

Ashrafi, Hedieh; Thiele, Aurélie C.

## Article

# A study of robust portfolio optimization with European options using polyhedral uncertainty sets

Operations Research Perspectives

**Provided in Cooperation with:**

Elsevier

*Suggested Citation:* Ashrafi, Hedieh; Thiele, Aurélie C. (2021) : A study of robust portfolio optimization with European options using polyhedral uncertainty sets, Operations Research Perspectives, ISSN 2214-7160, Elsevier, Amsterdam, Vol. 8, pp. 1-18, <https://doi.org/10.1016/j.orp.2021.100178>

This Version is available at:

<https://hdl.handle.net/10419/246438>

### Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

### Terms of use:

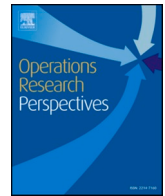
*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



<https://creativecommons.org/licenses/by-nc-nd/4.0/>



# A study of robust portfolio optimization with European options using polyhedral uncertainty sets

Hedieh Ashrafi, Aurélie C. Thiele\*

Engineering Management Information and Systems, Southern Methodist University, Dallas TX, USA

## ARTICLE INFO

### Keywords:

Robust optimization  
Portfolio management  
European options

## ABSTRACT

We consider the problem of maximizing the worst-case return of a portfolio when the manager can invest in stocks as well as European options on those stocks, and the stock returns are modeled using an uncertainty set approach. Specifically, the manager knows a range forecast for each factor driving the returns and a budget of uncertainty limiting the scaled deviations of these factors from their nominal values. Our goal is to understand the impact of options on the optimal portfolio allocation. We present theoretical results regarding the structure of that optimal allocation, in particular with respect to portfolio diversification. Specifically, we show that the presence of options only leads to limited diversification across the financial instruments available. We compare our robust portfolio to several benchmarks in numerical experiments and analyze how the optimal allocation varies with the budget of uncertainty. Our results indicate that our approach performs very well in practice.

## 1. Introduction

Markowitz [1] ushered in a new era in portfolio optimization when he articulated the investor's trade-off between risk and return in his landmark paper, where risk was measured by the portfolio variance and return by the portfolio's expected return. For a recent overview of quantitative methods in finance, the reader is referred to [2], which provides a comprehensive overview of financial engineering problems in terms of objectives, risk models and solution methods, and discusses the interplay between uncertainty representations and solution methods. In practice, managers have shown great interest in taking remedial action after uncertainty is revealed. This has led to the emergence of options as a risk management technique (see [3] and the references therein). Specifically, European call options (respectively, European put options) give its holder the right but not the obligation to purchase (respectively, sell) a share of stock at a pre-arranged price, called the strike price, at the option's expiration. The manager's recourse is to decide whether to exercise the option and thus buy or sell shares at a price more advantageous than the market price. When chosen appropriately, the availability of options helps mitigate risk.

In parallel, robust optimization has emerged as an effective tool to model uncertainty in environments where probabilistic descriptions of random variables are not available precisely [4,5]. Recent studies have shown the benefits of using robust optimization techniques in a wide

range of fields such as healthcare [6], power systems [7] and supply chain [8]. We follow here the approach to robust optimization presented in Bertsimas and Sim [5], which uses polyhedral sets centered at the nominal values of the uncertain parameters and the size of which is determined by a single parameter called the budget of uncertainty, limiting the number of uncertain parameters that can deviate from their nominal value. The bigger the budget of uncertainty, the larger the set of uncertain outcomes the manager is protecting himself against. Hence, in the [5] robust optimization setting, risk aversion is captured solely through the budget of uncertainty parameter. The main advantages of robust optimization over stochastic programming are that (i) it does not require the precise knowledge of the underlying probability distributions, and (ii) it is more computationally tractable. The reader is referred to [9] for a review of recent advances in robust optimization, including frameworks that combine robust optimization and stochastic programming in distributionally robust optimization.

A sizable literature exists on robust optimization in finance, especially without options. Goldfarb and Iyengar [10] and Tütüncü and Koenig [11] wrote landmark papers where they developed tractable robust optimization counterparts to famous frameworks in the finance literature. Garlappi et al. [12] provide a multi-prior approach to portfolio selection with parameter and model uncertainty when the multiple priors are characterized by confidence intervals around the estimated expected returns. They argue empirically that the resulting portfolios

\* Corresponding author.

E-mail addresses: [hashrafi@smu.edu](mailto:hashrafi@smu.edu) (H. Ashrafi), [athiele@smu.edu](mailto:athiele@smu.edu) (A.C. Thiele).

deliver a higher out-of-sample Sharpe ratio than classical portfolios. Cornuejols and Tütüncü [13], Pachamanova [14] and Kim et al. [15] describe several models for portfolio optimization under uncertainty that include robust optimization. Research on robust portfolio management focuses on the derivation of robust tractable reformulations for appropriate uncertainty sets, e.g., [16–22].

Among the scant robust optimization literature that considers options, Lutgens et al. [23] included European-style options as well as stocks in the Mean-Variance model. Subsequently, Zymler et al. [24] obtained strong guarantees on the worst-case portfolio return with the same model even when the actual stock returns fell outside of the uncertainty set and studied the trade-off between strong and weak guarantees (depending on stock price behavior) which leads to convex second-order cone programming problems. Zymler et al. [25] develop tractable approximations based on semidefinite programming for distributionally robust single and joint chance constraints, assuming that the first- and second-order moments as well as the support of the uncertain parameters are known. Then, Zymler et al. [26] addressed related computational difficulties by developing two tractable approximations to the problem of computing the VaR of a portfolio with derivatives when the first two moments of the underlying asset prices are known.

Matmoura and Penev [27] considers a dynamic risk optimization problem using stochastic programming and proposes an algorithmic procedure to optimize an option portfolio based on the minimization of higher-order coherent risk measures. Also, Davari-Ardakani et al. [28] utilizes options for mitigating market risk in a dynamic setting where the uncertainty return is modeled using a stochastic framework. Zhao and Palomar [29] applies the Markowitz mean-variance framework to option portfolio design by exploiting both first- and second-order statistics of option returns using stochastic differential equations and the delta-gamma approximation focused on the expected return of options. They provide two heuristic algorithms to solve the problem. A current sub-strand of the literature considers multistage portfolio optimization with stocks and options.

Goldfarb and Iyengar [10] and Chopra and Ziemba [30] examine the effect of errors in means, variances, and covariances on the optimal portfolio obtained by a robust approach. They conclude that the primary emphasis should be on gaining superior estimates of the mean, followed by good estimates of covariances because covariances are the least important in terms of their influence on the optimal allocation. Leung et al. [31] consider procedures to improve the estimation of the optimal portfolio return in the Markowitz framework.

The multistage adaptive approach holds particular appeal when affine policies are implemented, as documented in Bertsimas et al. [32]. Other papers such as [33] investigate the price of options and formulate the option pricing problem with the help of  $\epsilon$ -arbitrage and probability theory, e.g., the central limit theorem, by minimizing the worst-case replication error. Frey and Sin [34], DeMarzo et al. [35] also provide bounds for the price of options.

Subsequent streams of research have considered multi-stage models for portfolio selection under uncertainty. For instance, Shen and Zhang [36] provides robust portfolio selection based on a multi-stage scenario tree. Marzban et al. [37] extend the multi-period approach of [17] to consider options as well as stocks. Ling et al. [38] investigates robust multi-period portfolio selection based on downside risk with asymmetrically distributed uncertainty. Yin and Han [39] use a multi-stage stochastic program to allocate international assets with risk management via multi-stage stochastic programming.

**Contributions** In this paper, we investigate the combination of options and robust optimization to mitigate uncertainty in portfolio management, considering two models of uncertainty: (i) a very conservative setting where we optimize the worst-case portfolio return over all non-negative returns, also called the *strong-guarantee setting* because the portfolio return is guaranteed to be at least the value of the optimal objective, and (ii) a robust optimization setting where we optimize the

worst-case portfolio return over all non-negative returns belonging to a polyhedral set as in Bertsimas and Sim [5], centered at their nominal value and parametrized by a budget of uncertainty, also called the *weak-guarantee setting*, because the portfolio return is only guaranteed to be at least the value of the optimal objective if the actual stock returns do fall in the uncertainty set. While case (i) alone is likely too conservative to be of interest to the practical investor, the combination of (i) with (ii) to provide both a weak guarantee and a strong guarantee specifically for polyhedral uncertainty sets in a tractable, linear formulation is particularly appealing for investors facing large-scale portfolio problems in presence of high uncertainty. We are particularly interested in studying the impact of options on the structure of the optimal solution and comparing in numerical experiments the solutions obtained using polyhedral uncertainty sets with those obtained using ellipsoidal uncertainty sets in Zymler et al. [24].

Our results suggest, both through theoretical insights and numerical experiments showing detailed allocations and portfolio return profiles, that the use of options leads to limited diversification. Specifically, we prove that in the extremely conservative strong-guarantee setting, we invest in only one underlying asset at optimality, and in at most one call or stock and one put of that asset. Further, we prove that in the robust-optimization-based weak-guarantee setting, we invest in at most one call or stock and one put of each underlying asset that is included in the portfolio at optimality. Intuitively, diversification in robust portfolio optimization with polyhedral uncertainty sets occurs because the range forecasts of various stock returns overlap so that no stock returns outperforms all of the others. The availability of the options improves the effective range of some of the returns and makes some returns clearer “winners.” We also present an extension to multiple time periods using an open-loop approach.

The remainder of this paper is structured as follows. We present the problem setup and its tractable reformulation involving both the strong and weak guarantees in Section 2. We provide a detailed numerical study comparing our model with [24] in Section 3. Section 4 contains concluding remarks.

## 2. Portfolio model with European options using polyhedral uncertainty sets

We consider the following problem. The decision maker can invest in shares of stocks and European call and put options on these stocks with various strike prices. We consider a static setting throughout the paper, i.e., portfolio rebalancing is not allowed during the time window considered. The investor has one of the following objectives, which are increasingly difficult to formulate in a tractable manner and thus will be presented in sequence for greater readability of the paper:

**Strong-guarantee model** In the most pessimistic situation, the investor seeks to protect his portfolio against the most adverse realization of the uncertainty, when gross returns can take any non-negative value. (While this setting is very conservative, it is explored for instance in Cornuejols et al. [40] to determine when there exists arbitrage, i.e., the possibility to make infinite amounts of money, due to options mispricing. It is also useful in combination

with the weak-guarantee setting below.)

