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Working Paper

Dynamic Information Acquisition, Complementarity, and Market Liquidity

Working Paper, No. 2020-04

Provided in Cooperation with:

Department of Economics, Rutgers University

Suggested Citation: Cai, Zhifeng (2020) : Dynamic Information Acquisition, Complementarity, and Market Liquidity, Working Paper, No. 2020-04, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:

<https://hdl.handle.net/10419/246485>

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Dynamic Information Acquisition, Complementarity, and Market Liquidity

September 22, 2018

Abstract

This paper studies dynamic information acquisition in financial markets with information asymmetry. It first shows that multiplicity can arise in the information market due to a dynamic complementarity in information acquisition. It then characterizes interactions between information complementarity and market liquidity, in particular how market liquidity shapes information complementarity through the liquidity component in future stock returns. I find that i) information complementarity is always more prominent in low-volatility financial market equilibrium; ii) information complementarity can be more prominent with less persistent stock fundamental and/or more persistent stock supply and iii) regardless of the type of financial market equilibrium, public disclosure always makes information complementarity less prominent.

Keywords: Information acquisition; Financial markets; Dynamic complementarity; Multiplicity; Market Liquidity

1 Introduction

Information acquisition activities are pervasive in financial markets. Since [Grossman and Stiglitz \(1980\)](#) and [Hellwig \(1980\)](#), economists have been seeking to understand incentives behind investors' decision to acquire information as well as the nature of interactions among market participants. Most of this literature has focused on a static marketplace. The real-world financial market, on the other hand, is inherently dynamic where investors condition their information choices not only on what their peers know, but also on what information is available from the past, as well as what information will be incorporated into the stock price in the future. In a sense, dynamic information acquisition resembles a repeated game across investors at different points in time. Understanding the nature of dynamic coordination in information acquisition is important and may unveil some key forces that are absent when the information market is modeled as a static environment.

This paper seeks to understand coordination motives in information acquisition in a dynamic financial market with information asymmetry. It incorporates the static information-acquisition model of [Grossman and Stiglitz \(1980\)](#) into a dynamic noisy rational expectations framework (as in [Wang 1994](#); [Spiegel 1998](#); [Watanabe 2008](#)). In the model, there is a long-lived stock that pays a dividend each period. The dividend is stochastic and consists of a persistent component (the stock fundamental) and a noisy component. The stock's supply follows some mean-reverting process. Overlapping generations of investors, upon their birth, freely observe the entire history of stock prices, dividends, and public signals. They are then offered an opportunity to become informed, i.e., to observe the history of the stock fundamental at a cost.

In this environment, there are two main forces that shapes coordination motives in information choices. The first one is the classic static substitutability in information acquisition as in [Grossman and Stiglitz \(1980\)](#): as more peer investors get informed, the value of information decreases due to price learning. The second one is a dynamic complementarity in information acquisition: as more investors get informed in the future, the incentives to acquire information today increase, as the future resale stock price becomes more sensitive to the stock fundamental. This paper shows that

the dynamic complementarity may overpower the static substitutability and leads to an upward-sloping value of information as a function of steady-state share of informed investors. This suggests that multiplicity arises with appropriate level of information cost.

Previous literature has studied the dynamic complementarity in information acquisition in finite-horizon setups (Froot et al. 1992; Avdis 2016). Relative to the literature, this paper analyses interactions between information choices and market liquidity in an infinite-horizon overlapping-generation framework. The relation between information choice and liquidity is trivial in static, or finite-horizon models: more informed investors always imply a more liquid financial market, in the sense that liquidity traders have less price impact. In infinite-horizon models with overlapping-generations of investors, however, this relation becomes nontrivial: with more investors acquiring information, the financial market could become less liquid. This brings about interesting interactions between market liquidity and information choice incentives, to the extent that value of information is affected by the liquidity component in the future resale stock price. This is the *dynamic liquidity channel* that this paper will focus on.

The key contribution of the paper is to characterize information complementarity and its interaction with market liquidity under different types of financial market equilibria. It is well known that infinite-horizon overlapping-generation models with asymmetric information typically exhibit two financial market equilibria with different levels of stock market volatility (Spiegel, 1998; Bacchetta and Van Wincoop, 2006; Watanabe, 2008; Biais et al., 2010). The paper derives a unified necessary and sufficient condition under which dynamic information multiplicity arises across both types of financial market equilibrium.

Utilizing this condition, I first show that dynamic information multiplicity is always more likely to arise in the low-volatility equilibrium. That is, the low-volatility equilibrium is the more "fragile" equilibrium when endogenous information choice is incorporated. It is surprising given that the low-volatility equilibrium, when information choice is taken exogenous, is generally perceived to be the more stable and robust equilibrium.¹ The key mechanism that leads to this result is the dynamic

¹This paper does not take a stand on which financial market equilibrium one should select, as both high-volatility

liquidity channel. In the low-volatility equilibrium, more informed investors implies a more liquid financial market, or equivalently noise traders have less price impact. This implies that future stock return is less affected by liquidity trading, and thus fundamental information is more valuable, increasing the value of information. This makes dynamic coordination in information market relatively easy to achieve. In the high-volatility equilibrium, in contrast, more informed investors imply a less liquid financial market and thus a lower value of information. Hence, information multiplicity is less likely to arise in high-volatility equilibrium.

I then show that this dynamic liquidity channel offers new insights of comparative statics regarding dynamic information complementarity. For example, [Avdis \(2016\)](#) finds that a more persistent stock supply always makes information multiplicity less prominent. This is not true in the overlapping-generation framework, as a more persistent stock supply could strengthen the liquidity channel and makes an upward-sloping value of information easier to arise. I also examine the relation between information complementarity and fundamental persistence. One might expect this relation to be trivial as a more persistent stock fundamental implies a stronger intergenerational link which makes dynamic coordination easier to achieve. Surprisingly, the model predicts a non-monotonic relation between information multiplicity and fundamental persistence, again due to the liquidity channel.

Lastly, the paper considers issues related to policy. Would public disclosure reduce information fragility? The answer is yes: information fragility always becomes less prominent with more precise public signals. More importantly, this conclusion is robust to the selection of financial market equilibrium. Thus, for a regulator aiming to stabilize asset markets, disclosing more precise public information is helpful as it helps eliminate equilibrium multiplicity in information acquisition. This result contributes to the recent debate on the desirability of the regulatory effort to provide more precise public information, such as the Sarbanes-Oxley Act and, more recently, the Dodd-Frank Act.

Literature Review This paper is closely related to a set of papers that explore information and low-volatility equilibria has desirable features. See discussions in [Albagli \(2015\)](#) and [Banerjee \(2011\)](#).

multiplicity in dynamic financial markets (see [Froot et al. \(1992\)](#) and [Avdis \(2016\)](#)). The key differentiation of this paper is the overlapping-generation structure and the interactions between information choice and market liquidity that arise within the framework. This paper characterizes information multiplicity in both low-volatility and high-volatility financial market equilibrium. It also finds that the dynamic liquidity channel changes some predictions of the previous literature. For example, [Avdis \(2016\)](#) finds that information complementarity becomes less prominent with more persistent stock supply. This relation can be reversed in an overlapping generation framework due to the liquidity channel. [Banerjee and Breon-Drish \(2018\)](#) studies a dynamic information acquisition problem of a strategic trader in a continuous-time setup and shows that the optimal solution exhibits delay. This paper, on the other hand, focuses on the nature of coordination in a dynamic information market. [Cai \(2018\)](#) also examines dynamic information acquisition in overlapping-generation frameworks. Its focus, however, is on model dynamics and the interaction between information choice and (endogenous and exogenous) variations in uncertainty.

The paper is also related to the literature that studies exogenous asymmetric information trading models in an infinite horizon, pioneered by [Wang \(1993, 1994\)](#) and [Campbell and Kyle \(1993\)](#). It is particularly related to models that study overlapping generations of investors ([Spiegel, 1998](#); [Bacchetta and Van Wincoop, 2006](#); [Watanabe, 2008](#); [Biais et al., 2010](#); [Albagli, 2015](#)). Although the physical structure of my paper is very close to these papers, in my model the information acquisition choice is endogenous. [Dow and Gorton \(1994\)](#) study a dynamic overlapping-generations model with private information where, similar to this paper, a dynamic informational linkage is present: information gets incorporated into the price only if informed traders expect future traders to also impound their information in the price. Unlike this paper, however, it does not concern the issue of multiplicity.²

Information multiplicity also arises in static environments. For example, [Veldkamp \(2006a,b\)](#) generates complementary by embedding an increasing returns to scale information production sector into an otherwise standard noisy rational expectations model. [Ganguli and Yang \(2009\)](#) illustrate

²I thank a referee for pointing this out.

that complementarity may result when agents own private information about their endowment. [García and Strobl \(2011\)](#) illustrate how relative wealth concerns leads to herding on information and information acquisition multiplicity. [Goldstein and Yang \(2015\)](#) examines information complementarity arising due to the presence of diverse information. [Mele and Sangiorgi \(2015\)](#) explore market reactions to changes in uncertainty in a static model where investors are subject to Knightian uncertainty.

2 Model Economy

Time is discrete and runs from $-\infty$ to $+\infty$. The economy is populated by a continuum of overlapping generations risk-averse agents who consume a single consumption good. The good is treated as the numeraire. There are two assets in the economy: a bond in perfect elastic supply, paying a return R ;³ and a stock that pays a dividend

$$D_t = F_t + \varepsilon_t^D \tag{2.1}$$

each period. F_t is the persistent component of the dividend process. Later I call F_t the stock fundamental. The stock fundamental follows an AR(1) process:

$$F_t = \rho^F F_{t-1} + \varepsilon_t^F, 0 \leq \rho^F \leq 1. \tag{2.2}$$

The stock supply, x_t , follows an AR(1) process as well:

$$x_t = \rho^x x_{t-1} + \varepsilon_t^x, 0 \leq \rho^x \leq 1. \tag{2.3}$$

An interpretation of the stochastic stock supply is that there exists a group of liquidity traders who trade in the financial market for liquidity reasons. Thus, the sensitivity of stock prices with respect to stock supply can be interpreted as market liquidity, as it measures the price impact of

³Alternatively one can interpret the bond as a storage technology without nonnegative constraint.

these group of liquidity traders. This interpretation is also consistent with Kyle (1985).

As in Wang (1994), I assume that there is a public signal every period about the current fundamental:

$$S_t = F_t + \varepsilon_t^S. \quad (2.4)$$

The shock vector $\varepsilon_t = [\varepsilon_t^D, \varepsilon_t^F, \varepsilon_t^x, \varepsilon_t^S]$ is i.i.d. over time, with mean 0 and covariance matrix $diag(\sigma_D^2, \sigma_F^2, \sigma_x^2, \sigma_S^2)$.

Investors live for two periods.⁴ When they are born, they are endowed with a certain amount of wealth and also observe the entire history of the dividend and stock price. They are then offered an opportunity to acquire information at some cost χ . If they choose to acquire information, they also observe the history of the stock fundamental. I call investors who choose to acquire information the "informed" investors and the rest "uninformed." The information set of the generation- t uninformed is

$$\Omega_t^U = \{P_s, D_s, S_s\}_{s=-\infty}^t,$$

and that for the informed is

$$\Omega_t^I = \{P_s, D_s, S_s, F_s\}_{s=-\infty}^t.$$

As is standard in this class of models, an informed investor, observing the history of the fundamental and stock price (denoted by P_s), can perfectly deduce the stock supply. For uninformed investors, their conditional expectations are derived from Kalman filter equations. I use \hat{F} and \hat{x} to denote the conditional mean of the current fundamental and stock supply for the uninformed:

$$\hat{F}_t = E(F_t | \Omega_t^U) \quad (2.5)$$

$$\hat{x}_t = E(x_t | \Omega_t^U). \quad (2.6)$$

After the information acquisition stage, the financial market opens and trade occurs. After that, old investors exit and consume their wealth. The timeline is summarized in figure 1. The period- t

⁴An alternative, but equivalent model is that investors live forever but are myopic when making investment decisions.

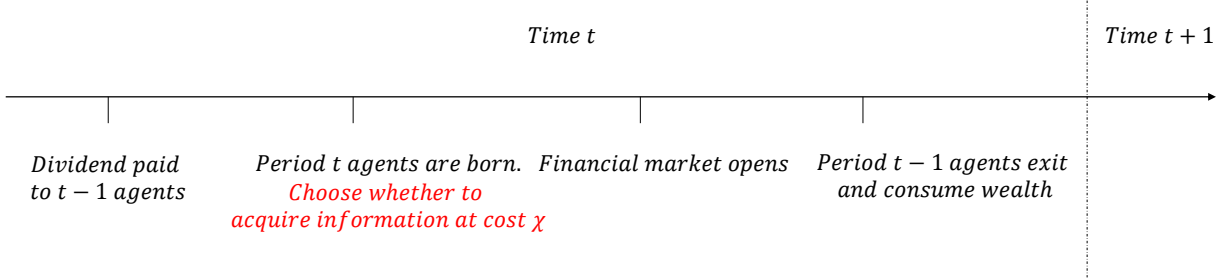


Figure 1: Timeline

born agents' problem is as follows. Upon birth, they make information acquisition choice:

$$\max\{W_t^I, W_t^U\},$$

where W_t^I denotes the expected utility of generation- t informed investors, and W_t^U denotes the expected utility for the generation- t uninformed. Then, conditional on the information set, they make their portfolio choice to maximize expected utility derived from terminal consumption:

$$\begin{aligned} W_t^i &= \max_{s,c} E(U(c)|\Omega_t^i) \\ c &\leq (D_{t+1} + P_{t+1} - RP_t)s + R(w - \mathbb{1}\{i = I\}\chi), \end{aligned} \tag{2.7}$$

where s denotes the number of stock shares to purchase and c denotes terminal consumption. As as standard in the literature, I assume that utility is exponential:

$$U(c) = -\exp(-\alpha c)$$

Where α is the risk-averse parameter.

I will focus on the stationary equilibrium where the pricing function is constant over time. Let λ be the steady-state share of informed investors.

Definition 2.1 Denote the state of the economy $\phi = \{\hat{F}, F, x\}$.

A steady state is $\{P(\phi), \lambda, \{s_i(\phi), c_i(\phi)\}_{i=U,I}\}$ s.t.

1. $s_i(\phi), c_i(\phi)$ solves the uninformed and informed agents' problem given $P(\phi)$.
2. The market clears: $\lambda s_I(\phi) + (1 - \lambda) s_U(\phi) = x(\phi)$.
3. $W_U = W_I$ if $\lambda \in (0, 1)$; if $\lambda = 0$, $W_U \geq W_I$; if $\lambda = 1$, $W_U \leq W_I$,

where \hat{F} is the conditional expectation defined by equation 2.5. The last condition guarantees that agents' information choice is optimal. For instance, if there is a positive fraction of both informed and uninformed investors ($\lambda \in (0, 1)$), it has to be the case that the expected utility of the informed and the expected utility of the uninformed are equalized.

It is challenging to solve noisy rational expectations models with general, potentially nonlinear, price functions. [Breon-Drish \(2015\)](#) shows that the linear equilibrium is the unique continuous equilibrium in the static model of [Grossman and Stiglitz \(1980\)](#). It therefore stands to reason that the dynamic model considered here has the same linear equilibrium as the unique continuous solution. I therefore focus on linear equilibria in which equilibrium stock price depends linearly on the (expected) stock fundamental and supply. That is, there exists a set of time-invariant coefficients $\{\bar{p}, p_{\hat{F}}, p_F, p_x\}$ such that

$$P_t = \bar{p} + p_{\hat{F}} \hat{F}_t + p_F F_t - p_x x_t. \quad (2.8)$$

2.1 Characterization

In this section I focus on characterizing how trading and dynamic learning works in this economy, taking agents' information choice as given. This paves the way to characterizing the information market in the next section. To begin, I define an exogenous-information steady state where the steady-state share of informed investors λ is given. The notion of steady-state equilibrium is the same as in Spiegel (1998), Watanabe (2008), and Biais et al. (2010):

Definition 2.2 *An exogenous-information steady state given λ is $\Phi(\lambda) = \{P(\phi), \lambda, \{s_i(\phi), c_i(\phi)\}_{i=U,I}\}$ such that it satisfies condition 1 and 2 in definition 2.1.*

In what follows, I start by characterizing beliefs of the investors in such an equilibrium.

