	M	M
EURO STOXX 50	A	A	A	A	M	M

Table 3: The models with the lowest relative MSE for the 14 indices under scrutiny and six forecast horizons $h = 1, 5, 10, 20, 50$, and 100 days (A = RV-ARFIMA, B = RV-BMSM, M = RV-MRW, X = other model)

Model confidence sets

We compute model confidence sets for the nine models under consideration using the methodology proposed by Hansen et al. (2011).[12] Whereas the empirical MSE and MAE values vary

[12]We employed the MFE Toolbox by Sheppard (2009) using MATLAB (MathWorks, 2018).

subject to the realized data sample, model confidence sets have the advantage of providing forecast assessments with a certain confidence level. By analogy with a $(1-\alpha)$ confidence interval, the model confidence set (MCS) is a random subset of models containing the most accurate model with a probability no less than $1-\alpha$. By definition, the 100% MCS is the set containing all models under scrutiny, i.e., the set of all models contains the best model with 100% probability.

We report *MCS p-values* in Appendix B. An MCS p-value of α corresponds to the pertinent model being contained in the $(1-\alpha)\times 100$ percent confidence set, i.e., the model confidence set one obtains at the $1-\alpha$ level of confidence. Choosing a higher level of confidence $(1-\alpha)$ implies a lower error probability when eliminating models. Hence, a higher confidence level tends to come along with a larger set of accepted models. The p-values then indicate at what confidence levels (i.e., one minus the p-value) a model is accepted. The results in Appendix B were computed based on 1,000 stationary bootstrap resamples of the original data based on the procedure in Politis and Romano (1994).[13]

By construction, the model with an MCS p-value of 100% is contained in all confidence sets, i.e., it provides the most accurate forecasts. Table 4 displays the models with MCS p-value of one based on the MSE as a loss function.

Overall, the MCS p-values confirm the tendencies we have seen in the results for the MSE and MAE criteria:

- First, in terms of MAE, RV-MRW is the best model throughout. Moreover, RV-MRW is the only model in the 99% MCS in eleven cases, indicating sizeable precision differences to the alternative models. For forecasting horizons beyond one day, hardly any other model is ever assigned a positive probability to be better, and with one exception, also at the 1-day horizon other models at best enter with very small p-values.[14] Overall, the results are quite uniform across the various indices of our study. It is noteworthy to underline how strongly the MCS test speaks in favor of the dominance of the RV-MRW: The p-values of mostly zero for the other models indicate that the superior performance of RV-MRW has been outside the complete bootstrap distribution for all other models.

- Second, in terms of MSE, RV-MRW provides most accurate forecasts for the longer horizons $h \geq 20$ days in the majority of cases. The picture is less clear for ten-day forecasts where both RV-MRW and RV-ARFIMA dominate the competition in about the same number of cases. One can observe a transition from RV-MRW dominating long term forecasts to RV-ARFIMA dominating middle- and short-term forecasts and RV-BMSM specialized in one-day forecasts. As a matter of fact, both RV-BMSM and RV-ARFIMA

[13] We have also experimented with a higher number of bootstrap replications as well as with the circular block bootstrap with practically no changes of the p-values.

[14] Interestingly, both RV-MRW and RV-ARFIMA provide most accurate 1-day forecasts in the case of S&P BSE Sensex. This is also the only index where another model other than the RV-MRW provides a positive although negligible MCS p-value at h = 5 days.

each provide the best forecasts in 50% of the cases for one-day forecasts, whereas RV-ARFIMA performs best at $h = 5$ days.

In the following, we take a closer look at the forecasting results in Appendix B in terms of MSE.[15]

- *50-day and 100-day forecasts*: RV-MRW is the only specification remaining in the model confidence set at a confidence level of 95 percent (remember: this is 1 minus the p-value of the marginally accepted model) at $h = 50$ and $h = 100$ days in the case of ten indices, suggesting vast forecast precision differences to the rest of the models.

- At the shorter forecasting horizons, other models also start entering the scene as successful competitors, along with RV-MRW. At the 20-day forecast horizon, the latter is still represented in any model confidence set in the case of 11 indices, while RV-ARFIMA enters 11 times at the 90 percent confidence level, and RV-BMSM enters 9 times.

- *one-day forecasts:* Going down in forecast horizons, the results become more and more balanced between these three until at the 1-day horizon all three models are represented in all 90 percent confidence sets (at the 80 percent level RV-MRW drops out 3 times and leaves only RV-ARFIMA and RV-BMSM).

To further scrutinize the performance of RV-BMSM at the one-day-horizon we also apply the superior predictive ability test by Hansen (2005) to check whether it is significantly outperformed by any of the alternative models in any of our time series.[16] As it turns out, we cannot reject the null hypothesis of non-inferiority of the RV-BMSM for any of the indices in the sample (significance level $\alpha = 5\%$).[17] As a matter of fact, also in the case of RV-ARFIMA taken as a benchmark we cannot find any evidence that this model is outperformed by any of its competitors.[18]

Forecast illustrations

The online appendix provides plots of the volatility forecasts and the reference quantity in the case of the indices DAX, FTSE 100, DJIA and Shanghai Composite. Overall, these plots illustrate the superiority of models based on RV. In particular, the superiority of RV-MRW is evident for the 100-day forecasts.

[15] When we estimated the parameters of RV-MRW with returns rather than RV, the outcome has been virtually the same with even a few more cases of dominance of this model at intermediate forecast horizons.

[16] The SPA test considers the null hypothesis that the benchmark model is not inferior to any of the alternative models against the alternative hypothesis that there exists a superior alternative model. Rejection of the null hypothesis at a significance level α would therefore indicate the existence of a better model in terms of the MSE or MAE criteria.

[17] When setting $\alpha = 10\%$ we can reject the null for the data on S&P BSE Sensex.

[18] We also tried to shed further light on the situation for 10-day forecasts. Both in the case of RV-MRW taken as a benchmark as well as when RV-ARFIMA is the benchmark model we cannot reject the null hypothesis for the SPA test.

	forecast horizon							forecast horizon						
index	1	5	10	20	50	100	index	1	5	10	20	50	100
DJIA	B	A	A	M	M	M	CAC 40	B	A	M	M	M	M
Nasdaq 100	B	A	A	M	M	M	DAX	B	M	M	M	M	M
S&P 500	B	A	M	M	M	M	FTSE MIB	A	A	A	M	M	M
							OMX Stockholm	B	B	M	M	M	M
Shanghai Composite	B	B	M	M	M	M	FTSE 100	A	X	A/X	A	M	M
HANG SENG	A	A	M	M	M	M	EURO STOXX 50	A	A	A	A	M	M
Nikkei 225	A	A	A	A	A	M							
NIFTY 50	A	B	M	M	M	M							
BSE Sensex	A	A	A	M	M	M							

Table 4: The models with MCS p-value of one based on the MSE loss function for the 14 indices under scrutiny and six forecast horizons $h = 1, 5, 10, 20, 50$, and 100 days (A = RV-ARFIMA, B = RV-BMSM, M = RV-MRW, X = other model).

In Figure 1 we take a closer look at the one-day forecasting performance of RV models. We can see that RV-MRW tends to underestimate phases of high volatility whereas providing best forecasts in phases of low volatility and a better fit of the relaxation of the volatility to its average level after any high-volatility spike. At the other end, RV-BMSM overestimates volatility during relatively calm phases, whereas providing better results during turbulent phases. RV-ARFIMA lies somewhere in between RV-MRW and RV-BMSM during turbulent times and closer to the behaviour of RV-BMSM during calm times.

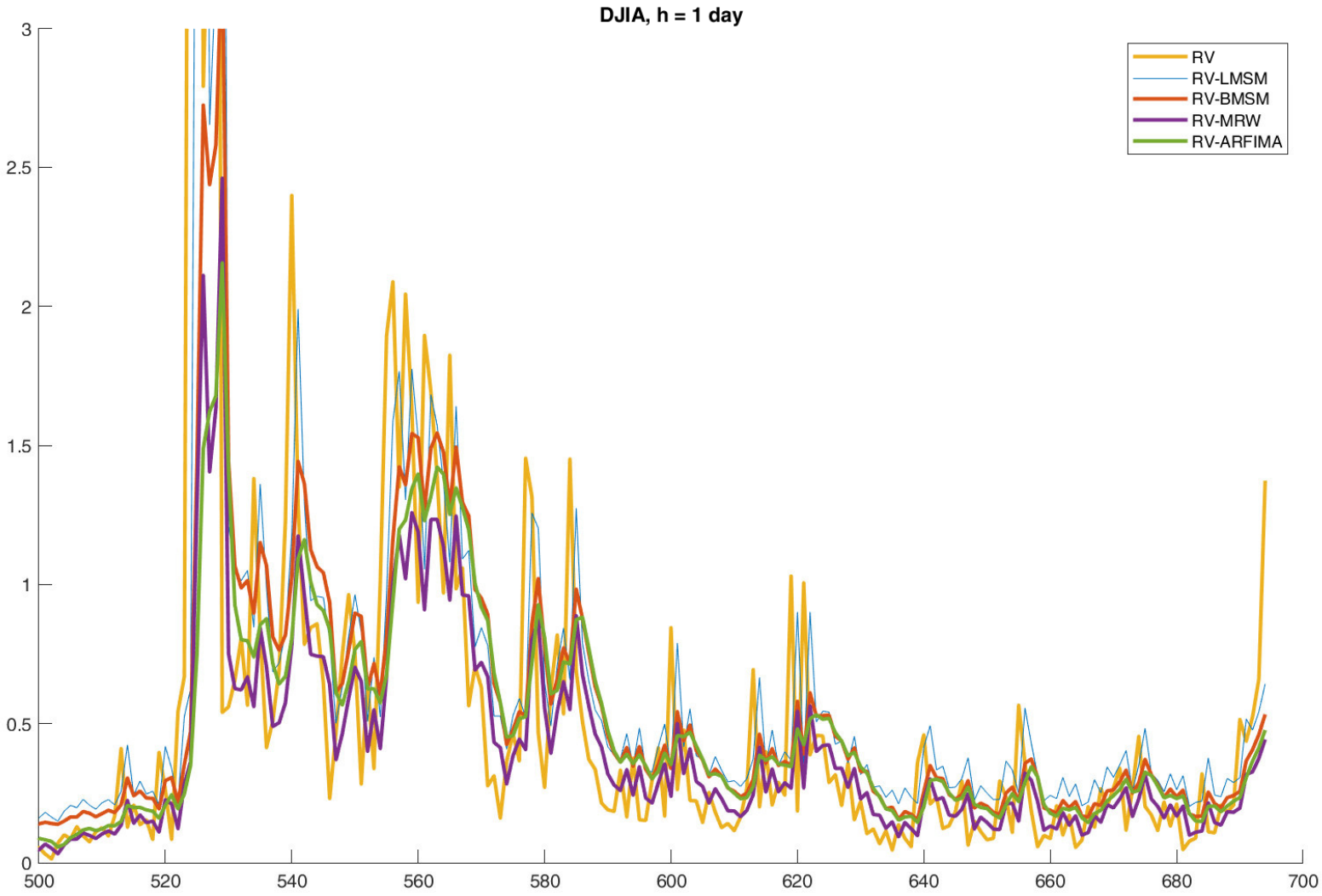

Figure 1: One-day ahead forecasts from the RV models for the DJIA index in comparison to RV, time period: January 2018 till October 2018.

Figure 1 also indicates superior results with the RV-LMSM during turbulent phases, the RV-LMSM forecasts appearing to replicate most accurately the true volatility dynamics, in particular the extent of fluctuations. Lux et al. (2014) actually reported superior forecasting capability for the RV-LMSM model during the turbulent period of the 2008–2009 financial crisis. Our forecast illustrations confirm these findings and additionally suggest similar performance for the RV-BMSM. Moreover, they point out the forecasting contribution of RV-MRW during calmer periods. These differences are the probable source of the better performance of RV-MRW for long horizons, and RV-BMSM and RV-ARFIMA for shorter ones: in the short run, the good fit during high-volatility periods will overcompensate the somewhat higher losses in tranquil periods, while for long horizons it is more valuable to better fit the relaxation of volatility (i.e. its pattern of mean reversion). For the future it would be therefore interesting to focus on the MSM and MRW model classes and shed further light on their forecasting performance as a function of the state of the market.

6 Conclusion

In this article we brought the little-known forecasting procedure by Duchon et al. (2012) into focus, a model extension of the baseline MRW introduced in Bacry et al. (2001). In spite of the sophisticated mathematical underpinnings, this forecast model stands out due to a couple of benefits:

- In the format adopted here it is an (only!) one-parameter forecast model.
- It exploits the RV dynamics for the calculation of volatility forecasts.
- The forecasting equation is straightforward to implement.
- The execution time for the computation of forecasts is negligible.