**Weak-guarantee model** The investor seeks to protect his portfolio against the most adverse realization of the uncertainty, when gross returns fall within an uncertainty set defined, as in Bertsimas and Sim [5], by range forecasts centered at the returns' nominal value and the cumulative scaled deviation is bounded above by a budget of uncertainty.

**Combined weak-and-strong guarantees model** In this setting, the investor considers the weak-guarantee model, optimizing the worst-case return over a polyhedral uncertainty set, and adds an insurance-type constraint where the absolute worst-case return (from the strong-guarantee model) must be at least a predefined fraction of the weak-guarantee objective. This model is also called Modified Insured Robust Optimization in Zymler et al. [24].

### 2.1. Strong-guarantee model

Here, the uncertainty set is the set of non-negative gross returns. the manager seeks to maximize his worst-case portfolio return measured over all possible non-negative gross returns, subject to fractions allocated in the various stocks and options summing to one and fractions being non-negative. Below, we formulate this problem in mathematical terms.

We will use the following notations:

#### Parameters

- $T$ : the time horizon,
- $n$ : the number of underlying assets considered,
- $n_i^+$ : the number of call options considered for asset  $i$ ,  $i = 1, \dots, n$ ,
- $n_i^-$ : the number of put options considered for asset  $i$ ,  $i = 1, \dots, n$ ,
- $S_i(0)$ : price of stock  $i$  at time 0 for asset  $i$ ,  $i = 1, \dots, n$ ,
- $R_i$ : return of stock  $i$  for all  $i$ ,  $i = 1, \dots, n$ ,
- $S_i(T)$ : price of stock  $i$  at time  $T$ , equal to  $R_i S_i(0)$  for all  $i$ ,  $i = 1, \dots, n$ ,
- $K_{ij}^+$ : strike price of stock  $i$  for call option  $j$  for all  $i = 1, \dots, n$  and  $j = 1, \dots, n_i^+$ ,
- $K_{ik}^-$ : strike price of stock  $i$  for put option  $k$  for all  $i = 1, \dots, n$  and  $k = 1, \dots, n_i^-$ ,
- $P_{ij}^+$ : price of call option  $j$  for stock  $i$  for all  $i = 1, \dots, n$  and  $j = 1, \dots, n_i^+$ ,
- $P_{ik}^-$ : price of put option  $k$  for stock  $i$  for all  $i = 1, \dots, n$  and  $k = 1, \dots, n_i^-$ ,

The (gross) return of stock  $i$  is defined as:  $R_i = S_i(T)/S_i(0)$  for all  $i$ .

The options' returns are given by:  $\max\left(0, \frac{S_i(T) - K_{ij}^+}{P_{ij}^+}\right)$  for call option  $j$  and

$\max\left(0, \frac{K_{ik}^- - S_i(T)}{P_{ik}^-}\right)$  for put option  $k$  with  $S_i(T) = R_i S_i(0)$  for all  $i, j, k$ . In particular, they are piecewise linear functions in the stock returns, with break points at  $\frac{K_{ij}^+}{S_i(0)}$  and  $\frac{K_{ik}^-}{S_i(0)}$ , respectively.

#### Decision variables

- $x_i$ : the fraction of the portfolio invested in shares of stock  $i$ ,  $i = 1, \dots, n$ ,
- $x_{ij}^+$ : the fraction of the portfolio invested in call option  $j$  for stock  $i$ ,  $i = 1, \dots, n$  and  $j = 1, \dots, n_i^+$ ,
- $x_{ik}^-$ : the fraction of the portfolio invested in buying put option  $k$  for stock  $i$ ,  $i = 1, \dots, n$  and  $k = 1, \dots, n_i^-$ .

**Mathematical model** The manager's problem of maximizing the worst case portfolio return at time  $T$  over all non-negative returns is given by:

$$\begin{aligned} \max_x \min_{R \geq 0} & \sum_{i=1}^n \left[ R_i x_i + \sum_{j=1}^{n_i^+} \max\left(0, \frac{R_i S_i(0) - K_{ij}^+}{P_{ij}^+}\right) x_{ij}^+ + \sum_{k=1}^{n_i^-} \max\left(0, \frac{K_{ik}^- - R_i S_i(0)}{P_{ik}^-}\right) x_{ik}^- \right] \\ \text{s.t.} & \sum_{i=1}^n \left[ x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ + \sum_{k=1}^{n_i^-} x_{ik}^- \right] = 1 \\ & x_i, x_{ij}^+, x_{ik}^- \geq 0 \quad \forall i, \forall j, \forall k. \end{aligned} \tag{1}$$

This problem admits a closed-form solution, as shown below.

**Theorem 1. (Worst-case problem)** *The worst-case portfolio return is equal to:*

$$Z^* = \max_i \max \left\{ \max_{j,k} \frac{K_{ik}^- - K_{ij}^+}{P_{ij}^+ + P_{ik}^-}, \max_k \frac{K_{ik}^-}{P_{ik}^- + S_i(0)} \right\} \tag{2}$$

**Proof:** We first consider the single-asset subproblem, where the manager only invests in one asset either through stocks, calls or puts. We observe that the manager will not invest in an option he does not exercise at optimality, because the objective would be strictly improved by allocating that money to a stock or an option that is exercised at optimality. (In our worst-case setting, we always know which options are exercised at optimality because there is no stochasticity.) Hence, the single-asset version of Problem (1) can be rewritten as:

$$\begin{aligned} \max_x \min_{R \geq 0} & \left[ R x + \sum_{j=1}^{n^+} \frac{RS(0) - K_j^+}{P_j^+} x_j^+ + \sum_{k=1}^{n^-} \frac{K_k^- - RS(0)}{P_k^-} x_k^- \right] \\ \text{s.t.} & x + \sum_{j=1}^{n^+} x_j^+ + \sum_{k=1}^{n^-} x_k^- = 1 \\ & x, x_j^+, x_k^- \geq 0, \quad \forall j, k, \end{aligned} \tag{3}$$

The slope of the inner function in  $R$  is:  $x + \sum_{j=1}^{n^+} \frac{S(0)}{P_j^+} x_j^+ - \sum_{k=1}^{n^-} \frac{S(0)}{P_k^-} x_k^-$ , which is non-decreasing in  $R$ . Therefore, the minimum over  $R$  is achieved either when this slope is equal to 0 or (if the slope is positive for all  $R \geq 0$ ) at  $R = 0$ . In both cases, this yields an objective of  $\sum_{k=1}^{n^-} \frac{K_k^- x_k^-}{P_k^-} - \sum_{j=1}^{n^+} \frac{K_j^+ x_j^+}{P_j^+}$ . But if  $x + \sum_{j=1}^{n^+} \frac{S(0)}{P_j^+} x_j^+ - \sum_{k=1}^{n^-} \frac{S(0)}{P_k^-} x_k^- > 0$ , there exists  $\epsilon > 0$  and indices  $j$  and  $k$  such that one can increase  $x_k^-$  by  $\epsilon$ , decrease  $x_j^+$  by  $\epsilon$  and still have a feasible solution, and the objective will be strictly improved by  $\left( \sum_{k=1}^{n^-} \frac{K_k^-}{P_k^-} + \sum_{j=1}^{n^+} \frac{K_j^+}{P_j^+} \right) \epsilon$ . Hence, this yields the following problem to solve:

$$\begin{aligned}
 \max \quad & \sum_{k=1}^{n^-} \frac{K_k^- x_k^-}{P_k^-} - \sum_{j=1}^{n^+} \frac{K_j^+ x_j^+}{P_j^+} \\
 \text{s.t.} \quad & x + \sum_{j=1}^{n^+} \frac{S(0)}{P_j^+} x_j^+ - \sum_{k=1}^{n^-} \frac{S(0)}{P_k^-} x_k^- = 0, \\
 & x + \sum_{j=1}^{n^+} x_j^+ + \sum_{k=1}^{n^-} x_k^- = 1, \\
 & x, x_j^+, x_k^- \geq 0 \quad \forall j, k.
 \end{aligned} \tag{4}$$

This is a linear programming problem with two equality constraints and a non-empty feasible set, so there exists an optimal solution with only two non-zero components. Because we must have  $x + \sum_{j=1}^{n^+} \frac{S(0)}{P_j^+} x_j^+ - \sum_{k=1}^{n^-} \frac{S(0)}{P_k^-} x_k^- = 0$ , the two non-zero components have to be one put and either one call or the stock and a put.

Hence, we know the optimal portfolio will consist of either calls and puts or stock shares and puts. When the optimal sub-portfolio consists of one put (denoted put  $k$  for some  $k$ ) and one call option (denoted call  $j$  for some  $j$ ) for a given stock, then the optimal worst-case return is:

$$Z^* = \max_{j,k} \frac{K_k^- - K_j^+}{P_j^+ + P_k^-}, \tag{5}$$

and the optimal allocations within the sub-portfolio are:

$$x^{*+} = \frac{P_j^+}{P_j^+ + P_k^-}, \quad x^{*-} = \frac{P_k^-}{P_j^+ + P_k^-}.$$

where  $j$  and  $k$  are those that achieve the maximum in Eq. (5).

This is because, when the two equality constraints in Problem (4) become (dropping the indices  $j$  and  $k$ ):

$$\begin{aligned}
 \frac{x^+}{P^+} &= \frac{x^-}{P^-}, \\
 x^+ + x^- &= 1,
 \end{aligned}$$

leading to Eq. (2.1). We find the optimal  $j$  and  $k$  by reinjecting the allocations into the objective and picking the  $j$  and  $k$  that achieve the highest objective value, yielding Eq. (5).

In a similar manner left to the reader, we can prove that, when the optimal sub-portfolio for a given asset consists of one put (denoted put  $k$  for some  $k$ ) on the stock and that stock itself, then the optimal worst-case return is:

$$Z^* = \max_k \frac{K_k^-}{P_k^- + S(0)}, \tag{6}$$

and the optimal sub-portfolio allocations in the stock and the put on that stock are, with  $k$  the index that achieves the optimum in Eq. (6):

$$x^* = \frac{S(0)}{P_k^- + S(0)}, \quad x_k^{*-} = \frac{P_k^-}{P_k^- + S(0)}$$

Because  $Z^*$  in Eq. (6) is always positive, it is suboptimal to only invest in the stock itself, for which the worst-case return is 0 in this framework.