Conditional Expectations of the uninformed investors

The uninformed investors form their beliefs observing the entire history of dividends, public signals, and equilibrium stock prices. Note that current prices contains information about both stock fundamental and stock supply. From the current stock price

$$P_t = \bar{p} + p_{\hat{F}} \hat{F}_t + p_F F_t - p_x x_t$$

the uninformed investors can infer a price signal:

$$S_{pt} = p_F F_t - p_x x_t$$

Note that \hat{F}_t is common knowledge. Crucially, given the price signal, stock fundamental F_t and stock supply x_t are perfectly positively correlated. This observation implies that the variance-covariance matrix for fundamental and supply is degenerate:

$$Cov(F_t, x_t | \Omega_t^U) = Cov\left(F_t, \frac{p_F F_t - S_t^P}{p_x} | \Omega_t^U\right) = \frac{p_F}{p_x} Var(F | \Omega_t^U) \quad (2.9)$$

$$Var(x_t | \Omega_t^U) = Var\left(\frac{p_F F_t - S_t^P}{p_x} | \Omega_t^U\right) = \left(\frac{p_F}{p_x}\right)^2 Var(F_t | \Omega_t^U). \quad (2.10)$$

This observation greatly simplifies the linear filtering problem faced by the uninformed investors, reducing it from two-dimensional (i.e dynamics of both F_t and x_t) to single-dimensional.⁵ Exploiting this property gives us the following proposition:

Proposition 2.1 *In an exogenous-information steady state $\Phi(\lambda)$ with price coefficients p_f, p_x :*

1. *the law of motion for $Var(F_t|\Omega_t^U)$ is characterized by*

$$\frac{1}{Var(F_{t+1}|\Omega_{t+1}^U)} = \frac{1}{Var(F_{t+1}|S_{pt+1}, \Omega_t^U)} + \frac{1}{\sigma_D^2} + \frac{1}{\sigma_S^2} \quad (2.11)$$

2. *the law of motion for $\hat{F}_t = E(F_t|\Omega_t^U)$ is characterized by*

$$\begin{aligned} \hat{F}_{t+1} &= \frac{Var(F_{t+1}|\Omega_{t+1}^U)}{Var(F_{t+1}|S_{pt+1}, \Omega_t^U)} \left[\rho^F \hat{F}_t + \frac{(\rho^F - \rho^x) \rho^F Var(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 Var(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} (S_{pt+1} - (\rho^F - \rho^x) \hat{F}_t - \rho^x S_{pt}) \right] \\ &+ \frac{Var(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} D_{t+1} + \frac{Var(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} S_{t+1} \end{aligned} \quad (2.12)$$

$$= f_1 \hat{F}_t + f_2 F_t + f_3 \varepsilon_{t+1}^F - f_4 \varepsilon_{t+1}^x + f_5 \varepsilon_{t+1}^D + f_6 \varepsilon_{t+1}^S \quad (2.13)$$

where $Var(F_{t+1}|S_{pt+1}, \Omega_t^U)$ denotes the conditional volatility of F_{t+1} upon observing the price signal S_{pt+1} but not dividend signal or public signal, and is given by:

$$Var(F_{t+1}|S_{pt+1}, \Omega_t^U) = (\rho^F)^2 Var(F_t|\Omega_t^U) + \sigma_F^2 - \frac{[(\rho^F - \rho^x) \rho^F Var(F_t|\Omega_t^U) + \sigma_F^2]^2}{(\rho^F - \rho^x)^2 Var(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2}$$

$f_i > 0$ are all functions of p_f and p_x .

The law of motion for $Var(F_t|\Omega_t^U)$ is similar to the Kalman Filter formula where the ex-post precision is the sum of the ex-ante precision plus the precision of signals. The only complication is that the price signal is correlated with the stock fundamental in a way that is different from white noises. Thus the Kalman Filter formula is not readily applicable⁶ and one needs to invoke the projection theorem of normally distributed variables to obtain the law of motion.

In a steady state, conditional volatility of stock fundamental $Var(F_t|\Omega_t^U)$ is time-invariant. Denote the steady-state volatility $Var(F|\Omega^U)$. Give equilibrium stock price, I can solve $Var(F|\Omega^U)$

⁵I thank an anonymous referee for suggesting to make it more explicit.

⁶It is readily applicable if one is willing to work in a two-dimensional space which involves matrix manipulation and is quite involved. Details are available in the appendix of previous versions of the paper.

from equation 2.11:

$$\frac{1}{\text{Var}(F|\Omega^U)} = \frac{1}{(\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2} - \frac{[(\rho^F - \rho^x)\rho^F \text{Var}(F|\Omega^U) + \sigma_F^2]^2}{(\rho^F - \rho^x)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} + \frac{1}{\sigma_D^2} + \frac{1}{\sigma_S^2} \quad (2.14)$$

This is the first restriction of exogenous-information steady state.

Excess Stock Return and Optimal Portfolios

Given the equilibrium price function and the uninformed investors' belief, one can derive the expression for excess stock return and optimal portfolios. The excess stock return consists of dividends and capital gains, less the interest cost of holding the stock:

$$Q_{t+1} = D_{t+1} + P_{t+1} - RP_t$$

Using the law of motion for F_{t+1} (equation 2.2), x_{t+1} (equation 2.3), and \hat{F}_{t+1} (equation 2.13), one can show that:

Lemma 2.1 *The excess stock return Q_{t+1} can be expressed as a linear combination of time- t variables and time- $t + 1$ innovations:*

$$Q_{t+1} = \bar{p} + e_1 \hat{F}_t - RP_t + e_2 F_t - e_3 x_t + e_4 \varepsilon_{t+1}^F - e_5 \varepsilon_{t+1}^x + e_6 \varepsilon_{t+1}^D + e_7 \varepsilon_{t+1}^S$$

Where the coefficients on fundamental and supply $\{e_2, e_3\}$ are given by:

1. $e_2 = \rho^F(1 + p_F) + \rho^F p_{\hat{F}} \left(\frac{\text{Var}(F|\Omega^U)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2} \right)$

2. $e_3 = \rho^x p_x$

One can see from the loading coefficient e_2 that the excess stock return depends on stock fundamental F through three channels: first, future dividend depends on the fundamental; second, future capital gain (stock price) depends on the fundamental; third, uninformed investors trade

upon dividend signal and public signal, which both depend on the stock fundamental. All three channels are discounted by the persistence parameter ρ^F .

$$e_2 = \rho^F \left(\underbrace{1}_{\text{dividend}} + \underbrace{p_F}_{\text{capital gain}} + \underbrace{p_{\hat{F}} \left(\frac{\text{Var}(F|\Omega^U)}{\sigma_D^2} + \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2} \right)}_{\text{signal to uninformed}} \right) \quad (2.15)$$

For simplicity, denote the ratio of variances $\theta_D = \frac{\text{Var}(F|\Omega^U)}{\sigma_D^2}$ and $\theta_S = \frac{\text{Var}(F|\Omega^U)}{\sigma_S^2}$.

Excess stock return depends on stock supply x only through the capital gain term as it does not enter into dividend nor signals.

$$e_3 = \rho^x p_x \quad (2.16)$$

Lemma 2.1 suggests that the excess stock return can be decomposed into the following three components in terms of information content:

$$Q_{t+1} = \underbrace{\bar{p} + e_1 \hat{F}_t - RP_t}_{\text{known to all}} + \underbrace{e_2 F_t - e_3 x_t}_{\text{known to informed only}} + \underbrace{e_4 \varepsilon_{t+1}^F - e_5 \varepsilon_{t+1}^x + e_6 \varepsilon_{t+1}^D + e_7 \varepsilon_{t+1}^S}_{\text{not known to either}} \quad (2.17)$$

The first component consists of constants, current stock prices and uninformed investors' belief \hat{F}_t . These are known to all agents in the economy. The second component consists of actual stock fundamental and actual stock supply. These information are known to only the informed investors. The third component consists of future noises that no one at period t could possibly know. Thus the conditional volatility of excess stock return for both the uninformed and the informed is given by:

$$V_t^U := \text{Var}(Q_{t+1}|\Omega^U) = \text{Var}(e_2 F_t - e_3 x_t + e_4 \varepsilon_{t+1}^F - e_5 \varepsilon_{t+1}^x + e_6 \varepsilon_{t+1}^D + e_7 \varepsilon_{t+1}^S | \Omega^U) \quad (2.18)$$

$$V_t^I := \text{Var}(Q_{t+1}|\Omega^I) = \text{Var}(e_4 \varepsilon_{t+1}^F - e_5 \varepsilon_{t+1}^x + e_6 \varepsilon_{t+1}^D + e_7 \varepsilon_{t+1}^S | \Omega^I) \quad (2.19)$$

Given the expected stock returns and conditional volatility, I can now derive the investors' optimal portfolio. As agents live for two periods and possess exponential utility, the optimal portfolio choice

is particularly simple:

$$s_t^i = \frac{E(Q_{t+1} | \Omega_t^i)}{V_t^i}$$

Market clearing condition implies that

$$\lambda s_t^I + (1 - \lambda) s_t^U = x_t$$

The price coefficients are then determined by matching coefficients so that the market clearing condition holds for any F, x, \hat{F} , given agents' belief. Combining the market clearing condition and the steady state equation characterizing conditional volatility, I obtain a full characterization of the exogenous-information steady state. To summarize, an exogenous information steady state given λ is a triple $(\text{Var}(F|\omega^U), p_F, p_x)$ characterized by three equations. One equation is derived from private agents' dynamic filtering problem whereas two other equations come from coefficients matching in the market clearing condition:

Proposition 2.2 *Given λ , an exogenous information steady state $(\text{Var}(F|\Omega^U), p_F, p_x)$ is fully characterized by:*

$$\frac{1}{\text{Var}(F|\Omega^U)} = \frac{1}{(\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 - \frac{[(\rho^F - \rho^x)\rho^F \text{Var}(F|\Omega^U) + \sigma_F^2]^2}{(\rho^F - \rho^x)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2}} + \frac{1}{\sigma_D^2} + \frac{1}{\sigma_S^2} \quad (2.20)$$

$$\alpha = \left[\lambda \frac{1}{V^I} + (1 - \lambda) \frac{1}{V^U} \right] (R - \rho^x) p_x \quad (2.21)$$

$$\frac{p_F}{p_x} = \lambda \frac{\rho^F + \frac{\rho^F}{R - \rho^F} f_2 + (\rho^F - \rho^x - f_2) p_F}{V^I} \quad (2.22)$$

Where V^I and V^U are conditional volatility of stock return for informed and uninformed investors defined by 2.19 and 2.18 and are functions of $(\text{Var}(F|\omega^U), p_F, p_x)$, expressions of which are given in the appendix. $p_{\hat{F}}$ is given by

$$p_{\hat{F}} = a - p_F \quad (2.23)$$

where $a = \frac{R}{R - \rho^x}$.

As is standard in the literature the sum of $p_{\hat{F}} + p_F$ is equal to $\frac{R}{R-\rho^x}$ regardless of λ . This is a standard property of asymmetric information trading model such as Wang (1994).

3 Value of Information

Given that I can fully characterize the exogenous information steady state conditional on the steady state share of informed investors λ , I now define the value of information as the ratio of the expected utilities for the informed and uninformed investors *net of information cost*:

Definition 3.1 *Denote the expected utility of the informed \hat{W}^I and uninformed \hat{W}^U net of information cost at each exogenous-information steady state $\Phi(\lambda)$. Define the value of information conditional on λ*

$$\pi(\lambda) = \hat{W}^U / \hat{W}^I,$$

where $\hat{W}^i, i = I, U$ are given by equation 2.7 with $\chi = 0$.

The value of information measures the expected gain from information acquisition. Comparing the benefit to the cost of acquiring information determines whether an exogenous-information steady state is a steady state in the benchmark economy:

Lemma 3.1 $\forall \lambda \in (0, 1)$, *an exogenous-information steady state $\Phi(\lambda)$ is a steady state if and only if*

$$\pi(\lambda) = \exp(\alpha R \chi).$$

For $\lambda = 0 (1)$, $\Phi(\lambda)$ is a steady state if and only if

$$\pi(\lambda) \leq (\geq) \exp(\alpha R \chi).$$

In a static environment as in Grossman and Stiglitz (1980), value of information is monotonically decreasing due to price learning, giving rise to a unique equilibrium. In this dynamic environment,

however, value of information can be upward-sloping due to a dynamic complementarity effect. To see this, I need to evaluate the slope of the value of information function. An elegant theoretical result in [Grossman and Stiglitz \(1980\)](#) for characterizing the value of information is that the value of information is equal to the (square root of the) ratio of conditional volatility of excess stock return faced by the uninformed and informed. This result carries over to this dynamic model. The crucial assumption here is that agents are, as in [Grossman and Stiglitz \(1980\)](#), two period lived. Thus, conditional on equilibrium price function, they solve exactly the same problem as in [Grossman and Stiglitz \(1980\)](#), yielding the same expression of expected utility, and hence the same expression for the value of information.

Proposition 3.1 *The value of information is equal to the square root of the ratio of conditional volatility of excess stock return faced by the uninformed and informed:*

$$\pi(\lambda) = \sqrt{\frac{V^U}{V^I}},$$

Where V^U and V^I are the steady-state conditional stock return volatility faced by uninformed and informed investors. I can express the value of information in terms of V^I and the difference in volatility defined as $\Delta V = V^U - V^I$:

$$\pi(\lambda) = \sqrt{\frac{V^U}{V^I}} = \sqrt{1 + \frac{\Delta V}{V^I}},$$

Thus, to evaluate the slope of $\pi(\lambda)$, one only need to evaluate the slope of $\frac{\Delta V}{V^I}$, or equivalently the sign of $\frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V^I} \frac{dV^I}{d\lambda}$. We summarize this observation into the following lemma:

Lemma 3.2

$$\text{sgn}\left(\frac{d\pi(\lambda)}{d\lambda}\right) = \text{sgn}\left(\underbrace{\frac{d\Delta V}{d\lambda}}_{\text{fundamental component}} - \underbrace{\frac{\Delta V}{V^I} \frac{dV^I}{d\lambda}}_{\text{liquidity component}}\right),$$

The lemma decomposes the slope of the value of information into two components. The first component, $\frac{d\Delta V}{d\lambda}$, is only affected by fundamental sensitivity p_f . Thus, I label it the fundamental

component. The second component, $\frac{\Delta V}{V^I} \frac{dV^I}{d\lambda}$, is mainly affected by variations in supply sensitivity p_x , and is therefore labelled liquidity component.

To see the dynamic complementarity effect, I first focus on the fundamental component $\frac{d\Delta V}{d\lambda}$. Write out expression of ΔV using equation 2.18 and 2.19:

$$\begin{aligned}\Delta V &= \text{Var}(Q_{t+1}|\Omega_t^U) - \text{Var}(Q_{t+1}|\Omega_t^I) \\ &= \text{Var}(e_2 F_t - e_3 x_t|\Omega_t^U) \\ &= e_2^2 \text{Var}(F_t|\Omega_t^U) + e_3^2 \text{Var}(x_t|\Omega_t^U) - 2e_2 e_3 \text{Cov}(F_t, x_t|\Omega_t^U).\end{aligned}\tag{3.1}$$

All the noise terms drop out because they are not known, nor are they correlated with current fundamental and supply. Information acquisition reduces the uncertainty associated to stock fundamental F_t and stock supply x_t . The first two variance terms reflect, respectively, that information is useful in reducing the uncertainty regarding stock fundamental and stock supply. The last correlation term is negative, reflecting that with high correlation, information is not that useful in guiding agent's portfolio choice because any signal that predicts good fundamental (suggesting buying the stock) also predicts excessive stock supply (suggesting shorting the stock). In what follows, I will characterize how change in λ affects each term:

$$\frac{d\Delta V}{d\lambda} = \frac{de_2^2 \text{Var}(F_t|\Omega_t^U)}{d\lambda} + \frac{de_3^2 \text{Var}(x_t|\Omega_t^U)}{d\lambda} - \frac{d2e_2 e_3 \text{Cov}(F_t, x_t|\Omega_t^U)}{d\lambda}.\tag{3.2}$$

The predictive role of stock fundamental

The first term $de_2^2 \text{Var}(F|\Omega^U)/d\lambda$ reflects how perturbations in λ affect the value of information through the predictive role of fundamental F . When the share of informed, λ , increases, two opposing forces affect the value of information about stock fundamental. On the one hand, classic substitutability says that more informed investors today implies a more informative current stock price. Thus, the conditional variance of fundamental $\text{Var}(F|\Omega^U)$ tends to decrease. On the other hand, since there are more informed investors in the future, the future stock price loads more

heavily on the fundamental, and thus the loading coefficient e_2 increases. One can see this effect by differentiate e_2 with respect to p_F :

$$\frac{\partial e_2}{\partial p_F} = \rho^F \left(1 - \frac{Var(F|\Omega^U)}{\sigma_D^2} - \frac{Var(F|\Omega^U)}{\sigma_S^2} \right) > 0$$

The "Kalman Gain", $\frac{Var(F|\Omega^U)}{\sigma_D^2} + \frac{Var(F|\Omega^U)}{\sigma_S^2}$, is always less than 1 because uninformed investors react less aggressively to noisier signals. When λ increases, more informed investors in the future implies that the excess stock return is more sensitive to future stock fundamental, hence current stock fundamental. This is *dynamic complementarity effect*. Note that the effect is discounted by ρ^F because loading on future fundamental gets discounted by the persistence parameter when translated into loading on current fundamental.