Additionally, we proposed a new estimation approach for the MRW based on the fitting of the volatility dynamics via GMM. To this end, we adopt the pertinent moment conditions provided in Bacry et al. (2008, 2013). Coupled with the forecasting procedure by Duchon et al. (2012), this results in the so called RV-MRW model. It is important to note, that the GMM estimation of λ^2 comes along with an additional estimate for the correlation length T, which is redundant in the case of the RV-MRW. Duchon et al. (2012) also proposed a new estimation procedure for λ^2 based on the fitting of a simple linear regression model, which is not used here, but can help speeding up the estimation.

We considered fourteen international stock market indices and six forecast horizons $1 \leq h \leq 100$ days. We evaluated rolling window forecasts of the most recent observations starting with January 2016 from the RV-MRW and eight alternative models. These include a new model extension of the BMSM model by Calvet and Fisher (2001) to the RV framework. The combination of optimal forecasts according to Bayes' rule and RV dynamics proves successful,

the RV-BMSM delivering outstanding one-day and five-day ahead forecasts in terms of the MSE criterium compared with the original BMSM for returns.

We compared predictive ability of all candidate models in terms of the empirical MSE and MAE as well as using model confidence sets following the methodology in Hansen et al. (2011) and the test of superior predictive ability by Hansen (2005). According to our findings, RV-MRW outperforms the alternative models including the popular RV-ARFIMA in a considerable number of cases: RV-MRW turns out to be *throughout* the best model for all forecast horizons using the MAE criterium as well as for long forecast horizons $h = 50$ and 100 days using the MSE. Moreover, the RV-MRW provides most accurate 20-day ahead forecasts in terms of MSE for the majority of indices. h = 10 appears intermediate with RV-MRW and RV-ARFIMA scoring best in about the same number of cases, whereas the latter dominates the competition in the short term.

It is worth stressing that RV-ARFIMA is a well-recognized model, the accepted industry standard of volatility forecasting using RV.[19] In view of this, the performance of RV-MRW over RV-ARFIMA is all the more significant. These results are extremely promising if we consider that this is the first empirical application of the RV-MRW. Moreover, whereas RV-ARFIMA forecasts are often a time consuming task, the RV-MRW stands out due to its fast execution and straightforward implementation.

Our forecast illustrations indicate superior results with the RV-BMSM during turbulent market phases, whereas the RV-MRW seems to dominate during calmer periods as it provides the best fit for the mean-reverting dynamics of volatility after episodes with very high volatility. For the future, it would be therefore interesting to focus on the MSM and MRW model classes and shed further light on their forecasting performance as a function of the state of the market.

[19] Numerous studies reported so far the superiority of RV-ARFIMA against various models, e.g., SV (stochastic volatility) (Koopman et al., 2005; Lux et al., 2014), models of the GARCH class (Koopman et al., 2005; Lux et al., 2014) or the heterogeneous autoregressive (HAR) model of Corsi (2009) (Hassler and Pohle, 2019).

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Appendix

A Relative empirical MSE and MAE of forecasts

Acronym	Model
Standard	
M1	LMSM
M2	BMSM
M3	MRW
M4	GARCH[1,1]
M5	GJR-GARCH[1,1,1]
Realized	
R1	RV-LMSM
R2	RV-BMSM
R3	RV-MRW
R4	RV-ARFIMA

Table A1: The models

	CAC 40									*DAX*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.400	0.353	0.336	0.433	0.617	0.209	**0.166**	0.171	0.167	0.333	0.336	0.269	0.312	0.423	0.156	**0.125**	0.127	**0.125**
5	0.428	0.368	0.350	0.460	0.635	0.221	**0.179**	0.180	**0.179**	0.362	0.357	0.277	0.334	0.441	0.160	0.129	**0.127**	0.129
10	0.449	0.384	0.352	0.466	0.610	0.259	0.182	**0.179**	0.182	0.396	0.389	0.287	0.351	0.433	0.185	0.135	**0.127**	0.133
20	0.484	0.442	0.373	0.505	0.586	0.315	0.184	**0.171**	0.181	0.463	0.455	0.312	0.407	0.443	0.230	0.140	**0.121**	0.133
50	0.520	0.544	0.377	0.570	0.535	0.423	0.144	**0.103**	0.138	0.576	0.583	0.343	0.521	0.458	0.334	0.148	**0.094**	0.127
100	0.590	0.699	0.434	0.727	0.555	0.576	0.176	**0.104**	0.158	0.695	0.732	0.398	0.702	0.505	0.462	0.196	**0.103**	0.156
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.573	0.427	0.433	0.413	0.430	0.250	0.211	**0.198**	0.214	0.480	0.450	0.410	0.398	0.423	0.221	0.189	**0.178**	0.188
5	0.602	0.485	0.456	0.447	0.465	0.332	0.233	**0.204**	0.233	0.539	0.503	0.429	0.424	0.450	0.282	0.212	**0.182**	0.206
10	0.623	0.526	0.473	0.473	0.487	0.404	0.248	**0.206**	0.244	0.586	0.552	0.449	0.459	0.477	0.343	0.232	**0.184**	0.219
20	0.654	0.593	0.503	0.534	0.537	0.490	0.267	**0.203**	0.255	0.648	0.621	0.477	0.520	0.520	0.420	0.262	**0.186**	0.232
50	0.703	0.712	0.550	0.666	0.625	0.625	0.303	**0.193**	0.270	0.744	0.740	0.526	0.666	0.615	0.549	0.319	**0.185**	0.257
100	0.754	0.825	0.610	0.826	0.715	0.746	0.359	**0.210**	0.314	0.823	0.842	0.575	0.818	0.688	0.660	0.391	**0.212**	0.306

Table A2: Relative empirical MSE and MAE values for CAC 40 and DAX for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	FTSE MIB									*OMX Stockholm*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.942	0.562	0.780	0.917	1.591	0.163	0.137	0.142	**0.136**	0.386	0.317	0.299	0.452	0.706	0.129	**0.100**	0.103	**0.100**
5	0.942	0.608	0.789	0.942	1.441	0.182	0.181	0.183	**0.179**	0.412	0.327	0.301	0.478	0.698	0.141	**0.102**	**0.102**	0.103
10	0.942	0.639	0.793	0.898	1.183	0.208	0.204	**0.203**	**0.203**	0.453	0.354	0.307	0.508	0.694	0.174	0.107	**0.103**	0.107
20	0.940	0.707	0.830	0.873	0.946	0.203	0.193	**0.184**	0.190	0.524	0.410	0.323	0.574	0.683	0.228	0.115	**0.102**	0.111
50	0.935	0.808	0.878	0.855	0.742	0.149	0.117	**0.085**	0.109	0.664	0.537	0.346	0.794	0.796	0.328	0.118	**0.074**	0.099
100	0.936	0.936	0.920	0.896	0.734	0.179	0.146	**0.092**	0.131	0.813	0.692	0.395	1.074	0.957	0.444	0.156	**0.076**	0.110
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.967	0.641	0.748	0.703	0.744	0.198	0.189	**0.177**	0.190	0.493	0.402	0.393	0.410	0.443	0.199	0.161	**0.151**	0.163
5	0.967	0.694	0.801	0.742	0.770	0.249	0.232	**0.208**	0.232	0.559	0.455	0.415	0.462	0.496	0.279	0.180	**0.150**	0.177
10	0.967	0.738	0.836	0.781	0.787	0.290	0.260	**0.222**	0.257	0.610	0.499	0.435	0.516	0.553	0.344	0.199	**0.154**	0.188
20	0.966	0.798	0.881	0.841	0.826	0.316	0.277	**0.222**	0.266	0.681	0.569	0.466	0.623	0.649	0.423	0.227	**0.158**	0.203
50	0.965	0.887	0.932	0.913	0.846	0.340	0.289	**0.197**	0.258	0.798	0.700	0.521	0.850	0.845	0.546	0.280	**0.152**	0.220
100	0.966	0.966	0.958	0.947	0.852	0.387	0.339	**0.224**	0.301	0.894	0.817	0.578	1.034	0.975	0.648	0.345	**0.166**	0.249

Table A3: Relative empirical MSE and MAE values for FTSE MIB and OMX Stockholm for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	FTSE 100									*EURO STOXX 50*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.922	0.914	0.913	0.913	0.915	1.309	0.909	0.916	**0.906**	0.858	0.841	0.846	0.862	0.928	1.166	0.822	0.829	**0.820**
5	0.935	0.926	0.925	0.933	**0.917**	1.041	0.923	0.923	0.919	0.872	0.852	0.857	0.882	0.934	0.938	0.829	0.832	**0.825**
10	0.941	0.933	0.929	0.943	**0.926**	0.989	0.929	0.929	**0.926**	0.878	0.855	0.857	0.885	0.934	0.898	0.837	0.837	**0.833**
20	0.948	0.942	0.933	0.955	0.943	0.963	0.928	0.928	**0.925**	0.893	0.872	0.864	0.900	0.946	0.880	0.837	0.836	**0.832**
50	0.955	0.949	0.928	0.959	0.944	0.946	0.928	**0.921**	0.922	0.914	0.890	0.864	0.905	0.924	0.885	0.833	**0.825**	0.828
100	0.968	0.965	0.933	0.965	0.940	0.946	0.936	**0.923**	0.925	0.946	0.926	0.881	0.933	0.913	0.906	0.846	**0.827**	0.837
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.580	0.486	0.454	0.459	0.425	0.466	0.363	**0.351**	0.360	0.550	0.465	0.475	0.458	0.486	0.399	0.308	**0.298**	0.312
5	0.628	0.539	0.485	0.498	0.472	0.505	0.406	**0.363**	0.396	0.602	0.511	0.495	0.486	0.519	0.449	0.340	**0.305**	0.341
10	0.664	0.583	0.505	0.532	0.494	0.544	0.437	**0.375**	0.420	0.643	0.550	0.513	0.509	0.540	0.500	0.357	**0.309**	0.354
20	0.709	0.648	0.531	0.601	0.548	0.577	0.466	**0.376**	0.438	0.700	0.612	0.541	0.564	0.587	0.558	0.376	**0.304**	0.365
50	0.781	0.751	0.557	0.725	0.600	0.628	0.526	**0.369**	0.461	0.788	0.719	0.587	0.683	0.675	0.655	0.412	**0.286**	0.382
100	0.851	0.850	0.598	0.866	0.640	0.678	0.597	**0.388**	0.506	0.872	0.824	0.654	0.835	0.771	0.745	0.472	**0.308**	0.437

Table A4: Relative empirical MSE and MAE values for FTSE 100 and EURO STOXX 50 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	Nasdaq 100									*S&P 500*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.294	0.134	0.127	0.117	0.132	0.063	**0.061**	0.065	0.066	0.233	0.212	0.191	0.211	0.232	0.165	**0.154**	0.162	0.166
5	0.332	0.198	0.151	0.161	0.193	0.108	0.095	0.095	**0.094**	0.345	0.309	0.264	0.302	0.364	0.272	0.206	0.206	**0.203**
10	0.358	0.234	0.163	0.178	0.213	0.118	0.098	**0.096**	**0.096**	0.393	0.354	0.282	0.333	0.390	0.333	0.221	**0.215**	0.217
20	0.394	0.288	0.172	0.195	0.222	0.130	0.097	**0.087**	0.089	0.441	0.406	0.281	0.352	0.380	0.420	0.215	**0.194**	0.205
50	0.471	0.427	0.204	0.262	0.262	0.172	0.115	**0.082**	0.089	0.566	0.565	0.317	0.460	0.413	0.625	0.253	**0.190**	0.215
100	0.549	0.585	0.243	0.361	0.310	0.222	0.147	**0.087**	0.096	0.677	0.731	0.350	0.595	0.452	0.823	0.308	**0.201**	0.232
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.520	0.297	0.294	0.248	0.263	0.130	0.119	**0.106**	0.116	0.397	0.325	0.304	0.301	0.305	0.218	0.173	**0.146**	0.168
5	0.553	0.372	0.304	0.288	0.306	0.209	0.166	**0.131**	0.146	0.488	0.413	0.348	0.360	0.371	0.379	0.229	**0.173**	0.200
10	0.577	0.422	0.322	0.311	0.332	0.245	0.183	**0.133**	0.150	0.542	0.470	0.371	0.396	0.403	0.475	0.259	**0.179**	0.212
20	0.609	0.492	0.343	0.347	0.365	0.290	0.209	**0.130**	0.155	0.605	0.548	0.397	0.453	0.445	0.588	0.300	**0.178**	0.228
50	0.672	0.631	0.393	0.453	0.449	0.369	0.267	**0.133**	0.169	0.716	0.706	0.454	0.604	0.553	0.763	0.385	**0.191**	0.261
100	0.729	0.751	0.440	0.565	0.521	0.435	0.328	**0.147**	0.189	0.797	0.830	0.494	0.731	0.615	0.895	0.463	**0.216**	0.295