It follows immediately by combining the results above that the optimal worst-case return for the single-asset subproblem is given by:

$$Z^* = \max \left\{ \max_{j,k} \frac{K_k^- - K_j^+}{P_j^+ + P_k^-}, \max_k \frac{K_k^-}{P_k^- + S(0)} \right\}. \tag{7}$$

We now return to the original worst-case problem where we have multiple underlying assets. Using Eq. (7) and introducing new decision variables  $\alpha_i$  as the fraction of the portfolio invested in asset  $i$  either by buying shares of the stock itself or by buying call options on that stock or by buying put options on that stock, we rewrite Problem (1) as:

$$\begin{aligned}
 \max \quad & \sum_{i=1}^n \max \left\{ \max_{j,k} \frac{K_{ik}^- - K_{ij}^+}{P_{ij}^+ + P_{ik}^-}, \max_k \frac{K_{ik}^-}{P_{ik}^- + S_i(0)} \right\} \alpha_i \\
 \text{s.t.} \quad & \sum_{i=1}^n \alpha_i = 1, \\
 & \alpha_i \geq 0, \quad \forall i.
 \end{aligned}$$

This is a linear problem over a simplex, whose optimal solution is achieved at a corner point, yielding a non-diversified optimal portfolio and Eq. (2) as the optimal objective.

Hence, at optimality, the manager who seeks to protect his portfolio against the most adverse realization of the uncertainty invests in only one asset, and for that asset invests in at most one call option and one put option, or at most one put option and the stock itself. Although implementing this approach will guarantee to the investor that his portfolio return will not fall below the optimal value in Eq. (2), it is extremely conservative because it assumes that the stock prices at the next time period can take any possible non-negative value, so that the gross returns can take any non-negative value as well. It is therefore natural to ask how the optimal strategy will change if we consider a more realistic description of uncertainty that uses our knowledge of the stock prices at the present time period and limits the possible outcomes at the next time period to a more realistic set of values. This is the purpose of the next section.

## 2.2. Weak guarantee model

### 2.2.1. Formulation

Here, we consider a more realistic modeling of uncertainty inspired by Bertsimas and Sim [5]. Specifically, we assume that the stock returns belong to a polyhedral set centered at the nominal values of the returns, the size of which is parametrized by a budget of uncertainty, and model correlation between stock returns using a factor model. The decision variables are the fractions of the portfolio invested in each financial instrument. The investor seeks to maximize the worst-case portfolio return over that polyhedral uncertainty set, subject to fractions summing to one and being non-negative. The difference with the strong-guarantee setting is that the previous uncertainty set consisted of all non-negative stock returns. In the current setting, we assume that option prices also are non-negative; however, this assumption is relaxed in Section 2.5.

We use the following notation, in addition to the one described in the previous section.

<b>Additional parameters:</b>	
$L$	the number of factors considered,
$\bar{R}_i$	nominal value of stock return $i$ for $i = 1, \dots, n$ ,
$\hat{R}_{il}$	maximum deviation of stock return $i$ from its nominal value due to factor $l$
	for $i = 1, \dots, n$ and $l = 1, \dots, L$ ,
$\Gamma$	budget of uncertainty (in $[0, L]$ ).

<b>Additional decision variables:</b>	
$z_l$	scaled deviation of factor $l$ from its nominal value for $l = 1, \dots, L$ .

Let denote  $\mathcal{R}$  the uncertainty set for the stock returns. We will define  $\mathcal{R}$  in terms of the scaled deviations of the stock returns from their nominal values, the sum of which cannot exceed the budget of uncertainty. In mathematical terms,  $\mathcal{R}$  is defined as:

$$\mathcal{R} = \left\{ R \mid \exists z, \sum_{l=1}^L |z_l| \leq \Gamma, |z_l| \leq 1, \forall l, R_i = \bar{R}_i + \sum_{l=1}^L \hat{R}_{il} z_l \quad \forall i, \right\}$$

The robust optimization problem is then formulated as:

$$\begin{aligned} \max_x \min_{R \in \mathcal{Z}} \sum_{i=1}^n & \left[ R_i x_i + \sum_{j=1}^{n_i^+} \max \left( 0, \frac{R_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) x_{ij}^+ + \sum_{k=1}^{n_i^-} \max \left( 0, \frac{K_{ik}^- - R_i S_i(0)}{P_{ik}^-} \right) x_{ik}^- \right] \\ \text{s.t.} \sum_{i=1}^n & \left[ x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ + \sum_{k=1}^{n_i^-} x_{ik}^- \right] = 1 \\ x_i, x_{ij}^+, x_{ik}^- & \geq 0 \forall i, j, k. \end{aligned} \tag{8}$$

This formulation is not yet tractable because both the outer maximization and the inner minimization are over continuous sets. To make the formulation efficiently solvable, we seek to obtain a single master maximization problem that would be linear or convex, to preserve tractability. While [5] has shown how to do this for problems linear in the decision variables and the uncertain parameters, the piecewise linear terms in options make the problem investigated here challenging. Theorem 2 shows how we can still obtain a tractable reformulation.

**Theorem 2.** (Tractable reformulation of the weak-guarantee problem) Problem (8) can be reformulated as the following linear problem:

$$\begin{aligned} \max \sum_{i=1}^n & \left[ \bar{R}_i x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \left( \frac{K_{ik}^- - \bar{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij}^+ \left( \frac{\bar{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\ & - \left( p\Gamma + \sum_{l=1}^L q_l \right) \text{s.t.} \sum_{i=1}^n \left[ x_i + \sum_{k=1}^{n_i^-} x_{ik}^- + \sum_{j=1}^{n_i^+} x_{ij}^+ \right] \\ & = 1, p + q_l \geq \sum_{i=1}^n \widehat{R}_{il} \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, p + q_l \\ & \geq \sum_{i=1}^n \widehat{R}_{il} \left( -x_i - \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, p, q_l, x_i, x_{ik}^-, x_{ij}^+ \\ & \geq 0, \forall i, j, k, l, \end{aligned} \tag{9}$$

**Proof:** The decision maker will not invest in options that have a zero payoff at optimality, because allocating that money from zero-payoff options to the stock or to options that have a positive payoff strictly improves the objective. (Since the setting is not stochastic, we know which options will have a positive payoff at optimality.) In other words, when  $\max \left( 0, \frac{K_{ik}^- - R_i S_i(0)}{P_{ik}^-} \right)$  or  $\max \left( 0, \frac{K_{ij}^+ - R_i S_i(0)}{P_{ij}^+} \right)$  is equal to zero for the worst-case  $R_i$ 's, the manager will not invest in call  $j$  or put  $k$  of stock  $i$  because he will prefer investing in the stock, which has non-negative gross return. Therefore, we can separate the nominal part of the portfolio return from the part subject to uncertainty in the objective as follows, with  $\mathcal{Z} = \{z | \sum_{l=1}^L |z_l| \leq \Gamma, |z_l| \leq 1, \forall l\}$ :

$$\begin{aligned} \max \sum_{i=1}^n & \left[ \bar{R}_i x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \left( \frac{K_{ik}^- - \bar{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij}^+ \left( \frac{\bar{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\ & - \max_{z \in \mathcal{Z}} \sum_{l=1}^L \left| \sum_{i=1}^n \widehat{R}_{il} \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \right| |z_l| \end{aligned}$$

We then invoke strong duality (see [41]) for the inner minimization problem in  $z$  to conclude.  $\square$

### 2.2.2. Optimal allocation

Theorem 3 provides structural insights into the optimal allocation.

**Theorem 3.** (Optimal allocation) The manager invests in at most one call and at most one put of the same underlying asset for each asset.

**Proof:** Consider the dual of Problem (9).

$$\begin{aligned} \min \alpha \\ \text{s.t.} \alpha + \sum_{i=1}^L \widehat{R}_{il} (\gamma_i^+ - \gamma_i^-) & \geq \bar{R}_i, \forall i, \\ \alpha + \sum_{i=1}^L \widehat{R}_{il} \frac{S_i(0)}{P_{ij}^+} (\gamma_i^+ - \gamma_i^-) & \geq \frac{\bar{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+}, \forall i, j, \\ \alpha - \sum_{i=1}^L \widehat{R}_{il} \frac{S_i(0)}{P_{ik}^-} (\gamma_i^+ - \gamma_i^-) & \geq \frac{K_{ik}^- - \bar{R}_i S_i(0)}{P_{ik}^-}, \forall i, k, \\ \sum_{i=1}^L (\gamma_i^+ + \gamma_i^-) & \leq \Gamma \\ \gamma_i^+ + \gamma_i^- & \leq 1, \forall i, \\ \gamma_i^+, \gamma_i^- & \geq 0, \forall i. \end{aligned} \tag{10}$$

We will prove that the manager invests in at most one call by contradiction. The proof for puts is similar and left to the reader. Assume that we have  $x_{ij} > 0$  and  $x_{ij'} > 0$  for some  $i, j, j'$ . By complementarity slackness:

$$\alpha + \sum_{i=1}^L \widehat{R}_{il} \frac{S_i(0)}{P_{ij}^+} (\gamma_i^+ - \gamma_i^-) = \frac{\bar{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+}$$

and

$$\alpha + \sum_{i=1}^L \widehat{R}_{il} \frac{S_i(0)}{P_{ij'}^+} (\gamma_i^+ - \gamma_i^-) = \frac{\bar{R}_i S_i(0) - K_{ij'}^+}{P_{ij'}^+}$$

Rearranging terms, this leads to:  $\alpha P_{ij}^+ + K_{ij}^+ = \alpha P_{ij'}^+ + K_{ij'}^+$  so that the optimal objective  $\alpha$  is given by:

$$\alpha = \frac{K_{ij'}^+ - K_{ij}^+}{P_{ij}^+ - P_{ij'}^+}.$$

Since the optimal objective is known as soon as the decision maker invests in two calls of the same stock, there exists an optimal primal solution for (9) where the manager only invests in those two call options (the dual constraints are not binding so the corresponding primal variables are zero). Writing Problem (9) in that case with  $x$  and  $x'$  the fractions invested in the two call options and dropping the subscript  $i$ , we obtain:

$$\max_x \left( \frac{\bar{r}S(0) - K^+}{P^+} \right) + x' \left( \frac{\bar{r}S(0) - K'^+}{P'^+} \right) - \left( \frac{S(0)}{P^+} x + \frac{S(0)'}{P'^+} x' \right) \cdot \max_{i=1}^L \left| \widehat{R}_{il} \right| z_l$$

$$\text{s.t. } \sum_{i=1}^L z_i \leq \Gamma \quad 0 \leq z_i \leq 1, \forall l, \quad \text{s.t. } x + x' = 1 \quad x, x' \geq 0.$$