Apart from the dynamic complementarity effect, I still need to evaluate the static substitutability effect. Observe two properties regarding the steady-state equations charactering the conditional volatility $Var(F|\Omega^U)$: first, λ does not enter into the equation directly. This implies that there is no direct impact of variations in λ on $Var(F|\Omega^U)$. Second, the price coefficients p_F and p_x enter the equation only through the square of the price ratio $(\frac{p_F}{p_x})^2$. Thus the derivatives always contain the price ratio $\frac{p_F}{p_x}$, which converges to 0 when $\lambda \rightarrow 0$. This implies that changes in λ cannot have indirect effect on the conditional volatility through changes in the price coefficients locally around $\lambda = 0$. These two properties taken together implies that the conditional volatility of stock fundamental is not affected by changes in λ , and thus it can be treated as a constant when λ is very close to 0.

To see this more explicitly, write equation 2.20 as $G\left(Var(F|\Omega^U), \left(\frac{p_F}{p_x}\right)^2\right) = 0$ for some differentiable function G . Total differentiate with respect to conditional volatility $Var(F|\Omega^U)$ (first term), price coefficients p_F, p_x (second term) and λ (last term):

$$\frac{\partial G}{\partial Var(F|\Omega^U)} dVar(F|\Omega^U) + \frac{\partial G}{\partial \left(\frac{p_F}{p_x}\right)^2} \left(\frac{2p_F}{p_x^2} dp_F - \frac{p_F}{p_x^3} dp_x \right) + \frac{\partial G}{\partial \lambda} d\lambda = 0$$

Note that $p_F \rightarrow 0$ when $\lambda \rightarrow 0$, thus the second term disappears. Also note that equation 2.20 does not depend on λ directly, so the last term disappears as well. Thus:

$$\begin{aligned} \frac{\partial G}{\partial \text{Var}(F|\Omega^U)} d\text{Var}(F|\Omega^U) &= 0 \\ \Rightarrow d\text{Var}(F|\Omega^U) &= 0 \end{aligned}$$

We summarize this observation into the following proposition

Proposition 3.2 (*Local Absence of Static Substitutability*) Suppose $\lambda \rightarrow 0$:

$$\frac{d\text{Var}(F|\Omega^U)}{d\lambda} \rightarrow 0$$

Thus, perturbing λ near $\lambda = 0$ does not affect the conditional uncertainty faced by the uninformed investors because it barely improves the precision of the price signal. In contrast, the magnitude of the dynamic complementarity effect is generally bounded away from zero. When λ is very small, the loading of stock price on the fundamental, p_F , is negligible. But the loading of excess stock return on the fundamental, e_2 , converges to some strictly positive number $\rho^F(1 + a(\theta_D + \theta_S)) > 0$. This is because of the presence of the interim dividend payout, as well as the fact that uninformed investors observe noisy signals about the fundamental. Thus we have:

$$\frac{de_2^2}{d\lambda} = 2(\rho^F)^2(1 + a(\theta_D + \theta_S))(1 - \theta_D - \theta_S) \frac{dp_F}{d\lambda} \text{ when } \lambda \rightarrow 0. \quad (3.3)$$

Combining the static substitutability (equation ??) and the dynamic complementarity (equation 3.3), one can show that the first term in equation 3.2 is always positive at the limit:

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \frac{d[e_2^2 \text{Var}(F|\Omega^U)]}{d\lambda} &= \underbrace{\lim_{\lambda \rightarrow 0} \frac{de_2^2}{d\lambda}}_{>0} \underbrace{\text{Var}(F|\Omega^U)}_{>0} + \lim_{\lambda \rightarrow 0} \underbrace{\frac{d\text{Var}(F|\Omega^U)}{d\lambda}}_{=0} e_2^2 \\ &= 2(\rho^F)^2(1 + a(\theta_D + \theta_S))(1 - \theta_D - \theta_S) \text{Var}(F|\Omega^U) \frac{dp_F}{d\lambda} \end{aligned} \quad (3.4)$$

The predictive role of supply x

The second term $de_3^2 Var(x|\Omega^U)/d\lambda$ in equation 3.2 captures the predictive role of supply x . As discussed in Avdis (2016), the equilibrium stock price becomes a noisier signal of supply when there are more informed investors. This force tends to increase the conditional supply uncertainty faced by the agents and thus increase the value of information. Note that both the uninformed and the informed observe the price signal $S^P = p_F F - p_x x$. Thus

$$Var(x|\Omega^U) = Var\left(\frac{p_F F - S^P}{p_x}|\Omega^U\right) = \left(\frac{p_F}{p_x}\right)^2 Var(F|\Omega^U). \quad (3.5)$$

As λ increases, the stock price becomes more sensitive to the fundamental, and thus the ratio $\frac{p_F}{p_x}$ increases in general. This tends to push up the conditional uncertainty of supply and thus increase the value of information.

This effect is absent locally around $\lambda = 0$ because its magnitude depends on the square of the price ratio $(\frac{p_F}{p_x})^2$. Thus, just like $Var(F|\Omega^U)$, the derivative of $Var(x|\Omega^U)$ with respect to λ is also proportional to $\frac{p_F}{p_x}$. As a result, it tends toward zero as λ tends toward zero:

$$\frac{dVar(x|\Omega^U)}{d\lambda} \rightarrow 0, \text{ as } \lambda \rightarrow 0.$$

Also note that as $p_F \rightarrow 0$, price signal becomes a perfect signal about stock supply. Thus, the conditional volatility of stock supply converges to 0 as well. Combining these observations, we conclude that the supply channel does not play any role locally around $\lambda = 0$:

$$\lim_{\lambda \rightarrow 0} \frac{d[e_3^2 Var(x|\Omega^U)]}{d\lambda} = \lim_{\lambda \rightarrow 0} \underbrace{\frac{de_3^2}{d\lambda} \underbrace{Var(x|\Omega^U)}_{=0}}_{=0} + \lim_{\lambda \rightarrow 0} \underbrace{\frac{dVar(x|\Omega^U)}{d\lambda}}_{=0} e_3^2 = 0 \quad (3.6)$$

The Covariance Term

The third term $-d2e_2e_3Cov(F, x|\Omega^U)/d\lambda$ in equation 3.2 reflects the fact that an increase in λ may reduce the value of information due to higher covariance between fundamental and supply. The logic is as follows. When λ increases, the price becomes a noisier signal of stock supply. This increases the conditional volatility of stock supply relative to the conditional volatility of stock fundamental and thus raises the conditional covariance between the fundamental and supply relative to the conditional volatility of stock fundamental. When the covariance increases, information about the fundamental is not that useful in guiding agent's portfolio choice because any signal that predicts good fundamental (suggesting buying the stock) also predicts excessive stock supply (suggesting shorting the stock).

To characterize the covariance term, we first substitute in expression 2.9 and expressions for e_2 and e_3 (equation 2.15 and 2.16):

$$\begin{aligned} 2e_2e_3Cov(F, x|\Omega^U) &= 2e_2e_3\frac{p_F}{p_x}Var(F|\Omega^U) \\ &= 2\rho^F(1+p_F+(a-p_F)(\theta_D+\theta_S))\rho^x p_x\frac{p_F}{p_x}Var(F|\Omega^U) \\ &= 2\rho^F\rho^x(1+p_F+(a-p_F)(\theta_D+\theta_S))p_FVar(F|\Omega^U) \end{aligned}$$

Note that when $\lambda \rightarrow 0$, $p_F \rightarrow 0$ and $\frac{dVar(F|\Omega^U)}{d\lambda} \rightarrow 0$. With these facts and the chain rule of differentiation, one get:

$$\frac{d2e_2e_3Cov(F_t, x_t|\Omega_t^U)}{d\lambda} \rightarrow 2\rho^F\rho^x(1+a(\theta_D+\theta_S))Var(F|\Omega^U)\frac{dp_F}{d\lambda} \quad (3.7)$$

Thus, this offsetting force depend on the product $\rho^F\rho^x$ as it measures the covariance of future fundamental and future supply. By combining equation 3.4 and 3.7 we are ready to derive the slope of the information gain component:

$$\frac{d\Delta V}{d\lambda} = 2\rho^F(1+a(\theta_D+\theta_S))Var(F|\Omega^U)((1-\theta_D-\theta_S)\rho^F-\rho^x)\frac{dp_F}{d\lambda} \quad (3.8)$$

Note that the slope of the fundamental component does not depend on how market liquidity

changes with λ . It only depends on how fundamental sensitivity varies with the share of informed investors. The next proposition shows that, regardless of the type of financial market equilibrium, fundamental sensitivity always increases with fraction of informed investors:

Proposition 3.3 *For both high-volatility and low-volatility equilibrium: as $\lambda \rightarrow 0$,*

$$\frac{dp_F}{d\lambda} \rightarrow \rho^F \frac{1 + a(\theta_D + \theta_S)}{\alpha V^I} p_x > 0 \quad (3.9)$$

Where V^I is the conditional volatility of excess stock return defined by equation 2.19.

Thus, all other terms in this equation are strictly positive except $(1 - \theta_D - \theta_S)\rho^F - \rho^x$. Thus I arrive at the following proposition:

Theorem 1

$$\frac{d\Delta V}{d\lambda} > 0 \text{ for } \lambda \text{ sufficiently small.}$$

if and only if

$$(1 - \theta_D - \theta_S)\rho^F > \rho^x$$

Theorem 1 provides sharp characterization of the fundamental component. The necessary and sufficient condition for the fundamental component to be upward sloping is that the (precision-adjusted) fundamental persistence is greater than the supply persistence. The intuition is the following. The dynamic complementarity says that when there are more informed investors in the future, future stock return would be more sensitive to the future stock fundamental, hence the current stock fundamental, increasing the value of information. When the fundamental is not very persistent, loading on future stock fundamental gets heavily discounted, reducing the dynamic complementarity. When signals available to uninformed agents are very precise, change in the share of informed investors does not lead to big change in the loading coefficients of stock fundamental. This also reduces the strength of dynamic complementarity. The condition also confirms the insight from Avdis (2016) that when the supply is also very persistent the information gain is less likely to be upward-sloping. Whether this component is upward sloping or not depends on a horseshoe

between the two forces and turns out one only needs to directly compare the two parameters to figure out the slope. Thus the model suggests that

Prediction 1 *Information multiplicity is more likely to arise when*

1. *the stock fundamental is more persistent than stock supply*
2. *Public signal is less precise.*

Note that the slope of the fundamental component does not depend on how market liquidity changes with λ . It only depends on how fundamental sensitivity varies with the share of informed investors. Does this imply that market liquidity is irrelevant for the value of information? The answer is No. And I now turn to analyzing how market liquidity impacts dynamic coordination motives in the information market.

3.1 Dynamic Liquidity Channel

In this section I characterize the liquidity component of the value of information.

$$\frac{\Delta V}{V^I} \frac{dV^I}{d\lambda}$$

Changes in λ can affect the value of information by directly affecting the *level* of uncertainty faced by the informed investors V^I . This is where liquidity comes into play. To see this, consider an increase in λ . If it raised the price impact of noise traders, then the future stock return uncertainty faced by informed investors would increase and thus information acquisition would become less appealing. This is the dynamic liquidity channel that works through expectations of future stock returns. I first state an existence theorem:

Proposition 3.4 *Let*

$$\Delta = \frac{(R - \rho^x)^2}{\alpha^2} - 4\sigma_x^2 \left((1 + a(\theta_D + \theta_S))^2 \left[(\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 \right] + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 \right)$$

If $\Delta \geq 0$, there exists two financial market equilibria characterized by different level of p_x at $\lambda = 0$ with values given by:

$$p_x = \frac{\frac{R - \rho^x}{\alpha} \pm \sqrt{\Delta}}{2\sigma_x^2} \quad (3.10)$$

This proposition is similar to the classic result of Spiegel (1998). Under appropriate assumptions, there exists two values of p_x consistent with equilibrium conditions. The equilibrium with the smaller root is called "low volatility equilibrium", whose unique limit corresponds to the equilibrium in Wang (1994) (Albagli (2015)). The other equilibrium is the "high volatility equilibrium". Both types of financial market equilibria have some appealing properties. This paper does not take a standing on which financial market equilibrium one should select, but rather provides characterization of information choice in both scenarios.

The first proposition states how p_x varies with λ :

Proposition 3.5 As $\lambda \rightarrow 0$:

$$\frac{dp_x}{d\lambda} \rightarrow -(1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F_t | \Omega_t^U) \frac{\rho^x + R}{\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2} \frac{dp_F}{d\lambda} \quad (3.11)$$

Where V^I is the conditional volatility of excess stock return defined by equation 2.19.

The sign of $\frac{dp_x}{d\lambda}$ depend on the sign of $\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2$. Plug in the expression of p_x from 3.10, one can see that

$$\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2 = \pm \sqrt{\Delta}$$

where Δ is defined in Proposition 3.4. This leads to the following proposition:

Proposition 3.6 The sign of $\frac{dp_x}{d\lambda}$ depends on the type of financial market equilibrium:

1. $\frac{dp_x}{d\lambda} < 0$ for low-volatility equilibrium
2. $\frac{dp_x}{d\lambda} > 0$ for high-volatility equilibrium

This proposition states that, in the low-volatility equilibrium, market becomes more liquid with more informed investors. The opposite is true in the high-volatility equilibrium. Thus, stock return uncertainty behaves differently under different financial market equilibrium, and this feeds back into the value of information, as stated in the following theorem:

Theorem 2 *When $\lambda \rightarrow 0$:*

1. $\frac{\partial V^I}{\partial p_F} \frac{\partial p_F}{\partial \lambda} < 0$

2. *The dynamic liquidity effect $\frac{\partial V^I}{\partial p_x} \frac{\partial p_x}{\partial \lambda}$ depends on the type of financial market equilibrium:*

- (a) *In a low-volatility equilibrium: $\frac{\partial V^I}{\partial p_x} \frac{\partial p_x}{\partial \lambda} < 0$*

- (b) *In a high-volatility equilibrium: $\frac{\partial V^I}{\partial p_x} \frac{\partial p_x}{\partial \lambda} > 0$*

From the theorem 2, one can draw a couple of conclusions. First, in a low volatility equilibrium, increases in λ increases p_F and reduces p_x , both reducing the uncertainty faced by the informed investors. In a high volatility equilibrium, however, both p_F and p_x are increased. This increases the uncertainty faced by the informed investors, making information acquisition less appealing. Thus, the value of information is more likely to be upward sloping in a low-volatility equilibrium. Therefore information multiplicity is more likely to arise in a low-volatility equilibrium than in high-volatility equilibrium.