Table A5: Relative empirical MSE and MAE values for Nasdaq 100 and S&P 500 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	DJIA									*Nikkei 225*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.285	0.300	0.274	0.317	0.383	0.265	**0.249**	0.266	0.269	0.812	0.742	0.759	0.859	0.873	0.814	0.624	0.633	**0.623**
5	0.448	0.422	0.377	0.460	0.587	0.378	0.317	0.319	**0.312**	0.830	0.765	0.781	0.896	0.911	0.707	0.641	0.650	**0.637**
10	0.484	0.460	0.390	0.484	0.596	0.415	0.337	0.333	**0.332**	0.858	0.803	0.800	0.932	0.936	0.670	0.646	0.655	**0.643**
20	0.521	0.510	0.387	0.505	0.570	0.475	0.338	**0.313**	0.324	0.882	0.832	0.801	0.964	0.924	0.638	0.624	0.629	**0.621**
50	0.617	0.654	0.423	0.615	0.577	0.637	0.402	**0.316**	0.342	0.919	0.886	0.798	0.971	0.895	0.610	0.598	0.594	**0.590**
100	0.693	0.789	0.448	0.743	0.586	0.790	0.480	**0.334**	0.368	0.978	0.971	0.826	1.055	0.927	0.594	0.588	**0.570**	0.576
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.412	0.366	0.339	0.349	0.361	0.250	0.214	**0.188**	0.206	0.693	0.612	0.650	0.645	0.649	0.303	0.259	**0.247**	0.259
5	0.518	0.451	0.384	0.414	0.432	0.388	0.277	**0.210**	0.234	0.756	0.669	0.675	0.695	0.701	0.344	0.287	**0.261**	0.281
10	0.568	0.504	0.402	0.452	0.459	0.471	0.319	**0.221**	0.253	0.797	0.715	0.694	0.741	0.742	0.351	0.299	**0.259**	0.288
20	0.625	0.583	0.425	0.513	0.504	0.574	0.375	**0.224**	0.273	0.848	0.776	0.715	0.817	0.798	0.357	0.305	**0.249**	0.284
50	0.721	0.737	0.484	0.677	0.618	0.738	0.490	**0.245**	0.319	0.920	0.879	0.744	0.950	0.880	0.366	0.330	**0.233**	0.285
100	0.783	0.850	0.515	0.809	0.676	0.858	0.585	**0.280**	0.365	0.982	0.975	0.784	1.059	0.931	0.371	0.362	**0.228**	0.298

Table A6: Relative empirical MSE and MAE values for DJIA and Nikkei 225 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	HANG SENG									*Shanghai Composite*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.319	0.335	0.248	0.252	0.288	0.114	**0.084**	0.087	**0.084**	0.631	0.367	0.425	0.438	0.489	0.307	**0.277**	0.285	0.280
5	0.364	0.373	0.254	0.268	0.304	0.106	**0.088**	0.089	**0.088**	0.678	0.377	0.471	0.470	0.543	0.266	**0.212**	0.216	0.214
10	0.409	0.416	0.267	0.287	0.323	0.117	0.093	**0.092**	**0.092**	0.732	0.401	0.509	0.492	0.559	0.264	0.195	**0.194**	0.202
20	0.472	0.479	0.282	0.307	0.336	0.133	0.093	**0.088**	0.090	0.821	0.452	0.594	0.564	0.628	0.299	0.166	**0.151**	0.176
50	0.599	0.622	0.313	0.363	0.364	0.178	0.103	**0.089**	0.094	0.977	0.588	0.742	0.710	0.740	0.391	0.158	**0.120**	0.180
100	0.729	0.786	0.360	0.456	0.412	0.230	0.122	**0.097**	0.105	1.100	0.742	0.869	0.893	0.874	0.503	0.192	**0.130**	0.216
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.500	0.482	0.428	0.387	0.416	0.160	0.141	**0.131**	0.141	0.763	0.418	0.490	0.429	0.449	0.272	0.231	**0.220**	0.235
5	0.556	0.539	0.430	0.407	0.436	0.203	0.158	**0.138**	0.154	0.802	0.472	0.526	0.463	0.484	0.340	0.248	**0.220**	0.251
10	0.600	0.585	0.444	0.428	0.456	0.238	0.170	**0.140**	0.162	0.839	0.520	0.561	0.493	0.513	0.396	0.260	**0.217**	0.260
20	0.659	0.650	0.465	0.463	0.489	0.283	0.181	**0.136**	0.164	0.894	0.593	0.624	0.570	0.589	0.470	0.282	**0.213**	0.273
50	0.758	0.767	0.504	0.545	0.552	0.360	0.214	**0.142**	0.178	0.974	0.718	0.719	0.727	0.736	0.589	0.337	**0.230**	0.311
100	0.844	0.876	0.546	0.643	0.613	0.428	0.253	**0.157**	0.201	1.038	0.833	0.804	0.898	0.888	0.684	0.398	**0.260**	0.362

Table A7: Relative empirical MSE and MAE values for HANG SENG and Shanghai Composite for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

	S&P BSE Sensex									*NIFTY 50*								
	MSE									MSE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.151	0.113	0.093	0.101	0.110	0.069	**0.051**	0.059	**0.051**	0.194	0.124	0.112	0.119	0.134	0.079	0.058	0.059	**0.057**
5	0.199	0.149	0.109	0.132	0.146	0.074	0.056	0.060	**0.055**	0.244	0.160	0.133	0.157	0.178	0.083	**0.062**	0.063	0.063
10	0.242	0.184	0.122	0.168	0.187	0.091	0.060	0.059	**0.056**	0.290	0.195	0.150	0.202	0.230	0.101	**0.064**	**0.064**	**0.064**
20	0.313	0.252	0.148	0.258	0.282	0.125	0.072	**0.060**	0.062	0.365	0.262	0.182	0.311	0.345	0.137	0.069	**0.066**	0.069
50	0.463	0.419	0.202	0.531	0.533	0.192	0.097	**0.049**	0.061	0.521	0.423	0.249	0.626	0.628	0.208	0.071	**0.057**	0.066
100	0.623	0.626	0.281	0.846	0.754	0.270	0.142	**0.053**	0.073	0.690	0.622	0.344	0.963	0.844	0.291	0.088	**0.062**	0.075
	MAE									MAE								
h	M1	M2	M3	M4	M5	R1	R2	R3	R4	M1	M2	M3	M4	M5	R1	R2	R3	R4
1	0.353	0.273	0.244	0.251	0.265	0.130	0.104	**0.101**	**0.101**	0.409	0.281	0.271	0.273	0.293	0.132	0.100	**0.094**	0.101
5	0.419	0.340	0.278	0.309	0.329	0.195	0.130	**0.105**	0.118	0.468	0.348	0.309	0.338	0.365	0.199	0.115	**0.100**	0.117
10	0.470	0.394	0.304	0.370	0.394	0.245	0.155	**0.108**	0.130	0.517	0.401	0.339	0.408	0.439	0.251	0.128	**0.105**	0.127
20	0.544	0.477	0.345	0.483	0.508	0.311	0.193	**0.111**	0.144	0.588	0.482	0.385	0.531	0.564	0.321	0.147	**0.110**	0.139
50	0.672	0.636	0.424	0.721	0.723	0.416	0.268	**0.114**	0.166	0.713	0.637	0.473	0.784	0.786	0.430	0.182	**0.113**	0.156
100	0.785	0.786	0.513	0.918	0.865	0.505	0.348	**0.131**	0.201	0.826	0.782	0.568	0.981	0.916	0.522	0.230	**0.128**	0.184

Table A8: Relative empirical MSE and MAE values for S&P BSE Sensex and NIFTY 50 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ measured against realized volatility.

B Model confidence sets

		CAC 40						DAX						FTSE MIB					
		1	5	10	20	50	100	1	5	10	20	50	100	1	5	10	20	50	100
MSE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0.04	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	**0.17**	0	0	0	0	0	0.04	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	**0.21**	0	0	0	0	0	0.04	0	0	0	0	0	0	**0.22**	**0.18**	0.04	0	0
	RV-LMSM	**0.21**	0	0	0	0	0	0.04	0	0	0	0	0	**0.40**	**0.95**	**0.76**	0.07	0	0
	RV-BMSM	**1.00**	**0.98**	**0.57**	0.05	0	0	**1.00**	**0.80**	**0.11**	0	0	0	**0.64**	**0.95**	**0.94**	**0.31**	0	0
	RV-MRW	**0.33**	**0.89**	**1.00**	**1.00**	**1.00**	**1.00**	**0.49**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**0.43**	**0.95**	**0.96**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	**0.84**	**1.00**	**0.69**	**0.24**	0	0	**0.94**	**0.80**	**0.29**	0.08	0	0	**1.00**	**1.00**	**1.00**	**0.44**	0	0
MAE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-MRW	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table B1: MCS p-values in terms of MSE and MAE based on 1,000 stationary bootstrap resamples for CAC 40, DAX and FTSE MIB for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ (forecasts in 90% MCS in bold type)

		OMX Stockholm						FTSE 100						EURO STOXX 50					
		1	5	10	20	50	100	1	5	10	20	50	100	1	5	10	20	50	100
MSE	LMSM	0	0	0	0	0	0	0	0.06	0.03	0.02	0	0	0	0.01	0	0	0	0
	BMSM	0.02	0	0	0	0	0	**0.35**	**0.18**	**0.26**	0.02	0	0	**0.15**	0.03	**0.29**	0.04	0.01	0
	MRW	0.02	0	0	0	0	0	0.05	**0.13**	**0.80**	0.02	0.05	0	0.07	0.03	**0.29**	0.07	0.02	0
	GARCH	0.02	0	0	0	0	0	**0.49**	**0.13**	**0.26**	0.02	0	0	**0.15**	0.01	**0.29**	0.04	0.01	0
	GJR-GARCH	0.02	0	0	0	0	0	**0.35**	**1.00**	**1.00**	0.02	0	0	0	0.01	0	0.04	0.01	0
	RV-LMSM	0.02	0	0	0	0	0	0	0.06	0.03	0.02	0	0	0	0.03	**0.29**	0.07	0	0
	RV-BMSM	**1.00**	**1.00**	**0.35**	0.02	0	0	**0.49**	**0.50**	**0.46**	**0.36**	0.05	0	**0.42**	**0.46**	**0.50**	**0.37**	**0.21**	0
	RV-MRW	**0.24**	**0.98**	**1.00**	**1.00**	**1.00**	**1.00**	**0.35**	**0.51**	**0.80**	**0.72**	**1.00**	**1.00**	**0.15**	**0.46**	**0.55**	**0.66**	**1.00**	**1.00**
	RV-ARFIMA	**0.79**	**0.97**	**0.37**	0.08	0	0	**1.00**	**0.87**	**1.00**	**1.00**	**0.89**	**0.73**	**1.00**	**1.00**	**1.00**	**1.00**	**0.78**	**0.44**
MAE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0.06	0	0	0	0	0
	RV-LMSM	0	0	0	0	0	0	0.04	0	0	0	0	0	0.09	0	0	0	0	0
	RV-BMSM	0	0	0	0	0	0	0.04	0	0	0	0	0	0.09	0	0	0	0	0
	RV-MRW	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	0	0	0	0	0	0	**0.25**	0	0	0	0	0	0.09	0	0	0	0	0

Table B2: MCS p-values in terms of MSE and MAE based on 1,000 stationary bootstrap resamples for OMX Stockholm, FTSE 100 and EURO STOXX 50 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ (forecasts in 90% MCS in bold type)

		DJIA						Nasdaq 100						S&P 500					
		1	5	10	20	50	100	1	5	10	20	50	100	1	5	10	20	50	100
MSE	LMSM	**0.54**	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0	0
	BMSM	**0.37**	0.01	0	0	0	0	0	0	0	0	0	0	**0.27**	0	0	0	0	0
	MRW	**0.54**	0	0	0.01	0.01	0.01	0	0	0	0	0	0	**0.39**	0	0	0	0	0
	GARCH	**0.37**	0	0	0	0	0	0.01	0.01	0	0	0	0	**0.27**	0	0	0	0	0
	GJR-GARCH	**0.37**	0	0	0.01	0.01	0	0.02	0	0	0	0	0	**0.39**	0	0	0	0	0
	RV-LMSM	**0.55**	0.01	0	0	0	0	**0.37**	0.05	0	0	0	0	**0.51**	0	0	0	0	0
	RV-BMSM	**1.00**	**0.65**	**0.90**	**0.49**	0.03	0.01	**1.00**	**0.74**	**0.57**	**0.32**	0.02	0	**1.00**	**0.76**	**0.84**	**0.36**	0.03	0
	RV-MRW	**0.54**	**0.65**	**0.91**	**1.00**	**1.00**	**1.00**	**0.37**	**0.74**	**0.90**	**1.00**	**1.00**	**1.00**	**0.51**	**0.76**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	**0.55**	**1.00**	**1.00**	**0.49**	0.03	0.01	**0.37**	**1.00**	**1.00**	**0.41**	0.02	0.03	**0.51**	**1.00**	**0.84**	**0.36**	0.03	0
MAE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-MRW	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table B3: MCS p-values in terms of MSE and MAE based on 1,000 stationary bootstrap resamples for DJIA, Nasdaq 100 and S&P 500 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ (forecasts in 90% MCS in bold type)