Hence the inner maximization in  $z$  is independent of  $x$  and  $x'$  and the outer maximization problem is linear in  $x$  and  $x'$  over a simplex. The optimal solution is thus attained when the manager only invests in the call option with the higher return.  $\square$

Intuitively, the investor does not invest in multiple calls or multiple puts of the same asset in the robust optimization framework because the worst-case portfolio return over the uncertainty set is strictly improved by combining the allocations into calls (respectively puts) of the same underlying asset into one well-chosen call (respectively put). This is because there is no stochasticity in the robust optimization framework, where the worst case is optimized over the uncertainty set, so the manager knows at optimality which options will be exercised. Once one call is in the money, i.e., the stock price is higher than the strike price, all calls with smaller strike price are in the money too. Similarly, once one put is in the money, i.e., the stock price is lower than the strike price, all puts with higher strike price are in the money too. So there is no advantage in having multiple calls or multiple puts on the same underlying asset in the robust optimization representation of uncertainty. Further, in robust optimization without options, the part of the objective function related to a specific asset always increases with that asset's return, so that the worst case is for a low return. Diversification then makes sense when the range forecasts of the returns overlap. In the case with options, the presence of puts means that the part of the objective function related to the underlying asset can be decreasing in the stock returns and the worst case is achieved where the slope of that part of the objective function change signs; also, the presence of options allows the decision maker to have a bigger impact on shaping the return of the portfolio so that the parts of the objective about various underlying assets may no longer overlap; that is why we have less diversification.

### 2.3. Combined weak-and-strong-guarantees model

Here we add the strong-guarantee constraint that we first considered in Section 2.1 and is inspired by [24], to the weak-guarantee model developed in Section 2.2. In this model, the investor seeks to maximize his worst-case return when the worst case is measured over a polyhedral uncertainty set for the stock returns ("reasonable worst case" for short), while ensuring that the absolute worst case (worst case measured over all non-negative stock returns) is at least a fraction  $\theta \in [0, 1]$  of the "reasonable worst case." Fractions (representing the fraction of the portfolio in a financial instrument) have to sum to one and be non-negative. That is, the investor seeks to solve:

$$\max z$$

$$\text{s.t. } \theta z \leq \sum_{i=1}^n \left[ R_i x_i + \sum_{j=1}^{n_i^+} \max \left( 0, \frac{R_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) x_{ij}^+ + \sum_{k=1}^{n_i^-} \max \left( 0, \frac{K_{ik}^- - R_i S(0)}{P_{ik}^-} \right) x_{ik}^- \right], \forall R \geq 0$$

$$\sum_{i=1}^n \left[ x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ + \sum_{k=1}^{n_i^-} x_{ik}^- \right] = 1$$

$$x_i, x_{ij}^+, x_{ik}^- \geq 0 \quad \forall i, j, k.$$

Problem (11) has an infinite number of constraints; however, Theorem 4 shows that it can be solved in a tractable manner as a linear programming problem with a finite number of constraints.

**Theorem 4. (Tractable reformulation)** Problem (11) can be reformulated as a linear problem:

$$\max z$$

$$\text{s.t. } z \leq \sum_{i=1}^n \left[ \widehat{R}_i x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \left( \frac{K_{ik}^- - \widehat{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij}^+ \left( \frac{\widehat{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right]$$

$$- \left( p\Gamma + \sum_{l=1}^L q_l \right) \theta z$$

$$\leq \sum_{i=1}^n \left( - \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{K_{ij}^+}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ik}^- \frac{K_{ik}^-}{P_{ik}^-} \right) x_i + \sum_{j=1}^{n_i^+} \frac{S_i(0)}{P_{ij}^+} x_{ij}^+ - \sum_{k=1}^{n_i^-} \frac{S_i(0)}{P_{ik}^-} x_{ik}^-$$

$$= 0 \quad \forall i, p + q_l \geq \sum_{i=1}^n \widehat{R}_{il} \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, p + q_l$$

$$\geq \sum_{i=1}^n \widehat{R}_{il} \left( -x_i - \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \sum_{i=1}^n \left[ x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \right. \\ \left. + \sum_{j=1}^{n_i^+} x_{ij}^+ \right]$$

$$= 1, p, q_l, x_i, x_{ik}^-, x_{ij}^+ \geq 0, \quad \forall i, j, k, l, \tag{12}$$

**Proof:** Using Theorem 2, Problem (11) can be formulated as:

$$\begin{aligned}
 \max \quad & z \\
 \text{s.t.} \quad & z \leq \sum_{i=1}^n \left[ \widehat{R}_i x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \left( \frac{K_{ik}^- - \overline{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij}^+ \left( \frac{\overline{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\
 & - \left( p\Gamma + \sum_{l=1}^L q_l \right) \theta z \\
 & \leq \sum_{i=1}^n \left[ \widehat{R}_i x_i + \sum_{j=1}^{n_i^+} \max \left( 0, \frac{R_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) x_{ij}^+ \right. \\
 & \quad \left. + \sum_{k=1}^{n_i^-} \max \left( 0, \frac{K_{ik}^- - R_i S_i(0)}{P_{ik}^-} \right) x_{ik}^- \right], \forall R \geq 0p + q_l \\
 & \geq \sum_{i=1}^n \widehat{R}_{il} \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, p + q_l \\
 & \geq \sum_{i=1}^n \widehat{R}_{il} \left( -x_i - \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \sum_{i=1}^n \left[ x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \right. \\
 & \quad \left. + \sum_{j=1}^{n_i^+} x_{ij}^+ \right] \\
 & = 1, p, q_l, x_i, x_{ik}^-, x_{ij}^+ \geq 0, \forall i, j, k, l,
 \end{aligned} \tag{13}$$

We then reformulate the right-hand side of the second constraint using Eq. (4) for multiple assets, leading to Problem (12).

□

#### 2.4. Extension to multi-period robust optimization

In this section, we incorporate multiple time periods to the weak-guarantee robust optimization formulation. The investor approaches the portfolio optimization problem as an open-loop problem, i.e., he acts upon the information available and re-solves the problem at each time period, incorporating the new information he has received. This means that at each time period, the investor takes only the first step of the optimal allocation strategy computed with the information up to that time period. In other words, he solves consecutive multi-period portfolio optimization problems with decreasing time horizons. The open-loop approach in robust optimization was first described in Bertsimas and Thiele [42] and has the advantage of being linear and computationally efficient. The linearity of the formulation is a significant advantage for large-scale problems. Closed-loop approaches such as adaptive or adjustable robust optimization as well as stochastic programming benchmarks are beyond the scope of this paper but represent a valuable direction for future work.

The multi-period portfolio optimization problem can be formulated as follows. There are  $n$  assets and  $T$  trading times. The investor's goal is to manage his portfolio of assets in a manner that maximizes his final wealth. The initial budget is \$1. After portfolio rebalancing, the decision variables about each stock, call and put are monetary values rather than fractions (because the budget after the initial time period may not be \$1). We consider a family of uncertainty budgets,  $\Gamma_t$  for  $t = 1, \dots, T$ , based on the manager's attitude toward risk and uncertainty. As in Bertsimas and Thiele [42], these budgets can be linear or concave in time to capture the canceling-out of uncertainty when many independent random variables are summed, in the spirit of the law of large numbers.

The problem can be formulated stylistically as follows:

Maximize robust worst-case return over the polyhedral uncertainty set at time  $T$ ,

Subject to Initial Monetary Budget: Initial portfolio allocations sum to 1,

Subject to Dynamics Equations: Amount allocated at the beginning of time  $t$  is at most the robust worst-case return over the polyhedral uncertainty set at time  $t - 1$ ,

Subject to Auxiliary Constraints: this group of constraints connects the auxiliary variables generated from writing the dual problem and using strong duality to the amount of uncertainty and the allocation at time  $t$ ,

Subject to Non-negativity Constraints: all decision variables are non-negative.

We have seen in the previous section that the worst-case portfolio return for a generic time period is given by:

$$\begin{aligned}
 & \sum_{i=1}^n \left[ \widehat{R}_i x_i + \sum_{k=1}^{n_i^-} x_{ik}^- \left( \frac{K_{ik}^- - \overline{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij}^+ \left( \frac{\overline{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\
 & - \left( p\Gamma + \sum_{l=1}^L q_l \right)
 \end{aligned}$$

with the auxiliary constraints:

$$p + q_l \geq \sum_{i=1}^n \widehat{R}_{il} \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l,$$

$$p + q_l \geq \sum_{i=1}^n \widehat{R}_{il} \left( -x_i - \sum_{j=1}^{n_i^+} x_{ij}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ik}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l,$$

It is straightforward to add indices  $t$  for  $t = 1, \dots, T$  and obtain the following formulation, which we state without proof:

*Multi-stage weak-guarantee formulation*

$$\begin{aligned}
 \max \quad & \sum_{i=1}^n \left[ \widehat{R}_{iT} x_{iT} + \sum_{k=1}^{n_i^-} x_{ikT}^- \left( \frac{K_{ik}^- - \overline{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ijT}^+ \left( \frac{\overline{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\
 & - \left( p_T \Gamma_T + \sum_{l=1}^L q_{lT} \right) \\
 \text{s.t.} \quad & \sum_{i=1}^n \left[ x_{i1} + \sum_{k=1}^{n_i^-} x_{ik1}^- + \sum_{j=1}^{n_i^+} x_{ij1}^+ \right] = 1 \\
 & \sum_{i=1}^n \left[ x_{it} + \sum_{k=1}^{n_i^-} x_{ikt}^- + \sum_{j=1}^{n_i^+} x_{ijt}^+ \right] \leq \\
 & \sum_{i=1}^n \left[ \widehat{R}_{i,t-1} x_{i,t-1} + \sum_{k=1}^{n_i^-} x_{ik,t-1}^- \left( \frac{K_{ik}^- - \overline{R}_i S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij,t-1}^+ \left( \frac{\overline{R}_i S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\
 & - \left( p_{t-1} \Gamma_{t-1} + \sum_{l=1}^L q_{l,t-1} \right), \forall t, \\
 & p_t + q_{lt} \geq \sum_{i=1}^n \widehat{R}_{ilt} \left( x_{it} + \sum_{j=1}^{n_i^+} x_{ijt}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ikt}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, t, \\
 & p_t + q_{lt} \geq \sum_{i=1}^n \widehat{R}_{ilt} \left( -x_{it} - \sum_{j=1}^{n_i^+} x_{ijt}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ikt}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, t, \\
 & p_t, q_{lt}, x_{it}, x_{ik,t}^-, x_{ij,t}^+ \geq 0, \forall i, j, k, l, t.
 \end{aligned} \tag{14}$$



It is possible to add strong-guarantee constraints to the formulation as in the single-period case, which is left to the reader. Further, the theorem below shows that the same structural insights we derived in [Theorem 3](#) for the single-stage case holds for multiple stages.

