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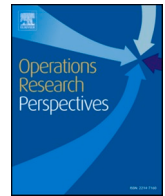
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An integrated manufacturer-buyer chain with bounded production cycle length

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ABSTRACT

A mathematical formulation is developed for modeling a joint two-echelon supply chain consisting of a single manufacturer (i.e., vendor) who continuously produces a production lot of size Q items with constant production rate $P \geq D$, where D is the given demand rate set by the retailer (i.e., buyer). The unit production cost $c_V(P)$ is assumed to be a linearly decreasing function of the production rate P . This assumption may hold as long as economies of scale are in effect and production rate does not exceed very high level which entails heavy wear associated with expensive equipment. The optimization problem of minimizing the joint total cost is solved analytically. The proposed formulation avoids scenarios under which the production rate approaches the demand rate (i.e., producing almost without breaks) and scenarios under which the production cycle length (at the manufacturer's warehouse) is unbounded. Very long production cycle lengths inevitably mean very large lots, which are associated with immense and inefficient warehouses. Moreover, they might be infeasible due to scheduled maintenance, holidays or end weeks. We prove that the optimal solution can consist of no more than two switching points along the shipment-size axis n .

1. Introduction

Joint manufacturer-buyer coordination offers operational and monetary benefits. Scenarios under which the production cycle length (at the manufacturer's warehouse) is bounded did not receive sufficient attention in existing literature of modeling the integrated manufacturer-buyer problem.

1.1. A bound on the production cycle and applications in the real world

A well-known strategy practiced in real world applications in order to reduce the cycle time is achieving higher productivity. Based on data collected through 1576 observations in a Serbian company that manufactures electrical and electronic equipment for motor vehicles, Brkić et al. [4] suggest using control charts on that sample. They show that the production cycle was reduced in 2012 by 28.53 percent, compared with 2011, while production time was shortened by 19.17 percent.

Outcomes of some pertinent studies on the cycle time reduction are presented by Lathashankar et al. [19]. In a case study of reputed construction equipment manufacturing company of India [19] it was observed that the demand for the 4Ton compactors remained unmet as there were no dedicated fixtures for the tack welding operation of those

compactors. A new design of a dedicated fixture for tack welding operation of front and rear chassis so as to reduce the cycle time of tack welding operation was developed and generated by using CATIA V5 R20 modeling software.

A case study in the gear box manufacturing industry is presented by Gnanavel et al. [12]. The motor tilting and clamping the motor with proper alignment are not only consuming more time, but also the cause of frequent injuries to the labors and give fatigue to workers. It was planned to design supporting equipment instead of using standard supporting blocks, crane, and eliminate the practice of manually guessed tilt and checking of the angle of tilt, motor clamping time etc. The new equipment has increased the average testing rate from 208 gear boxes per month to 302 gear boxes per month.

The above applications as well as many others are very useful in facing limited production cycles. Yet, even after developing an efficient process of technological improvement that reduces the cycle time, a bound on the production cycle length may still exist. In addition, very long production cycle lengths might be infeasible due to scheduled maintenance, holidays, or weekends. Sarker and Babu [22] showed that in the case where products can only be stored in the inventory for a limited time (e.g., perishables), it may be beneficial either to reduce the production cycle or increase the production rate to shorten the storage

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time of the products and to avoid spoilage. We suggest an operational research approach that aims to select the best production rate (along with shipment frequency and size) in terms of the overall integrated cost within the frame of the joint buyer-manufacturer problem.

1.2. Mathematical models of the joint buyer-manufacturer problem

Coordination among supply chain participants is necessary to achieve better performance [17]. The class of problems that are associated with minimizing the total system cost, coordinating the inventory replenishment, and establishing long-term relationships between players are known as joint economic lot sizing (JELS) problems or integrated production-inventory models. Ben-Daya et al. [3] presented a comprehensive and up-to-date review of the joint economic lot sizing problem and also provided some extensions of this important problem. The JELS problem has received considerable attention in recent years in the literature. David and Eben-Chaime [6] analyzed a simplified integrated vendor-buyer supply chain consisting of a single item, a single manufacturer, and a single buyer. Further, they assumed a just-in-time (JIT) manufacturer and that the demand of the customer is continuous and follows a fixed rate, a fixed unit production cost, and a given, finite production rate P . The main contribution of their model is that it provides explicit optimal solutions for the shipment frequency and shipment size. Chu and Leon [5] studied the problem of coordinating the single-vendor multiple-buyer (SVMB) inventory system under restricted information sharing. Glock and Ries [8] analyzed the case of a buyer sourcing a product from multiple suppliers under stochastic demand and a lead time that varies linearly with the lot size. Hoque [14] also analyzed the vendor-buyer integrated production-inventory model under random lead-time. His-formulation considers a transport capacity constraint. Wu and Zhao [26] formulated a collaborative mechanism for optimizing the replenishment schedule under trade credit and a shelf-life constraint in a two-echelon supply chain. In order to determine the optimal replenishment schedule and the equitable trade credit period, they proposed an efficient solution procedure. Torres et al. [25] examined the effects of parameter variations on proposed performance measurements (e.g., joint cost, holding cost, ordering cost) in a manufacturer-buyer VMI (vendor-managed inventory)-coordinated scheme. A supply chain consisting of a single manufacturer and a single retailer was analyzed by Shah [24]. She assumed that units in the warehouse are subject to deterioration at a constant rate. The demand was assumed to be a decreasing function of the selling price and an increasing function of the credit period offered by the retailer to the customers. The manufacturer was assumed to follow a lot-for-lot production strategy. An algorithm utilizing optimality conditions and a Maple software program was then described to find the best strategy, which involves maximizing the joint profit for the case where the credit period and production rate are given constants. Astanjin and Sajadieh [2] investigated supply-chain coordination by developing two models, involving coordinated and independent decision-making for an imperfect manufacturing system with a probabilistic defect rate. Using a primal-dual interior point method, algorithms to find the optimal solution were suggested for both models. Gautam et al. [10] developed an inventory model for a vendor - buyer system where the production process employed at the vendor's side is assumed to be imperfect and yields defectives. An integrated imperfect production model for buyers and vendors was proposed by Khanna et al. [18]. In this study, they considered a warranty policy for imperfect items and maintenance policy for the production system.

1.3. Variable production rate

The aforementioned papers, as well as many others that analyze the coordinated vendor-buyer problem, assume a given finite production

rate. A variable production rate, however, is a key factor in increasing the flexibility of a supply network, as it allows a firm to respond to changes in customer orders without an associated major increase in cost or reduction in quality. In Glock [7], who addressed a single-stage production system, a convex function $C_p(P)$ is attributed to the dependence of the unit production cost on the production rate. The tacit argument for the convexity property is that at low rates of production, economies of scale are in effect: workers slow down while machines are running at a lower speed, meaning that costs are incurred when capacity is limited by slow production. At a certain, very low production rate, however, the unit production cost increases because workers have to work overtime; thus machines have less time for preventive maintenance. Very high production rates entail heavy wear associated with more powerful and expensive equipment; hence the unit production cost starts to increase. A relatively small proportion of the literature on the joint manufacturer-buyer problem has considered the production rate as a decision variable [1]. Sajadieh and Larsen [21] dealt with the case where the retailer and the manufacturer face random demand and random yield, respectively. Their formulation as a Markov decision process is unusual in the literature as it allows a finite random discrete production rate P for the manufacturer. Marchi et al. [20] presented a joint economic lot-size model that allows investments financed cooperatively by the members of the supply chain, assuming a decreasing, inverse-proportional relationship for the function $C_p(P)$. Jauhari et al. [15] proposed an iterative procedure to find the optimal solutions to a JELS problem under fuzzy demand and inspection errors. Aldurgam et al. [1] proposed a numerical line search algorithm to solve a JELS model with stochastic demand and a shipment constraint where the production rate at the manufacturer can be varied. Note that a variable production rate was also studied by Jauhari et al. [15], Sarkar et al. [23], Gautam et al. [11], and Kamna et al. [16]. The unit production cost function $C_p(P)$ was assumed to be convex. Sarkar et al. [23] analyzed how the flexibility of the production rate affects the product quality as well the entire supply-chain cost under a single-setup multiple-delivery policy. They used classical optimization techniques to develop an iterative search within a solution algorithm.

1.4. Contribution

Table 1 summarizes the work most relevant to the suggested problem and classifies papers according to key characteristics. The present study considers the supply chain addressed by David and Eben-Chaime [6], and follows a similar approach to that presented in their paper; however, it seeks to derive explicit optimal solutions for a more general problem. Similar to the work of Jauhari et al. [15] and Aldurgam et al. [1], we seek to find the optimal production rate. However, none of these studies derived explicit solutions, nor did they consider a bound on the length of the production cycle.

An approximated solution to the constrained integrated manufacturer-buyer supply problem was recently published by Herbon [13]. They assume the unit cost is independent of the production rate. We postulate a linear approximation for the dependence of the unit production cost on the production rate, $C_p(P)$, in order to facilitate the mathematical analysis, and seek the optimal production rate P . This assumption is reasonable for a wide range of production rates, and especially when the production rate is bounded from below, as is assumed in our model. Our work contributes to the literature in the following respects:

- (1) Our solution approach is mostly analytical. Analytical approaches have an advantage over enumerative search procedures, which are commonly suggested in the existing literature (e.g., [1, 20, 23]), as they yield theoretical insights into the problem under consideration.

Table 1

A comparison of the proposed model with related work.

Papers	Adjustable production rate	Shipment freq.	Bounding production cycle	Unit production cost	Solution method	Additional characteristics
David, I., Eben-Chaime (2008)	NO	Integer	NO	Const	Explicit	NA
Glock, C.H. [7]	Yes	Integer	NO	Convex	Solution procedure	Unequal-sized batch shipments
Sajadieh and Larsen [21]	Yes (Integer)	Given	NO	Const	MDP solution procedure	Random demand
Shah [24]	NO	Contin.	NO	Const	Solution procedure	Deteriorating items
Marchi et al. [20]	NO	Integer	NO	Decreasing, inverse-proportional	Enumerative procedure	Joint financing of investments
Jauhari et al. [15]	Yes	Integer	NO	Convex	Iterative procedure	Fuzzy demand and inspection errors
Aldurgam et al. [1]	Yes	Integer	NO	Convex	Numerical line search algorithm	Stochastic demand and a shipment constraint
Sarkar et al. [23]	Yes	Integer	NO	Convex	Iterative procedure	Multiple buyers and imperfect production
Gautam and Khanna [9]	NO	Integer	NO	Const	Solution procedure	Set-up cost reduction
Gautam et al. [10]	NO	Given	NO	Const	Explicit	Carbon emissions; Imperfect production
Herbon [13]	Yes	Integer	Yes	Const	Approx.	Buyer's and Manufacture's unit holding cost are different
Proposed model	Yes	Integer	Yes	Linear	Explicit	Lower bound on production rate

- (2) Unlike the case in which the production cycle length is unbounded, for the bounded problem the specific problem parameters might impose infeasibility.
- (3) Interestingly, a bound on the production cycle length enables optimal selection of intermediate production rates, thus avoiding the utilization of solely optimal endpoint production rates (e.g., production always at the maximal rate) which characterizes the unbounded problem.
- (4) We show that in the special case where the unit cost is independent of the production rate (for an unbounded production cycle length), the demand-to-production rate ratio does not depend on the shipment quantity.
- (5) The results indicate that the unit cost sensitivity has a dichotomous effect on decision variables and on the optimal joint cost.