Prediction 2 *Information multiplicity arises in high-volatility equilibrium implies that it also arises in low-volatility equilibrium. The inverse is not necessarily true.*

One might wonder what is the magnitude of the dynamic liquidity channel. The next proposition shows that it can be the dominating force shaping the value of information as its magnitude can converge to infinity:

Proposition 3.7 *Let $\bar{\rho}^F$ and $\bar{\rho}^x$ be the upper bound at which financial market equilibrium exists (that is, $\Delta \geq 0$). As either $\rho^F \rightarrow \bar{\rho}^F$ or $\rho^x \rightarrow \bar{\rho}^x$, the magnitude of noise trading effect goes to*

infinity: $\left| \frac{\partial V^I}{\partial p_x} \frac{\partial p_x}{\partial \lambda} \right| \rightarrow \infty$

The proposition says that the dynamic complementarity effect can be arbitrarily large when the stock fundamental or stock supply is sufficiently persistent. The proof of the proposition hinges on the following observation. From equation 3.11, when $\lambda \rightarrow 0$,

$$\frac{dp_x}{d\lambda} \rightarrow (1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F_t | \Omega_t^U) \frac{-\rho^x - R}{\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2} \frac{dp_F}{d\lambda} \quad (3.12)$$

If ρ^F or ρ^x is pushed to its upper bound, the denominator $\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2$ converges to zero, and thus $\frac{dp_x}{d\lambda}$ goes to infinity.

To what follows, I first state a necessary and sufficient condition under which information multiplicity arises. I then use this condition to explore numerically the role of each force in determining the strength of information multiplicity:

Theorem 3 *The value of information is upward-sloping*

$$\left. \frac{d\pi(\lambda)}{d\lambda} \right|_{\lambda=0} > 0$$

if and only if

$$\left[(1 - \theta_D - \theta_S) \rho^F - \rho^x \right] + \frac{\Delta V}{V^I} \left(\rho^F (1 - (\theta_D + \theta_S)) + p_x \sigma_x^2 \frac{\rho^x + R}{\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2} \right) > 0 \quad (3.13)$$

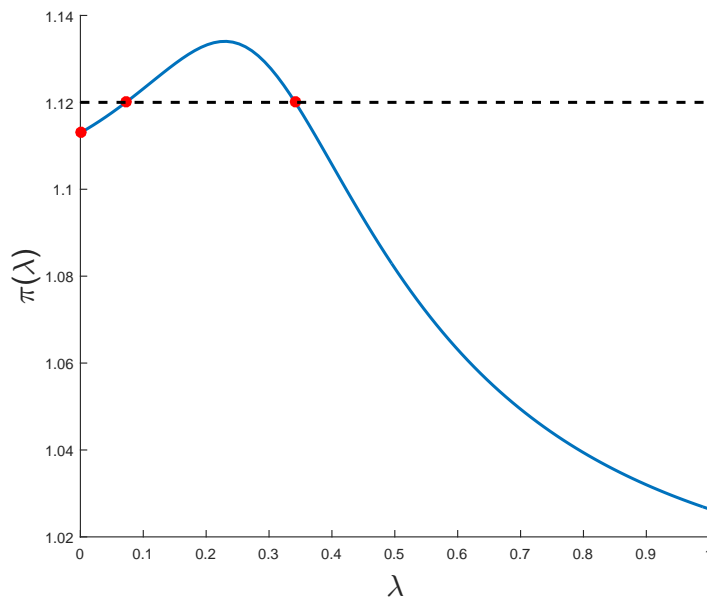
Where V^I is the conditional stock return volatility for informed investors at $\lambda = 0$. ΔV is the information gain component at $\lambda = 0$. p_x is the loading of stock price on noisy supply, given by equation 3.10.

Theorem 3 combines the fundamental component and the liquidity component and provides a complete characterization of the dynamic complementarity in information acquisition. The first part of the condition 3.13, $(1 - \theta_D - \theta_S) \rho^F - \rho^x$, comes from the information gain component whereas the rest comes from the volatility component. In particular, the dynamic liquidity effect is captured by the last term $p_x \sigma_x^2 \frac{\rho^x + R}{\frac{1}{\alpha}(R - \rho^x) - 2p_x \sigma_x^2}$. As shown in 3.7, this term could dominate other

effects under certain conditions. Thus, in the next section, I conduct numerical exercises to study these forces jointly.

4 Numerical Experiments

In this experiment I closely follow the calibration strategy in [Albagli \(2015\)](#). Volatility of stock fundamental σ_F^2 is set to 1 as well as the dividend volatility. Volatility of public signal is set to 1 as well. Volatility of stock supply σ_x^2 is set to match an annual turnover rate of 10%. Risk averse parameter α is set to 1. The risk free rate is set to 1.05. I start by examining the case where stock fundamental is quite persistent $\rho^F = 0.75$ whereas the stock supply is relatively transient $\rho^x = 0.15$. I plot the value of information as a function of share of informed investors λ :



The blue curve depicts numerically solved $\pi(\lambda)$. The black dashed line depicts the information cost. Red dots are numerically solved steady states. Graph depicts a situation where the value of information is locally increasing when λ is sufficiently small, and thus for appropriate level of information cost there exists multiple steady states. Parameter values: $\alpha = 1, R = 1.05, \rho^F = 0.75, \rho^x = 0.15, \sigma_F^2 = 1, \sigma_D^2 = 1, \sigma_x^2 = 0.01, \sigma_S^2 = 1$.

Figure 2: The value of information $\pi(\lambda)$

The graph depicts a situation where the the value of information is locally increasing when λ is sufficiently small, and thus for appropriate level of information cost there exists multiple steady states. Next, I conduct comparative statics exercises to see how the slope of value of information at $\lambda = 0$ changes with various model parameters.

The fundamental persistence ρ^F

In the first experiment, I keep all other parameters at benchmark value and vary the fundamental persistence ρ^F from 0.4 to 0.8. The first row of the figure plots slope of value of information $\frac{d\pi}{d\lambda}$ at 0. Left panel depicts high-volatility equilibrium and right panel depicts low-volatility equilibrium. The shaded area is the multiplicity region where the derivative is positive. The derivative is then decomposed into the following three parts:

$$\underbrace{\frac{d\Delta V}{d\lambda}}_{\text{Fundamental Component}} - \underbrace{\frac{\Delta V}{V^I} \left(\overbrace{\frac{\partial V^I}{\partial p_F} \frac{\partial p_F}{\partial \lambda}}^{\text{via } p_F} + \overbrace{\frac{\partial V^I}{\partial p_x} \frac{\partial p_x}{\partial \lambda}}^{\text{via } p_x} \right)}_{\text{Liquidity Component}} \quad (4.1)$$

The first part is the fundamental component which is unaffected by market liquidity. The second part captures how stock return uncertainty is affected by fundamental sensitivity. This part does not play significant role. The last part is the dynamic liquidity channel where variations in market liquidity affects condition stock return volatility and impacts the value of information. Results are shown in figure 3.

First of all, one can see that information fragility is more likely to rise in the low-volatility equilibrium (larger shaded area). Second, in term of the relation between information multiplicity and fundamental persistence, at low-volatility equilibrium (right panel), information fragility is always more prominent with more persistent stock fundamental, as both the fundamental component and the liquidity component are upward sloping. This is consistent with prediction 1. Turning to the high-volatility equilibrium reveals a different pattern: there is a non-monotonic relation between fundamental persistence and multiplicity: the slope of value of information increases initially, but then drop sharply for sufficiently high stock fundamental. The drop is entirely driven by the dynamic liquidity channel (last row, left column), namely how λ affects conditional volatility though

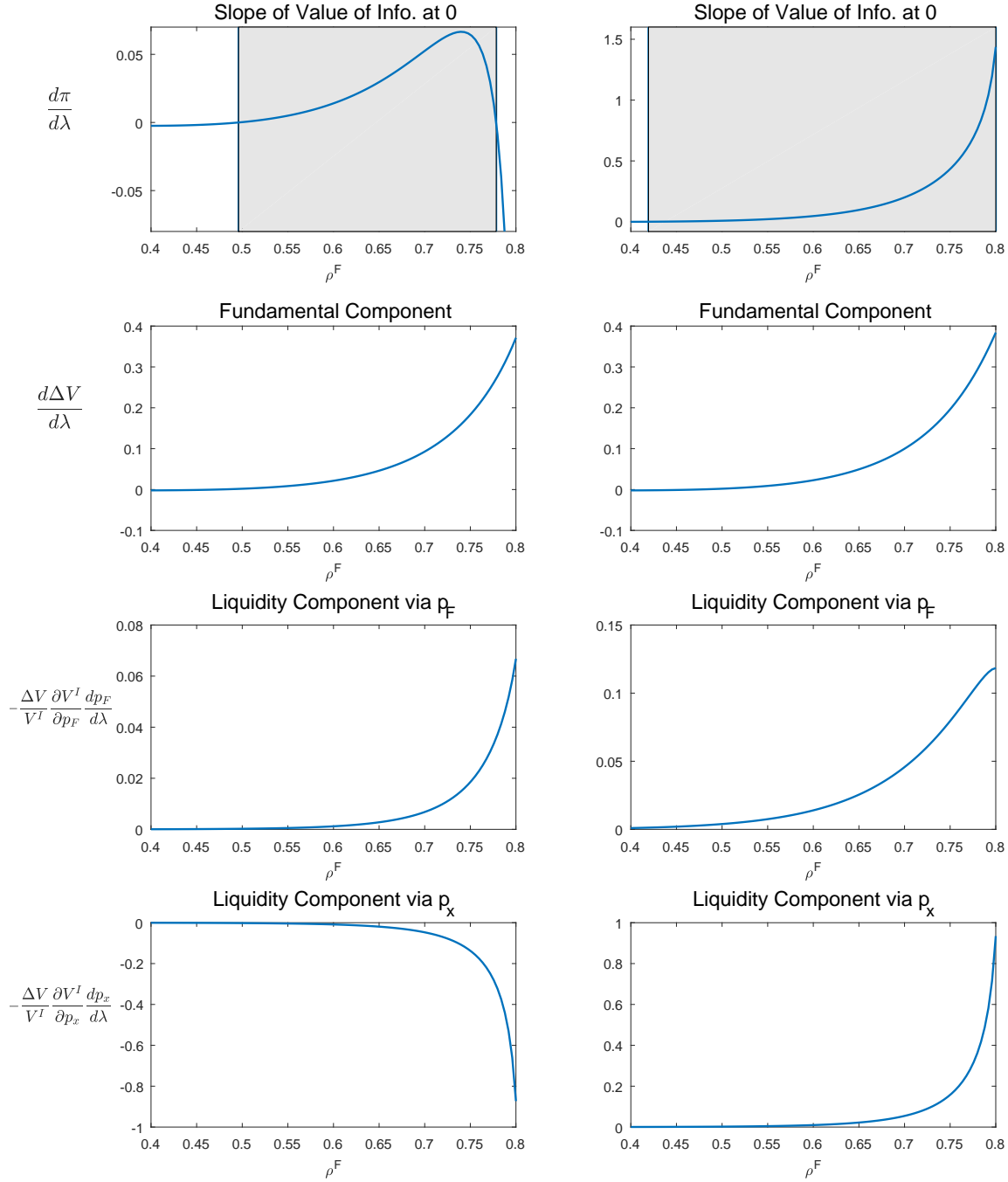
p_x . As λ increases, stock price loads more heavily onto supply noise in the high-volatility equilibrium, p_x increases. The increase in p_x is more significant with persistent fundamental, which amplifies the information channel. The increase in p_x then raises V^I , the conditional volatility of stock return faced by the information investors, making information acquisition unfavorable. Thus, the model's prediction regarding ρ^F depends on the type of financial market equilibrium. In particular, at high-volatility equilibrium, more persistent stock fundamental could make information coordination harder to achieve.

The fundamental persistence ρ^x

In the second experiment, I experiment with variations of supply persistence. Results are shown in figure 4. First of all, similar to figure 3 the shaded area is bigger under low-volatility equilibrium, suggesting that information multiplicity is more pronounced. Second, in the low-volatility equilibrium, there exists a nonmonotonic relation between supply persistence and information fragility: increasing the stock supply persistence reduces the slope of value of information initially, but for sufficiently big values of ρ^x the trend is reversed. Decomposing the derivative into three components, one can see the the reversal is driven by the liquidity channel. In the low-volatility equilibrium greater value of λ reduces loading of stock price on supply noise: p_x (see proposition 3.6). This reduces the conditional volatility faced by informed investors, increasing the value of information. This effect is more prominent with greater supply persistence. In a high-volatility equilibrium, since increase in λ increases the value of p_x , the volatility component predicts the same trends as the information gain component. Thus increase in ρ^x generates a monotonic decreasing pattern in volatility equilibrium.

The general lesson here is that the overlapping-generation structure changes some of the existing findings in the literature. In particular, information multiplicity can be more prominent with less persistent fundamental, and with more persistent stock supply. This is summarized into the following prediction:

Prediction 3 *Information multiplicity can be more prominent:*



The figure plots slope of value of information $\frac{d\pi}{d\lambda}$ at 0 for a range of fundamental persistence ρ^F . Left panel depicts high-volatility equilibrium and right panel depicts low-volatility equilibrium. The shaded area is the multiplicity region where the derivative is positive. The derivative is then decomposed into three components according to equation 4.1: the information gain component, volatility component with partial derivative through p_x , and volatility component with partial derivative through p_x .

Figure 3: Comparative Statics I: ρ^F

1. *with less persistent stock fundamental in the high-volatility equilibrium*

2. *with more persistent stock supply in the low-volatility equilibrium*

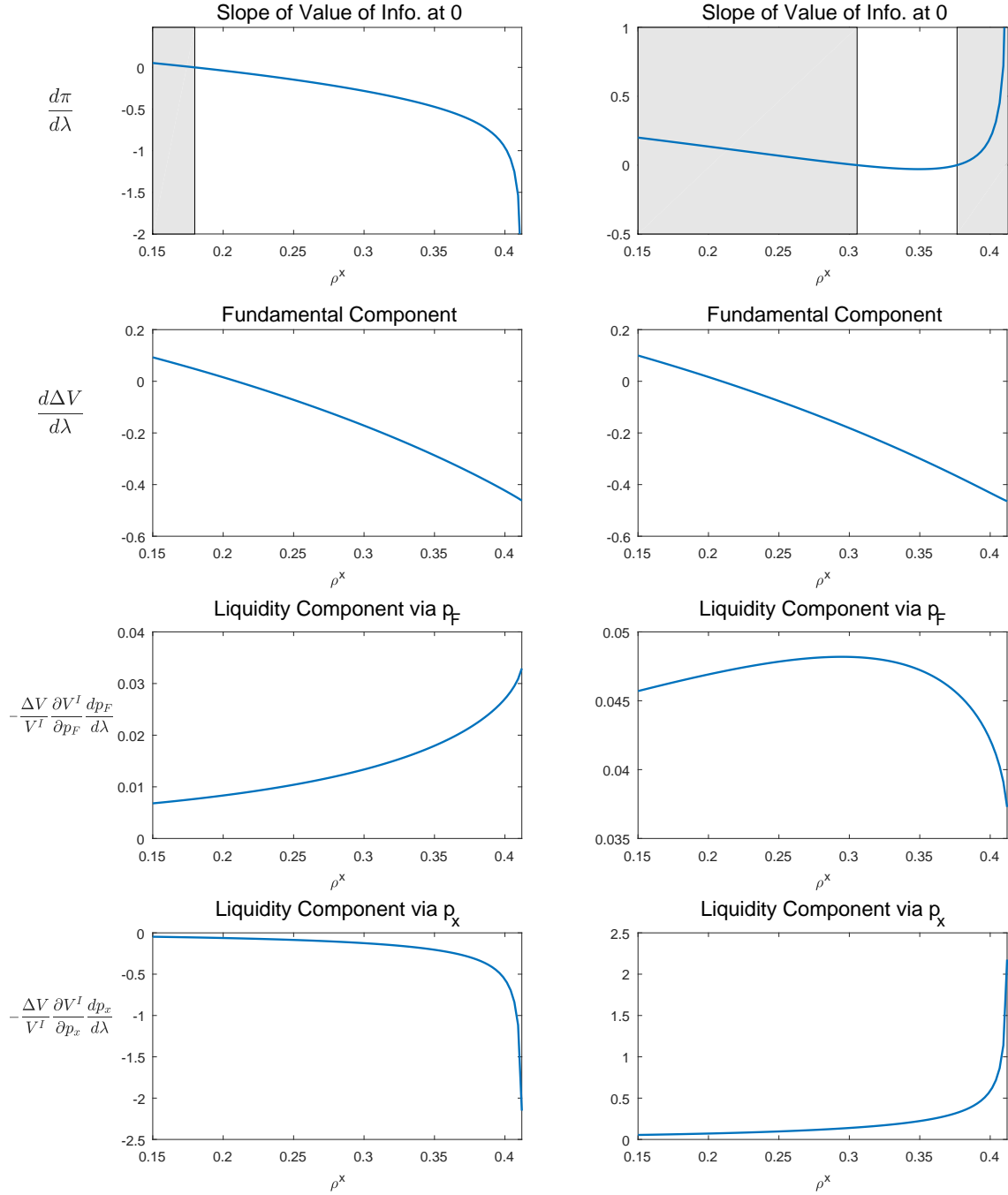
Public precision σ_S^2

In the previous two experiments, I show that prediction of the model depends on which financial market equilibrium is selected. Somewhat surprisingly, when it gets to comparative statics with respect to public signal precision σ_S^2 , this is no longer the case. The result is shown in figure 5. Both low-volatility and high-volatility equilibrium predicts that increasing the precision of public signal (i.e. reducing its variance σ_S^2) tends to reduce the information multiplicity. The reason is that the strength of the liquidity channel is bounded even when the precision of the public signal goes to zero, as there is still the dividend signal providing reasonably precise information about stock fundamental. This implies that the value of Δ is bounded away from zero. Thus, the overall slope of the value of information closely follows the fundamental component.