**Theorem 5. (Optimal allocation for multi-stage problem)** *At each time period, it is optimal to invest in at most one call and at most one put of the same underlying asset.*

**Proof:** We prove it is true at time  $T$  and proceed backward to prove that it is true for all time periods  $1 \leq t \leq T$ .

To analyze the optimal allocation at time  $T$  we assume that the problem has been solved to optimality and so we set the decision variables at times  $t < T$  to their optimal values, denoted by star superscripts. Let  $A_T$  the allocation budget available at time  $T$ , defined by:

$$A_T = \sum_{i=1}^n \left[ \bar{R}_{i,T-1} x_{i,T-1}^* + \sum_{k=1}^{n_i^-} x_{ik,T-1}^* \left( \frac{K_{ik}^- - \bar{R}_{i,T-1} S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij,T-1}^* \left( \frac{\bar{R}_{i,T-1} S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] - \left( p_{T-1}^* \Gamma_{T-1} + \sum_{l=1}^L q_{l,T-1}^* \right)$$

The problem at time  $T$  can then be formulated as:

$$\begin{aligned} \max \quad & \sum_{i=1}^n \left[ \bar{R}_{iT} x_{iT} + \sum_{k=1}^{n_i^-} x_{ikT}^- \left( \frac{K_{ik}^- - \bar{R}_{iT} S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ijT}^+ \left( \frac{\bar{R}_{iT} S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\ & - \left( p_T \Gamma_T + \sum_{l=1}^L q_{lT} \right) \\ \text{s.t.} \quad & \sum_{i=1}^n \left[ x_{iT} + \sum_{k=1}^{n_i^-} x_{ikT}^- + \sum_{j=1}^{n_i^+} x_{ijT}^+ \right] \leq A_T, \\ & p_T + q_{lT} \geq \sum_{i=1}^n \hat{R}_{iIT} \left( x_{iT} + \sum_{j=1}^{n_i^+} x_{ijT}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ikT}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \\ & p_T + q_{lT} \geq \sum_{i=1}^n \hat{R}_{iIT} \left( -x_{iT} - \sum_{j=1}^{n_i^+} x_{ijT}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ikT}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \\ & p_T, q_{lT}, x_{iT}, x_{ikT}^-, x_{ijT}^+ \geq 0, \quad \forall i, j, k, l. \end{aligned}$$

We then use the result in [Theorem 3](#) where the last time period here is the single period of the single-stage case and the structural results we had previously derived (about investing in at most one call and at most one put) still hold. It is easy to see that the optimal objective of Problem (5) increases with  $A_T$ , since the feasible set of the maximization problem increases when  $A_T$  increases, which we need in subsequent parts of the proof.

Moving on to time  $t, t \leq T - 1$ , we want  $A_{t+1}$  to be as large as possible so that the investor's terminal wealth will be in turn as large as possible. Again, we set the decision variables for times before  $t$  to their optimality and define:

$$A_t = \sum_{i=1}^n \left[ \bar{R}_{i,t-1} x_{i,t-1} + \sum_{k=1}^{n_i^-} x_{ik,t-1}^- \left( \frac{K_{ik}^- - \bar{R}_{i,t-1} S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ij,t-1}^+ \left( \frac{\bar{R}_{i,t-1} S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] - \left( p_{t-1} \Gamma_{t-1} + \sum_{l=1}^L q_{l,t-1} \right)$$

So we solve:

$$\begin{aligned} \max \quad & A_{t+1} = \sum_{i=1}^n \left[ \bar{R}_{it} x_{it} + \sum_{k=1}^{n_i^-} x_{ikt}^- \left( \frac{K_{ik}^- - \bar{R}_{it} S_i(0)}{P_{ik}^-} \right) + \sum_{j=1}^{n_i^+} x_{ijt}^+ \left( \frac{\bar{R}_{it} S_i(0) - K_{ij}^+}{P_{ij}^+} \right) \right] \\ & - \left( p_t \Gamma_t + \sum_{l=1}^L q_{lt} \right) \\ \text{s.t.} \quad & \sum_{i=1}^n \left[ x_{it} + \sum_{k=1}^{n_i^-} x_{ikt}^- + \sum_{j=1}^{n_i^+} x_{ijt}^+ \right] \leq A_t, \\ & p_t + q_{lt} \geq \sum_{i=1}^n \hat{R}_{ilt} \left( x_{it} + \sum_{j=1}^{n_i^+} x_{ijt}^+ \frac{S_i(0)}{P_{ij}^+} - \sum_{k=1}^{n_i^-} x_{ikt}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \\ & p_t + q_{lt} \geq \sum_{i=1}^n \hat{R}_{ilt} \left( -x_{it} - \sum_{j=1}^{n_i^+} x_{ijt}^+ \frac{S_i(0)}{P_{ij}^+} + \sum_{k=1}^{n_i^-} x_{ikt}^- \frac{S_i(0)}{P_{ik}^-} \right) \quad \forall l, \\ & p_t, q_{lt}, x_{it}, x_{ikt}^-, x_{ijt}^+ \geq 0, \quad \forall i, j, k, l. \end{aligned}$$

We then use the result in [Theorem 3](#) to prove that we still invest in at most one call and at most one put of the underlying asset at optimality. The objective function of the problem increases in  $A_t$  so we can move on to time  $t - 1$  where we will seek to maximize  $A_t$  and repeat until we reach  $t = 1$ .

### 2.5. Extension to negative option prices

The economic situation in 2020 has made it apparent that there is a need for the operations research literature to also include the case when option prices become negative. A negative call price indicates that the option writer pays the option purchaser to take the option. This is not a normal occurrence in well-functioning financial markets but can happen in periods of great crisis or uncertainty. Longstaff [\[43\]](#) observed that market prices for callable Treasury bonds often imply negative values for the implicit call option. They considered a variety of possible explanations for these negative values including the Treasury's track record in calling bonds optimally, tax-related effects, tax-timing options, and bond liquidity.

In order to capture this abnormal market condition as well as the situation of well-functioning financial markets, we rewrite the formulation as Problem (15) below, where the decision variables are the number of shares in each financial instrument. In this model, the option prices can take negative as well as positive (or zero) values and the objective function maximizes the total wealth instead of total return. Then, the constraint regarding the summation of decision variables is a budget constraint with the initial budget of  $B$  dollars.

**Table 1**  
Benchmarks.

Model	Type	$\Gamma$	$\theta$	$p$	$q$
1	MVO	-	-	-	-
2	Deterministic	0	0	0	0
3	Proposed RPO	1	0	-	-
4	Proposed RPO	2	0	-	-
5	Proposed RPO	3	0	-	-
6	Deterministic IRPO	0	0.5	0	0
7	Proposed IRPO	1	0.5	-	-
8	Proposed IRPO	2	0.5	-	-
9	[24]'s RPO	-	0	0.5	0.7
10	[24]'s IRPO	-	0.5	0.5	0.7
11	Worst case	$\geq 4$	1	1	1
12	Max diversified	0	0	0	0
13	RPO without options	1	-	-	-
14	RPO without options	2	-	-	-

**Table 2**  
Results for our proposed approach and benchmarks.

Model	Optimal solution
1	26 stocks: around 3% each
2	call3 BA: 1
3	call1 AXP: 0.0286, call1 V: 0.169, call3 BA: 0.149, call5 UNH: 0.134, put1 GS: 0.015 put3 IBM: 0.168, put3 MRK: 0.022, put3 WBA: 0.314
4	call1 AXP : 0.107, call1 V : 0.0272, call2 HD: 0.0483 call3 BA: 0.078, call3 PG : 0.024, call5 UNH: 0.139 put2 CAT : 0.024, put2 MMM: 0.116, put3 IBM: 0.062 put3 MRK: 0.0614, put3 WBA: 0.192, put4 TUX : 0.117
5	call1 AXP: 0.0841, call2 CVX : 0.051, call2 HD: 0.0597, put2 CAT: 0.023, call3 BA: 0.053, call3 MSFT: 0.05, call3 PG : 0.019 call5 UNH : 0.098, put2 MMM: 0.0358 put3 IBM: 0.03, put3 KO: 0.0353, put3 MRK: 0.0458 put3 WBA: 0.145, put4 TUX: 0.0914, put4 XOM: 0.176, put5 WMT: 0.0004
6	put3 WBA: 0.2419, call 3 WBA: 0.4348, call3 AXP: 0.333
7	call5 UNH: 0.027, call1 V: 0.0594, call2 CAT: 0.0158, call3 MSFT: 0.0649, call3 JPM: 0.0315, call1 VZ: 0.013, call2 HD: 0.046, put3 AXP: 0.06, put3 WBA: 0.29, call1 AXP: 0.0177
8	call3 WBA: 0.637, put3 WBA : 0.363
9	call1 V: 0.13, call3 MSFT: 0.083, call3 BA: 0.21, call5 UNH: 0.103, put2 MMM: 0.11, put3 IBM: 0.079, put3 WBA: 0.165
10	call5 UNH: 0.156, call1 V: 0.1552, call1 AXP: 0.058, call3 WBA: 0.072, call3 BA: 0.334, put3 IBM: 0.08, put3 WBA: 0.1431
11	call3 WBA: 0.637, put3 WBA : 0.363
12	call in all stocks: 0.033 each
13	UNH: 0.0926, V: 0.125, AXP: 0.029, WMT: 0.109, MSFT: 0.095 JPM: 0.0962, HD: 0.105, AAPL: 0.08, BA: 0.078, NKE: 0.072, CSCO: 0.07
14	UNH: 0.056, PFE: 0.049, V: 0.084, CAT: 0.0158, JNJ: 0.08, AXP: 0.0318, WMT: 0.068, MCD: 0.08, MSFT: 0.063, JPM: 0.054, VZ: 0.051, HD: 0.053, APPL: 0.06, BA: 0.062, NKE: 0.062, COSCO: 0.038

$$\begin{aligned}
 \max z \\
 s.t. \quad z \leq \sum_{i=1}^n \left[ R_i S_i(0) x_i + \sum_{j=1}^{n_i^+} (R_i S_i(0) - K_{ij}^+) x_{ij}^+ + \sum_{k=1}^{n_i^-} (K_{ik}^- - R_i S_i(0)) x_{ik}^- \right], \forall R \in \mathcal{R} \\
 \theta z \leq \sum_{i=1}^n \left[ R_i S_i(0) x_i + \sum_{j=1}^{n_i^+} (R_i S_i(0) - K_{ij}^+) x_{ij}^+ + \sum_{k=1}^{n_i^-} (K_{ik}^- - R_i S_i(0)) x_{ik}^- \right], \forall R \geq 0 \\
 \sum_{i=1}^n \left[ S_i(0) x_i + \sum_{j=1}^{n_i^+} p_{ij}^+ x_{ij}^+ + \sum_{k=1}^{n_i^-} p_{ik}^- x_{ik}^- \right] \leq B \\
 x_i, x_{ij}^+, x_{ik}^- \geq 0 \quad \forall i, j, k.
 \end{aligned} \tag{15}$$

Once we have redefined the problem as above and made the change in defining the decision variables, it is easy to reformulate Problem (15) in a tractable manner using the techniques outlined above, leading to Theorem 6.