		Shanghai Composite						HANG SENG						Nikkei 225					
		1	5	10	20	50	100	1	5	10	20	50	100	1	5	10	20	50	100
MSE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	**0.23**	0	0	0	0	0	0	0	0	0	0	0	0.01	0.02	0	0	0	0
	MRW	**0.14**	0	0	0	0	0	0	0	0	0	0	0	0.01	0.02	0.01	0	0	0
	GARCH	**0.22**	0	0	0	0	0	0	0.01	0	0	0	0	0.01	0.02	0.01	0	0	0
	GJR-GARCH	**0.23**	0	0	0	0	0	0	0.01	0	0	0	0	0.01	0.02	0	0	0	0
	RV-LMSM	**0.26**	0	0	0	0	0	**0.16**	0.01	0	0	0	0	**0.30**	**0.17**	**0.26**	**0.18**	0	0
	RV-BMSM	**1.00**	**1.00**	**0.95**	0.02	0	0	**0.87**	**0.63**	**0.75**	**0.17**	0.01	0	**0.67**	**0.17**	**0.26**	**0.29**	0.02	0.01
	RV-MRW	**0.32**	**0.66**	**1.00**	**1.00**	**1.00**	**1.00**	**0.17**	**0.63**	**1.00**	**1.00**	**1.00**	**1.00**	**0.30**	**0.17**	**0.26**	**0.29**	**0.58**	**1.00**
	RV-ARFIMA	**0.32**	**0.66**	**0.56**	0.02	0	0	**1.00**	**1.00**	**0.79**	**0.37**	**0.14**	0.04	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**0.42**
MAE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	RV-LMSM	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0	0
	RV-BMSM	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0	0
	RV-MRW	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	0.01	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0	0	0

Table B4: MCS p-values in terms of MSE and MAE based on 1,000 stationary bootstrap resamples for Shanghai Composite, HANG SENG and Nikkei 225 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ (forecasts in 90% MCS in bold type)

		S&P BSE Sensex						NIFTY 50					
		1	5	10	20	50	100	1	5	10	20	50	100
MSE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0
	RV-LMSM	0.04	0	0	0	0	0	0.06	0	0	0	0	0
	RV-BMSM	**0.25**	**0.49**	0.08	0.01	0	0	**0.23**	**1.00**	**0.82**	**0.37**	0	0
	RV-MRW	**0.25**	**0.49**	**0.65**	**1.00**	**1.00**	**1.00**	**0.15**	**0.50**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	**1.00**	**1.00**	**1.00**	**0.56**	0	0	**1.00**	**0.50**	**0.75**	**0.37**	0	0
MAE	LMSM	0	0	0	0	0	0	0	0	0	0	0	0
	BMSM	0	0	0	0	0	0	0	0	0	0	0	0
	MRW	0	0	0	0	0	0	0	0	0	0	0	0
	GARCH	0	0	0	0	0	0	0	0	0	0	0	0
	GJR-GARCH	0	0	0	0	0	0	0	0	0	0	0	0
	RV-LMSM	0	0	0	0	0	0	0	0	0	0	0	0
	RV-BMSM	0.03	0	0	0	0	0	0	0	0	0	0	0
	RV-MRW	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	RV-ARFIMA	**1.00**	0.06	0	0	0	0	0	0	0	0	0	0

Table B5: MCS p-values in terms of MSE and MAE based on 1,000 stationary bootstrap resamples for S&P BSE Sensex and NIFTY 50 for each of the forecasting horizons $h = 1, 5, 10, 20, 50, 100$ (forecasts in 90% MCS in bold type)

Online Appendix

C Descriptive statistics and ACF plots

	CAC 40			DAX		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-8.5224	0	0	-9.1840	0	0
Maximum	10.4387	10.4387	108.9668	12.0269	12.0269	144.6452
Mean	-0.0026	0.9919	2.0051	0.0116	1.0168	2.1459
St. dev.	1.4162	1.0107	5.2749	1.4650	1.0546	5.9264
Skewness	-0.0912	2.5592	8.2363	-0.0693	2.6551	10.0735
Kurtosis	7.9187	13.7733	102.6995	8.6282	15.4356	164.6713
Kolmogorov	0.0711	0.1630	0.3517	0.0754	0.1673	0.3584
Anderson-Darling	52.8402	Inf	Inf	57.7865	Inf	Inf
LM(10)	14.0831***	231.1299	72.0212	12.7178***	241.0852	83.8153
LM(20)	24.4982***	260.4079	107.7548	22.7285***	282.9052	121.5224
Q(10)	45.0817	$3.0387 \cdot 10^3$	$2.4817 \cdot 10^3$	34.5965	$3.0363 \cdot 10^3$	$2.2630 \cdot 10^3$
Q(20)	68.4696	$5.2068 \cdot 10^3$	$4.0622 \cdot 10^3$	61.7681	$5.3797 \cdot 10^3$	$3.9195 \cdot 10^3$

Table C1: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	FTSE MIB			OMX Stockholm		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-13.3314	0.0003	0	-9.4641	0	0
Maximum	10.6840	13.3314	177.7273	10.0345	10.0345	100.6904
Mean	-0.0017	1.1502	2.4771	0.0223	0.9077	1.7582
St. dev.	1.5742	1.0745	6.1453	1.3260	0.9667	5.1194
Skewness	-0.3324	2.3141	13.8337	-0.1236	2.8730	9.3261
Kurtosis	7.1508	14.8484	331.7234	9.4876	16.6958	126.5329
Kolmogorov	0.0553	0.1418	0.3430	0.0826	0.1736	0.3653
Anderson-Darling	14.0467	Inf	Inf	48.6564	Inf	Inf
LM(10)	11.3520***	135.9017	80.3553	7.8280***	178.3371	63.8230
LM(20)	18.4488***	154.3032	94.0454	17.0726***	196.4177	81.5258
Q(10)	21.2855*	616.4022	280.9847	23.6824	$1.9907 \cdot 10^3$	$1.2056 \cdot 10^3$
Q(20)	29.6668**	975.2137	385.0340	43.8423	$3.4164 \cdot 10^3$	$2.0468 \cdot 10^3$

Table C2: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	FTSE 100			EURO STOXX 50		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-8.9263	0	0	-8.7698	0.0003	0
Maximum	9.4849	9.4849	89.9628	10.5536	10.5536	111.3777
Mean	0.0015	0.7940	1.3231	-0.0082	0.9987	2.0640
St. dev.	1.1504	0.8324	3.8476	1.4368	1.0329	5.4002
Skewness	-0.1655	2.8609	10.4733	-0.0883	2.5106	8.0896
Kurtosis	9.4553	17.2889	170.2831	7.8424	13.1764	102.2814
Kolmogorov	0.0746	0.1699	0.3653	0.0721	0.1667	0.3509
Anderson-Darling	61.7084	Inf	Inf	57.8040	Inf	Inf
LM(10)	13.5088***	187.9792	51.9822	16.1015**	257.7858	84.6218
LM(20)	20.5577***	218.3115	62.6131	27.5558***	296.6760	124.2736
Q(10)	49.9655	$3.8500 \cdot 10^3$	$3.2875 \cdot 10^3$	48.2402	$3.1558 \cdot 10^3$	$2.4593 \cdot 10^3$
Q(20)	71.1369	$6.4026 \cdot 10^3$	$5.3733 \cdot 10^3$	77.9246	$5.4984 \cdot 10^3$	$3.9450 \cdot 10^3$

Table C3: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	Nasdaq 100			S&P 500		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-10.2244	0.0004	0	-9.6884	0	0
Maximum	13.2782	13.2782	176.3111	10.6420	10.6420	113.2527
Mean	0.0125	1.0606	2.4772	0.0138	0.7926	1.4264
St. dev.	1.5740	1.1630	7.0445	1.1944	0.8935	4.5776
Skewness	-0.0385	2.6941	9.4697	-0.2135	3.0792	12.0075
Kurtosis	9.0869	14.9721	147.6574	11.3085	19.8087	221.2207
Kolmogorov	0.0929	0.1808	0.3623	0.0949	0.1873	0.3775
Anderson-Darling	Inf	Inf	Inf	Inf	Inf	Inf
LM(10)	9.0483***	325.9599	117.6129	15.5739***	193.8749	60.8058
LM(20)	21.8339***	358.8629	154.7066	26.5868***	205.1128	69.5981
Q(10)	33.4801	5.1603	3.0959	57.0401	4.6532	3.8425
Q(20)	81.8256	9.2174	5.2702	116.6969	8.0912	6.7098

Table C4: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	DJIA			Nikkei 225		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-8.6146	0.0004	0	-12.1110	0.0003	0
Maximum	10.5321	10.5321	110.9255	13.2346	13.2346	175.1542
Mean	0.0173	0.7525	1.2633	0.0039	1.0659	2.2643
St. dev.	1.1240	0.8350	4.0251	1.5049	1.0623	6.5540
Skewness	-0.1313	3.0711	12.4570	-0.4253	2.8507	12.4866
Kurtosis	11.1616	20.0176	245.9869	9.3804	19.2647	236.6990
Kolmogorov	0.0898	0.1836	0.3766	0.0638	0.1577	0.3646
Anderson-Darling	Inf	Inf	Inf	Inf	Inf	Inf
LM(10)	14.6274***	189.4704	61.8189	5.2452***	90.2202	28.5276
LM(20)	27.0501***	206.3352	73.3744	10.4588***	111.4646	44.7105
Q(10)	54.5787	$4.4314 \cdot 10^3$	$3.3496 \cdot 10^3$	16.5629**	$2.1811 \cdot 10^3$	$3.0598 \cdot 10^3$
Q(20)	106.3169	$7.7023 \cdot 10^3$	$5.9201 \cdot 10^3$	26.8295***	$3.4778 \cdot 10^3$	$4.3361 \cdot 10^3$

Table C5: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	HANG SENG			Shanghai Composite		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-13.5820	0.0003	0	-9.2114	0	0
Maximum	13.4068	13.5820	184.4714	9.5055	9.5055	90.3537
Mean	0.0089	1.0194	2.1867	0.0146	1.0781	2.4925
St. dev.	1.4789	1.0713	6.9391	1.5789	1.1534	6.5153
Skewness	-0.0945	3.1033	13.7719	-0.3466	2.4828	6.7846
Kurtosis	11.0703	22.1246	281.0459	7.8450	12.2079	64.1061
Kolmogorov	0.0725	0.1705	0.3761	0.0846	0.1748	0.3508
Anderson-Darling	Inf	Inf	Inf	68.8514	251.5708	Inf
LM(10)	4.8956***	136.9238	39.4746	19.1047*	286.8208	120.4308
LM(20)	11.4321***	207.2606	80.8078	38.5591	350.9026	154.4136
Q(10)	17.7882**	$3.1426 \cdot 10^3$	$2.9677 \cdot 10^3$	38.9358	$1.8402 \cdot 10^3$	$0.9430 \cdot 10^3$
Q(20)	33.9504*	$5.5104 \cdot 10^3$	$4.5471 \cdot 10^3$	69.3472	$3.2261 \cdot 10^3$	$1.5305 \cdot 10^3$

Table C6: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

	S&P BSE Sensex			NIFTY 50		
	Returns	Absolute	Squared	Returns	Absolute	Squared
Minimum	-11.8092	0.0001	0.0000	-13.0539	0	0
Maximum	16.1147	16.1147	259.6823	16.2255	16.2255	263.2681
Mean	0.0401	1.0140	2.1537	0.0404	1.0105	2.1315
St. dev.	1.4672	1.0610	6.7736	1.4596	1.0539	7.0260
Skewness	-0.2077	3.0308	17.4062	-0.3275	3.1894	18.8194
Kurtosis	10.9237	21.9720	529.8937	11.9130	25.1130	562.5971
Kolmogorov	0.0781	0.1694	0.3750	0.0769	0.1686	0.3806
Anderson-Darling	Inf	Inf	Inf	Inf	Inf	Inf
LM(10)	20.0483*	234.3613	92.4134	22.4616*	224.9059	85.2925
LM(20)	34.2159*	269.6858	125.1892	39.3264	261.8864	122.4319
Q(10)	46.8564	$3.2505 \cdot 10^3$	$1.3885 \cdot 10^3$	50.1953	$2.8315 \cdot 10^3$	$1.1226 \cdot 10^3$
Q(20)	74.4427	$5.2772 \cdot 10^3$	$2.0267 \cdot 10^3$	84.1268	$4.4817 \cdot 10^3$	$1.5568 \cdot 10^3$

Table C7: Descriptive statistics. Note: In the case of test statistics we asterisked the cases when we fail to reject the null hypothesis: *p-value> 0.01, **p-value> 0.05, ***p-value> 0.10.