2. Supply-chain description and problem formulation

This section presents the integrated inventory and production system that is analyzed in this paper. The mathematical formulation of the problem and the assumptions are introduced. Consider a two-echelon supply chain where a single manufacturer (i.e., vendor) continuously produces a production lot of size Q items with constant production rate, $D \leq P \leq U$, where D is the given demand rate set by the retailer (i.e., buyer) and U is the maximal production rate. The demand rate is equal to the demand that is driven by the market and is observed by the single buyer. We define the demand-to-production rate ratio $r = D/P$, which facilitates the presentation of the formulas.

Produced items associated with the lot are stored in the manufacturer's warehouse and are delivered to the retailer's warehouse (corresponding to the simultaneous sudden changes in inventory levels in both warehouses shown in Fig. 1) in n (integer) equal in-transit shipments; that is, each shipment is of size $q = Q/n$. As is commonly practiced, each production lot incurs a set-up cost of K that is assumed to be independent of its size.

The additional key assumptions of the suggested model are as follows:

- A1 The two players cooperate, i.e., they share information and have signed a pre-agreement about the process of dividing the total profit.
- A2 Each player prohibits shortages.
- A3 The production cycle length is bounded by a given length T_p .

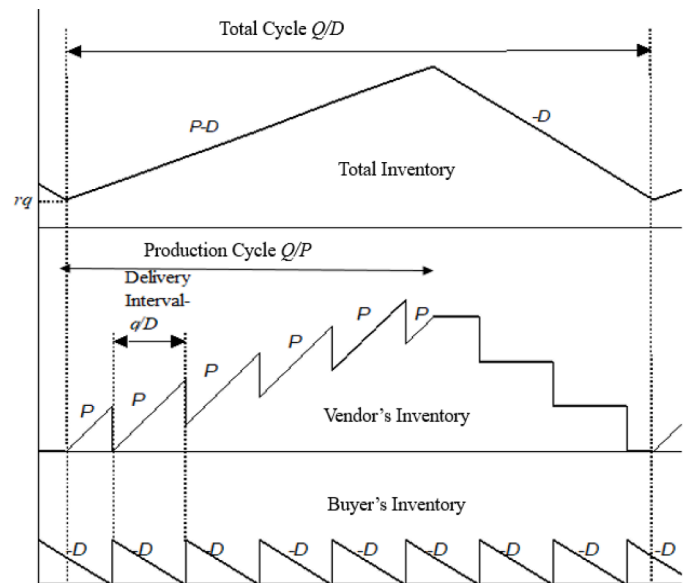


Fig. 1. Inventory levels at the buyer's and the vendor's, and the total inventory.

A4 The unit production cost, $c_v(P)$, is a linearly decreasing function of the production rate P .

Thus, the total cost per unit time is

$$C(P, Q, n) = Dc_v(P) + \frac{DK}{Q} + \frac{k_v D \cdot n}{Q} + h_v \left(\frac{D}{P} \frac{Q}{n} + Q \left/ 2 \left(1 - \frac{D}{P} \right) - \frac{Q}{2n} \right) + \frac{k_B D \cdot n}{Q} + h_B \frac{Q}{2n} \tag{1}$$

where k_B and k_v denote the cost of a single in-transit shipment associated with the buyer and the vendor, respectively and h_B and h_v denote the holding cost of a unit per unit-time associated with the buyer and the vendor, respectively. We assume the unit cost $c_v(P)$ is a linearly decreasing function of the production rate P with rate α . This assumption may hold as long as economies of scale are in effect and the production rate does not exceed so high level that they entail heavy wear associated with expensive equipment. We note that the convex unit cost

function $c_V(P) = a/P + bP$ would be more accurate with respect to a linear function for very high production rates, but suffers a drawback of approaching infinite unit cost for very low values of P . Ours simplification is relatively reasonable for a wide domain of production rates and also allows for analytical treatment of the problem. Keeping in mind that $P \geq D$, the unit cost is,

$$c_V(P) = c_V - \alpha(P - D), \tag{2}$$

where the unit cost sensitivity α , keeps $0 < \alpha < c_V/(U - D)$ in order to prevent negative values of $c_V(P)$ and c_V is the maximal unit cost under feasible production designs (i.e., when $P = D$). This function approximates the unit cost until some maximal value of the production rate, P_M , after which a possible increase may be observed (e.g., due to too much tools scrap, or due to a very expensive production capacity). Accordingly, we assume $P_M > U$. The general problem considered in this paper is:

Problem (P1):

$$\begin{aligned} & \min_{r,q,n} C(r, q, n), \\ & s.t \\ & (3.1) D/U \leq r \leq r_{\max} \\ & (3.2) \frac{mq}{D} \leq T_p \\ & q > 0 \\ & n \geq 1, \text{ Integer} \end{aligned} \tag{3}$$

where $C(r, q, n)$ is given in (1) and accordingly,

$$C(r, q, n) = D(c_V - \alpha D(1/r - 1)) + \frac{DK}{nq} + \frac{kD}{q} + h_V \left(r q + n q / 2(1 - r) - \frac{q}{2} \right) + h_B \frac{q}{2}, \tag{4}$$

where we define $k = k_B + k_V$ without losing generality, in order to simplify tracking. The suggested formulation considers two additional constraints. The first excludes scenarios under which continuous production is realized. To represent this constraint, we denote r_{\max} as a bound on r and accordingly there is a minimal production rate, $P = P_{\min}$. The second excludes scenarios under which the production cycle length (at the manufacturer's warehouse) is unbounded. Unbounded production cycle lengths are associated with oversized and inefficient warehouses. Furthermore, unbounded production cycles might not even be feasible due to scheduled maintenance, holidays or end weeks. To avoid these scenarios, we confine the production cycle duration and denote the upper bound as T_p . Optimization problem (P1) is non-linear and has mixed integer variables.

3. Optimization with unbounded production cycle length

This section addresses a relaxed problem of formulation (2). This solution strategy enables intermediate insights to be obtained, while the

$$C(q, n) = D(c_V - \alpha D(1/r^*(q, n) - 1)) + \frac{DK}{nq} + \frac{kD}{q} + h_V \left(r^*(q, n) q + n q / 2(1 - r^*(q, n)) - \frac{q}{2} \right) + h_B \frac{q}{2} \tag{10}$$

optimal solution serves as a lower bound to solutions of the original problem. We consider problem (P2) as a relaxed formulation of problem (P1):

Problem (P2):

$$\begin{aligned} & \min_{r,q,n} C(r, q, n). \\ & s.t \\ & (5.1) D/U \leq r \leq r_{\max} \\ & q > 0 \\ & n \geq 1, \text{ Integer} \end{aligned} \tag{5}$$

3.1. Optimization of the demand-to-production rate ratio r given n and q

Following (4),

$$\partial C(r, q, n) / \partial r = h_V q(1 - n/2) + \alpha \frac{D^2}{r^2}. \tag{6}$$

The first-order optimality condition (FOC) is

$$h_V q(1 - n/2) + \alpha \frac{D^2}{r^2} = 0, q > 0, n \geq 1, \tag{7}$$

which has the solution

$$r^{up}(n, q) = D \sqrt{\frac{\alpha}{h_V q(n/2 - 1)}}, q > 0, n \geq 3. \tag{8}$$

We begin by stating the conclusion:

$$r^*(q, n) = \begin{cases} D/U & n \leq 2 \\ r_{\max} & n \geq 3 \text{ and } r^{up}(n, q) \leq D/U \\ \operatorname{argmin}_{r=D/U, r_{\max}} \{C(r, q, n)\} & n \geq 3 \text{ and } D/U < r^{up}(n, q) < r_{\max} \\ D/U & n \geq 3 \text{ and } r_{\max} \leq r^{up}(n, q) \end{cases}$$

We summarize the above in an explicit form:

Proposition 1. For a given delivery frequency n (integer) and shipment quantity q (problem (P2)), the optimal demand-to-production rate ratio r is

$$r^*(q, n) = \begin{cases} D/U & n \leq 2 \\ r_{\max} & n \geq 3 \text{ and } r^{up}(n, q) \leq D/U \\ r_{\max} & \left\{ n \geq 3 \text{ and } D/U < r^{up}(n, q) < r_{\max} \text{ and } h_V q(1 - n/2)(r_{\max} - D/U) < \alpha D((D/r_{\max}) - U) \right\} \\ D/U & \left\{ n \geq 3 \text{ and } D/U < r^{up}(n, q) < r_{\max} \text{ and } h_V q(1 - n/2)(r_{\max} - D/U) \geq \alpha D((D/r_{\max}) - U) \right\} \\ D/U & n \geq 3 \text{ and } r_{\max} \leq r^{up}(n, q) \end{cases} \tag{9}$$

Proof. For $n \leq 2$, $\partial C(r, q, n) / \partial r > 0$; thus, $r^*(q, n) = D/U$. The other four cases result from the position of the maximal point $r^{up}(n, q)$ relative to the endpoints $r = D/U$ and $r = r_{\max}$, and from the value of $C(r, q, n)$ at the endpoints $r = D/U$ and $r = r_{\max}$. \square

Of particular interest is the case where the unit cost is independent of the production rate, i.e., when $\alpha = 0$. In this case, $r^*(q, n) = r^*(n)$, meaning the demand-to-production rate ratio r does not depend on $r^{up}(n, q)$ and eventually does not depend on the shipment quantity q . Consequently, $r^*(q, n)$ is applied and

$\min_{r,q,n} C(r, q, n)$ may be written now as $\min_n \left\{ \min_{q>0} C(q, n) \right\}$. Utilizing (8) and (9), $r^*(n, q)$ becomes

$$r^*(q, n) = \begin{cases} D/U, & n \leq 2 \\ r_{\max}, & n \geq 3 \text{ and } q > \frac{\alpha U^2}{h_V \left(\frac{n}{2} - 1\right)} \\ r_{\max}, & \left\{ \begin{array}{l} n \geq 3 \text{ and } \frac{\alpha D^2}{h_V(n/2-1)(r_{\max})^2} < q \leq \frac{\alpha U^2}{h_V(n/2-1)} \text{ and} \\ h_V q(1-n/2)(r_{\max} - D/U) < \alpha D((D/r_{\max}) - U) \end{array} \right\} \\ D/U, & \left\{ \begin{array}{l} n \geq 3 \text{ and } \frac{\alpha D^2}{h_V(n/2-1)(r_{\max})^2} < q \leq \frac{\alpha U^2}{h_V(n/2-1)} \text{ and} \\ h_V q(1-n/2)(r_{\max} - D/U) \geq \alpha D((D/r_{\max}) - U) \end{array} \right\} \\ D/U, & n \geq 3 \text{ and } q \leq \frac{\alpha D^2}{h_V(r_{\max})^2(n/2-1)} \end{cases} \quad (11)$$

Following (11), $r^*(n, q)$ can only take on two possible endpoint values, $r = D/U$ and $r = r_{\max}$. The following technical lemma argues that there always exists a specific threshold defined as the *switching point along the q-axis* after which the optimal demand-to-production rate ratio alters.