**Theorem 6. (Tractable reformulation)** Problem (15) can be

reformulated as a linear problem:

$$\begin{aligned}
 \max z \\
 s.t. \quad z \leq \sum_{i=1}^n \left[ \bar{R}_i S_i(0) x_i + \sum_{k=1}^{n_i^-} x_{ik}^- (K_{ik}^- - \bar{R}_i S_i(0)) + \sum_{j=1}^{n_i^+} x_{ij}^+ (\bar{R}_i S_i(0) - K_{ij}^+) \right] \\
 - \left( p\Gamma + \sum_{l=1}^L q_l \right) \theta z \\
 \leq \sum_{i=1}^n \left( - \sum_{j=1}^{n_i^+} x_{ij}^+ K_{ij}^+ + \sum_{k=1}^{n_i^-} x_{ik}^- K_{ik}^- \right) x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ - \sum_{k=1}^{n_i^-} x_{ik}^- = 0 \quad \forall i, p + q_l \\
 \geq \sum_{i=1}^n \hat{R}_{il} S_i(0) \left( x_i + \sum_{j=1}^{n_i^+} x_{ij}^+ - \sum_{k=1}^{n_i^-} x_{ik}^- \right) \quad \forall l, p + q_l \\
 \geq \sum_{i=1}^n \hat{R}_{il} S_i(0) \left( -x_i - \sum_{j=1}^{n_i^+} x_{ij}^+ + \sum_{k=1}^{n_i^-} x_{ik}^- \right) \quad \forall l, \sum_{i=1}^n \left[ S_i(0) x_i + \sum_{j=1}^{n_i^+} p_{ij}^+ x_{ij}^+ \right. \\
 \left. + \sum_{k=1}^{n_i^-} p_{ik}^- x_{ik}^- \right] \\
 \leq Bp, q_l, x_i, x_{ik}^-, x_{ij}^+ \geq 0, \quad \forall i, j, k, l,
 \end{aligned} \tag{16}$$

The proof is similar to the proof of Theorem 4 and therefore is left to the reader.

### 3. Numerical experiments

In this section we investigate the practical performance of the proposed approach, compared to the following benchmarks.

1. MVO: Classical Markowitz portfolio optimization without options,
2. [24]'s RPO, IRPO: robust portfolio optimization (weak guarantee) and insured robust portfolio optimization (combined weak and strong guarantees) as described in Zymler et al. [24]
3. Diversified max: Invest in the highest expected return instrument of

all underlying assets in the same proportion  $1/n$ :  $\max_i \left( \bar{R}_i,$

$$\max_j \frac{\bar{R}_j S_j(0) - K_{jj}^+}{p_j^+}, \max_k \frac{K_{kk}^- - \bar{R}_k S_k(0)}{p_{kk}^-} \right),$$

which means that for each asset  $i$ , among its all available call options, all put options, and the stock itself, we pick the instrument with the highest return.

4. The worst-case approach (maxmin problem where  $R \geq 0$ ),
5. RPOW: the robust optimization approach without options (for the same  $\Gamma$  as the approach with options).

Further, Table 1 shows the different parameters for the two types of uncertainty sets we consider to describe the random returns: ellipsoidal and polyhedral uncertainty sets.  $p, q$  refer to the parametrization of the ellipsoidal uncertainty set and  $\Gamma$  to the parametrization of the polyhedral uncertainty set.

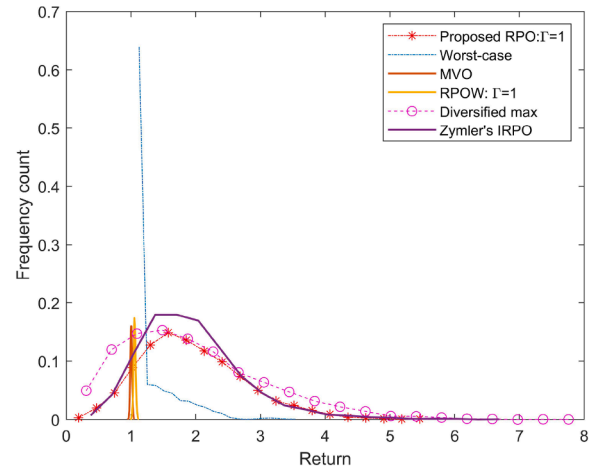
We consider the following metrics to compare the portfolio return  $R$  obtained from the various models presented in Table 1:

1. Sharpe Ratio with a riskless rate of 0%:  $\frac{\text{mean}(R) - 1}{\text{std}(R)}$
2.  $\text{VaR}_\alpha = \min\{r | \text{Pr}(R \leq r) \geq \alpha\}, \alpha = 0.1$
3. ES (Expected Shortfall, a.k.a CVaR)  $= E[R | R \geq \text{VaR}_\alpha]$
4. CE (Certainty Equivalent)  $= U^{-1}(E(U(R)))$  where  $U(a) = \log(a + 1), U^{-1}(a) = e^a - 1$

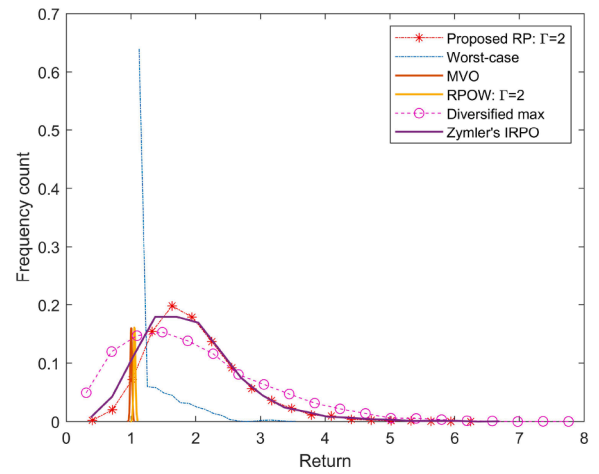
These metrics are explained in further detail in Zhao and Palomar [29].

**Table 3**  
Allocation table for Data Set 1.

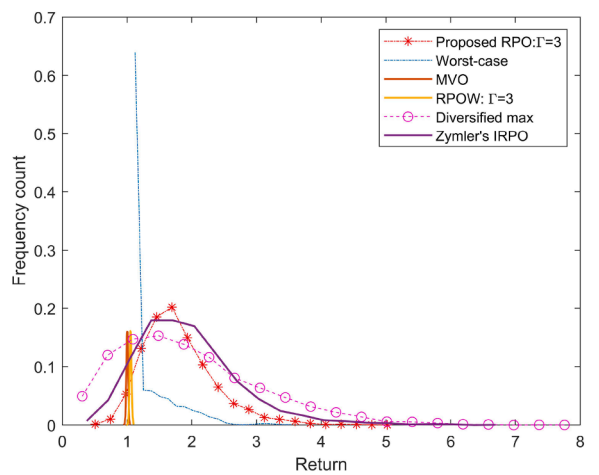
Model	UNH	PFE	MRK	V	CAT	JNJ	AXP	WMT	MMM	MSFT	IBM	XOM	JPM	PG	GS	TUX	VZ	HD	WBA	CVX	AAPL	BA	NKE	CSCO
1	s	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	c5	-	s	s	-	-	-	-	-	s	s	-	s	-	s	-	s	-	s	-	-	c3	-	-
3	c5	-	p3	c1	-	-	c1	-	-	-	p3	-	-	-	p1	-	-	-	p1	-	-	-	s	-
4	c5	-	p3	c2	p2	-	c1	-	p2	-	p3	-	-	-	p4	-	-	c2	p3	-	-	c3	-	-
5	c5	-	p3	-	p2	-	c1	-	p2	c3	p3	p4	-	c3	-	-	-	c2	p3	c2	-	-	c3	-
6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	p3, c3	-	-	-	-	-
7	c5	-	-	c1	c2	-	c1, p3	-	-	c3	-	-	c3	-	-	p4	c1	c2	p3	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-	p3	-	-	-	-	-	-	-	p3, c3	-	-	-	-	-
9	c5	-	-	c1	-	-	-	-	p2	c3	-	-	-	-	-	-	-	c2	p3, c3	-	-	-	-	-
10	c5	-	-	c1	-	-	c1	-	-	-	p4	-	-	-	-	-	-	-	p3, c3	-	-	c3	-	-
11	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	p3, c3	-	-	-	-	-
12	c5	-	c3	c1	c2	-	c3	c2	c3	c2	c5	c4	c3	c3	c2	c2	c1	c2	c3	c2	-	-	c2	c2
13	s	-	-	s	-	-	s	s	-	s	-	-	s	-	-	-	-	s	-	-	-	-	-	s
14	s	-	-	s	-	-	s	s	s	s	-	-	s	-	-	-	-	s	-	-	-	-	-	s



(a)  $\Gamma = 1$



(b)  $\Gamma = 2$



(c)  $\Gamma = 3$

**Fig. 1.** The comparison of return distribution of methods with different uncertainty budget parameters on the portfolio return ( $0 \leq \Gamma \leq 4$ ) for the first data set.

**Table 4**

Performance evaluation for Data Set 1 (bolded: all models with Sharpe ratio higher than 1, those underlined are our proposed models, not underlined are [24]'s).