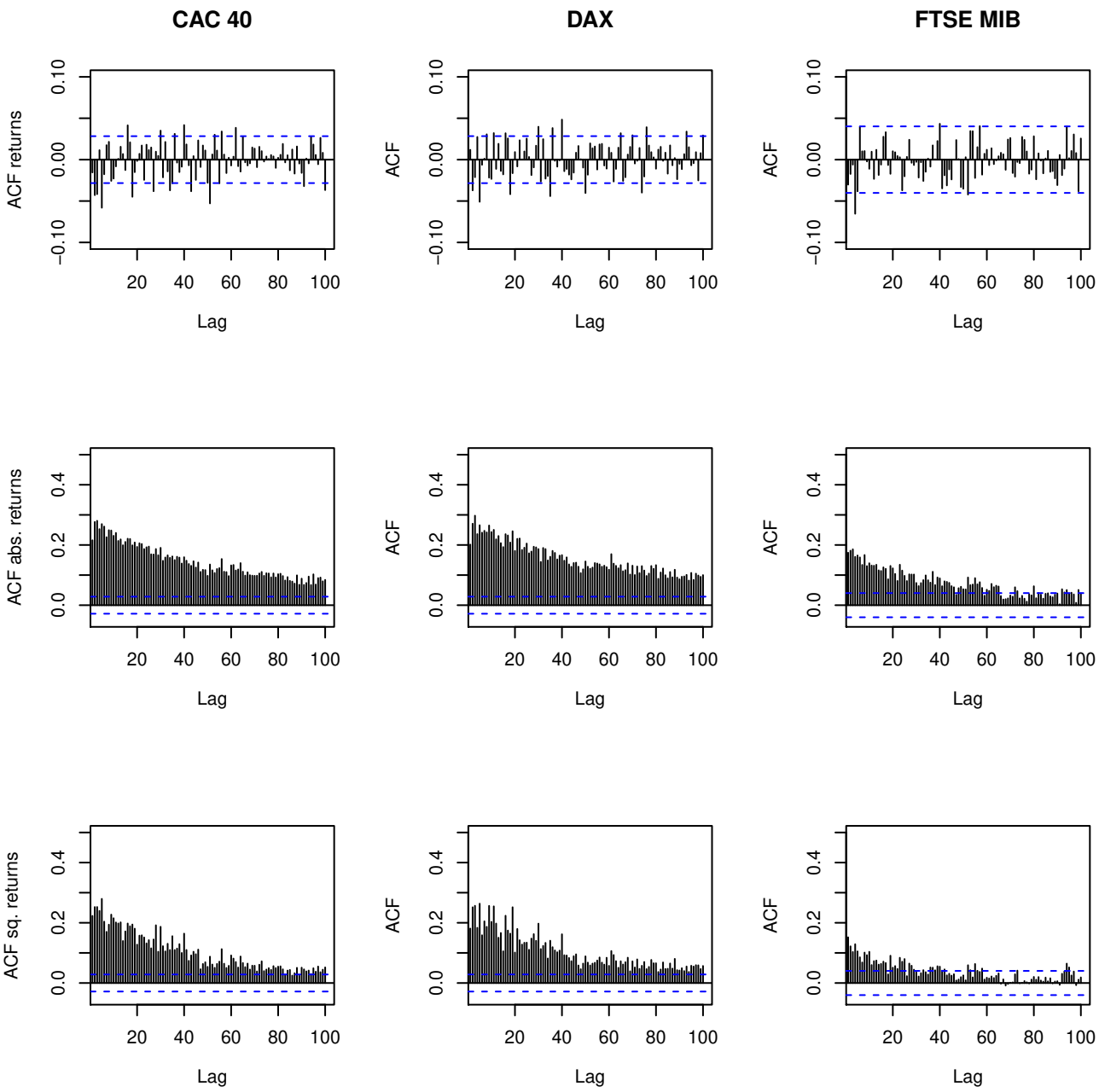

Figure C1: ACF plots

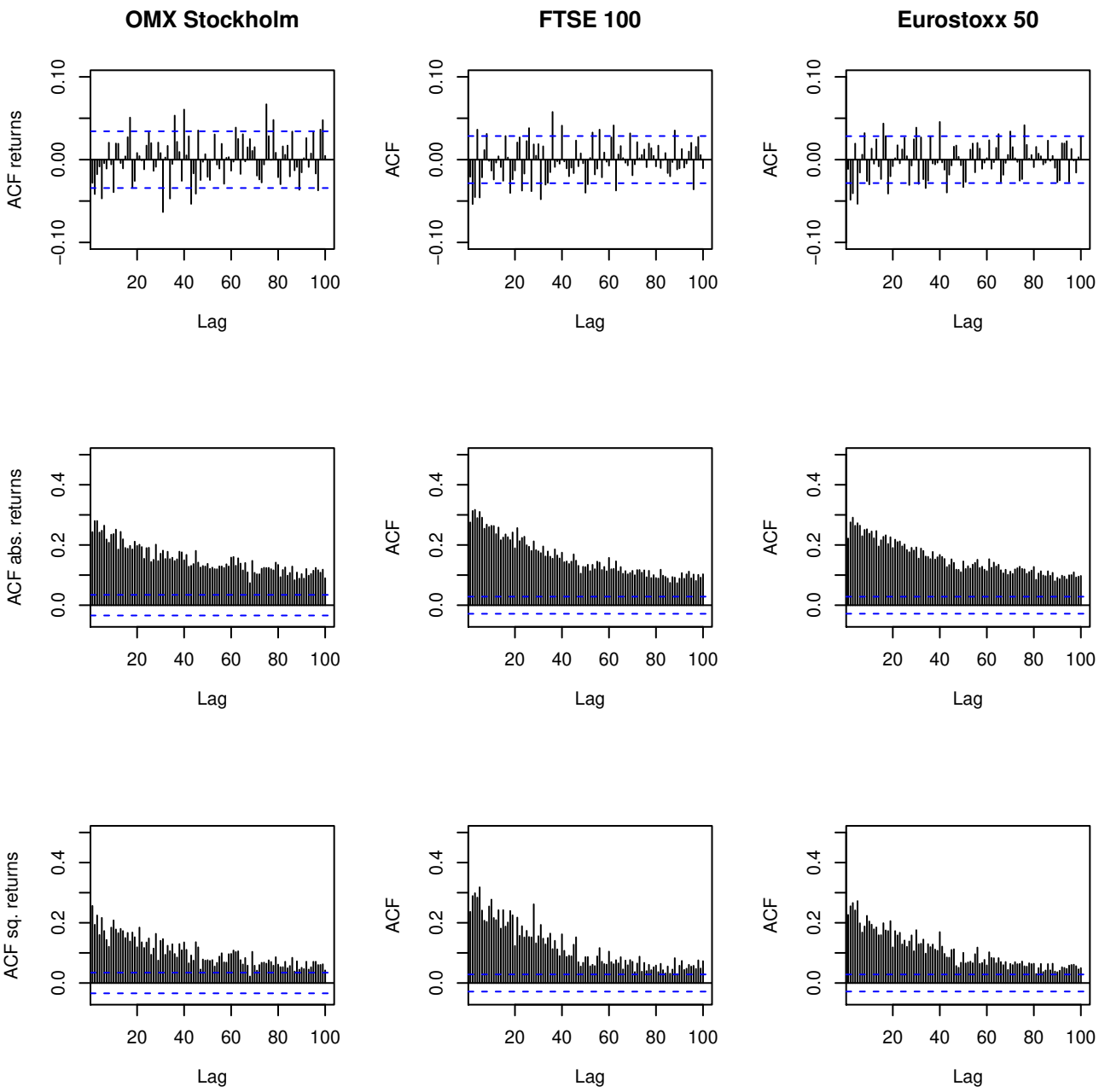

Figure C2: ACF plots

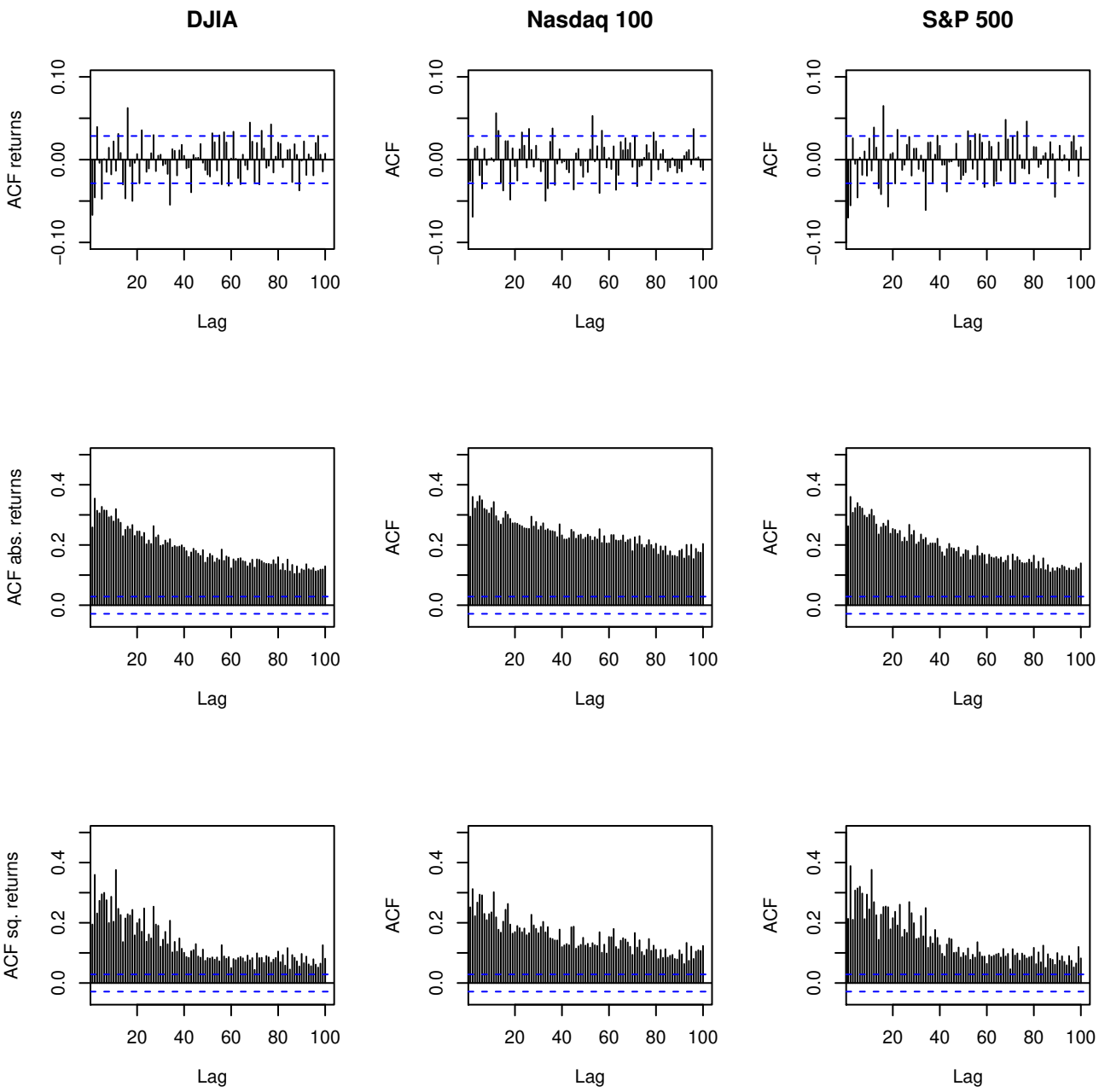

Figure C3: ACF plots

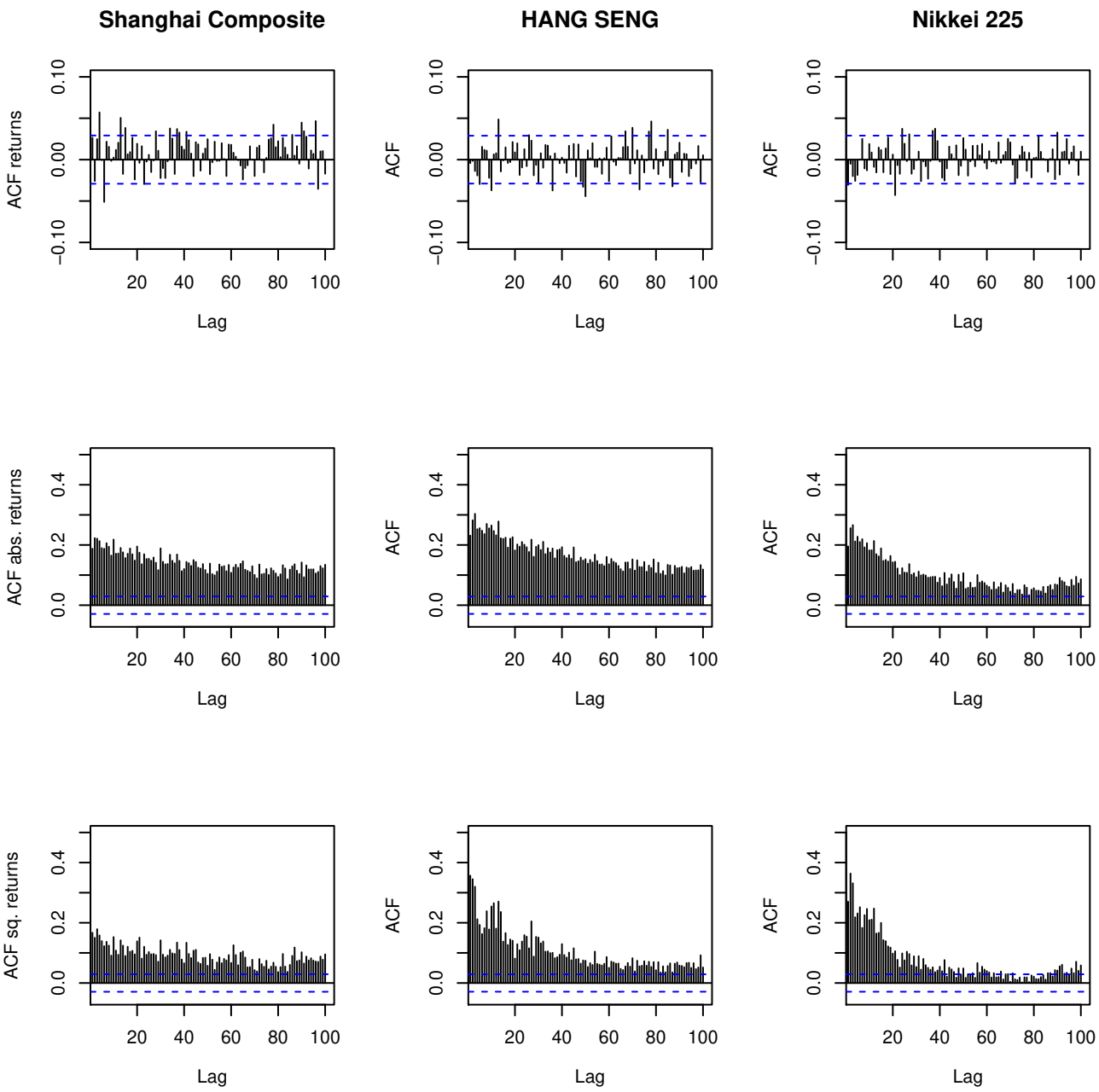

Figure C4: ACF plots

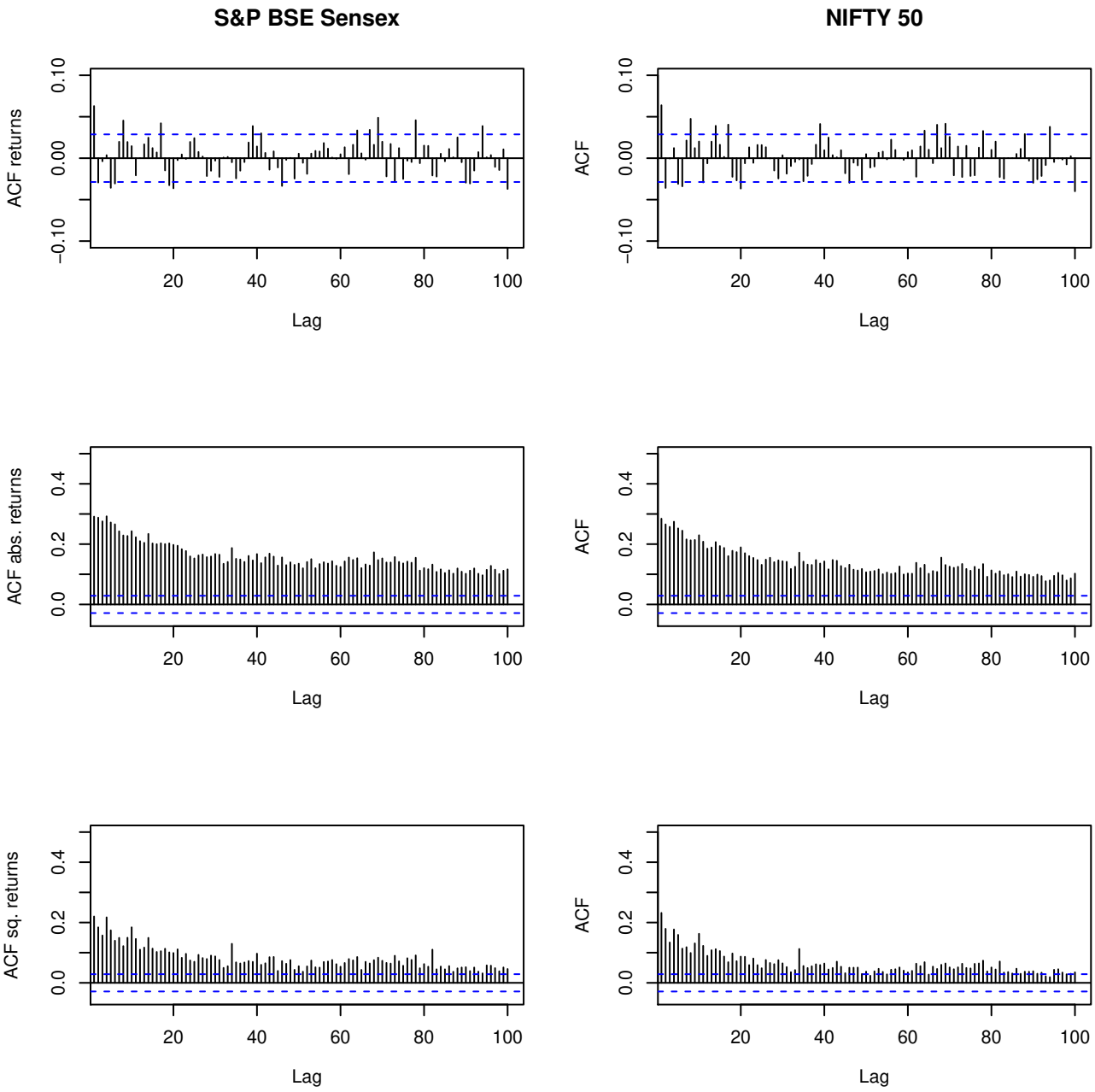

Figure C5: ACF plots

D Estimation results

	GARCH			GJR-GARCH				RV-ARFIMA					
	$\widehat{\omega}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\omega}$	$\widehat{\alpha}$	$\widehat{\varsigma}$	$\widehat{\beta}$	$\widehat{d}$	$\widehat{\mu}$	$\widehat{\phi_1}$	$\widehat{\phi_2}$	$\widehat{\phi_4}$	$\widehat{\theta}$
CAC 40	0.022	0.087	0.904	0.026		0.152	0.910	0.494	-0.196	0.091			0.359
	(0.004)	(0.006)	(0.007)	(0.003)		(0.012)	(0.008)	(0.008)	(2.481)	(0.059)			(0.057)
DAX	0.024	0.091	0.899	0.029		0.151	0.909	0.496	0.009	-0.238	-0.052		
	(0.004)	(0.007)	(0.008)	(0.003)		(0.011)	(0.008)	(0.004)	(3.341)	(0.016)	(0.016)		
FTSE MIB	0.078	0.072	0.898	0.093		0.142	0.890	0.467	-0.072				0.112
	(0.021)	(0.009)	(0.014)	(0.019)		(0.020)	(0.017)	(0.027)	(0.968)				(0.038)
OMX Stockholm	0.039	0.124	0.859	0.045	0.026	0.196	0.854	0.488	-0.612				0.234
	(0.006)	(0.008)	(0.009)	(0.005)	(0.004)	(0.013)	(0.008)	(0.014)	(2.088)				(0.025)
FTSE 100	0.016	0.109	0.881	0.019		0.166	0.898	0.496	-0.331				0.218
	(0.003)	(0.007)	(0.008)	(0.002)		(0.012)	(0.008)	(0.005)	(3.063)				(0.017)
EURO STOXX 50	0.021	0.087	0.905	0.024		0.156	0.910	0.499	0.581	0.103			0.389
	(0.004)	(0.006)	(0.007)	(0.003)		(0.011)	(0.007)	(0.000)	(7.301)	(0.051)			(0.046)
DJIA	0.016	0.098	0.890	0.016		0.161	0.904	0.494	-0.516	-0.102			0.147