Lemma 1. Given $n \geq 3$, there exists a single switching point along the q axis, $q = q_s^1(n)$, where $r^*(q, n)$ switches from D/U to r_{\max} and

$$q_s^1(n) = \frac{\alpha DU}{h_V r_{\max}(n/2-1)} \quad (12)$$

Proof. See Appendix A.

Following Lemma 1 and (11),

$$r^*(q, n) = \begin{cases} D/U & n \leq 2 \\ D/U & n \geq 3 \text{ and } q \leq q_s^1(n) \\ r_{\max} & n \geq 3 \text{ and } q > q_s^1(n) \end{cases} \quad (13)$$

The demand-to-production rate ratio associated with the highest inefficiency is defined as $r_*(n, q)$ - that is, the scenario under which the maximal costs are obtained (i.e., the worst possible scenario). Thus, $r_*(n, q) = \min\{r_{\max}, r^{up}(n, q)\}$, $q > 0, n \geq 3$ when $r^{up}(n, q) > D/U$ and $r_*(n, q) = D/U, q > 0, n \geq 3$ when $r^{up}(n, q) \leq D/U$. Interestingly, $r_*(n, q)$ serves as an upper bound on production inefficiency. Consequently, one can evaluate the potential savings due to shifting from the worst possible scenario to any production state.

3.2. Optimizing the shipment size q given n

The first derivative of the objective with respect to the variable q , $q \neq q_s^1(n)$, is

$$\begin{aligned} \frac{\partial C(r^*(n, q), q, n)}{\partial q} &= \alpha D^2 (1/(r^*(q, n))^2) \frac{\partial r^*(n, q)}{\partial q} - \frac{DK}{nq^2} - \frac{kD}{q^2} + \\ h_V \left(r^*(q, n) + q \frac{\partial r^*(n, q)}{\partial q} + n \left[2(1 - r^*(q, n)) - nq \right] \frac{\partial r^*(n, q)}{\partial q} - \frac{1}{2} \right) &+ \frac{h_B}{2} \end{aligned}$$

The FOC for the problem $\min_{q>0} C(q, n)$ is

$$\begin{aligned} \alpha D^2 (1/(r^*(q, n))^2) \frac{\partial r^*(n, q)}{\partial q} - \frac{DK}{nq^2} - \frac{kD}{q^2} + \\ h_V \left(r^*(q, n) + q \frac{\partial r^*(n, q)}{\partial q} + n \left[2(1 - r^*(q, n)) - nq \right] \frac{\partial r^*(n, q)}{\partial q} - \frac{1}{2} \right) &+ \frac{h_B}{2} = 0 \end{aligned}$$

Following (13), $r^*(q, n)$ is a stepwise constant function of q , that is, $\partial r^*(n, q)/\partial q = 0$ (excluding at the transition point $q = q_s^1(n)$), and the FOC for the problem $\min_{q>0} C(q, n)$ reduces to

$$-\frac{DK}{nq^2} - \frac{kD}{q^2} + h_V \left(\left(1 - \frac{n}{2}\right) r^*(q, n) + \frac{n}{2} - \frac{1}{2} \right) + \frac{h_B}{2} = 0, n \geq 1, q \neq q_s^1(n), \quad (14)$$

where $r^*(q, n)$ is given in (13). Taking into account that $\partial^2 C(r^*(n, q), q, n)/\partial q^2|_{q \neq q_s^1(n)} > 0$, we summarize the above analysis by the following:

Proposition 2. For a given delivery frequency n (integer), the problem (P2) $\min_{q>0} C(q, n)$ has a minimal global solution $q^*(n)$ specified below:

a) For $n \leq 2$,

$$q^*(n) = \sqrt{\frac{D \left(\frac{k}{n} + k\right)}{h_V \left(\left(1 - \frac{n}{2}\right) \frac{D}{U} + \frac{n}{2} - \frac{1}{2} \right) + \frac{h_B}{2}}} \quad (15)$$

b) Denote by $q_{r_{\max}}^*(n)$ and $q_{D/U}^*(n)$ the points resulting from the minimum of $C(r^*(n, q) = r_{\max}, q, n)$ and $C(r^*(n, q) = D/U, q, n)$, respectively. To simplify the notations, we also denote $q_{D/U}^s(n) = \min(q_{D/U}^*(n), q_s^1(n))$ and $q_{r_{\max}}^s(n) = \min(q_{r_{\max}}^*(n), q_s^1(n))$. For $n \geq 3$ the optimal shipment size is given by

$$q^*(n) = \underset{\left\{ \left(r^*(n, q) = D/U, q_{D/U}^*(n) \right), \left(r^*(n, q) = r_{\max}, q_{r_{\max}}^*(n) \right) \right\}}{\operatorname{argmin}} C(r^*(n, q), q, n), \quad (16)$$

where

$$\begin{aligned} q_{r_{\max}}^*(n) &= \sqrt{\frac{D \left(\frac{k}{n} + k\right)}{h_V \left(\left(1 - \frac{n}{2}\right) r_{\max} + \frac{n}{2} - \frac{1}{2} \right) + \frac{h_B}{2}}} \text{ and } q_{D/U}^*(n) \\ &= \sqrt{\frac{D \left(\frac{k}{n} + k\right)}{h_V \left(\left(1 - \frac{n}{2}\right) \frac{D}{U} + \frac{n}{2} - \frac{1}{2} \right) + \frac{h_B}{2}}} \end{aligned} \quad (16.1)$$

Proof. For $n \leq 2$, $q^*(n)$ in (15) is the only solution of FOC (14) and always exists.

For $n \geq 3$, we conclude from (13) that at $q = q_s^1(n)$, the cost function $C(r^*(n, q), q, n)$ is not always differentiable with variable q . Following the convexity of $C(r^*(n, q), q, n)$ with variable q , we are left with the two options on the right-hand side of (16). □

Interestingly, coefficient α does not directly affect the optimal in-transit shipment size, $q^*(n)$. It acts indirectly, by affecting which one of the three values $q_{r_{\max}}^*(n)$, $q_{D/U}^*(n)$ or $q_s^1(n)$ is eventually chosen. In order to find $q^*(n)$, one should consider $q = q_{r_{\max}}^*(n)$ only if it is greater than

$q_s^1(n)$; otherwise $q_s^1(n)$ is chosen. One should consider $q = q_{D/U}^*(n)$ only if it is smaller than $q_s^1(n)$; otherwise $q_s^1(n)$ is chosen. Each feasible solution should be evaluated by inserting it into $C(q, n)$.

3.3. Optimizing the number of shipments n

In this sub-section we use the above explicit results for $r^*(q, n)$ and $q^*(n)$ to reveal certain properties the objective $C(n)$, which then allows an efficient search for the optimal number of shipments n^* , to be developed. For $n \leq 2$, assigning $r^*(q, n) = D/U$ and using the expression for $q^*(n)$ in (15),

$$C(r = D/U, q^*(n), n \leq 2) = D(c_V - \alpha D(U/D - 1)) + \sqrt{2D\left(\frac{K}{n} + k\right)[h_V((2-n)D/U + n - 1) + h_B]}$$

The condition that $C(1) < C(2)$ is

$$4D(K + k)[h_V(D/U) + h_B](2D(K + 2k)[h_V + h_B]$$

For $n \geq 3$,

$$C(r = D/U, q^*(n, D/U), n) = D(c_V - \alpha D(U/D - 1)) + \frac{DK}{q^*(n, D/U)} + \frac{kD}{q^*(n, D/U)} + h_V\left(q^*(n, D/U)D/U + nq^*(n, D/U)\right) / 2(1 - D/U) - \frac{q^*(n, D/U)}{2} + h_B \frac{q^*(n, D/U)}{2} \tag{17.1}$$

$$C(r = r_{\max}, q^*(n, r_{\max}), n) = D(c_V - \alpha D(1/r_{\max} - 1)) + \frac{DK}{nq^*(n, r_{\max})} + \frac{kD}{q^*(n, r_{\max})} + h_V\left(r_{\max}q^*(n, r_{\max}) + nq^*(n, r_{\max})\right) / 2(1 - r_{\max}) - \frac{q^*(n, r_{\max})}{2} + h_B \frac{q^*(n, r_{\max})}{2} \tag{17.2}$$

where $q^*(n, D/U) = \min(q_{D/U}^*(n), q_s^1(n))$ and $q^*(n, r_{\max}) = \max(q_{r_{\max}}^*(n), q_s^1(n))$. Therefore, $C(n) = \min\{C(r = D/U, q^*(n, D/U), n), C(r = r_{\max}, q^*(n, r_{\max}), n)\}$.

The optimal shipment size is obtained, by definition, from

$$C(r = D/U, q_{D/U}^*(n), n) = C_1 + \sqrt{2D\left(\frac{K}{n} + k\right)\left[\sqrt{h_V(2D/U + n(1 - D/U) - 1) + h_B}\right]}$$

$\operatorname{argmin}_{n>0} C(q^*(n), r^*(n, q), n)$. To avoid a full search, that is, to find the optimal n^* in an efficient manner, we start by analyzing the objective. The following result states that once the optimal demand-to-production rate ratio is $r^*(q^*(n), n) = r_{\max}$ along the n -axis, it remains unchanged for higher values of n . The following technical lemma supports Lemma 3 in showing that the optimal demand-to-production rate ratio is a two-

stepwise production rate function of n .