This result provides an interesting perspective on recent policy attempting to provide more precise public information. It says, for a regulator seeking to stabilize asset prices, it is *universally* desirable to disclose more precise public information because it helps to eliminate information multiplicity, regardless of the type of financial market equilibrium.

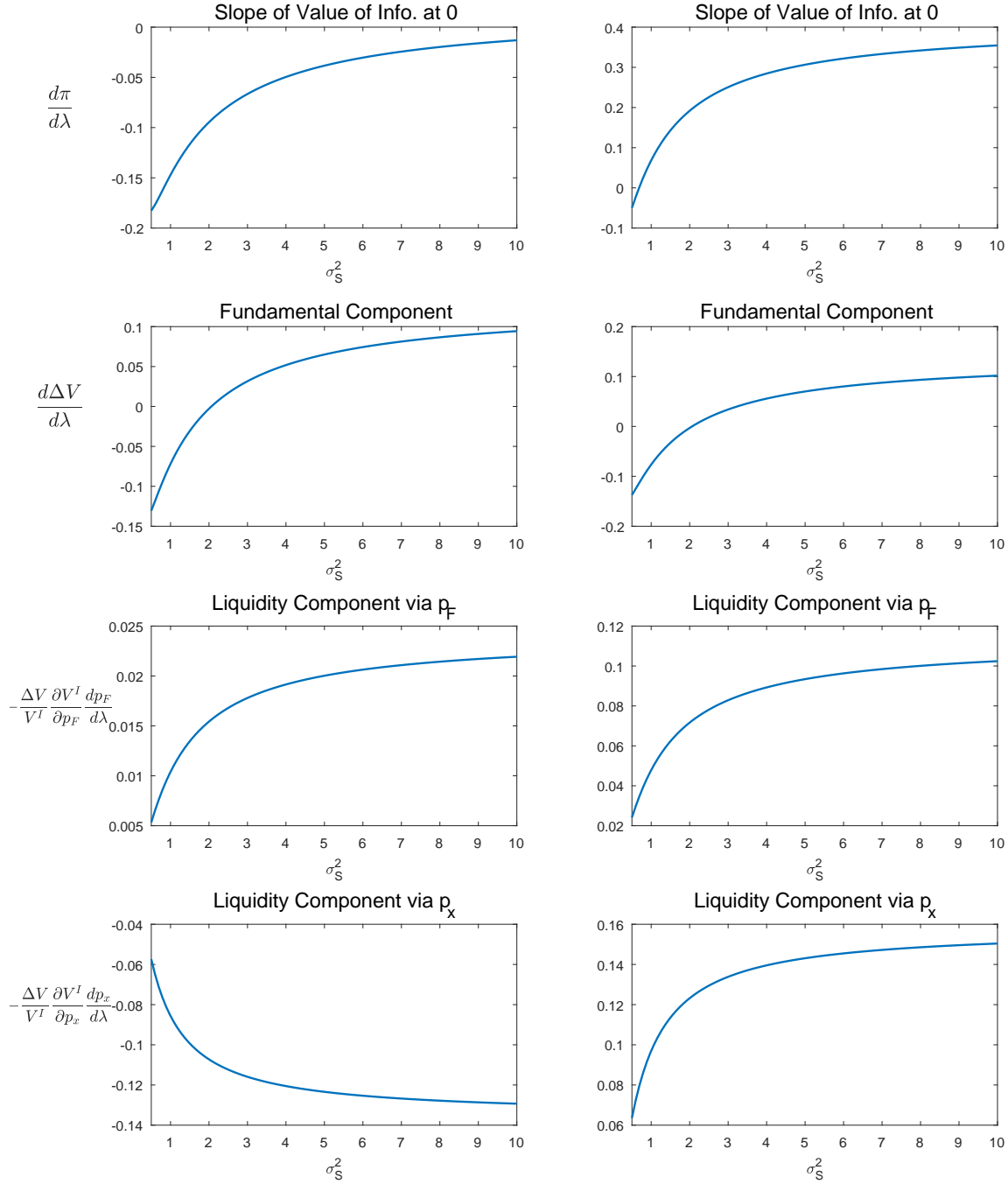
5 Conclusion

This paper studies dynamic information acquisition in a financial market with information asymmetry, with an emphasis on characterizing the relation between information choices and market liquidity. This relation is trivial in static, or finite-horizon settings: as more investors become informed, market liquidity improves. In infinite-horizon models with overlapping-generations of investors, however, this relation becomes nontrivial: with more investors acquiring information, the financial market could become less liquid. This brings about interesting interactions between market liquidity and information choice incentives, to the extent that value of information is affected



The figure plots slope of value of information $\frac{d\pi}{d\lambda}$ at 0 for a range of supply persistence ρ^F . Left panel depicts high-volatility equilibrium and right panel depicts low-volatility equilibrium. The shaded area is the multiplicity region where the derivative is positive. The derivative is then decomposed into three components according to equation 4.1: the information gain component, volatility component with partial derivative through p_x , and volatility component with partial derivative through p_x .

Figure 4: Comparative Statics II: ρ^x



The figure plots slope of value of information $\frac{d\pi}{d\lambda}$ at 0 for a range of supply persistence ρ^F . Left panel depicts high-volatility equilibrium and right panel depicts low-volatility equilibrium. The shaded area is the multiplicity region where the derivative is positive. The derivative is then decomposed into three components according to equation 4.1: the information gain component, volatility component with partial derivative through p_x , and volatility component with partial derivative through p_x .

Figure 5: Comparative Statics II: σ_S^2

by the liquidity component in the future resale stock price.

The paper analyzes how the dynamic liquidity channel shapes investors' information incentives. I find that i) information complementarity is always more prominent in low-volatility financial market equilibrium; ii) information complementarity can be more prominent with less persistent stock fundamental and/or more persistent stock supply and iii) regardless of the type of financial market equilibrium, public disclosure always makes information complementarity less prominent. The theory provides a unified necessary and sufficient condition under which dynamic information multiplicity arises. This condition can be tested empirically either for aggregate US stock markets or for individual stocks. I leave it to future research.

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A Appendix

Proof of Proposition 2.1

The target of this proof is to find $Var(F_{t+1}|\Omega_{t+1}^U)$ and $E(F_{t+1}|\Omega_{t+1}^U)$ as functions of $Var(F_t|\Omega_t^U)$ and $E(F_t|\Omega_t^U)$. Note that

$$\Omega_{t+1}^U = \{S_{pt+1}, D_{t+1}, S_{t+1}\} \cup \Omega_t^U$$

Where S_{pt+1} denotes the price signal

$$S_{pt+1} = F_{t+1} - \frac{p_x}{p_F} x_{t+1}$$

The proof is proceeded in a sequential fashion. That is, we first derive $Var(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)$ and $E(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)$: beliefs conditional on the price signal only. Note that the price signal can be expressed as a linear combination of F_t, S_{pt} , and time- $t+1$ noises:

$$\begin{aligned} S_{pt+1} &= F_{t+1} - \frac{p_x}{p_F} x_{t+1} \\ &= F_{t+1} - \frac{p_x}{p_F} (\rho^x(x_t) + \varepsilon_{t+1}^x) \\ &= F_{t+1} - \frac{p_x}{p_F} \left(\rho^x \frac{p_F}{p_x} (F_t - S_{pt}) + \varepsilon_{t+1}^x \right) \\ &= F_{t+1} - \rho^x (F_t - S_{pt}) - \frac{p_x}{p_F} \varepsilon_{t+1}^x \\ &= F_{t+1} - \rho^x F_t - \frac{p_x}{p_F} \varepsilon_{t+1}^x + \rho^x S_{pt} \\ &= \rho^F F_t + \varepsilon_{t+1}^F - \rho^x F_t - \frac{p_x}{p_F} \varepsilon_{t+1}^x + \rho^x S_{pt} \\ &= (\rho^F - \rho^x) F_t + \varepsilon_{t+1}^F - \frac{p_x}{p_F} \varepsilon_{t+1}^x + \rho^x S_{pt} \end{aligned}$$

Also write out the expression for future fundamental F_{t+1}

$$F_{t+1} = \rho^F F_t + \varepsilon_{t+1}^F$$

Thus, conditional on Ω_t^U , F_{t+1} and S_{pt+1} are jointly normally distributed with mean:

$$\begin{bmatrix} \rho^F E(F_t|\Omega_t^U) \\ (\rho^F - \rho^x) E(F_t|\Omega_t^U) + \rho^x S_{pt} \end{bmatrix}$$

And variance-covariance matrix

$$\begin{bmatrix} (\rho^F)^2 Var(F_t|\Omega_t^U) + \sigma_F^2 & (\rho^F - \rho^x) \rho^F Var(F_t|\Omega_t^U) + \sigma_F^2 \\ (\rho^F - \rho^x) \rho^F Var(F_t|\Omega_t^U) + \sigma_F^2 & (\rho^F - \rho^x)^2 Var(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2 \end{bmatrix}$$

Thus, we can invoke the projection theorem with normal variables to obtain the conditional distribution of F_{t+1} given $\{S_{pt+1}\} \cup \Omega_t^U$:

$$\begin{aligned}
& \text{Var} \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right) \\
= & \text{Var} \left(F_{t+1} | \Omega_t^U \right) - \frac{\text{Cov} \left(F_{t+1}, S_{pt+1} | \Omega_t^U \right)^2}{\text{Var} \left(F_{t+1}, S_{pt+1} | \Omega_t^U \right)} \\
= & \left(\rho^F \right)^2 \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2 - \frac{\left[\left(\rho^F - \rho^x \right) \rho^F \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2 \right]^2}{\left(\rho^F - \rho^x \right)^2 \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2 + \left(\frac{p_x}{p_F} \right)^2 \sigma_x^2} \\
& E \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right) \\
= & E \left(F_{t+1} | \Omega_t^U \right) + \frac{\text{Cov} \left(F_{t+1}, S_{pt+1} | \Omega_t^U \right)}{\text{Var} \left(F_{t+1}, S_{pt+1} | \Omega_t^U \right)} \left(S_{pt+1} - E \left(S_{pt+1} | \Omega_t^U \right) \right) \\
= & \rho^F E \left(F_t | \Omega_t^U \right) + \frac{\left(\rho^F - \rho^x \right) \rho^F \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2}{\left(\rho^F - \rho^x \right)^2 \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2 + \left(\frac{p_x}{p_F} \right)^2 \sigma_x^2} \left(S_{pt+1} - \left(\rho^F - \rho^x \right) E \left(F_t | \Omega_t^U \right) - \rho^x S_{pt} \right)
\end{aligned}$$

Next we need to also incorporate the dividend signal and the public signal. Note that these signals are F_{t+1} plus white noises $(\varepsilon_{t+1}^D, \varepsilon_{t+1}^S)$. Thus, standard Bayesian updating formula for normal variable applies where the precision of the posterior variable is the sum of the precision of ex-ante variable and the precision of the signals:

$$\begin{aligned}
\frac{1}{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)} &= \frac{1}{\text{Var} \left(F_{t+1} | \{S_{pt+1}, D_{t+1}, S_{t+1}\} \cup \Omega_t^U \right)} \\
&= \frac{1}{\text{Var} \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right)} + \frac{1}{\sigma_D^2} + \frac{1}{\sigma_S^2}
\end{aligned}$$

Thus we obtain the law of motion for $\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)$.

The posterior mean is a weighted average of the ex-ante mean and signals:

$$\begin{aligned}
E \left(F_{t+1} | \Omega_{t+1}^U \right) &= E \left(F_{t+1} | \{S_{pt+1}, D_{t+1}, S_{t+1}\} \cup \Omega_t^U \right) \\
&= \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\text{Var} \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right)} E \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right) + \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\sigma_D^2} D_{t+1} + \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\sigma_S^2} S_{t+1} \\
&= \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\text{Var} \left(F_{t+1} | \{S_{pt+1}\} \cup \Omega_t^U \right)} \left[\frac{\rho^F E \left(F_t | \Omega_t^U \right)}{\left(\rho^F - \rho^x \right)^2 \text{Var} \left(F_t | \Omega_t^U \right) + \sigma_F^2 + \left(\frac{p_x}{p_F} \right)^2 \sigma_x^2} \left(S_{pt+1} - \left(\rho^F - \rho^x \right) E \left(F_t | \Omega_t^U \right) - \rho^x S_{pt} \right) \right] \\
&\quad + \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\sigma_D^2} D_{t+1} + \frac{\text{Var} \left(F_{t+1} | \Omega_{t+1}^U \right)}{\sigma_S^2} S_{t+1}
\end{aligned}$$

Thus I get equation 2.12. Substituting in expression for

$$\begin{aligned}
S_{pt+1} &= \left(\rho^F - \rho^x \right) F_t + \varepsilon_{t+1}^F - \frac{p_x}{p_F} \varepsilon_{t+1}^x + \rho^x S_{pt} \\
D_{t+1} &= F_{t+1} + \varepsilon_{t+1}^D = \rho^F F_t + \varepsilon_{t+1}^F + \varepsilon_{t+1}^D \\
S_{t+1} &= F_{t+1} + \varepsilon_{t+1}^S = \rho^F F_t + \varepsilon_{t+1}^F + \varepsilon_{t+1}^S
\end{aligned}$$

Rearrange and collect terms, we obtain:

$$E \left(F_{t+1} | \Omega_{t+1}^U \right) = f_1 E \left(F_t | \Omega_t^U \right) + f_2 F_t + f_3 \varepsilon_{t+1}^F - f_4 \varepsilon_{t+1}^x + f_5 \varepsilon_{t+1}^D + f_6 \varepsilon_{t+1}^S$$

where

$$\begin{aligned}
f_1 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \left(\rho^F - \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} (\rho^F - \rho^x) \right) \\
f_2 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} (\rho^F - \rho^x) + \left(\frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} \right) \rho^F \\
f_3 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} \\
f_4 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} \frac{p_x}{p_F} \\
f_5 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} \\
f_6 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2}
\end{aligned}$$

Thus we obtain equation 2.13.

Proof of proposition 2.2

With exponential utility, the decision rule for informed and uninformed investors $s_t^i, i = I, U$ are given by:

$$s_t^i = \frac{E(Q_{t+1}|\Omega_t^i)}{\text{Var}(Q_{t+1}|\Omega_t^i)}$$

Given the expression for Q_{t+1}

$$Q_{t+1} = \bar{p} + e_1 \hat{F}_t + e_2 F_t - e_3 x_t + e_4 \varepsilon_{t+1}^F - e_5 \varepsilon_{t+1}^x + e_6 \varepsilon_{t+1}^D + e_7 \varepsilon_{t+1}^S - RP_t$$

we can derive expectation and variance of Q_{t+1} for both informed and uninformed investors.