	(0.002)	(0.007)	(0.007)	(0.002)		(0.010)	(0.007)	(0.008)	(2.782)	(0.066)			(0.068)
Nasdaq 100	0.017	0.079	0.913	0.021		0.132	0.922	0.494	-0.118				0.213
	(0.003)	(0.007)	(0.007)	(0.003)		(0.011)	(0.008)	(0.008)	(2.530)				(0.018)
S&P 500	0.017	0.094	0.894	0.019		0.162	0.901	0.494	-0.591	-0.128			0.164
	(0.002)	(0.007)	(0.007)	(0.002)		(0.012)	(0.008)	(0.007)	(2.902)	(0.056)			(0.057)
Shanghai Composite	0.030	0.075	0.915	0.031	0.062	0.026	0.915	0.478	0.107				0.141
	(0.004)	(0.005)	(0.005)	(0.004)	(0.006)	(0.007)	(0.005)	(0.017)	(1.303)				(0.025)
HANG SENG	0.017	0.066	0.925	0.025	0.021	0.080	0.924	0.487	-0.408				0.314
	(0.003)	(0.005)	(0.006)	(0.003)	(0.006)	(0.008)	(0.007)	(0.014)	(1.566)				(0.022)
Nikkei 225	0.046	0.106	0.877	0.057	0.047	0.106	0.874	0.449	-0.490				0.188
	(0.008)	(0.008)	(0.009)	(0.008)	(0.007)	(0.009)	(0.011)	(0.021)	(0.775)				(0.029)
NIFTY 50	0.050	0.118	0.863	0.062	0.047	0.138	0.857	0.482	-0.575			0.046	0.157
	(0.006)	(0.007)	(0.008)	(0.006)	(0.007)	(0.012)	(0.008)	(0.017)	(1.611)			(0.016)	(0.024)
S&P BSE Sensex	0.045	0.114	0.868	0.053	0.047	0.128	0.866	0.477	-0.197				0.109
	(0.005)	(0.007)	(0.007)	(0.005)	(0.006)	(0.011)	(0.007)	(0.018)	(1.362)				(0.025)

Table D1: Estimation results. Note: ML parameter estimates for GARCH, GJR-GARCH and RV-ARFIMA. We sometimes drop the ARCH parameter α from the GJR-GARCH model in order to improve goodness of fit. Most of the times RV-ARFIMA[0,d,1] fitted best wheras in some cases alternative specifications with autoregressive terms were necessary. In-sample period between first observation available and 2015/12.