Lemma 2. Under $h_B \geq h_V(1 - 2r_{\max})$, if there exists n' , where $r^*(n', q^*(n')) = r_{\max}$, then $q^*(n) = \max\{q_{r_{\max}}^*(n), q_s^1(n)\}$ and $r^*(n, q^*(n)) = r_{\max}$ for $n > n'$.

Proof. $q_s^1(n)$ is strictly decreasing with n (see (12)) and currently there are two possible scenarios, $q_{r_{\max}}^*(n') \geq q_s^1(n')$ and $q_{r_{\max}}^*(n') < q_s^1(n')$, complying with $r^*(n', q^*(n')) = r_{\max}$. See Appendix B for completing the proof.

We conclude from Lemma 2 that there exists no more than one

switching point $n_s \geq 3$ along the n -axis where the optimal demand-to-production rate ratio alters from D/U to r_{\max} . Therefore, the objective as a function of n is

$$C(r^*(n, q), q^*(n), n \geq 1) = \begin{cases} C(r = D/U, q^*(n, r = D/U), n) & n < n_s \\ C(r = r_{\max}, q^*(n, r = r_{\max}), n) & n \geq n_s \end{cases} \tag{18}$$

Following (18), the switching point n_s^c (the continuous point) is the root of

$$C(r = D/U, q^*(n, D/U), n) = C(r = r_{\max}, q^*(n, r_{\max}), n) \tag{19}$$

We summarize the above by:

Lemma 3. Under an unlimited production cycle, the switching point (integer) n_s is $n_s = \lfloor n_s^c \rfloor + 1$, where $\lfloor x \rfloor$ denotes the "integer part" (rounding down of x).

Define

where

$$C_1 \equiv D(c_V - \alpha D(U/D - 1));$$

$$C(r = D/U, q^*(n, D/U) = q_s^1(n), n) = D(c_v - \alpha D(U/D - 1)) + \frac{DKh_v r_{\max}(n/2 - 1)}{\alpha nDU} + \frac{kDh_v r_{\max}(n/2 - 1)}{\alpha DU} + h_v \frac{\alpha DU}{h_v r_{\max}(n/2 - 1)} \left(D/U + \frac{n}{2}(1 - D/U) - \frac{1}{2} \right) + h_B \frac{\alpha DU}{2h_v r_{\max}(n/2 - 1)}$$

$$C(r = r_{\max}, q_{r_{\max}}^*(n), n) = C_2 + \sqrt{2D \left(\frac{K}{n} + k \right) \left[\sqrt{(h_v(2r_{\max} + n(1 - r_{\max}) - 1) + h_B)} \right]}$$

where $C_2 \equiv D(c_v - \alpha D(1/r_{\max} - 1))$, and

$$C(r = r_{\max}, q^*(n, r_{\max}) = q_s^1(n), n) = D(c_v - \alpha D(1/r_{\max} - 1)) + \frac{DKh_v r_{\max}(n/2 - 1)}{\alpha nDU} + \frac{kDh_v r_{\max}(n/2 - 1)}{\alpha DU} + h_v \frac{\alpha DU}{h_v r_{\max}(n/2 - 1)} \left(r_{\max} + \frac{n}{2}(1 - r_{\max}) - \frac{1}{2} \right) + h_B \frac{\alpha DU}{2h_v r_{\max}(n/2 - 1)}$$

The following technical lemma assists in specifying $C(n)$, which is composed of the above four functions (in the general case).

Lemma 4.

The cost functions $C(r=D/U, q_{D/U}^*(n), n), C(r=D/U, q^*(n, D/U) = q_s^1(n), n), C(r=r_{\max}, q_{r_{\max}}^*(n), n)$ and $C(r=r_{\max}, q^*(n, r_{\max}) = q_s^1(n), n)$ are quasi-convex functions with the continuous variable n .

Proof. See Appendix C.

Following Lemma 4, each of the curves $C(r = D/U, q_{D/U}^*(n), n), C(r = D/U, q^*(n, D/U) = q_s^1(n), n), C(r = r_{\max}, q_{r_{\max}}^*(n), n)$ and $C(r = r_{\max}, q^*(n, r_{\max}) = q_s^1(n), n)$ is a unimodal function of variable n having a single global minimum. We denote the minimal global points (integers) of the above four costs functions (all are unimodal) by $n_{D/U}^*, n_{D/U}^*(q_s^1), n_{r_{\max}}^*$ and $n_{r_{\max}}^*(q_s^1)$, accordingly. These minima may be found through any common procedure for a one-dimensional search. Looking at $C(n)$, we are left with two local minimal points, one within $n \in [1, n_s - 1]$ and the other within $n \in [n_s, \infty)$. The following theorem summarizes the overall optimization search when the production cycle is unlimited.

Theorem 1. The optimal shipment frequency n^* (of problem (P2)) is specified by the following procedure:

Step 1. If $4D(K + k)[h_v(D/U) + h_B] \geq 2D(K + 2k)[h_v + h_B]$, then $C_{1,2} = C(1)$ and $n_{1,2} = 1$; otherwise, $C_{1,2} = C(2)$ and $n_{1,2} = 2$.

Step 2. Solve (19) to obtain n_s^c and switching point (integer) $n_s = \lfloor n_s^c \rfloor + 1$.

Step 3. $n_{D/U} = \operatorname{argmin}_{\{\min(n_{D/U}^*, n_s - 1), \min(n_{D/U}^*(q_s^1), n_s - 1)\}} C(n)$.

Step 4. $n_{r_{\max}} = \operatorname{argmin}_{\{\max(n_{r_{\max}}^*, n_s), \max(n_{r_{\max}}^*(q_s^1), n_s)\}} C(n)$.

Step 5. $n^* = \operatorname{argmin}_{n_{1,2}, n_{r_{\max}}, n_{D/U}} C(n)$.

Step 6. End.

Any optimal solution to the problem with an unlimited production cycle length that is also feasible under the general problem (P1) is also optimal for the general problem. In order to highlight the importance of the intermediate theoretical results obtained in this section, Figs. 2 and 3 present, respectively, the optimal size of the in-transit shipments q and the integrated cost C , both as a function of the number of in-transit shipments, n , for the following set of parameter values: $D = 200$ (daily), $U = 500$ (daily), $K = 5000$, $k_v=50$, $k_B=50$, $h_v=10$, $h_B=10$, $r_{\max}=0.75$ and $\alpha = 0.03$. The optimal solution for the unbounded production cycle length is $n^* = 8$, $r^* = 0.4$ (i.e., $P^* = 500$), and $q^* = 71.9$, with accordingly an optimal cost (excluding fixed costs) of $C^* = 2229.89$. As shown in Lemma 2, the optimal demand-to-production rate ratio starts at $r^*(n) = D/U$ for $n < n_s$ and shifts to $r^*(n) = r_{\max}$ for $n \geq n_s$, where $n_s = 32$. The stepwise decreasing function of $q^*(n)$ and the unimodality of $C^*(n)$ with n are shown in Figs. 2 and 3, accordingly.

4. Optimization under a bound on the production cycle length

In the previous section intermediate results were developed. Those analytical results have their own importance since, under an unlimited production cycle length (as is generally assumed in the existing literature), the optimal demand-to-production rate ratio is composed of only a

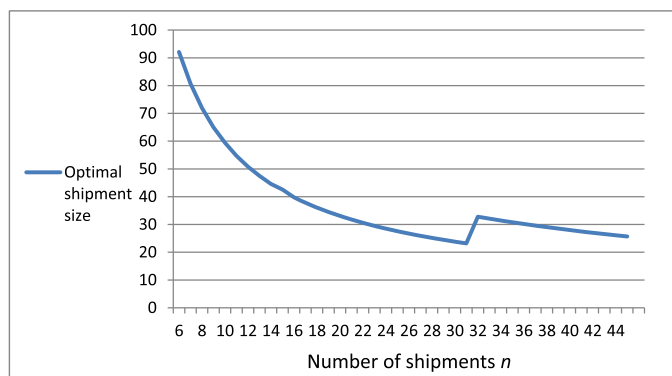


Fig. 2. Optimal shipment size $q^*(n)$ as a function of the number of shipments.

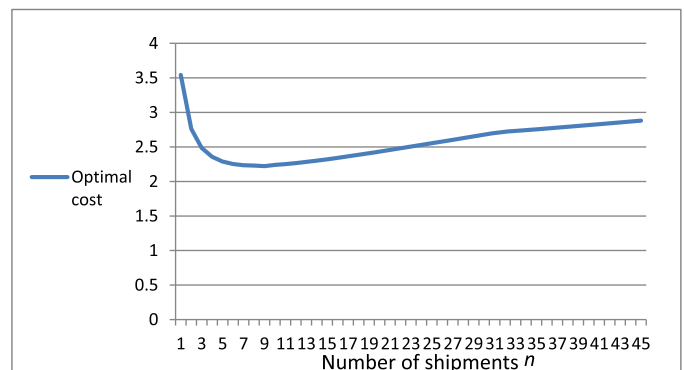


Fig. 3. Optimal cost (10^3 \$) per unit time, $C^*(n)$, as a function of the number of shipments.

single switching point along the shipment frequency axis, n . This property simplified the analysis to find the optimal $r^*(n)$ and $q^*(n)$ given n . Reverting to the data in the numerical example presented in Section 3, but assuming that the bound on the production cycle length is reduced to $T_p = 4$, does not alter the optimal solution (i.e., $n^* = 8, P^* = 500$ and $q^* = 71.9$); however some of the values of the shipment sizes n presented in Figs. 2 and 3 for the unbounded problem become infeasible under the original problem (P1). In particular, for $n \geq 34$, the constraint $r \leq \frac{DT_p}{nq}$ is violated. Of course, further decreasing T_p eventually alters the optimal solution as well. For example, when $T_p = 1$, for $n \geq 4$, the constraint $r \leq \frac{DT_p}{nq}$ is violated and the optimal solution (under the postulate of not violating this constraint) is $n^* = 7, P^* = 500$ and $q^* = 71.43$.

We now solve the general problem (P1), which considers the constraint on the production cycle length, $\frac{mq}{D} \leq T_p$, together with the existing constraint on the demand-to-production rate ratio r . Thus, $r^*(n, q) \leq \min\left\{\frac{DT_p}{nq}, r_{\max}\right\}$.