For informed investors:

$$\begin{aligned}
E(Q_{t+1}|\Omega_t^I) &= \bar{p} + e_1 \hat{F}_t + e_2 F_t - e_3 x_t - RP_t \\
V^I &= \text{Var}(Q_{t+1}|\Omega_t^I) = e_4^2 \sigma_F^2 + e_5^2 \sigma_x^2 + e_6^2 \sigma_D^2 + e_7^2 \sigma_S^2
\end{aligned}$$

For uninformed investors:

$$\begin{aligned}
E(Q_{t+1}|\Omega_t^U) &= \bar{p} + e_1 \hat{F}_t + e_2 \hat{F}_t - e_3 \hat{x}_t - RP_t \\
&= \bar{p} + e_1 \hat{F}_t + e_2 \hat{F}_t - e_3 \left(\frac{p_F}{p_x} \hat{F}_t - \frac{p_F}{p_x} F_t + x_t \right) - RP_t \\
&= \bar{p} + \left(e_1 + e_2 - e_3 \frac{p_F}{p_x} \right) \hat{F}_t + e_3 \frac{p_F}{p_x} F_t - e_3 x_t - RP_t
\end{aligned}$$

Where we substitute out \hat{x}_t using the relation:

$$S_{pt} = F_t - \frac{p_x}{p_F} x_t = \hat{F}_t - \frac{p_x}{p_F} \hat{x}_t$$

This relation is obtained because both uninformed and informed investors observe the price signal S_{pt} . The conditional volatility of stock return faced by uninformed investors is given by:

$$\begin{aligned} V^U &= \text{Var} \left(Q_{t+1} | \Omega_t^U \right) \\ &= e_2^2 \text{Var} \left(F_t | \Omega_t^U \right) + e_3^2 \text{Var} \left(x_t | \Omega_t^U \right) - 2e_2e_3 \text{Cov} \left(F_t, x_t | \Omega_t^U \right) \\ &\quad + e_4^2 \sigma_F^2 + e_5^2 \sigma_x^2 + e_6^2 \sigma_D^2 + e_7^2 \sigma_S^2 \end{aligned}$$

With expressions for $E(Q_{t+1} | \Omega_t^I)$, $E(Q_{t+1} | \Omega_t^U)$, V^I , V^U , we obtain the decision rule for both types of investors. We turn next to the market clearing condition:

$$\lambda s_t^I + (1 - \lambda) s_t^U = x$$

Substitue in demand function for informed and uninformed investors:

$$\lambda \frac{\bar{p} + e_1 \hat{F}_t + e_2 F_t - e_3 x_t - R P_t}{V^I} + (1 - \lambda) \frac{\bar{p} + \left(e_1 + e_2 - e_3 \frac{p_F}{p_x} \right) \hat{F}_t + e_3 \frac{p_F}{p_x} F_t - e_3 x_t - R P_t}{V^U} = x$$

We also know that the equilibrium price function is

$$P_t = \bar{p} + p_{\hat{F}} \hat{F}_t + p_F F_t - p_x x_t$$

Thus we can match coefficient in front of \hat{F}_t , F_t and x_t in a standard way. This gives us three equations:

$$\hat{F}_t : \lambda \frac{e_1 - R p_{\hat{F}}}{\alpha V^I} + (1 - \lambda) \frac{\left(e_1 + e_2 - e_3 \frac{p_F}{p_x} \right) - R p_{\hat{F}}}{\alpha V^U} = 0 \quad (\text{A.1})$$

$$F_t : \lambda \frac{e_2 - R p_F}{\alpha V^I} + (1 - \lambda) \frac{e_3 \frac{p_F}{p_x} - R p_F}{\alpha V^U} = 0 \quad (\text{A.2})$$

$$x_t : \lambda \frac{-e_3 + R p_x}{\alpha V^I} + (1 - \lambda) \frac{-e_3 + R p_x}{\alpha V^U} = 1 \quad (\text{A.3})$$

Given the three equations, we first show that

$$p_{\hat{F}} + p_F = \frac{\rho^F}{R - \rho^F}$$

Add up equation A.1 and A.2:

$$\lambda \frac{e_1 - R p_{\hat{F}} + e_2 - R p_F}{\alpha V^I} + (1 - \lambda) \frac{e_1 + e_2 - R p_{\hat{F}} - R p_F}{\alpha V^U} = 0$$

Factor out $\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U}$:

$$e_1 - R p_{\hat{F}} + e_2 - R p_F = 0$$

Note that $e_1 + e_2$ is given by:

$$\begin{aligned} &e_1 + e_2 \\ &= p_{\hat{F}} f_1 + \rho^F (1 + p_F) + p_{\hat{F}} f_2 \\ &= p_{\hat{F}} (f_1 + f_2) + \rho^F (1 + p_F) \end{aligned}$$

Plug in expressions for f_1 and f_2 :

$$\begin{aligned} e_1 + e_2 &= p_{\hat{F}}(f_1 + f_2) + \rho^F(1 + p_F) \\ &= p_{\hat{F}} \left(\frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \rho^F + \left(\frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} \right) \rho^F \right) + \rho^F(1 + p_F) \end{aligned}$$

Using equation 2.11, the first term collapses to $p_{\hat{F}}\rho^F$. Thus:

$$e_1 + e_2 = p_{\hat{F}}\rho^F + \rho^F(1 + p_F)$$

Thus

$$\begin{aligned} e_1 - Rp_{\hat{F}} + e_2 - Rp_F &= 0 \\ \rho^F p_{\hat{F}} + \rho^F(1 + p_F) - Rp_{\hat{F}} - Rp_F &= 0 \\ \rho^F + (\rho^F - R)(p_F + p_{\hat{F}}) &= 0 \\ p_F + p_{\hat{F}} &= \frac{\rho^F}{R - \rho^F} \end{aligned}$$

Thus we only need to solve for p_F and p_x to obtain the equilibrium price function. Now focus on equation A.2:

$$\lambda \frac{e_2 - Rp_F}{\alpha V^I} + (1 - \lambda) \frac{e_3 \frac{p_F}{p_x} - Rp_F}{\alpha V^U} = 0$$

Plug in expression of e_2 and e_3 :

$$\begin{aligned} \lambda \frac{\rho^F(1 + p_F) + p_{\hat{F}}f_2 - Rp_F}{\alpha V^I} + (1 - \lambda) \frac{\rho^x p_F - Rp_F}{\alpha V^U} &= 0 \\ \lambda \frac{\rho^F(1 + p_F) + p_{\hat{F}}f_2 - \rho^x p_F + \rho^x p_F - Rp_F}{\alpha V^I} + (1 - \lambda) \frac{\rho^x p_F - Rp_F}{\alpha V^U} &= 0 \end{aligned}$$

Collect terms related to $\rho^x p_F - Rp_F$ and move to the right hand side:

$$\lambda \frac{\rho^F(1 + p_F) + p_{\hat{F}}f_2 - \rho^x p_F}{\alpha V^I} = \left[\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) p_F$$

Substitute out $p_{\hat{F}}$:

$$\lambda \frac{\rho^F(1 + p_F) + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 - \rho^x p_F}{\alpha V^I} = \left[\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) p_F$$

Thus

$$\lambda \frac{\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 + (\rho^F - \rho^x - f_2) p_F}{\alpha V^I} = \left[\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) p_F \quad (\text{A.4})$$

Now turn to equation A.3:

$$\lambda \frac{-e_3 + Rp_x}{\alpha V^I} + (1 - \lambda) \frac{-e_3 + Rp_x}{\alpha V^U} = 1$$

Plug in e_3 :

$$\begin{aligned} \lambda \frac{-\rho^x p_x + Rp_x}{\alpha V^I} + (1 - \lambda) \frac{-\rho^x p_x + Rp_x}{\alpha V^U} &= 1 \\ \left[\lambda \frac{1}{\alpha V^I} + (1 - \lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) p_x &= 1 \end{aligned} \quad (\text{A.5})$$

Divide equation A.4 by A.5, one get

$$\lambda \frac{\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F\right) f_2 + (\rho^F - \rho^x - f_2) p_F}{\alpha V^I} = \frac{p_F}{p_x} \quad (\text{A.6})$$

Thus, equation A.5 and A.6 solves for coefficients p_F and p_x given beliefs of the agents. The two equations plus the law of motion for belief jointly pin down an exogenous information steady state given λ .

Proof of Proposition 3.1

We will show that, at the exogenous-information steady state $\Phi(\lambda)$, the value of information

$$\pi(\lambda) = \frac{\hat{W}^U}{\hat{W}^I} = \sqrt{\frac{V^U}{V^I}}$$

This is an extension of Theorem 2 in Grossman and Stiglitz (1980). Plug agents' budget constraint: $c_t = (D_{t+1} + P_{t+1} - RP_t)s$ into the utility function, we obtain the expected utility of each type of agent conditional on the realized market price P_t :

$$\hat{W}^i(P_t) = \max_s EU((D_{t+1} + P_{t+1} - RP_t)s | \Omega_t^i)$$

Given CARA utility and normally distributed random variables:

$$\begin{aligned} \hat{W}^i(P_t) &= \max_s EU((D_{t+1} + P_{t+1} - RP_t)s | \Omega_t^i) \\ &= \max_s EU(-e^{-(D_{t+1} + P_{t+1} - RP_t)s} | \Omega_t^i) \\ &= \max_s -\exp[-\alpha(E[D_{t+1} + P_{t+1} - RP_t | \Omega_t^i]s - \frac{1}{2}\alpha s^2 \text{Var}(D_{t+1} + P_{t+1} - RP_t))] \end{aligned} \quad (\text{A.7})$$

Hence, maximizing over the objective function is equivalent to maximizing

$$\max_s E[D_{t+1} + P_{t+1} - RP_t | \Omega_t^i]s - \frac{1}{2}\alpha s^2 \text{Var}(D_{t+1} + P_{t+1} - RP_t | \Omega_t^i)$$

Solve for optimal s^* :

$$s^{i*} = \frac{E[D_{t+1} + P_{t+1} - RP_t | \Omega_t^i]}{\alpha \text{Var}(D_{t+1} + P_{t+1} - RP_t | \Omega_t^i)}$$

Plug back into the original objective function:

$$\hat{W}^i(P_t) = -\exp\left[-\frac{1}{2} \frac{(E[D_{t+1} + P_{t+1} | \Omega_t^i] - RP_t)^2}{\text{Var}(Q_{t+1} | \Omega_t^i)}\right]$$

Let

$$h = \text{Var}(Q_{t+1} | \Omega_t^U) - \text{Var}(Q_{t+1} | \Omega_t^I) > 0 \quad (\text{A.8})$$

The reason why h is greater than 0 is that the information set of the uninformed investors is more coarse than that of the informed investors. Taking the ex-ante conditional expectation of the informed $\hat{W}^I(P)$ with respect to the uninformed's information set

Ω_t^U :

$$\begin{aligned}
E[\hat{W}^I(P_t)|\Omega_t^U] &= E\left[-e^{-\frac{1}{2} \frac{(E[D_{t+1}+P_{t+1}|\Omega_t^I]-RP_t)^2}{\text{Var}(Q_{t+1}|\Omega_t^I)}} \middle| \Omega_t^U\right] \\
&= E\left[-e^{-\frac{1}{2} \frac{(E[D_{t+1}+P_{t+1}|\Omega_t^I]-RP_t)^2}{h} \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}} \middle| \Omega_t^U\right] \\
&= E\left[-e^{-\frac{1}{2} \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}} z^2 \middle| \Omega_t^U\right],
\end{aligned}$$

where $z = \frac{(E[D_{t+1}+P_{t+1}|\Omega_t^U]-RP_t)}{\sqrt{h}}$.

Thus, by the moment-generating function of a noncentral chi-squared distribution (formula A21 of [Grossman and Stiglitz \(1980\)](#)):

$$\begin{aligned}
E[\hat{W}^I(P_t)|\Omega^U] &= \frac{1}{\sqrt{1 + \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}}} \exp\left(\frac{-E[z|\Omega_t^U]^2 \frac{1}{2} \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}}{1 + \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}}\right) \\
&= \sqrt{\frac{\text{Var}(Q_{t+1}|\Omega_t^I)}{\text{Var}(Q_{t+1}|\Omega_t^U)}} \exp\left(\frac{-E[z|\Omega^U]^2 \frac{1}{2} \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}}{1 + \frac{h}{\text{Var}(Q_{t+1}|\Omega_t^I)}}\right) \\
&= \sqrt{\frac{\text{Var}(Q_{t+1}|\Omega_t^I)}{\text{Var}(Q_{t+1}|\Omega_t^U)}} W_U(P_t)
\end{aligned}$$

Integrating on both sides with respect to the current stock price P_t , one gets:

$$\hat{W}^I = \sqrt{\frac{\text{Var}(Q_{t+1}|\Omega_t^I)}{\text{Var}(Q_{t+1}|\Omega_t^U)}} \hat{W}^U$$

Or

$$\frac{\hat{W}^I}{\hat{W}^U} = \sqrt{\frac{V_t^I}{V_t^U}}$$

Note that the time script t does not matter in a stationary environment.

Proof of Proposition 3.3 and 3.5

Before we get to the proof, it is useful to show the following lemma which is about how the conditional volatility V^I, V^U and the return coefficients $\{e_i\}$ change with p_F, p_x respectively when $\lambda \rightarrow 0$:

Lemma A.1 As $\lambda \rightarrow 0$:

$$\begin{aligned}
e_2 &\rightarrow \rho^F (1 + a(\theta_D + \theta_S)) \\
\frac{\partial e_2}{\partial p_F} &\rightarrow \rho^F (1 - (\theta_D + \theta_S)) \\
\frac{\partial e_2}{\partial p_x} &\rightarrow 0 \\
e_3 &\rightarrow \rho^x p_x \\
\frac{\partial e_3}{\partial p_F} &\rightarrow 0 \\
\frac{\partial e_3}{\partial p_x} &\rightarrow \rho^x \\
e_4 &\rightarrow 1 + a(\theta_D + \theta_S) \\
\frac{\partial e_4}{\partial p_F} &\rightarrow 1 - (\theta_D + \theta_S) \\
\frac{\partial e_4}{\partial p_x} &\rightarrow 0 \\
e_5 &\rightarrow p_x \\
\frac{\partial e_5}{\partial p_F} &\rightarrow 0 \\
\frac{\partial e_5}{\partial p_x} &\rightarrow 1 \\
e_6 &\rightarrow a\theta_D + 1 \\
\frac{\partial e_6}{\partial p_F} &\rightarrow -\theta_D \\
\frac{\partial e_6}{\partial p_x} &\rightarrow 0 \\
e_7 &\rightarrow a\theta_S \\
\frac{\partial e_7}{\partial p_F} &\rightarrow -\theta_S \\
\frac{\partial e_7}{\partial p_x} &\rightarrow 0
\end{aligned}$$

$$\begin{aligned}
V^I &\rightarrow (1 + a(\theta_D + \theta_S))^2 \sigma_F^2 + (p_x)^2 \sigma_x^2 + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 \\
\frac{\partial V^I}{\partial p_F} &\rightarrow -2(\rho^F)^2 (1 + a(\theta_D + \theta_S)) \text{Var}(F|\Omega^U) ((1 - (\theta_D + \theta_S))) \\
\frac{\partial V^I}{\partial p_x} &\rightarrow 2p_x \sigma_x^2 \\
V^U &\rightarrow (1 + a(\theta_D + \theta_S))^2 \left((\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 \right) + (p_x)^2 \sigma_x^2 + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 + (1 + a(\theta_D + \theta_S))^2 \\
\frac{\partial V^U}{\partial p_F} &\rightarrow -2(1 + a(\theta_D + \theta_S)) \rho^x \rho^F \text{Var}(F|\Omega^U) \\
\frac{\partial V^U}{\partial p_x} &\rightarrow 2p_x \sigma_x^2 \\
\Delta V &\rightarrow (\rho^F)^2 (1 + a(\theta_D + \theta_S))^2 \text{Var}(F|\Omega^U) \\
\frac{\partial \Delta V}{\partial p_F} &\rightarrow 2(\rho^F + a(\theta_D + \theta_S) \rho^F) (\rho^F - (\theta_D + \theta_S) \rho^F - \rho^x) \text{Var}(F|\Omega^U) \\
\frac{\partial \Delta V}{\partial p_x} &\rightarrow 0
\end{aligned}$$