Model	mean ( $\bar{R} - 1$ )	std	Sharpe ratio	$VaR(\alpha = 0.1)$	$CVaR_\alpha$	$U^{-1}(E(U(R)))$
1	0.003	0.014	0.246	0.913	1.003	1.003
2	1.474	2.484	0.594	0.903	3.617	1.719
3	0.952	0.816	<b><u>1.166</u></b>	0.901	2.053	1.843
4	0.942	0.713	<b><u>1.322</u></b>	0.901	2.027	1.910
5	0.801	0.563	<b><u>1.422</u></b>	0.909	1.818	1.749
6	1.115	2.139	0.521	0.915	2.661	1.734
7	0.632	0.488	<b><u>1.296</u></b>	0.904	1.649	1.590
8	0.297	0.393	0.755	0.911	1.297	1.269
9	0.981	0.891	<b>1.101</b>	0.910	2.019	1.945
10	0.942	0.810	<b>1.162</b>	0.911	2.016	1.841
11	0.297	0.393	0.755	0.911	1.297	1.269
12	0.955	1.117	0.855	0.906	2.234	1.761
13	0.006	0.017	0.370	0.947	1.007	1.006
14	0.005	0.014	0.361	0.940	1.005	1.005

3.1. Model evaluation using Data Set 1

In this section, we consider two years of bi-weekly stock price data from Yahoo! Finance for 30 different components of the Dow Jones Index downloaded on September 13, 2018. Stock prices and expected returns are shown in Table 13. For each asset we include 5 calls and 5 puts. The strike prices are provided in Appendix Table 12. Because Yahoo! Finance does not provide the prices of European options, the options' prices are computed using the Black-Scholes formula [3]. The  $\hat{R}_{it}$  parameters are obtained from the Cholesky decomposition of the correlation matrix of the returns. We do note that the assumptions for Black-Scholes option pricing may not be satisfied and that it may underestimate extreme market moves and therefore option prices. An interesting future research direction would be to develop an option pricing theory that would be aligned with robust optimization and would help hedge against extreme market moves, which is outside the scope of this paper.

Two years of bi-weekly stock prices are used as historical data to simulate 50,000 return scenarios using the Gaussian Copula distribution as the joint distribution of the correlated stocks' returns and assume an initial budget of \$1. The simulated return values which are the uncertain parameters in our model have the same mean and covariance as the historical data we used.

3.1.1. Portfolio allocation

Table 2 below shows the optimal solutions for all benchmarks using Data Set 1. From a portfolio allocation perspective, varying the budget parameter  $\Gamma$  leads to 5 different solutions(models 2,3,4,5,11) and the most diversified solution corresponds to model 5( $\Gamma = 3$ ), when 7 calls and 9 puts are invested in, involving 16 different underlying assets. We also observe that some stocks only appear in a single model at optimality, through a call option (JPM, PG, VZ, CVX) or a put (XOM, GS), some stocks appear in several models and always through the same option (UNH, MRK, MMM, MSFT, TUX, HD, BA), some stocks appear in several models through two options (V, CAT, AXP, IBM), and one stock (WBA) appears in several models through three options. In the case where a stock appears through two or three options, one of the options appears only once. The spectrum of solutions is much more manageable to analyze than a naive approach where the investor has all possible puts, calls and stocks available for investment, as the overwhelming and paralyzing effect of too much choice has been documented in Iyengar and Lepper [44]. Hence, these numerical experiments suggest that the use of (European) options in conjunction with a modeling of the uncertainty using uncertainty sets present significant benefits for portfolio managers by limiting downside risk while allowing upside risk potential and also by keeping the amount of diversification easier to grasp by

**Table 5**

Optimal solution for Data Set 2.

Model	Result
1	NDXP: 0.118, SPX: 0.402, RUT: 0.201, AAPL: 0.152
2	call 4 of NDXP
3	put15 of RUT:0.434, call1 of NDXP:0.566
4	put15 of RUT:0.383, call1 of NDXP:0.613,call15 of AAPL:0.0042
5	RUT:0.97, put3 of RUT: 0.0273
6	stock3: 0.57,put15 of stock3: 0.0161,call4 of stock1: 0.41
7	stock3: 0.487, put15 of RUT:0.221, call1 of NDXP:0.2695
8	put15 of RUT:0.204, call1 of NDXP:0.306,put15 of AAPL:0.0021
9	put15 of RUT:0.4132, call1 of NDXP:0.5868
10	put15 of RUT:0.2169, call1 of NDXP:0.2886,stock3:0.4945
11	RUT:0.97, put3 of RUT: 0.03
12	call 4 of NDXP: 0.2, stock2,3,4,5: 0.2 each
13	NDXP: 0.145, SPX: 0.483, RUT: 0.22, AAPL: 0.1505
14	NDXP: 0.118, SPX: 0.402, RUT: 0.1257, AAPL: 0.152, IBM: 0.15

**Table 6**

Allocation table for Data Set 2.

Model	NDXP	SPX	RUT	AAPL	IBM
1	s	s	s	s	s
2	c4	-	-	-	-
3	c1	-	p15	-	-
4	c1	-	p15	c15	-
5	-	-	s, p3	-	-
6	c4	-	s, p15	p15	-
7	c1	-	s	p15	-
8	c1	-	p15	p15	-
9	c1	-	s, p15	-	-
10	c1	-	s, p15	-	-
11	-	-	s, p3	-	-
12	c4	s	s	s	s
13	s	s	s	s	-
14	s	s	s	s	s

managers.

For greater clarity, we also present in Table 3 a summary table showing which calls and puts are invested in (positive allocation) for each model and each underlying asset, without providing the exact allocation amounts. This table shows not only the diversification reduction in our proposed RPO but also shows this diversification reduction in Zymler et al. [24]'s IRPO. This suggests that diversification reduction may follow from applying robust optimization along with options, independently of the specific type of the uncertainty set. Some stocks, such as PFD, JNJ, WMT, AAPL, NKE, and CSCO, are only chosen in the robust optimization models without options.

3.1.2. Portfolio performance

Fig. 1 shows that the robust approach with different values of the uncertainty budget  $\Gamma$  outperforms other methods in the sense that it allows for far more upside risk with only a small increase in downside risk(the right tail of our models are lower than the benchmarks). In contrast, the diversified max approach has comparable upside risk but higher downside risk. Also, in Fig. 1, we observe that the optimal portfolio in the worst-case approach exhibits little volatility, as it is represented by a mostly vertical line on the graph. This is because when the portfolio consists of a call and a put and both are in the money with slight differences in options' prices and allocations, then the portfolio value is (dropping the index  $i$  for convenience as we know that in this case it is optimal to invest only in one asset):

$$R(S(0) \left( \frac{x^+}{P^+} - \frac{x^-}{P^-} \right) + \frac{x^- K^-}{P^-} - \frac{x^+ K^+}{P^+}$$

where  $\left( \frac{x^+}{P^+} - \frac{x^-}{P^-} \right)$  is close to zero. Hence, the portfolio value depends little on  $R$  in this case.

As  $\Gamma$  increases, the right tail of the portfolio value diminishes as the

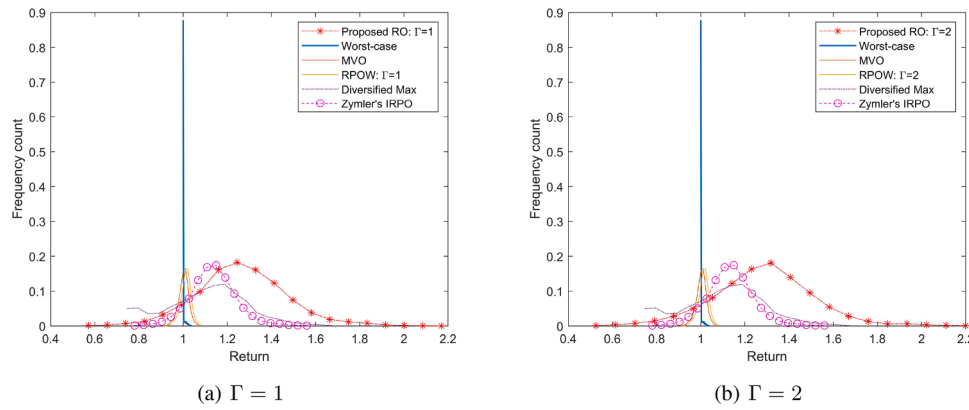


Fig. 2. The comparison of different methods with the robust optimization approach with different uncertainty budget parameter on the portfolio return ( $0 \leq \Gamma \leq 4$ ).

Table 7

Performance evaluation for Data Set 2 (bolded: all models with Sharpe ratio higher than 1, those underlined are our proposed models, not underlined are [24]’s).

Model	mean ( $\bar{R} - 1$ )	std	Sharpe ratio	VaR( $\alpha = 0.1$ )	CVaR $_{\alpha}$	$U^{-1}(E(U(R)))$
1	0.002	0.022	0.098	0.908	1.002	1.002
2	0.510	0.831	0.614	0.902	1.861	1.350
3	0.260	0.202	<b>1.284</b>	0.917	1.284	1.256
4	0.280	0.226	<b>1.240</b>	0.914	1.313	1.275
5	0.001	0.004	0.246	0.900	1.001	1.001
6	0.209	0.340	0.613	0.905	1.335	1.350
7	0.127	0.098	<b>1.287</b>	0.914	1.131	1.256
8	0.144	0.115	<b>1.252</b>	0.900	1.150	1.275
9	0.271	0.212	<b>1.276</b>	0.912	1.121	1.115
10	0.134	0.105	<b>1.272</b>	0.909	1.141	1.133
11	0.001	0.004	0.246	0.900	1.001	1.001
12	0.104	0.185	0.560	0.906	1.001	1.096
13	0.003	0.022	0.125	0.903	1.003	1.003
14	0.002	0.022	0.098	0.920	1.002	1.002

manager becomes more conservative. In this example,  $\Gamma = 1$  or 2 achieves good performance by having an upside tail similar to that in the diversified max case and much lower downside risk. Hence, we believe the methodology holds much promise for investors.

The various performance metrics are computed for all models in Table 4. Based on the results in Table 4, we observe that our robust models 4, 5 and 7 have the highest Sharpe ratio, i.e., we achieve a better risk-to-reward tradeoff. Also, these models have large ES and CE values, although a slightly larger potential loss in VaR in comparison with other benchmarks.