4.1. Optimization of the demand-to-production rate ratio r given n and q

In the case where $\frac{DT_p}{nq} < D/U$, it means that the production lot Q cannot be accomplished within the production cycle length T_p ; that is, there is no optimal solution. We define $r_{T_p}^{\max}(q, n) \equiv \min\left(r_{\max}, \frac{DT_p}{nq}\right)$. Thus, given a feasible couple (q, n) ,

$$r_B^*(q, n) = \begin{cases} D/U & n \leq 2 \\ r_{T_p}^{\max}(q, n) & n \geq 3 \text{ and } r^{\mu p}(n, q) \leq D/U \\ \operatorname{argmin}_{r=D/U, r_{T_p}^{\max}(q, n)} \{C(r, q, n)\} & n \geq 3 \text{ and } D/U < r^{\mu p}(n, q) < r_{T_p}^{\max}(q, n) \\ D/U & n \geq 3 \text{ and } r_{T_p}^{\max}(q, n) \leq r^{\mu p}(n, q) \end{cases} \quad (20)$$

where $r_B^*(q, n)$ represents the demand-to-production rate ratio r for the bounded problem. More explicitly,

$$r_B^*(q, n) = \begin{cases} D/U & n \leq 2 \\ r_{T_p}^{\max}(q, n) & n \geq 3 \text{ and } D \sqrt{\frac{\alpha}{h_v q(n/2-1)}} \leq D/U \\ \operatorname{argmin}_{r=D/U, r_{T_p}^{\max}(q, n)} \{C(r, q, n)\} & n \geq 3 \text{ and } D \sqrt{\frac{\alpha}{h_v q(n/2-1)}} < r_{T_p}^{\max}(q, n) \\ D/U & n \geq 3 \text{ and } r_{T_p}^{\max}(q, n) \leq D \sqrt{\frac{\alpha}{h_v q(n/2-1)}} \end{cases} \quad (21)$$

Following the result obtained by the analysis of the relaxed problem in Section 3, we axiomatically assume the existence of a switching point along the q -axis $q_s(n)$, i.e., shifting from the demand-to-production rate ratio D/U to $r_{T_p}^{\max}(q, n)$. Starting from the hypothetical equation $C\left(r = \frac{D}{U}, q, n\right) = C\left(r = \frac{DT_p}{nq}, q, n\right)$:

$$h_v q(1 - n/2) \left(\frac{DT_p}{nq} - D/U\right) = \alpha D((nq/T_p) - U)$$

After several algebraic manipulations,

$$q = \frac{h_v(2-n)DUT_p^2 + 2\alpha nDU^2T_p}{(2\alpha DUn^2 + h_v nT_p(2-n)D)}$$

$$q = \frac{UT_p(h_v(2-n)T_p + 2\alpha nU)}{n(h_v(2-n)T_p + 2\alpha nU)}$$

$$q = \frac{UT_p}{n}$$

In order for the optimal demand-to-production rate ratio to switch directly from $r = D/U$ to $r = DT_p/nq$ (i.e., as shown in the third line in (21)) when q increases, $q(n)$ should exceed the value UT_p/n . Such a scenario is infeasible since it implies that even when production occurs under the maximal rate, the bound on the production cycle length T_p is such that the cycle is not long enough to produce the entire production lot $Q = nq$. The switching point is obtained from the second line in (21):

$$q_s^1(n) = \begin{cases} \frac{\alpha DU}{h_v r_{\max}(n/2-1)} & r_{T_p}^{\max}(q, n) = r_{\max} \\ \frac{DT_p}{nr_{\max}} & r_{T_p}^{\max}(q, n) = \frac{DT_p}{nq} \end{cases} \quad (22)$$

Define the intersection point $q_s^2(n) = \frac{DT_p}{nr_{\max}}$ where $r_{T_p}^{\max}(q, n)$ becomes $\frac{DT_p}{nq}$. We summarize without proof in the following proposition:

Proposition 3. For a given delivery frequency n (integer) and shipment quantity q for problem (P1), the optimal demand-to-production rate ratio is

$$r_B^*(q, n) = \begin{cases} D/U & n \leq 2 \\ D/U & n \geq 3 \text{ and } q \leq q_s^1(n) \\ r_{\max} & n \geq 3 \text{ and } q_s^1(n) < q \leq q_s^2(n) \\ \frac{DT_p}{nq} & n \geq 3 \text{ and } q > q_s^2(n) \end{cases} \quad (23)$$

The demand-to-production rate ratio in (23) generalizes that derived in (13). In contrast to the case where the cycle length is unlimited, under which the optimal production rate can take only endpoints, the general problem may also choose intermediate production rates. Furthermore, unlike the case of the unbounded cycle length, given a couple (q, n) , the possibility of infeasible solutions arises.

As in the case of the relaxed problem in sub-Section 3.1, we address the worst possible production scenario, that is, the production rate that is most inefficient, for the more general problem.

Proposition 4. For problem (P1),

(a) if $\frac{DT_p}{nq} < D/U$, then there is no feasible production rate.

(b) if $n \leq 2$, then $r_*(n, q) = \min\left\{\frac{DT_p}{nq}, r_{\max}\right\}$.

Otherwise (i.e., if $n \geq 3$):

(c) The demand-to-production rate ratio associated with the highest inefficiency, $r_*(n, q)$ is given by

$$r_*(n, q) = \begin{cases} \min\{r_{\max}, r^{\mu p}(n, q)\} & \frac{DT_p}{nq} \geq r_{\max} \text{ and } r^{\mu p}(n, q) > D/U \\ D/U & \frac{DT_p}{nq} \geq r_{\max} \text{ and } r^{\mu p}(n, q) \leq D/U \\ \frac{DT_p}{nq} & \frac{DT_p}{nq} < r_{\max} \text{ and } r^{\mu p}(n, q) > \frac{DT_p}{nq} \\ r^{\mu p}(n, q) & \frac{DT_p}{nq} < r_{\max} \text{ and } \frac{DT_p}{nq} \geq r^{\mu p}(n, q) > D/U \\ D/U & \frac{DT_p}{nq} < r_{\max} \text{ and } r^{\mu p}(n, q) \leq D/U \end{cases} \quad (24)$$

where $r^{\mu p}(n, q)$ is given in (8).

Proof. $\frac{DT_p}{nq} < D/U$ means $UT_p < nq$, i.e., the production cycle time is insufficient to produce the production lot $Q = nq$. For $n \leq 2$, $\partial C(r, q, n)/\partial r > 0$; thus, $r^*(q, n) = D/U$ and $r_s(n, q) = \min\left\{\frac{DT_p}{nq}, r_{\max}\right\}$. The other five cases result from the concavity of $C(r, q, n)$ with the demand-to-production rate ratio r and from the positions of the endpoints $r = D/U$ and $r = r_{\max}$ relative to the points $r^{up}(n, q)$ and $\frac{DT_p}{nq}$. \square

Our main objective is to seek the minimal cost thus, we now go back to computer $^*(q, n)$.

4.2. Optimizing the number of shipments n

We start by seeking the optimal shipment size q given $n, q_B^*(n)$. For the two options under which the demand-to-production rate ratio is fixed, the formulas for $q_{r_{\max}}^*(n)$ and $q_{D/U}^*(n)$ in (16.1) represent the associated global minimal values (which might be infeasible). Considering that $\partial^2 C(r^*(q, n), q, n)/\partial q^2|_{q=q_s^1(n)} > 0$, we summarize the above analysis by the following:

Proposition 5. For problem (P1), denote by $q_{r_{\max}}^*(n), q_{D/U}^*(n)$ the minimal points of $C(r_B^*(n, q) = r_{\max}, q, n)$ and $C(r_B^*(n, q) = D/U, q, n)$, respectively, as presented in (16.1). The problem has a minimal global solution $q_B^*(n)$, specified below:

For $n \leq 2$, the shipment size is given by

$$q_B^*(n) = \min\left(q_{D/U}^*(n), q_s^1(n), UT_p/n\right). \tag{24.1}$$

(b) To simplify notations we also denote

$$q_{D/U}^{s, TP}(n) = \min\left(q_{D/U}^*(n), q_s^1(n), UT_p/n\right) \text{ when } r_{TP}^{\max}(q, n) = r_{\max};$$

$$\text{and } q_{r_{\max}}^{s, TP}(n) = \min\left(\max\left(q_{r_{\max}}^*(n), q_s^1(n)\right), UT_p/n\right) \text{ when } r_{TP}^{\max}(q, n) = r_{\max};$$

$$\text{and } q_{DT_p/nq}^{s, TP}(n) = \min\left(\max\left(q_{DT_p/nq}^*(n), q_s^2(n)\right), UT_p/n\right) \text{ when } r_{TP}^{\max}(q, n) = \frac{DT_p}{nq}.$$

For $n \geq 3$, the shipment size is given by

$$q_B^*(n) = \underset{\left\{ \left(q_{D/U}^{s, TP}(n), r^*(n, q) = D/U \right), \left(q_{r_{\max}}^{s, TP}(n), r^*(n, q) = r_{\max} \right), \left(q_{DT_p/nq}^{s, TP}(n), r^*(n, q) = \frac{DT_p}{nq} \right) \right\}}{\operatorname{argmin}} C(r^*(n, q), q, n). \tag{24.2}$$

We now seek the optimal number of shipments, $r_B^*(n)$. We denote the optimal cost given n by $C_B^*(n)$, which is obtained by inserting the optimal shipment size $q_B^*(n)$ and the associated optimal demand-to-production rate ratio $r_B^*(n, q)$ into (4). Lemma 2, which implies that when reaching the demand-to-production rate ratio $r = r_{\max}$, the ratio remains unchanged for all greater n , does not hold for the general problem. When n increases, there is a possibility that $r_{TP}^{\max}(q, n) = r_{\max}$ alters into $r_{TP}^{\max}(q, n) = \frac{DT_p}{nq}$ and that the optimal demand-to-production rate ratio alters into $r = \frac{DT_p}{nq}$ (according to Proposition 3). Once the ratio reaches $r = \frac{DT_p}{nq}$, however, it remains unchanged, as is shown in the following proposition:

Proposition 6. Under $h_B \geq h_V(1 - 2r_{\max})$, for problem (P1), the optimal demand-to-production rate ratio $r_B^*(n)$ is a quasi-concave function of n .

Proof. See Appendix D.