To prove the lemma we start by examining each e_i one by one. We start with e_2 :

$$e_2 = \rho^F (1 + p_F) + p_F f_2 = \rho^F (1 + p_F) + (a - p_F) f_2$$

we first focus on the expression of f_2

$$\begin{aligned} f_2 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} (\rho^F - \rho^x) + \left(\frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} \right) \rho^F \\ &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{[(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \left(\frac{p_x}{p_F}\right)^2}{[(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \left(\frac{p_x}{p_F}\right)^2 + \sigma_x^2} (\rho^F - \rho^x) + \left(\frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} \right) \rho^F \end{aligned}$$

Note that f_2 depends on p_F and p_x only through the square of the price ratio $\left(\frac{p_x}{p_F}\right)^2$, and thus, its derivative with respect to p_F and p_x will have a term $\frac{p_x}{p_F} \rightarrow 0$ as $\lambda \rightarrow 0$. We conclude:

$$\begin{aligned} f_2 &\rightarrow (\theta_D + \theta_S) \rho^F \\ \frac{\partial f_2}{\partial p_F} &\rightarrow 0 \\ \frac{\partial f_2}{\partial p_x} &\rightarrow 0 \end{aligned}$$

Thus

$$\begin{aligned} e_2 &\rightarrow \rho^F (1 + a(\theta_D + \theta_S)) \\ \frac{\partial e_2}{\partial p_F} &= \rho^F - f_2 - p_F \frac{\partial f_2}{\partial p_F} \rightarrow \rho^F (1 - \theta_D - \theta_S) \\ \frac{\partial e_2}{\partial p_x} &\rightarrow 0 \end{aligned}$$

For e_3 it is pretty straightforward as it depends only on p_x in linear way:

$$\begin{aligned} e_3 &= \rho^x p_x \rightarrow \rho^x p_x \\ \frac{\partial e_3}{\partial p_F} &\rightarrow 0 \\ \frac{\partial e_3}{\partial p_x} &\rightarrow \rho^x \end{aligned}$$

For e_4

$$e_4 = 1 + p_F + p_{\hat{F}} f_3$$

We need to first examine f_3

$$f_3 = \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2}$$

Again, f_3 depends on p_F and p_x only through the square of the price ratio $\left(\frac{p_x}{p_F}\right)^2$, and thus, its derivative with respect to p_F and p_x will have a term $\frac{p_x}{p_F} \rightarrow 0$ as $\lambda \rightarrow 0$. Thus

$$\begin{aligned} f_3 &\rightarrow \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_D^2} + \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\sigma_S^2} = \theta_D + \theta_S \\ \frac{\partial f_3}{\partial p_F} &\rightarrow 0 \\ \frac{\partial f_3}{\partial p_x} &\rightarrow 0 \end{aligned}$$

The property that the function depends on p_F and p_x only through the ratio holds for f_5 and f_6 as well. Thus it is easy to

show that

$$\begin{aligned} e_4 &\rightarrow 1 + a(\theta_D + \theta_S) \\ \frac{\partial e_4}{\partial p_F} &\rightarrow 1 - \theta_D - \theta_S \\ \frac{\partial e_4}{\partial p_x} &\rightarrow 0 \end{aligned}$$

$$\begin{aligned} e_6 &\rightarrow a\theta_D + 1 \\ \frac{\partial e_6}{\partial p_F} &\rightarrow -\theta_D \\ \frac{\partial e_6}{\partial p_x} &\rightarrow 0 \end{aligned}$$

$$\begin{aligned} e_7 &\rightarrow a\theta_S \\ \frac{\partial e_7}{\partial p_F} &\rightarrow -\theta_S \\ \frac{\partial e_7}{\partial p_x} &\rightarrow 0 \end{aligned}$$

We omit the proof for these three coefficients. Details are available upon request. Now we need to examine e_5 :

$$e_5 = p_x + p_{\bar{F}} f_4$$

With

$$f_4 = \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2}{(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2 + \left(\frac{p_x}{p_F}\right)^2 \sigma_x^2} \frac{p_x}{p_F}$$

Multiply both the denominator and numerator with $\left(\frac{p_F}{p_x}\right)^2$:

$$\begin{aligned} f_4 &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{[(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \left(\frac{p_F}{p_x}\right)^2}{[(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \left(\frac{p_F}{p_x}\right)^2 + \sigma_x^2} \frac{p_x}{p_F} \\ &= \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{[(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \frac{p_F}{p_x}}{[(\rho^F - \rho^x)^2 \text{Var}(F_t|\Omega_t^U) + \sigma_F^2] \left(\frac{p_F}{p_x}\right)^2 + \sigma_x^2} \end{aligned}$$

Now note that there is a term $\frac{p_F}{p_x}$. Thus its derivative with respect to p_F is not in general zero:

$$\begin{aligned} f_4 &\rightarrow 0 \\ \frac{\partial f_4}{\partial p_F} &\rightarrow \frac{\text{Var}(F_{t+1}|\Omega_{t+1}^U)}{\text{Var}(F_{t+1}|\{S_{pt+1}\} \cup \Omega_t^U)} \frac{[(\rho^F - \rho^x) \rho^F \text{Var}(F_t|\Omega_t^U) + \sigma_F^2]}{p_x \sigma_x^2} \\ \frac{\partial f_4}{\partial p_x} &\rightarrow 0 \end{aligned}$$

Thus

$$\begin{aligned} e_5 &\rightarrow p_x \\ \frac{\partial e_5}{\partial p_F} &= -f_4 - p_F \frac{\partial f_4}{\partial p_F} \end{aligned}$$

Note that $p_F \rightarrow 0$. Thus

$$\begin{aligned} \frac{\partial e_5}{\partial p_F} &\rightarrow 0 \\ \frac{\partial e_5}{\partial p_x} &\rightarrow 1 \end{aligned}$$

Now we are ready to derive expressions for the conditional volatility V^I and V^U .

Start with V^I :

$$V^I = e_4^2 \sigma_F^2 + e_5^2 \sigma_x^2 + e_6^2 \sigma_D^2 + e_7^2 \sigma_S^2$$

Plugging in expressions for e_4, e_5, e_6, e_7 and take limit, it is easy to show that:

$$V^I \rightarrow (1 + a(\theta_D + \theta_S))^2 \sigma_F^2 + (p_x)^2 \sigma_x^2 + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2$$

For $\frac{\partial V^I}{\partial p_F}$, one also needs to plug in the derivatives:

$$\begin{aligned} \frac{\partial V^I}{\partial p_F} &= 2e_4 \frac{\partial e_4}{\partial p_F} \sigma_F^2 + 2e_5 \frac{\partial e_5}{\partial p_F} \sigma_x^2 + 2e_6 \frac{\partial e_6}{\partial p_F} \sigma_D^2 + 2e_7 \frac{\partial e_7}{\partial p_F} \sigma_S^2 \\ &\rightarrow 2(1 + a(\theta_D + \theta_S))(1 - (\theta_D + \theta_S)) \sigma_F^2 - 2(1 + a\theta_D) \theta_D \sigma_D^2 - 2(a\theta_S) \theta_S \sigma_S^2 \\ &= 2(1 + a(\theta_D + \theta_S))(1 - (\theta_D + \theta_S)) \sigma_F^2 - 2(1 + a\theta_D) \frac{Var(F|\Omega^U)}{\sigma_D^2} \sigma_D^2 - 2(a\theta_S) \left(\frac{Var(F|\Omega^U)}{\sigma_S^2} \right) \sigma_S^2 \\ &= 2(1 + a(\theta_D + \theta_S))(1 - (\theta_D + \theta_S)) \sigma_F^2 - 2(1 + a(\theta_D + \theta_S)) Var(F|\Omega^U) \\ &= 2(1 + a(\theta_D + \theta_S)) \left[(1 - (\theta_D + \theta_S)) \sigma_F^2 - Var(F|\Omega^U) \right] \\ &= 2(1 + a(\theta_D + \theta_S)) \left[(1 - (\theta_D + \theta_S)) \sigma_F^2 - Var(F|\Omega^U) \right] \\ &= 2(1 + a(\theta_D + \theta_S)) \left[\left(\frac{Var(F|\Omega^U)}{(\rho^F)^2 Var(F|\Omega^U) + \sigma_F^2} \right) \sigma_F^2 - Var(F|\Omega^U) \right] \\ &= 2(1 + a(\theta_D + \theta_S)) Var(F|\Omega^U) \left[\frac{\sigma_F^2}{(\rho^F)^2 Var(F|\Omega^U) + \sigma_F^2} - 1 \right] \\ &= -2(\rho^F)^2 (1 + a(\theta_D + \theta_S))(1 - (\theta_D + \theta_S)) Var(F|\Omega^U) \end{aligned}$$

Similarly for

$$\begin{aligned} \frac{\partial V^I}{\partial p_x} &= 2e_4 \frac{\partial e_4}{\partial p_x} \sigma_F^2 + 2e_5 \frac{\partial e_5}{\partial p_x} \sigma_x^2 + 2e_6 \frac{\partial e_6}{\partial p_x} \sigma_D^2 + 2e_7 \frac{\partial e_7}{\partial p_x} \sigma_S^2 \\ &\rightarrow 2p_x \sigma_x^2 \end{aligned}$$

To derive expression for V^U , we first derive expression for the information gain ΔV :

$$\Delta V = Var(e_2 F_t - e_3 x_t | \Omega_t^U)$$

Plug in expressions of e_2 and e_3 :

$$\Delta V = Var\left(\left(\rho^F (1 + p_F) + p_{\hat{F}} f_2\right) F_t - \rho^x p_x x_t | \Omega_t^U\right)$$

Note that we can substitute out $p_x x_t$ using the price signal:

$$\begin{aligned} S_{pt} &= p_F F_t - p_x x_t \\ p_x x_t &= p_F F_t - S_{pt} \end{aligned}$$

Plug in

$$\begin{aligned} \Delta V &= \text{Var} \left(\left(\rho^F (1 + p_F) + p_{\hat{F}} f_2 \right) F_t - \rho^x (p_F F_t - S_{pt}) \mid \Omega_t^U \right) \\ &= \text{Var} \left(\left(\rho^F (1 + p_F) + p_{\hat{F}} f_2 \right) F_t - \rho^x p_F F_t \mid \Omega_t^U \right) \end{aligned}$$

As the price signal is in the information set of the uninformed. Thus

$$\begin{aligned} \Delta V &= \text{Var} \left(\left(\rho^F (1 + p_F) + p_{\hat{F}} f_2 - \rho^x p_F \right) F_t \mid \Omega_t^U \right) \\ &= \left(\rho^F (1 + p_F) + (a - p_F) f_2 - \rho^x p_F \right)^2 \text{Var} \left(F \mid \Omega^U \right) \end{aligned}$$

Thus

$$\Delta V \rightarrow \left(\rho^F + a(\theta_D + \theta_S) \rho^F \right)^2 \text{Var} \left(F \mid \Omega^U \right) = \left(\rho^F \right)^2 (1 + a(\theta_D + \theta_S))^2 \text{Var} \left(F \mid \Omega^U \right)$$

Now examine the derivatives

$$\frac{\partial \Delta V}{\partial p_F} = 2 \left(\rho^F (1 + p_F) + (a - p_F) f_2 - \rho^x p_F \right) \left(\rho^F - f_2 + (a - p_F) \frac{\partial f_2}{\partial p_F} - \rho^x \right) \text{Var} \left(F \mid \Omega^U \right)$$

Here we omit the derivative with respect to $\text{Var} \left(F \mid \Omega^U \right)$ as we will take $\lambda \rightarrow 0$:

$$\begin{aligned} \frac{\partial \Delta V}{\partial p_F} &\rightarrow 2 \left(\rho^F + a(\theta_D + \theta_S) \rho^F \right) \left(\rho^F - (\theta_D + \theta_S) \rho^F - \rho^x \right) \text{Var} \left(F \mid \Omega^U \right) \\ &= 2 \left(\rho^F + a(\theta_D + \theta_S) \rho^F \right) \left((1 - \theta_D - \theta_S) \rho^F - \rho^x \right) \text{Var} \left(F \mid \Omega^U \right) \end{aligned}$$

$$\frac{\partial \Delta V}{\partial p_x} = 0$$

As p_x does not show up in the expression of ΔV .

Now we are ready to derive expression for V^U :

$$V^U = V^I + \Delta V$$

Thus

$$\begin{aligned} V^U &\rightarrow \left(\rho^F \right)^2 (1 + a(\theta_D + \theta_S))^2 \text{Var} \left(F \mid \Omega^U \right) + (1 + a(\theta_D + \theta_S))^2 \sigma_F^2 + (p_x)^2 \sigma_x^2 + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 \\ &= (1 + a(\theta_D + \theta_S))^2 \left[\left(\rho^F \right)^2 \text{Var} \left(F \mid \Omega^U \right) + \sigma_F^2 \right] + (p_x)^2 \sigma_x^2 + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial V^U}{\partial p_F} &= \frac{\partial V^I}{\partial p_F} + \frac{\partial \Delta V}{\partial p_F} \\ &\rightarrow -2 \left(\rho^F \right)^2 (1 + a(\theta_D + \theta_S)) (1 - (\theta_D + \theta_S)) \text{Var} \left(F \mid \Omega^U \right) + 2 \rho^F (1 + a(\theta_D + \theta_S)) \left((1 - \theta_D - \theta_S) \rho^F - \rho^x \right) \text{Var} \left(F \mid \Omega^U \right) \\ &= 2 \rho^F (1 + a(\theta_D + \theta_S)) \text{Var} \left(F \mid \Omega^U \right) \left[(-1 + \theta_D + \theta_S) \rho^F + (1 - \theta_D - \theta_S) \rho^F - \rho^x \right] \\ &= -2 \rho^F \rho^x (1 + a(\theta_D + \theta_S)) \text{Var} \left(F \mid \Omega^U \right) < 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial V^U}{\partial p_x} &= \frac{\partial V^I}{\partial p_x} + \frac{\partial \Delta V}{\partial p_x} \\ &\rightarrow 2p_x \sigma_x^2\end{aligned}$$

With the lemma, we are ready to prove proposition ???. First note that as $\lambda \rightarrow 0$, $Var(F|\Omega^U)$ does not change with λ . Thus we only need to characterize the derivative of p_F and p_x with respect to λ from the two market-clearing coefficient matching equations:

$$\left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) p_x - 1 = 0 \quad (\text{A.9})$$

$$\lambda \frac{\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 + (\rho^F - \rho^x - f_2) p_F}{\alpha V^I} - \frac{p_F}{p_x} = 0 \quad (\text{A.10})$$

Total differentiate the two equations with respect to p_F, p_x , and λ :

$$\begin{aligned}0 &= \left[\lambda \frac{-1}{\alpha (V^I)^2} \frac{\partial V^I}{\partial p_F} + (1-\lambda) \frac{-1}{\alpha (V^U)^2} \frac{\partial V^U}{\partial p_F} \right] (R - \rho^x) p_x dp_F \\ &+ \left\{ \left[\lambda \frac{1}{\alpha V^I} + (1-\lambda) \frac{1}{\alpha V^U} \right] (R - \rho^x) + \left[\lambda \frac{-1}{\alpha (V^I)^2} \frac{\partial V^I}{\partial p_x} + (1-\lambda) \frac{-1}{\alpha (V^U)^2} \frac{\partial V^U}{\partial p_x} \right] (R - \rho^x) p_x \right\} dp_x \\ &+ \left[\frac{1}{\alpha V^I} - \frac{1}{\alpha V^U} \right] (R - \rho^x) p_x d\lambda\end{aligned}$$

$$\begin{aligned}0 &= \left[\lambda \frac{\left[\rho^F - f_2 + (\rho^F - \rho^x - f_2) - p_F \frac{\partial f_2}{\partial p_F} \right] V^I - \left[\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 + (\rho^F - \rho^x - f_2) p_F \right] \frac{\partial V^I}{\partial p_F} - \frac{1}{p_x}}{\alpha (V^I)^2} \right] dp_F \\ &+ \left[\lambda \frac{- \left[\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 + (\rho^F - \rho^x - f_2) p_F \right] \frac{\partial V^I}{\partial p_x} + \frac{p_F}{(p_x)^2}}{\alpha (V^I)^2} + \frac{p_F}{(p_x)^2} \right] dp_x \\ &+ \frac{\rho^F + \left(\frac{\rho^F}{R - \rho^F} - p_F \right) f_2 + (\rho^F - \rho^x - f_2) p_F}{\alpha V^I} d\lambda\end{aligned}$$