	LMSM		RV-LMSM		BMSM		RV-BMSM			MRW			RV-MRW	
	$\widehat{\lambda}$	$\widehat{\sigma}$	$\widehat{\lambda}$	$\widehat{\sigma}$	$\widehat{m_0}$	$\widehat{\sigma}$	$\widehat{m_0}$	$\widehat{\phi}$	$\widehat{\xi}$	$\widehat{\lambda^2}$	$\widehat{\ln T}$	$\widehat{\ln \sigma}$	$\widehat{\lambda^2}$	$\widehat{\ln T}$
CAC 40	0.006	1.363	0.078	1.435	1.279	1.479	1.264	0.130	0.297	0.014	9.528	0.277	0.014	9.780
	(0.026)	(0.081)	(0.003)	(0.047)	(0.009)	(0.052)	(0.007)	(0.025)	(0.005)	(0.005)	(2.011)	(0.054)	(0.002)	(1.099)
DAX	0.047	1.489	0.059	1.417	1.288	1.531	1.274	0.252	0.245	0.016	9.235	0.249	0.013	11.790
	(0.023)	(0.085)	(0.002)	(0.057)	(0.010)	(0.056)	(0.006)	(0.024)	(0.005)	(0.005)	(1.872)	(0.061)	(0.002)	(1.882)
FTSE MIB	0	1.616	0.053	1.213	1.223	1.690	1.240	0.228	0.228	0.019	4.615	0.476	0.017	5.862
	(0.031)	(0.074)	(0.002)	(0.049)	(0.019)	(0.109)	(0.015)	(0.080)	(0.008)	(0.005)	(0.887)	(0.040)	(0.002)	(0.677)
OMX Stockholm	0.043	1.450	0.094	1.301	1.305	1.430	1.310	0.143	0.314	0.021	9.731	0.329	0.019	10.665
	(0.027)	(0.088)	(0.004)	(0.085)	(0.013)	(0.075)	(0.011)	(0.063)	(0.007)	(0.005)	(1.489)	(0.060)	(0.003)	(1.325)
FTSE 100	0.014	1.178	0.068	1.048	1.299	1.215	1.276	0.130	0.247	0.027	7.700	0.092	0.012	11.962
	(0.022)	(0.078)	(0.002)	(0.075)	(0.009)	(0.047)	(0.008)	(0.036)	(0.005)	(0.005)	(0.756)	(0.067)	(0.002)	(1.560)
EURO STOXX 50	0.027	1.493	0.069	1.407	1.288	1.485	1.270	0.205	0.314	0.014	11.426	0.311	0.015	10.387
	(0.022)	(0.076)	(0.004)	(0.063)	(0.009)	(0.052)	(0.008)	(0.032)	(0.011)	(0.004)	(2.415)	(0.055)	(0.002)	(1.212)
DJIA	0.100	1.111	0.077	1.204	1.315	1.211	1.288	0.127	0.273	0.020	8.665	-0.082	0.011	13.830
	(0.026)	(0.075)	(0.003)	(0.061)	(0.011)	(0.051)	(0.008)	(0.037)	(0.005)	(0.007)	(1.535)	(0.069)	(0.002)	(1.980)
Nasdaq 100	0.005	1.536	0.064	1.270	1.296	1.646	1.285	0.248	0.249	0.010	24.967	0.401	0.014	14.319
	(0.021)	(0.090)	(0.002)	(0.078)	(0.009)	(0.048)	(0.007)	(0.029)	(0.005)	(0.005)	(9.918)	(0.073)	(0.002)	(1.887)
S&P 500	0.052	1.222	0.093	1.321	1.314	1.293	1.292	0.076	0.306	0.023	9.046	0.051	0.012	15.499
	(0.024)	(0.089)	(0.002)	(0.052)	(0.011)	(0.057)	(0.008)	(0.048)	(0.005)	(0.006)	(1.270)	(0.072)	(0.002)	(2.446)
Shanghai Composite	0.005	1.729	0.058	1.510	1.296	1.667	1.273	0.310	0.238	0.018	11.389	0.526	0.015	16.050
	(0.023)	(0.068)	(0.002)	(0.051)	(0.012)	(0.074)	(0.008)	(0.037)	(0.006)	(0.004)	(2.153)	(0.030)	(0.002)	(4.067)
HANG SENG	0.038	1.541	0.074	1.208	1.284	1.597	1.246	0.032	0.290	0.007	19.975	0.255	0.009	15.064
	(0.023)	(0.101)	(0.002)	(0.050)	(0.012)	(0.067)	(0.008)	(0.036)	(0.005)	(0.006)	(12.877)	(0.062)	(0.002)	(2.121)
Nikkei 225	0.051	1.586	0.089	0.985	1.270	1.624	1.284	0.124	0.308	0.010	8.878	0.275	0.013	9.388
	(0.023)	(0.092)	(0.003)	(0.075)	(0.012)	(0.074)	(0.010)	(0.044)	(0.006)	(0.006)	(2.770)	(0.055)	(0.002)	(1.054)
NIFTY 50	0.020	1.565	0.080	1.297	1.290	1.575	1.310	0.087	0.248	0.025	7.132	0.435	0.016	10.756
	(0.023)	(0.078)	(0.003)	(0.050)	(0.012)	(0.075)	(0.007)	(0.039)	(0.006)	(0.005)	(0.819)	(0.051)	(0.002)	(1.334)
S&P BSE Sensex	0.033	1.543	0.084	1.286	1.292	1.592	1.288	0.244	0.249	0.026	7.484	0.436	0.022	7.859
	(0.022)	(0.082)	(0.003)	(0.061)	(0.012)	(0.073)	(0.008)	(0.026)	(0.005)	(0.005)	(0.884)	(0.050)	(0.002)	(0.759)

Table D2: Estimation results. Note: GMM estimates for LMSM, RV-LMSM, MRW and RV-MRW, ML estimates for BMSM and RV-BMSM. In-sample period between first observation available and 2015/12. In order to prevent zero estimates for λ, the LMSM in-sample period slightly deviates in the case of four indices: we use 2000/10 – 2015/12 for FTSE 100, Nasdaq 100 and Shanghai Composite, and 2003/03 – 2015/12 for CAC 40.

E Forecast illustrations

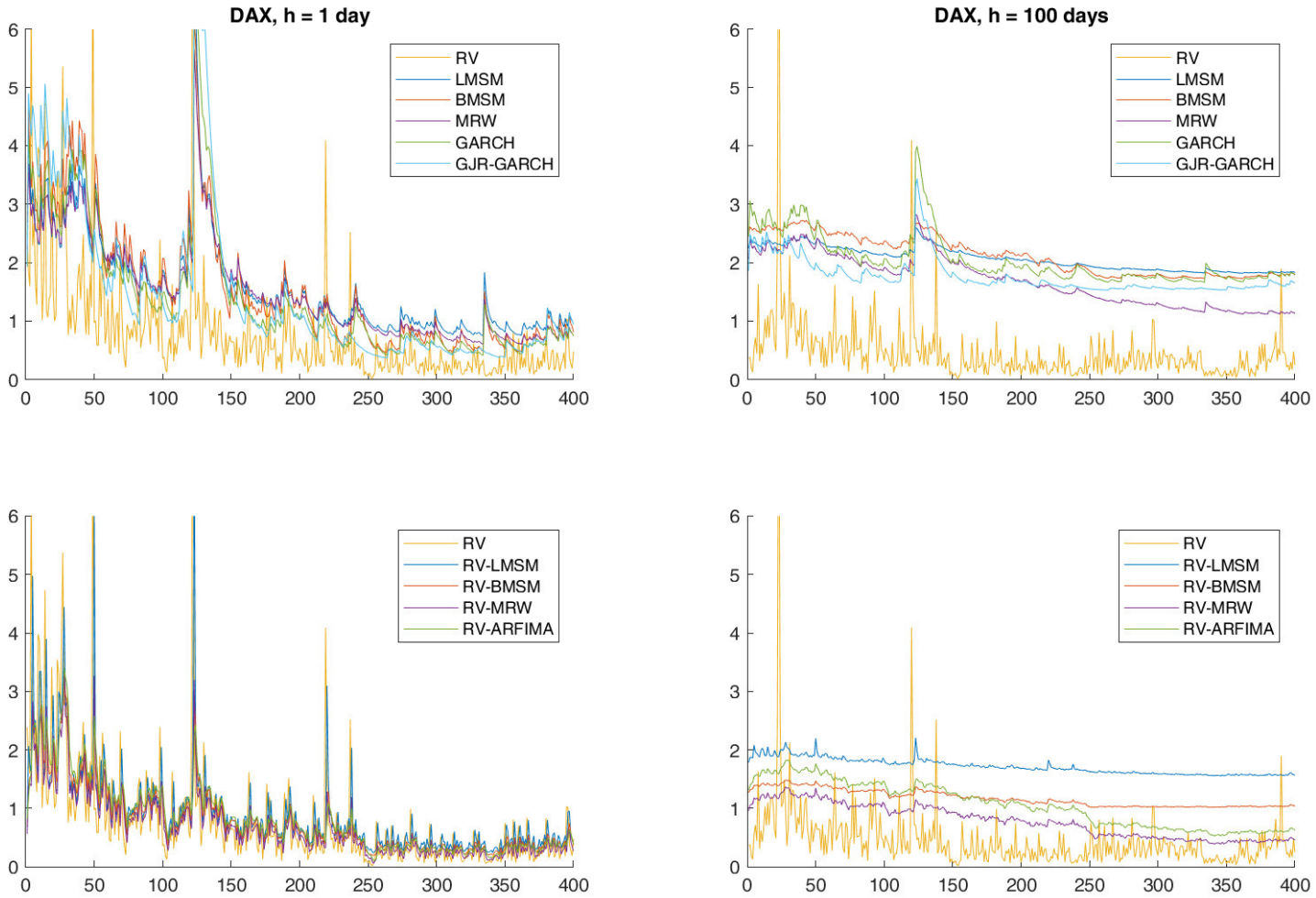

Figure E1: Forecasts for the DAX index at horizons $h = 1$ day (plots on the left) and $h = 100$ days (plots on the right) in comparison to realized volatility (RV), time period: 2016/01 – 2017/08. We distinguish between models based on returns (upper plots) and RV models (lower plots).

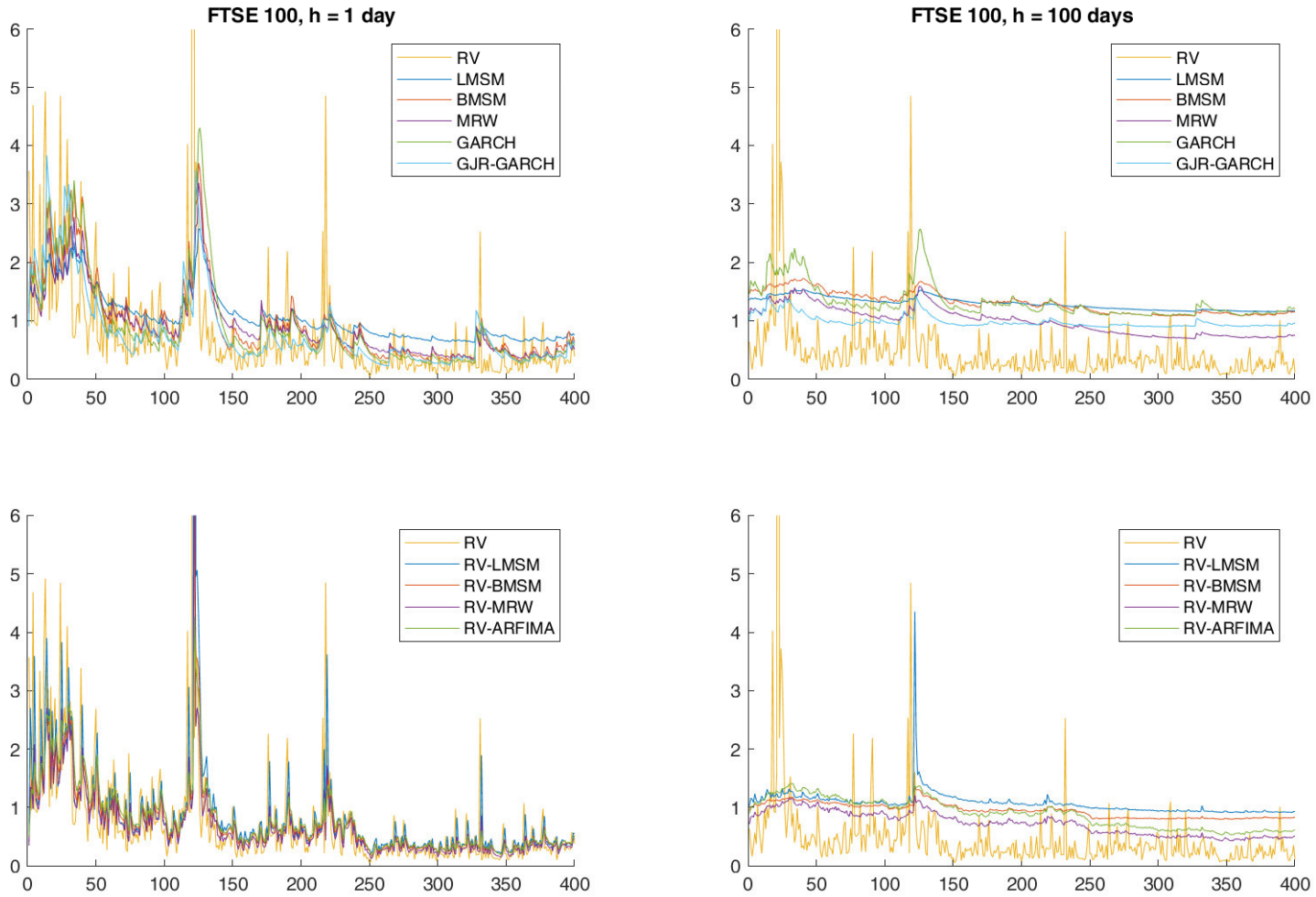

Figure E2: Forecasts for the FTSE 100 index at horizons $h = 1$ day (plots on the left) and $h = 100$ days (plots on the right) in comparison to RV, time period: 2016/01 – 2017/08. We distinguish between models based on returns (upper plots) and RV models (lower plots).