We conclude from Proposition 6 that in general, there are no more

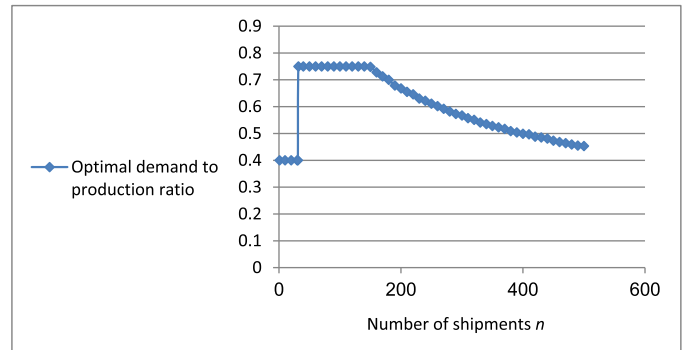


Fig. 4. Optimal demand-to-production rate ratio $r^*(n)$ as a function of the number of shipments.

than two switching points: the first, $n_{s, B}^1$, from $r = D/U$ into $r = r_{\max}$, and the second, $n_{s, B}^2$, from $r = r_{\max}$ into $r = \frac{DT_p}{nq}$. Therefore, the objective as a function of n is

$$C_B(r_B^*(n, q), q_B^*(n), n \geq 1) = \begin{cases} C_B(I, n) & n < n_{s, B}^1, B \\ C_B(II, n) & n_{s, B}^1, B \leq n \leq n_{s, B}^2, B \\ C_B(III, n) & n > n_{s, B}^2, B \end{cases} \tag{25}$$

where

$$C_B(I, n) \equiv C_B\left(r = D/U, q_B^*(n) = \min\left(q_{D/U}^*(n), q_s^1(n), UT_p/n\right), n\right) \tag{25.1}$$

$$C_B(II, n) \equiv C_B\left(r = r_{\max}, q_B^*(n) = \min\left(\max\left(q_{r_{\max}}^*(n), q_s^1(n)\right), UT_p/n\right), n\right) \tag{25.2}$$

$$C_B(III, n) \equiv C_B\left(r = \frac{DT_p}{nq}, q_B^*(n) = \min\left(\max\left(q_{DT_p/nq}^*(n), q_s^2(n)\right), UT_p/n\right), n\right), \tag{25.3}$$

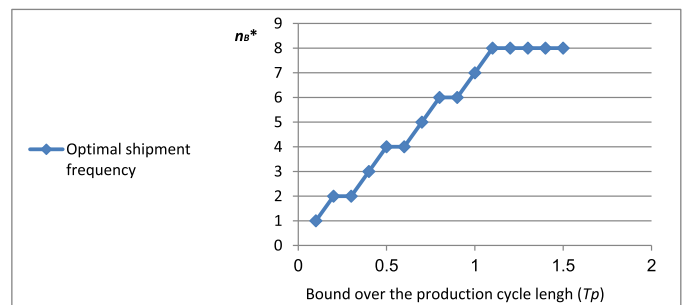


Fig. 5. Optimal shipment frequency n_s^* as a function of the bound on the production cycle length.

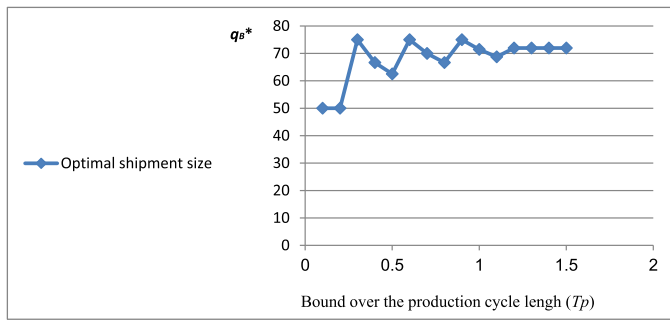


Fig. 6. Optimal shipment size q_B^* as a function of the bound on the production cycle length.

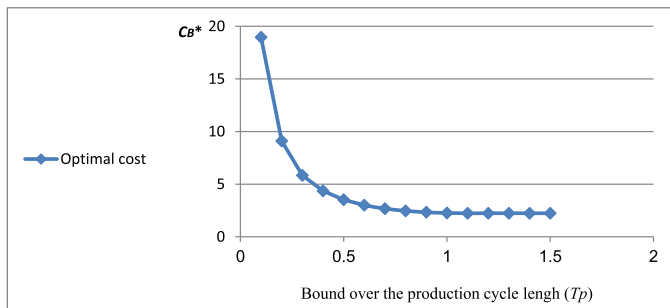


Fig. 7. Optimal cost ($10^3\$$) per unit time, $C_B^*(n)$, as a function of the bound on the production cycle length.

where

$$q_{D/r_p}^*(n) = \sqrt{\frac{D\left(\frac{k}{n} + k\right)}{h_V\left(\eta/2 - \frac{1}{2}\right) + \frac{h_B}{2} - \alpha D\left(\frac{n}{T_p}\right)}}, \quad (25.4)$$

Following (25), the continuous switching points $n_{s,B}^{1,c}$ and $n_{s,B}^{2,c}$, where $n_{s,B}^1 = \lfloor n_{s,B}^{1,c} \rfloor + 1$ and $n_{s,B}^2 = \lfloor n_{s,B}^{2,c} \rfloor + 1$, are the roots (if they exist) of $C_B(I, n) = C_B(II, n)$ and of $C_B(II, n) = C_B(III, n)$, respectively.

The following technical lemma assists in specifying $C_B(n)$, which is composed of the above three functions (in the general case). Due to the similarity with Lemma 4, in which the quasi-convexity is shown, we present the following lemma without proof.

Lemma 5. *The cost functions $C_B(I, n)$, $C_B(II, n)$ and $C_B(III, n)$ are quasi-convex functions with the continuous variable n .*

Following Lemma 5, each of the five curves $C_B(r = D/U, q_B^*(n) = q_{D/U}^*(n, n))$, $C_B(r = D/U, q_B^*(n) = q_s^1(n, n))$, $C_B(r = r_{max}, q_B^*(n) = q_{r_{max}}^*(n, n))$,

$C_B(r = r_{max}, q_B^*(n) = q_s^1(n, n))$, and $C_B\left(r = \frac{DT_p}{nq}, q_B^*(n) = q_s^2(n, n)\right)$ is a unimodal function with a single global minimum. We denote the minimal global points (integers) of the above five cost functions (all are unimodal) by $n_{D/U}^*$, $n_{D/U}^1$, $n_{r_{max}}^*$, $n_{r_{max}}^1$, and $n_{DT_p/nq}^*$, accordingly, and they can all be found through any common one-dimensional search procedure. Looking at $C_B(n)$, we are left with three local minimal points, one within $n \in [1, n_{s,B}^1 - 1]$, the second within $n \in [n_{s,B}^1, n_{s,B}^2 - 1]$, and the third within $n \in [n_{s,B}^2, \infty)$. The following theorem summarizes the overall optimization when the production cycle is unlimited.

Theorem 2. *The optimal shipment frequency n_B^* for problem (P1) is specified by the following procedure:*

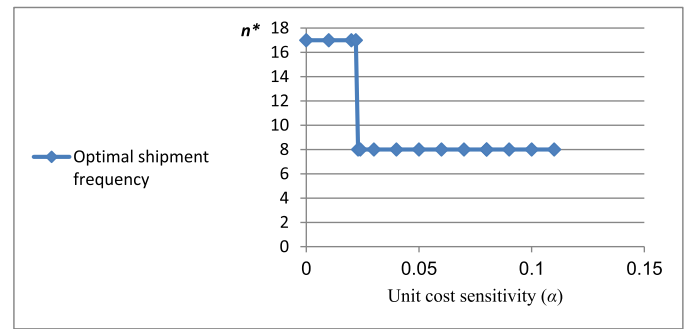


Fig. 8. Optimal shipment frequency n^* as a function of the unit cost sensitivity.

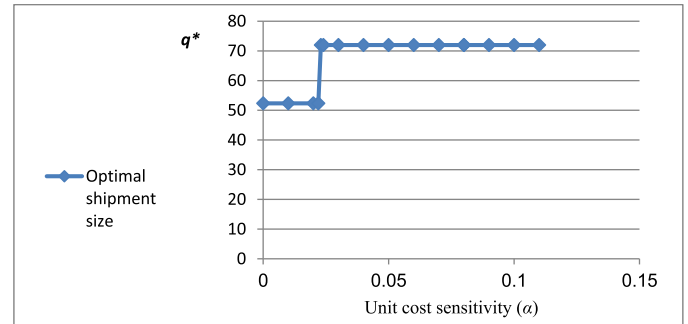


Fig. 9. Optimal shipment size q^* as a function of the unit cost sensitivity.

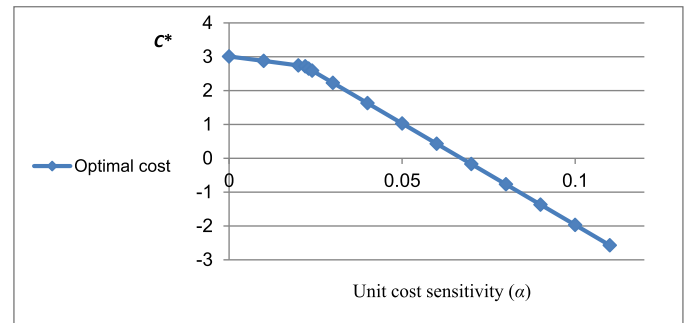


Fig. 10. Optimal cost ($10^3\$$) per unit time, $C^*(n)$, as a function of the unit cost sensitivity.

- Step 1. If $C_B(r_B^*(q, 1) = D/U, q_B^*(1), 1) < C_B(r_B^*(q, 2) = D/U, q_B^*(2), 2)$, then $C_{1,2} = C_B(1)$ and $n_{1,2} = 1$; otherwise, $C_{1,2} = C_B(2)$ and $n_{1,2} = 2$.
- Step 2. Solve $C_B(I, n) = C_B(II, n)$ and $C_B(II, n) = C_B(III, n)$ to obtain $n_{s,B}^{1,c}$ and $n_{s,B}^{2,c}$, respectively; then the integer switching points are $n_{s,B}^1 = \lfloor n_{s,B}^{1,c} \rfloor + 1$ and $n_{s,B}^2 = \lfloor n_{s,B}^{2,c} \rfloor + 1$, respectively.
- Step 3. $n_{D/U} = \operatorname{argmin} C_B(n)$.
- Step 4. $n_{r_{max}} = \operatorname{argmin} C_B(n)$.
- Step 5. $n_{DT_p/nq} = \operatorname{argmin} C_B(n)$.
- Step 6. $n_B^* = \operatorname{argmin}_{n_{1,2}, n_{r_{max}}, n_{D/U}, n_{DT_p/nq}} C_B(n)$.
- Step 7. End.

Obviously, once the optimal shipment frequency n_B^* is obtained, the decision maker can utilize, (24.1) and (24.2) in Proposition 5 to obtain q_B^* and later (23) in Proposition 3 to obtain r_B^* .