Take $\lambda \rightarrow 0$ and hence $p_F \rightarrow 0$:

$$\begin{aligned}0 &= \frac{-1}{\alpha (V^U)^2} \frac{\partial V^U}{\partial p_F} (R - \rho^x) p_x dp_F + \left\{ \frac{1}{\alpha V^U} (R - \rho^x) - \frac{1}{\alpha (V^U)^2} \frac{\partial V^U}{\partial p_x} (R - \rho^x) p_x \right\} dp_x + \left[\frac{1}{\alpha V^I} - \frac{1}{\alpha V^U} \right] (R - \rho^x) p_x d\lambda \\ &= -\frac{1}{p_x} dp_F + \frac{\rho^F + a(\theta_D + \theta_S) \rho^F}{\alpha V^I} d\lambda\end{aligned}$$

From the second equation one immediately see that

$$\frac{dp_F}{d\lambda} = \rho^F \frac{1 + a(\theta_D + \theta_S)}{\alpha V^I} p_x$$

This completes the proof of Proposition 3.3. The derivation of $\frac{dp_x}{d\lambda}$ is as follows. From the first equation:

$$\frac{dp_x}{d\lambda} = \frac{\frac{1}{\alpha(V^U)^2} \frac{\partial V^U}{\partial p_F} (R - \rho^x) p_x \frac{dp_F}{d\lambda} - \left[\frac{1}{\alpha V^I} - \frac{1}{\alpha V^U} \right] (R - \rho^x) p_x}{\frac{1}{\alpha V^U} (R - \rho^x) - \frac{1}{\alpha(V^U)^2} \frac{\partial V^U}{\partial p_x} (R - \rho^x) p_x}$$

eliminate α and $R - \rho^x$ from both the numerator and denominator:

$$\frac{dp_x}{d\lambda} = \frac{\frac{1}{(V^U)^2} \frac{\partial V^U}{\partial p_F} p_x \frac{dp_F}{d\lambda} - \left[\frac{1}{V^I} - \frac{1}{V^U} \right] p_x}{\frac{1}{V^U} - \frac{1}{(V^U)^2} \frac{\partial V^U}{\partial p_x} p_x}$$

When $\lambda \rightarrow 0$, equation A.5 becomes:

$$\begin{aligned} \frac{1}{\alpha V^U} (R - \rho^x) p_x &= 1 \\ p_x &= \frac{\alpha V^U}{(R - \rho^x)} \end{aligned}$$

Plug p_x into $\frac{dp_x}{d\lambda}$:

$$\begin{aligned} \frac{dp_x}{d\lambda} &= \frac{\frac{1}{(V^U)^2} \frac{\partial V^U}{\partial p_F} \frac{\alpha V^U}{(R - \rho^x)} \frac{dp_F}{d\lambda} - \left[\frac{1}{V^I} - \frac{1}{V^U} \right] \frac{\alpha V^U}{(R - \rho^x)}}{\frac{1}{V^U} - \frac{1}{(V^U)^2} \frac{\partial V^U}{\partial p_x} \frac{\alpha V^U}{(R - \rho^x)}} \\ &= \frac{\frac{\partial V^U}{\partial p_F} \frac{\alpha}{(R - \rho^x)} \frac{dp_F}{d\lambda} - \left[\frac{V^U}{V^I} - 1 \right] \frac{\alpha V^U}{(R - \rho^x)}}{1 - \frac{\partial V^U}{\partial p_x} \frac{\alpha}{(R - \rho^x)}} \\ &= \frac{\frac{\partial V^U}{\partial p_F} \frac{dp_F}{d\lambda} - \frac{\Delta V}{V^I} V^U}{\frac{(R - \rho^x)}{\alpha} - \frac{\partial V^U}{\partial p_x}} \end{aligned}$$

Now plug in expression for $\frac{\partial V^U}{\partial p_F}$, $\frac{\partial V^U}{\partial p_x}$, $\frac{dp_F}{d\lambda}$ and ΔV :

$$\frac{dp_x}{d\lambda} = \frac{-2\rho^F \rho^x (1 + a(\theta_D + \theta_S)) \text{Var}(F|\Omega^U) \rho^F \frac{1+a(\theta_D+\theta_S)}{\alpha V^I} p_x - (\rho^F)^2 (1+a(\theta_D+\theta_S))^2 \text{Var}(F|\Omega^U) V^U}{\frac{(R - \rho^x)}{\alpha} - 2p_x \sigma_x^2}$$

Factor out $\frac{(\rho^F)^2 (1+a(\theta_D+\theta_S))^2}{V^I} \text{Var}(F|\Omega^U)$

$$\frac{dp_x}{d\lambda} = \frac{(\rho^F)^2 (1 + a(\theta_D + \theta_S))^2}{V^I} \text{Var}(F|\Omega^U) \frac{-2\rho^x \frac{1}{\alpha} p_x - V^U}{\frac{(R - \rho^x)}{\alpha} - 2p_x \sigma_x^2}$$

Now use the relation $V^U = \frac{1}{\alpha} (R - \rho^x) p_x$:

$$\begin{aligned} &= \frac{(\rho^F)^2 (1 + a(\theta_D + \theta_S))^2}{V^I} \text{Var}(F|\Omega^U) \frac{-2\rho^x \frac{1}{\alpha} p_x - \frac{1}{\alpha} (R - \rho^x) p_x}{\frac{(R - \rho^x)}{\alpha} - 2p_x \sigma_x^2} \\ &= - \frac{(\rho^F)^2 (1 + a(\theta_D + \theta_S))^2}{\alpha V^I} \text{Var}(F|\Omega^U) p_x \frac{2\rho^x + R - \rho^x}{\frac{(R - \rho^x)}{\alpha} - 2p_x \sigma_x^2} \end{aligned}$$

Note that $\frac{dp_F}{d\lambda} = \rho^F \frac{1+a(\theta_D+\theta_S)}{\alpha V^I} p_x$, plug in this expression:

$$\begin{aligned} \frac{dp_x}{d\lambda} &= -(1+a(\theta_D+\theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{2\rho^x + R - \rho^x}{\frac{(R-\rho^x)}{\alpha} - 2p_x \sigma_x^2} \frac{dp_F}{d\lambda} \\ &= -(1+a(\theta_D+\theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{\rho^x + R}{\frac{(R-\rho^x)}{\alpha} - 2p_x \sigma_x^2} \frac{dp_F}{d\lambda} \end{aligned}$$

Proof of Theorem 1

To begin, we write out the partial derivatives:

$$\frac{d\Delta V}{d\lambda} = \frac{\partial \Delta V}{\partial p_F} \frac{dp_F}{d\lambda} + \frac{\partial \Delta V}{\partial p_x} \frac{dp_x}{d\lambda}$$

Plugging in $\frac{\partial \Delta V}{\partial p_F}$ and $\frac{\partial \Delta V}{\partial p_x}$ from lemma A.1 and $\frac{dp_F}{d\lambda}$ and $\frac{dp_x}{d\lambda}$ from proposition ??:

$$\frac{d\Delta V}{d\lambda} = 2 \left(\rho^F + a(\theta_D + \theta_S) \rho^F \right) \left(\rho^F - (\theta_D + \theta_S) \rho^F - \rho^x \right) \text{Var}(F|\Omega^U) \rho^F \frac{1+a(\theta_D+\theta_S)}{\alpha V^I} p_x \quad (\text{A.11})$$

All other terms are strictly positive for generic parameter values except

$$\rho^F - (\theta_D + \theta_S) \rho^F - \rho^x$$

Thus $\frac{d\Delta V}{d\lambda} > 0$ if and only if $(1 - (\theta_D + \theta_S)) \rho^F - \rho^x > 0$

Proof of Proposition 3.4

Take equation A.5 and take $\lambda \rightarrow 0$:

$$(R - \rho^x) p_x = \alpha V^U$$

Plug in value of V^U from lemma A.1, rearrange:

$$\sigma_x^2 (p_x)^2 - \frac{(R - \rho^x)}{\alpha} p_x + (1 + a(\theta_D + \theta_S))^2 \left((\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 \right) + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 + (1 + a(\theta_D + \theta_S))^2 = 0$$

This is a quadratic equation from which one can solve for p_x :

$$p_x = \frac{\frac{R-\rho^x}{\alpha} \pm \sqrt{\Delta}}{2\sigma_x^2}$$

Where

$$\Delta = \left(\frac{(R - \rho^x)}{\alpha} \right)^2 - 4\sigma_x^2 \left[(1 + a(\theta_D + \theta_S))^2 \left((\rho^F)^2 \text{Var}(F|\Omega^U) + \sigma_F^2 \right) + (1 + a\theta_D)^2 \sigma_D^2 + (a\theta_S)^2 \sigma_S^2 + (1 + a(\theta_D + \theta_S))^2 \right]$$

[The proof of Proposition 3.5 is combined with that of Proposition 3.3.]

Proof of Proposition 3.6

From proposition 3.5:

$$\frac{dp_x}{d\lambda} = -(1+a(\theta_D+\theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{\rho^x + R}{\frac{(R-\rho^x)}{\alpha} - 2p_x \sigma_x^2} \frac{dp_F}{d\lambda}$$

All other terms are strictly positive except (note that $\frac{dp_F}{d\lambda} > 0$)

$$\frac{(R - \rho^x)}{\alpha} - 2p_x\sigma_x^2$$

We also know from proposition 3.4 that

$$p_x = \frac{\frac{R - \rho^x}{\alpha} \pm \sqrt{\Delta}}{2\sigma_x^2}$$

Thus at the low volatility equilibrium

$$\begin{aligned} & \frac{(R - \rho^x)}{\alpha} - 2 \frac{\frac{R - \rho^x}{2\alpha} - \sqrt{\Delta}}{2\sigma_x^2} \sigma_x^2 \\ = & \frac{(R - \rho^x)}{\alpha} - \left(\frac{R - \rho^x}{2\alpha} - \sqrt{\Delta} \right) = \sqrt{\Delta} \geq 0 \end{aligned}$$

Likewise at the high volatility equilibrium

$$\frac{(R - \rho^x)}{\alpha} - 2p_x\sigma_x^2 = -\sqrt{\Delta} \leq 0$$

Thus

$$\begin{aligned} \frac{dp_x}{d\lambda} & \leq 0 \text{ if it is low-volatility equilibrium} \\ \frac{dp_x}{d\lambda} & \geq 0 \text{ if it is high-volatility equilibrium} \end{aligned}$$

Proof of Theorem 2

In view of proposition 3.6 and 3.5, in order to prove Theorem 2 it suffices to show that $\frac{\partial V^I}{\partial p_F} < 0$ and $\frac{\partial V^I}{\partial p_x} > 0$. This can be seen from lemma A.1:

$$\begin{aligned} \frac{\partial V^I}{\partial p_F} & \rightarrow -2 \left(\rho^F \right)^2 (1 + a(\theta_D + \theta_S)) \text{Var}(F|\Omega^U) ((1 - (\theta_D + \theta_S))) < 0 \\ \frac{\partial V^I}{\partial p_x} & \rightarrow 2p_x\sigma_x^2 > 0 \end{aligned}$$

Proof of Proposition 3.7

Manipulate equation 3.10, we have

$$\frac{R - \rho^x}{\alpha} - 2p_x\sigma_x^2 = \pm\sqrt{\Delta}$$

Where the sign depends on the type of the financial market equilibrium. The sign is positive in low-volatility equilibrium and is negative in high-volatility equilibrium.

We also know from lemma A.1 that as $\lambda \rightarrow 0$:

$$\frac{\partial V^I}{\partial p_x} \rightarrow 2p_x\sigma_x^2$$

And from equation 3.11:

$$\frac{dp_x}{d\lambda} = -(1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{\rho^x + R}{\frac{(R - \rho^x)}{\alpha} - 2p_x\sigma_x^2} \frac{dp_F}{d\lambda}$$

Plug in expression for $\frac{R - \rho^x}{\alpha} - 2p_x\sigma_x^2$:

$$\frac{dp_x}{d\lambda} = -(1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{\rho^x + R}{\pm\sqrt{\Delta}} \frac{dp_F}{d\lambda}$$

Thus the absolute value of $\frac{\partial V^I}{\partial p_x} \frac{dp_x}{d\lambda}$ is

$$\left| \frac{\partial V^I}{\partial p_x} \frac{dp_x}{d\lambda} \right| = 2p_x \sigma_x^2 (1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{\rho^x + R}{\sqrt{\Delta}} \frac{dp_F}{d\lambda}$$

As ρ^x and ρ^F are pushed to its boundary, $\sqrt{\Delta} \rightarrow 0$ whereas all other terms are bounded. Thus we have

$$\left| \frac{\partial V^I}{\partial p_x} \frac{dp_x}{d\lambda} \right| \rightarrow \infty$$

We still need to show that such boundary exists. This can be done by observing that Δ is monotonically decreasing with respect to both ρ^x and ρ^F and for appropriate values of (ρ^x, ρ^F) , $\Delta > 0$.

Proof of Theorem 3

Note that to show the slope of $\frac{d\pi(\lambda)}{d\lambda} > 0$, it suffices to show the slope of

$$\frac{d\frac{\Delta V}{V^I}}{d\lambda} > 0$$

Or equivalently

$$\frac{d\Delta V}{d\lambda} - \frac{\Delta V}{V^I} \frac{dV^I}{d\lambda}$$

From equation A.11:

$$\frac{d\Delta V}{d\lambda} = 2\rho^F (1 + a(\theta_D + \theta_S)) (\rho^F - (\theta_D + \theta_S)\rho^F - \rho^x) \text{Var}(F|\Omega^U) \rho^F \frac{1 + a(\theta_D + \theta_S)}{\alpha V^I} p_x$$

Also

$$\frac{dV^I}{d\lambda} = \frac{\partial V^I}{\partial p_F} \frac{dp_F}{d\lambda} + \frac{\partial V^I}{\partial p_x} \frac{dp_x}{d\lambda}$$

Plug in the derivatives from lemma A.1 and proposition ??:

$$\begin{aligned} &= \frac{\partial V^I}{\partial p_F} \frac{dp_F}{d\lambda} + \frac{\partial V^I}{\partial p_x} \frac{dp_x}{d\lambda} \\ &= -2(\rho^F)^2 (1 + a(\theta_D + \theta_S)) (1 - (\theta_D + \theta_S)) \text{Var}(F|\Omega^U) \rho^F \frac{1 + a(\theta_D + \theta_S)}{\alpha V^I} p_x \\ &\quad - 2p_x \sigma_x^2 (1 + a(\theta_D + \theta_S)) \rho^F \text{Var}(F|\Omega^U) \frac{2\rho^x + R - \rho^x}{\alpha} \rho^F \frac{1 + a(\theta_D + \theta_S)}{\alpha V^I} p_x \end{aligned}$$

Thus

$$\frac{d\Delta V}{d\lambda} = 2\text{Var}(F|\Omega^U) (\rho^F)^2 \frac{(1 + a(\theta_D + \theta_S))^2}{\alpha V^I} p_x \left[\rho^F - (\theta_D + \theta_S)\rho^F - \rho^x + \frac{\Delta V}{V^I} \left(\rho^F (1 - (\theta_D + \theta_S)) + p_x \sigma_x^2 \frac{\rho^x + R}{\alpha} \right) \right]$$

All other terms are strictly positive except

$$\rho^F - (\theta_D + \theta_S)\rho^F - \rho^x + \frac{\Delta V}{V^I} \left(\rho^F (1 - (\theta_D + \theta_S)) + p_x \sigma_x^2 \frac{\rho^x + R}{\alpha} \right)$$

Thus $\frac{d\Delta V}{d\lambda} > 0$ if and only if

$$\rho^F - (\theta_D + \theta_S)\rho^F - \rho^x + \frac{\Delta V}{VI} \left(\rho^F (1 - (\theta_D + \theta_S)) + p_x \sigma_x^2 \frac{\rho^x + R}{\frac{(R - \rho^x)}{\alpha} - 2p_x \sigma_x^2} \right) > 0$$