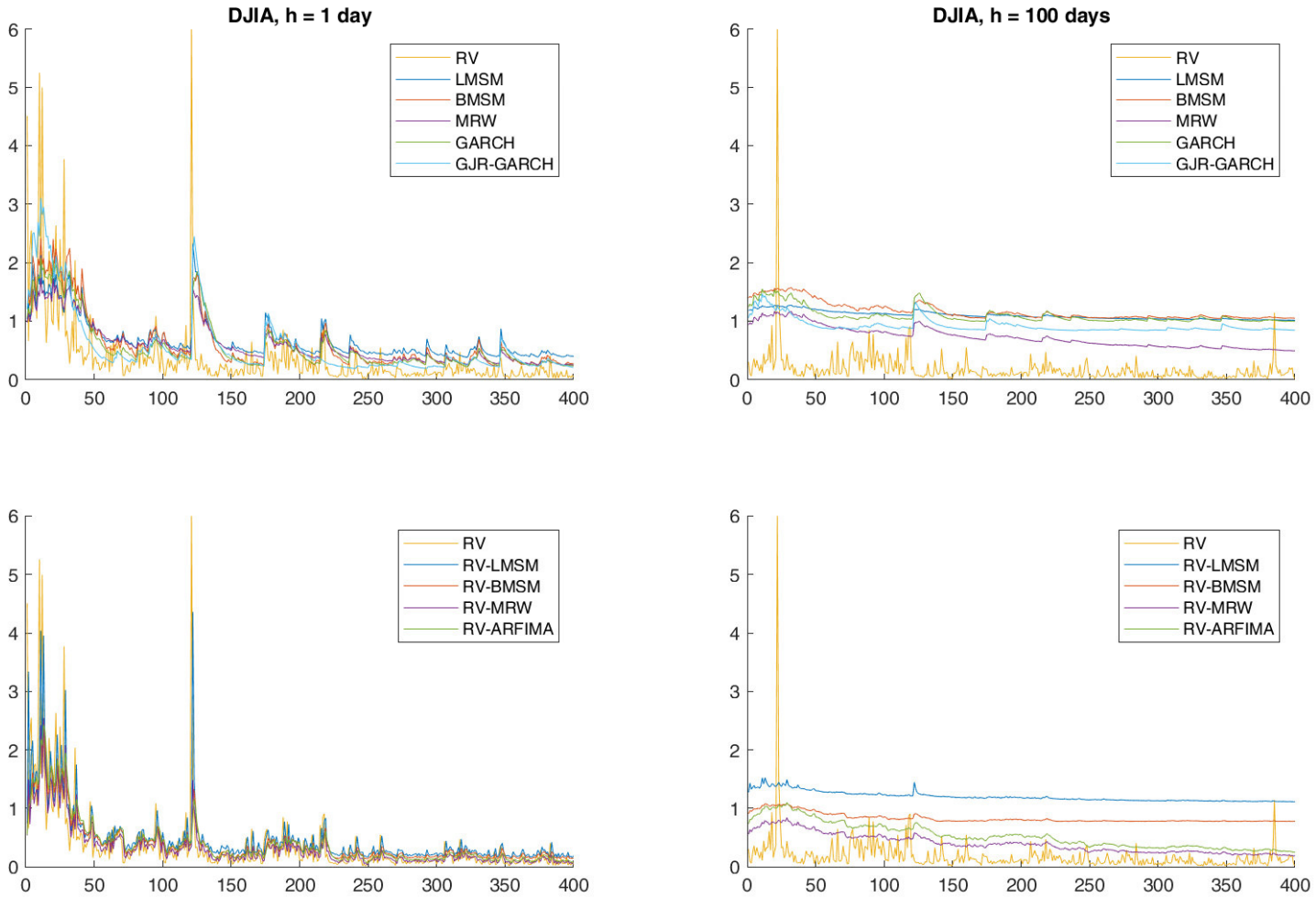

Figure E3: Forecasts for the DJIA index at horizons $h = 1$ day (plots on the left) and $h = 100$ days (plots on the right) in comparison to RV, time period: 2016/01 – 2017/08. We distinguish between models based on returns (upper plots) and RV models (lower plots).

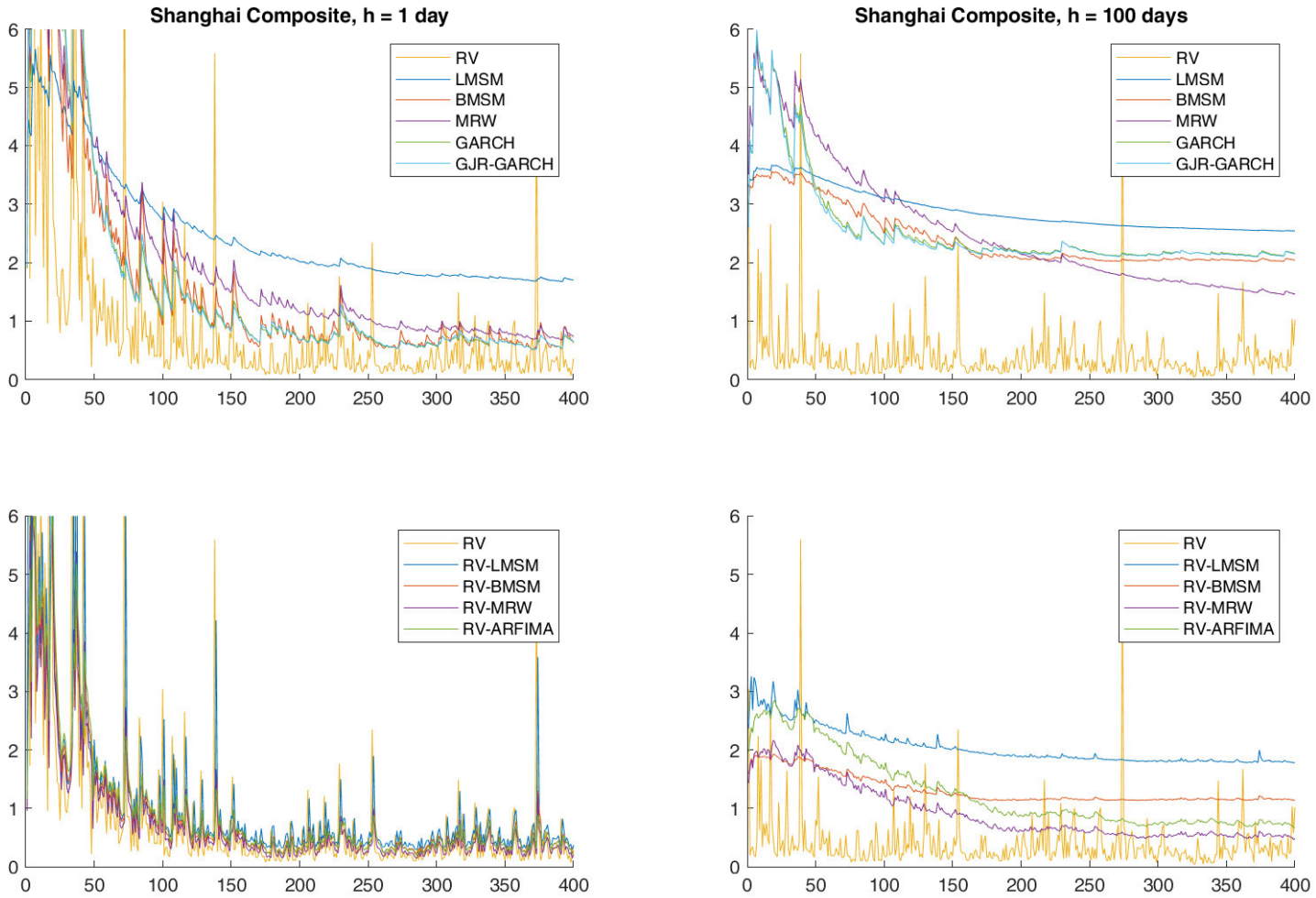

Figure E4: Forecasts for the Shanghai Composite index at horizon $h = 1$ day (plots on the left) and $h = 100$ days (plots on the right) in comparison to RV, time period: 2016/01 – 2017/08. We distinguish between models based on returns (upper plots) and RV models (lower plots).

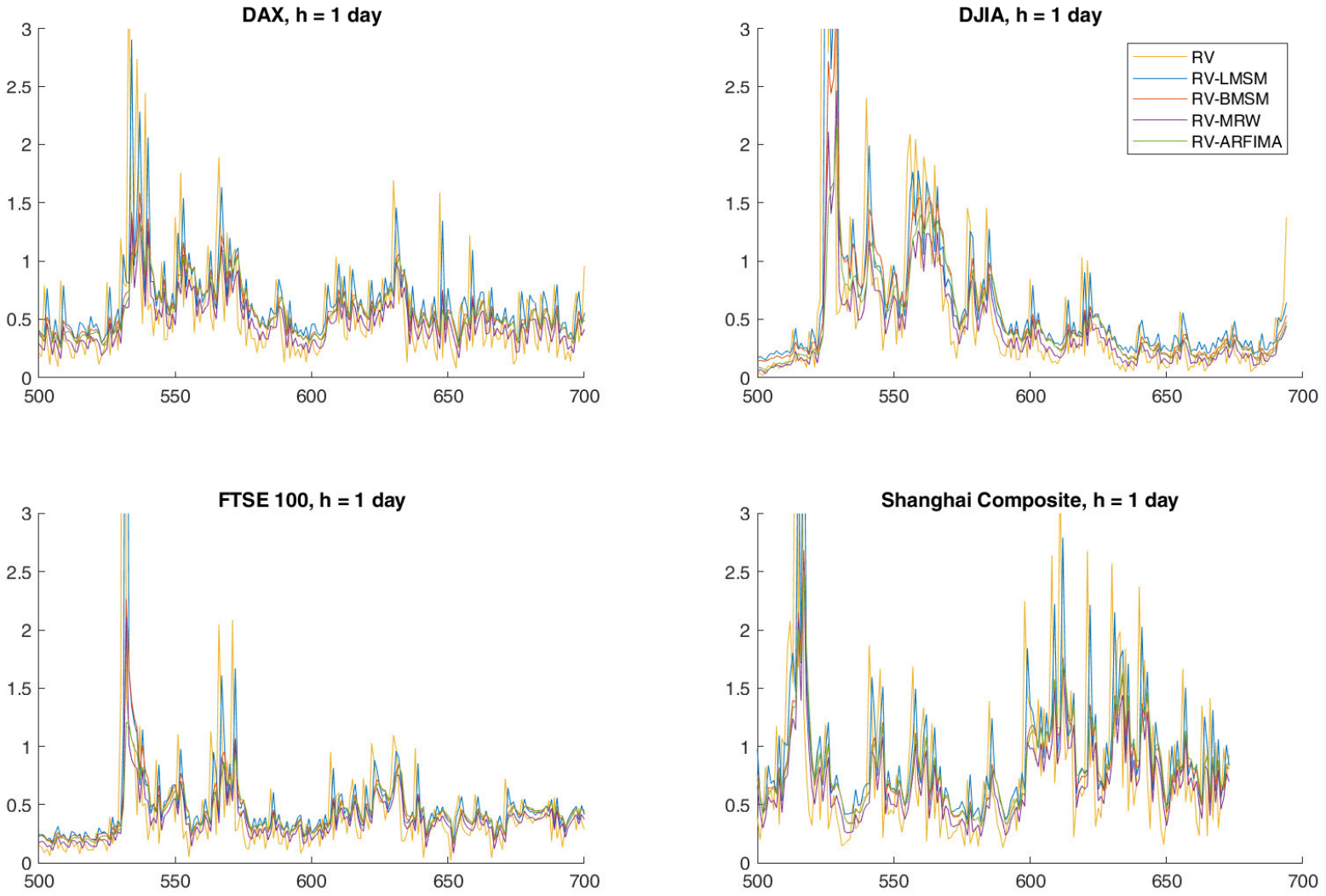

Figure E5: One-day ahead forecasts from the RV models for the indices DAX, FTSE 100, DJIA and Shanghai Composite for the out-of-sample period January 2018 till October 2018 in comparison to RV.