5. Numerical illustration and sensitivity of key parameters

In this section, we demonstrate numerically the solution of the proposed problem and highlight its significance. A sensitivity analysis of the key parameters of the problem is also presented. In particular, the bound on the production cycle length and the rate of decrease of the unit cost $c_v(P)$ with the production rate P are investigated. The general problem analyzed in Section 4 allows no more than two switching points, with shipment frequency n and this is demonstrated in the following subsection.

5.1. Switching points with shipment frequency n

Here we present three examples. The first demonstrates two switching points, the second demonstrates one switching point, and the third no switching points. Utilizing the data of the example in Section 3, i.e., with $T_p = 4$, the optimal demand-to-production rate ratio (see also Fig. 4) is:

$$r^*(n) = \begin{cases} 0.4 & n < 32 \\ 0.75 & 32 \leq n \leq 148 \\ [0.4, 0.75] & n > 149 \end{cases}$$

Omitting the restriction about the production cycle length, i.e., assigning $T_p = \infty$, the optimal demand-to-production rate ratio is $r^*(n) = \begin{cases} 0.4 & n < 32 \\ 0.75 & n \geq 32 \end{cases}$. Decreasing the bound on the production cycle length to $T_p = 1$ changes the optimal policy into $r^*(n) = 0.4, n \geq 1$.

5.2. The effect of the bound on the production cycle length T_p

A restriction on the production cycle length might be imposed in real life. In such scenarios, the managerial alternatives include outsourcing, increasing production rates (e.g., additional shifts and machines), and working with smaller production lots (which incurs significant fixed costs). The monetary penalties (associated with total costs) as well as the service levels (associated with the number of shipments and their size) are the main performance measures that are affected by the bound. The optimal demand-to-production rate ratio is 0.4 (i.e., $P = 500$) for all instances (when $T_p \leq 1.5$). Figs. 5–7 present the optimal shipment frequency n , the optimal shipment size q , and the optimal joint cost C_B for different values of T_p .

Figs. 5–7 show that bounds on the production cycle length that exceed 1.2 days are not active, i.e., the optimal solution does not alter with T_p . For bounds below the threshold of approximately 1.2 days, the production cycle length coincides with the imposed bound. After this threshold, the ratio between the actual production cycle length and the bound decreases with T_p quite sharply. For example, when $T_p = 1.5$, the value of this ratio is 0.768. Fig. 7 shows an exponential increase in the optimal joint cost with a decreasing bound on the production cycle length, T_p . We conclude that it is worthwhile for managers to make significant efforts to relax the bound on the production cycle length. Fig. 5 shows a non-increasing response of the optimal shipment frequency (i.e., n) to decreases in the bound on the production cycle length, T_p . Of particular interest is the periodic but decreasing trend seen for the optimal shipment size q when the bound on T_p becomes stricter (see Fig. 6). It can be explained by the fact that both n and the combined value $Q = nq$ increase (or, more accurately, do not decrease) with T_p . Since the value of n (integer) sometimes does not alter with decreasing T_p , the shipment size q must adjust itself to comply with the active constraint.

It is worth noting that for all instances of the bound on the produc-

tion cycle length T_p , the optimal production rate is maximal, that is, $P^* = U$, and that for the special case where $T_p = 0.2$ days, two optimal solutions are obtained with the optimal cost of 9100\$. The two alternatives are $(n_B^* = 1, q_B^* = 100)$ and $(n_B^* = 2, q_B^* = 50)$. In such a scenario, the decision makers may choose the best alternative based on considerations other than the joint total cost, such as the storage capacity in the buyer's warehouse or the vendor's flexibility in splitting the production lot into two shipments.

5.3. The effect of unit cost sensitivity α

Our model assumes that the unit production cost $c_v(P)$ is a linearly decreasing function of the production rate P , with the rate of decrease denoted by α . Such a decrease can be linked to the relatively high costs associated with higher production durations. Increasing P means shortening the production time, and in most cases, also decreasing the unit cost. This of course may only hold true for limited increases in the production rate, as significant monetary investments are likely to be needed to achieve very high production rates. We go back to the almost unbounded scenario (e.g., $T_p = 6$) and investigate the effect of changing the unit cost sensitivity α . Figs. 8–10 present the shipment frequency n , the optimal shipment size q , and the optimal joint cost C for different values of α .

Figs. 8–10 indicate that the unit cost sensitivity has a dichotomous effect on decision variables and on the optimal cost. When $\alpha = 0.023$, the optimal shipment frequency n^* jumps from 8 to 17, and the optimal shipment size q^* jumps from 52.36 to 71.96. The optimal cost decreases linearly with the unit cost sensitivity, but the slope alters and suddenly increases at the value of $\alpha = 0.023$. Unlike the effect of the bound on the production cycle length, increasing α changes the optimal production rate from $P^* = 266.66$ (the minimal value) to $P^* = 500$ (the maximal value).

6. Conclusions

6.1. Summary

This paper analyzes an integrated manufacturer-buyer supply chain with bounded production cycle length and discrete shipment size. The proposed formulation avoids scenarios under which the production rate approaches the demand rate (that is, producing almost without breaks) and scenarios under which the production cycle length (at the manufacturer's warehouse) is unbounded. For the case of the unbounded production cycle length, we observe a stepwise decreasing function for the optimal shipment size $q^*(n)$ and unimodality of the joint cost function $C^*(n)$ with n , as shown in Figs. 2 and 3, accordingly. Unlike the case in which the production cycle length is unbounded, for the bounded problem, the specific problem parameters might impose infeasibility.

Utilizing the optimality properties of the solution, we optimally solve the bounded problem, obtain intermediate explicit expressions, and develop an efficient convex searching procedure to find the shipment frequency. In particular, we prove that the optimal solution can have no more than two switching points along the shipment-size axis n . We find, for the special case where the unit cost is independent of the production rate (i.e., when $\alpha = 0$), that $r^*(q, n) = r^*(n)$, meaning that the demand-to-production rate ratio r does not depend on the shipment quantity q . Interestingly, we find an upper bound on the production inefficiency. Consequently, one can evaluate the potential savings due to shifting from the worst possible scenario to any other production state, including the actual one. Numerical examples indicate that smaller bounds on the production cycle length simultaneously increase the fluctuations of the optimal shipment size q^* and decrease the optimal shipment size n^* . Such trends imply smaller production lots Q^* and accordingly greater production set-up costs. Numerical examples also indicate that the unit cost sensitivity has a dichotomous effect on decision variables and on the

optimal cost.

6.2. Managerial implications and future research

The effect of the bound on the production cycle length has both theoretical and practical implications. Numerical illustration shows that decreasing the bound on the production cycle length can substantially increase the optimal joint cost. This observation should encourage manufacturers to invest monetary efforts in relaxing this limitation as much as possible, for example, by ensuring that there is sufficient capacity. Another possible strategy to cope with the bound on the production cycle length is to outsource some of the operations planned for the period when the machines are idle and, as a result, to increase the bound on the production cycle. A third possibility would be to increase the production rate by outsourcing some of the work, thus avoiding the sharp expected increase in the total costs. Future research could compare these alternatives.

Contrary to what one might expect, imposing a bound on the production cycle length does not only have negative consequences. In the case of the unlimited bound, only the endpoints of the demand-to-production rate ratio are optimal; i.e., only two values, the highest demand-to-production rate ratio $r^*(n) = D/U$ and the lowest one, $r^*(n) = r_{\max}$, may be chosen. Theoretically, production under the maximal value, $P = U$ is feasible; however, practically speaking, such a policy

may be unstable and risky for long durations. In the bounded case, on the other hand, intermediate values for the demand-to-production rate ratio as a function of shipment size n may also be selected.

All numerical examples presented in Section 5 show that the optimal demand-to-production rate ratio is always $r^* = D/U$. Since setting $\alpha = 0$ instead of $\alpha = 0.03$ in the dataset presented in Section 3 produced $r^* = r_{\max}$, we conclude that the unit cost sensitivity is a key parameter in changing the optimal demand-to-production rate ratio.

We presented examples where multiple solutions are optimal. A more formal analysis that investigates these scenarios and introduces the conditions under which multiple optimal solutions exist is suggested for future research. Such research will enable decision makers to choose the best policy among available solutions, not only based on total cost, but also on other criteria. The suggested model has several limitations which are mainly related to the relatively simplified supply chain addressed in this paper. Obviously, many other research directions could follow from the current work. These include: analyzing the case where the players do not cooperate; analyzing different models of the unit production cost function; and modeling the supply chain when the demand is considered random.

Declaration of Competing Interest

None.

Appendix A. Proof of Lemma 1

We show existence of the switching point $q_s^1(n)$ by addressing the two endpoints in (11). First, we show that

$$h_V q(1 - n/2)(r_{\max} - D/U) \geq \alpha D((D/r_{\max}) - U) \text{ when } q = \frac{\alpha D^2}{h_V (r_{\max})^2 (n/2 - 1)}$$

$$h_V \frac{\alpha D^2}{h_V (r_{\max})^2 (n/2 - 1)} (1 - n/2)(r_{\max} - D/U) \geq \alpha D((D/r_{\max}) - U)$$

$$D(r_{\max} - D/U) \leq r_{\max} U(r_{\max} - D/U).$$

$U \geq \frac{D}{r_{\max}}$, which is true. Second, we show that

$$h_V q(1 - n/2)(r_{\max} - D/U) < \alpha D((D/r_{\max}) - U) \text{ when } q = \frac{\alpha U^2}{h_V (\frac{n}{2} - 1)}$$

$$h_V \frac{\alpha U^2}{h_V (\frac{n}{2} - 1)} (1 - n/2)(r_{\max} - D/U) < \alpha D((D/r_{\max}) - U).$$

$$U^2(D/U - r_{\max}) < D((D/r_{\max}) - U)$$

$$U^2(D/U - r_{\max}) < D \frac{U}{r_{\max}} (D/U - r_{\max})$$

$U \geq \frac{D}{r_{\max}}$, which is true. Thus, the switching point is the root of

$$h_V q(1 - n/2)(r_{\max} - D/U) = \alpha D((D/r_{\max}) - U), \tag{A.1}$$

that is, $q_s^1(n) = \frac{\alpha D U}{h_V r_{\max} (n/2 - 1)}$. \square

Appendix B. Proof of Lemma 2

By definition of $q_{\max}^*(n)$ and $q_s^1(n)$ we need to show that the condition in which the optimal demand-to-production rate ratio under which $r^*(n, q^*(n)) = r_{\max}$ also remains for $n > n'$. We start by assuming $q_{\max}^*(n') \geq q_s^1(n')$, i.e.,

$$\sqrt{\frac{D \left(\frac{k}{n'} + k \right)}{h_V \left(\left(1 - \frac{n'}{2} \right) r_{\max} + \frac{n'}{2} - \frac{1}{2} \right) + \frac{h_V}{2}}} \geq \frac{2\alpha D U}{h_V r_{\max} (n' - 2)}$$

$$\frac{2D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n' - 2)]^2}{h_V(r_{\max}(2 - n') + n' - 1) + h_B} \geq 4\alpha^2 D^2 U^2 \tag{B.1}$$

Given that there exists $n = n'$ such that (B.1) is true, implying that $r^*(n', q^*(n')) = r_{\max}$, it is sufficient to show that

$$\frac{2D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n - 2)]^2}{h_V(r_{\max}(2 - n) + n - 1) + h_B} \geq 4\alpha^2 D^2 U^2 \text{ for } n \geq n'.$$

For $n = n'$ the claim is true. We show that the derivative of the left-hand side is positive (the derivative of the right-hand side is 0).

$$\begin{aligned} \frac{\partial}{\partial n} \left[\frac{2D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n - 2)]^2}{h_V(r_{\max}(2 - n) + n - 1) + h_B} \right] = & \\ \frac{[h_V(r_{\max}(2 - n) + n - 1) + h_B] \left[2D\left(\frac{-K}{n^2}\right)[h_V r_{\max}(n - 2)]^2 + 4h_V r_{\max} D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n - 2)] \right]}{[h_V(r_{\max}(2 - n) + n - 1) + h_B]^2} & \\ - \frac{h_V(1 - r_{\max}) \left[2D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n - 2)]^2 \right]}{[h_V(r_{\max}(2 - n) + n - 1) + h_B]^2} & \\ - [h_V(r_{\max}(2 - n) + n - 1) + h_B] \left[2D\left(\frac{-K}{n^2}\right)[h_V r_{\max}(n - 2)] + 4h_V r_{\max} D\left(\frac{K}{n} + k\right) \right] & \\ - h_V(1 - r_{\max}) \left[2D\left(\frac{K}{n} + k\right)[h_V r_{\max}(n - 2)] \right] & \\ - \frac{[h_V(r_{\max}(2 - n) + n - 1) + h_B] [2DK[h_V r_{\max}(2 - n)] + 4h_V r_{\max} D(Kn + kn^2)] -}{h_V(1 - r_{\max}) [2D(Kn + kn^2)[h_V r_{\max}(n - 2)]]} & \\ - \frac{[h_V n(1 - r_{\max})][2h_V r_{\max}][2DK] + [h_V n(1 - r_{\max})][2h_V r_{\max} D(kn^2)] +}{h_V(1 - r_{\max})[2h_V r_{\max}][2D(Kn)] + [h_V(1 - r_{\max})][h_V r_{\max}(2)][2D(kn^2)] +} & \\ - \frac{[2h_V r_{\max} - h_V + h_B][2h_V r_{\max}][2DK] + [2h_V r_{\max} - h_V + h_B][2h_V r_{\max} D(Kn)] +}{[2h_V r_{\max} - h_V + h_B][4h_V r_{\max} Dkn^2]} > 0 & \end{aligned}$$

For the opposite direction, i.e., assuming $q_{r_{\max}}^*(n') < q_s^1(n')$ and $r^*(n', q^*(n')) = r_{\max}$, then $q^*(n) = q_s^1(n)$ at least for $n = n', n' + 1, n' + 2, \dots, n''$ (since $q_s^1(n)$ strictly decreases with n), after which $q_{r_{\max}}^*(n'' + 1) > q_s^1(n'' + 1)$. Therefore, $q^*(n) = q_{r_{\max}}^*(n)$ also for $n > n''$ following the former scenario. We conclude that in both cases, $q^*(n) \geq q_s^1(n)$, and according to (13), $r^*(q, n) = r_{\max}$. \square

Appendix C. Proof of Lemma 4

We start with $C(r = r_{\max}, q_{r_{\max}}^*(n), n)$ and take the first derivative with respect to variable n .

$$\begin{aligned} C'(r = r_{\max}, q_{r_{\max}}^*(n), n) = & \left[2D\left(\frac{K}{n} + k\right) \right]^{1/2} \frac{1}{2} h_V(1 - r_{\max}) [(h_V(2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{-1/2} - \\ \frac{1}{2} \left[2D\left(\frac{K}{n} + k\right) \right]^{-1/2} \left(\frac{2DK}{n^2} \right) & [(h_V(2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{1/2} \end{aligned}$$

The FOC becomes:

$$\left[2D\left(\frac{K}{n} + k\right) \right] h_V(1 - r_{\max}) = \left(\frac{2DK}{n^2} \right) [(h_V(2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] \tag{C.1}$$

The second derivative becomes:

$$C''(r = r_{\max}, q_{r_{\max}}^*(n), n) = - \left[2D \left(\frac{K}{n} + k \right) \right]^{\frac{1}{2}} \frac{1}{4} h_V (1 - r_{\max}) h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{-3/2} -$$

$$\frac{1}{2} \left[2D \left(\frac{K}{n} + k \right) \right]^{-1/2} \left(\frac{2DK}{n^2} \right) \frac{1}{2} h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{-1/2} -$$

$$\frac{1}{2} \left[2D \left(\frac{K}{n} + k \right) \right]^{-1/2} \left\{ \left(\frac{2DK}{n^2} \right) \frac{1}{2} h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{-1/2} - \right.$$

$$\left. \left(\frac{4DK}{n^3} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{1/2} \right\} -$$

$$\left\{ \frac{1}{4} \left[2D \left(\frac{K}{n} + k \right) \right]^{-3/2} \left(\frac{2DK}{n^2} \right) \right\} \left(\frac{2DK}{n^2} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^{1/2}$$

We now seek the sign of $C''(r_{\max}, n)$. By multiplying $C''(r_{\max}, n)$ by the positive function $[(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] \left[2D \left(\frac{K}{n} + k \right) \right]$, it becomes

$$-\frac{1}{4} \left[2D \left(\frac{K}{n} + k \right) \right]^2 h_V (1 - r_{\max}) h_V (1 - r_{\max}) -$$

$$\left[D \left(\frac{K}{n} + k \right) \right] \left(\frac{2DK}{n^2} \right) h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] +$$

$$\left[D \left(3 \frac{K}{n} + 4k \right) \right] \left(\frac{DK}{n^3} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^2$$

Utilizing (C.1) by assigning the FOC in the above expression,

$$= -\frac{1}{4} \left[\left(\frac{2DK}{n^2} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] \right]^2 - \left[D \left(\frac{K}{n} + k \right) \right] \left(\frac{2DK}{n^2} \right) h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] +$$

$$\left[D \left(3 \frac{K}{n} + 4k \right) \right] \left(\frac{DK}{n^3} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^2 = \left[D \left(\frac{K}{n} + k \right) \right] \left(\frac{2DK}{n^2} \right) h_V (1 - r_{\max}) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)] +$$

$$\left[D \left(2 \frac{K}{n} + 4k \right) \right] \left(\frac{DK}{n^3} \right) [(h_V (2r_{\max} + n(1 - r_{\max}) - 1) + h_B)]^2 > 0$$

Thus, each point that is the root of the FOC is a local minimum. A similar procedure for $C(r = D/U, q_{D/U}^*(n), n)$ results in the same conclusion.

We show the convexity of

$$C \left(r = r_{\max}, q_s^1(n) = \frac{\alpha DU}{h_V r_{\max} (n/2 - 1)}, n \right) = D(c_V - \alpha D(1/r_{\max} - 1)) + \frac{DK h_V r_{\max} (n/2 - 1)}{\alpha DU} + \frac{k D h_V r_{\max} (n/2 - 1)}{\alpha DU} +$$

$$h_V \left(\frac{\alpha DU}{h_V r_{\max} (n/2 - 1)} D / U + \frac{n \alpha DU}{2 h_V r_{\max} (n/2 - 1)} (1 - D/U) - \frac{\alpha DU}{2 h_V r_{\max} (n/2 - 1)} \right) + h_B \frac{\alpha DU}{2 h_V r_{\max} (n/2 - 1)}$$

$$C' \left(r = r_{\max}, q_s^1(n) = \frac{\alpha DU}{h_V r_{\max} (n/2 - 1)}, n \right) = \frac{DK h_V r_{\max}}{2 \alpha DU} + \frac{k D h_V r_{\max}}{2 \alpha DU} - \frac{\alpha DU}{r_{\max} (n - 2)^2} (2 - 2D/U) -$$

$$\left(\frac{2 \alpha DU}{r_{\max} (n - 2)^2} (D/U - 1/2) \right) - h_B \frac{\alpha DU}{h_V r_{\max} (n - 2)^2}$$

$$C'' \left(r = r_{\max}, q_s^1(n) = \frac{\alpha DU}{h_V r_{\max} (n/2 - 1)}, n \right) = \frac{2}{(n - 2)^3} \frac{\alpha DU}{r_{\max}} (2 - 2D/U) +$$

$$\left(\frac{4 \alpha DU}{r_{\max} (n - 2)^3} (D/U - 1/2) \right) + h_B \frac{2 \alpha DU}{h_V r_{\max} (n - 2)^3} > 0$$

A similar procedure for $C(r = D/U, q_s^1(n), n)$ results in the same conclusion. \square

Appendix D. Proof of Proposition 6

We divide the proof into two scenarios:

If there does not exist any n' under which $r_{T_p}^{\max}(q_B^*(n'), n') = \frac{DT_p}{n' q_B^*(n')}$, the claim is proved in Lemma 2 (i.e., starts with $r_B^*(n) = D/U$ and alters into $r_B^*(n) = r_{\max}$).

In the case where $n = n'$, the demand-to-production rate ratio $r_B^*(q^*(n'), n') = \frac{DT_p}{n' q_B^*(n')}$ is optimal, thus we show that for all $n > n'$, the optimal demand-to-production rate ratio remains as $r_B^*(q^*(n), n) = \frac{DT_p}{n q_B^*(n)}$. If $r_{T_p}^{\max}(q_B^*(n'), n') = \frac{DT_p}{n' q_B^*(n')}$, it is left to show that $n q_B^*(n)$ does not decrease with n . Considering the components in the third line of (24.2), i.e., $q_B^*(n) = \min(\max(q_{DT_p/nq}^*(n), q_s^2(n)), UT_p/n)$, and since $n q_{DT_p/nq}^*(n)$ increases with n and $n q_s^2(n)$ does not decrease with n , the proof is complete.

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