3.2. Model evaluation using data set 2

In this section we evaluate our model using real data including stock and option data from the Bloomberg Terminal. The trading date is Jan. 3, 2020 and we choose 5 equities as the underlying stocks: NASDAQ 100 index (NDX), S&P 500 index (SPX), Russell 2000 index (RUT), Apple Inc (AAPL), and International Business Machines Corporation (IBM). We also choose 150 options whose expiration date is 35 days after the trading date.

3.2.1. Portfolio allocation

Table 5 shows the optimal solution for different models.

In Table 6, we observe fundamental differences in the optimal allocations obtained by the robust optimization with and without options. The robust optimization models without options (Model 13 with  $\Gamma = 1$  and Model 14 with  $\Gamma = 2$ ) lead to diversified solutions with 4 and 5 stocks, respectively. The robust optimization models with options

(Models 3 to 11) invest in at most 3 assets, whether through stocks, puts or calls. Among those 9 models, 6 invest directly in a stock and that is always RUT. The remaining models only invest in calls and puts: always c1 of NDXP and p15 of RUT and, for one model, p15 of AAPL. This suggests that, when the decision maker seeks to protect himself from the worst-case return, well-chosen options make it less necessary to diversify the portfolio for protection against risk.

We also observe that in Data Set 1 where option prices were computed using Black–Scholes option pricing, we never invested in the underlying stock, but do so in Data Set 2 where we use real option prices from the Bloomberg terminal. This suggests that options were attractively priced and perhaps underpriced in Data Set 1.

3.2.2. Portfolio evaluation

Fig. 2 shows that the proposed robust approach has the highest upside risk and lowest downside risk in comparison with other approaches since the cumulative distribution of the returns in our approach is more pushed to the right (towards higher returns) than others. Table 7 also documents this fact.

In Table 7, we observe that Models 3 and 7 outperform the benchmark portfolios; specifically, they exhibit the highest Sharpe ratio.

3.3. Additional experiments

Below we perform additional experiments. For ease of comparison we present the results for Data Sets 1 and 2 next to each other.

3.3.1. Connection with utility theory

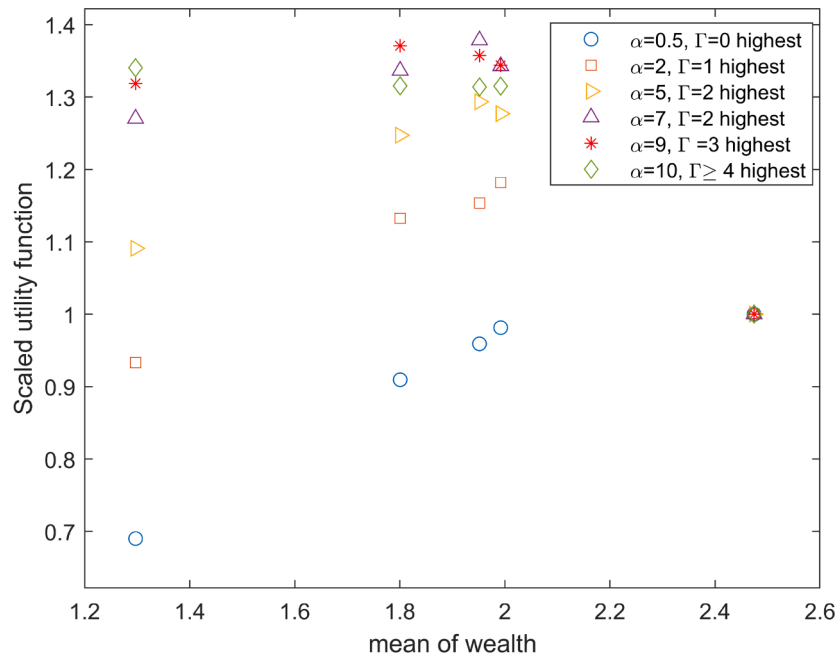
Because  $\Gamma$  is often interpreted in the robust optimization literature as a measure of the investor’s risk aversion, it is natural to investigate how the proposed framework fits with the classical utility theory of [45]. Specifically, we study which uncertainty budget  $\Gamma$  leads to the robust solution with the highest expected utility on simulated returns, when we vary the parameters defining the utility function. We consider two types of utility functions:

Constant Absolute Risk Aversion:  $U_1(w) = 1 - \exp(1 - \alpha w)$

Hyperbolic Absolute Risk Aversion:  $U_2(w) = \frac{1 - \alpha}{\alpha} \left( \frac{\alpha w}{1 - \alpha} + b \right)^\alpha$

(The coefficient of absolute risk aversion of a utility function  $U$  is defined as  $-U''(w)/U'(w)$ .)

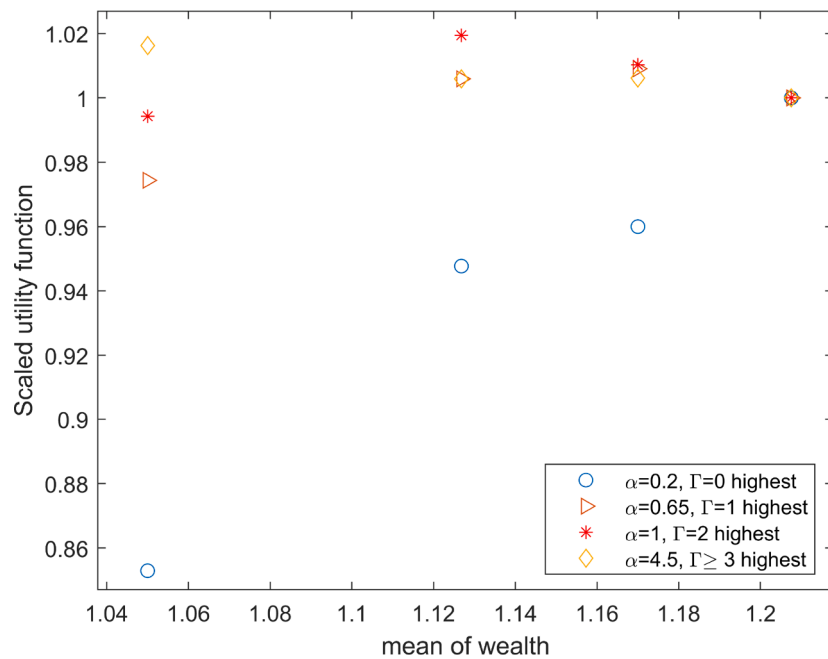
Fig. 3 presents the models with budgets  $\Gamma = 0, 1, 2, 3, 4$ . (a) and (b) are for Data Set 1 and (c) and (d) are for Data Set 2. (b) and (d) show the optimal  $\Gamma$  for various parameter ranges while (a) and (c) provide a more comprehensive picture of how the allocations obtained with various budgets of uncertainty compare to each other. Each symbol (circle, triangle, etc) is shown five times on Subfigures (a) and (c), once for each



(a) Utility values of different uncertainty budgets



(b) Data Set 1

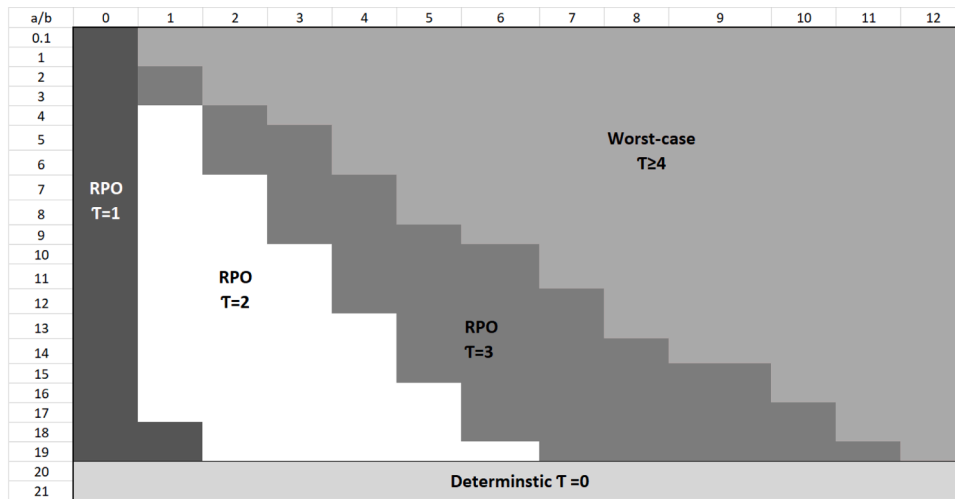


(c) Utility values of different uncertainty budgets

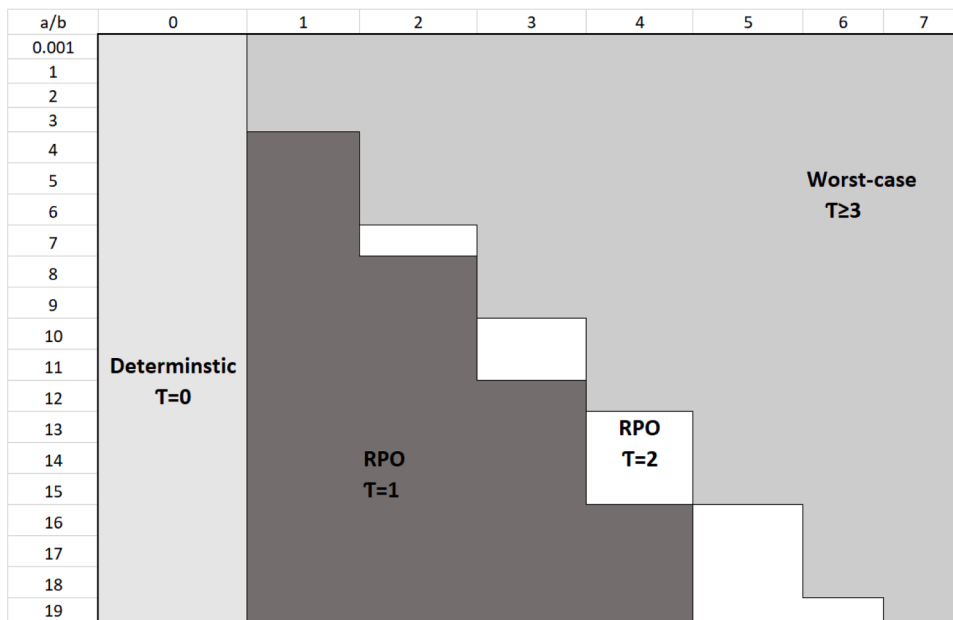


(d) Data Set 2

Fig. 3. Results for constant absolute risk aversion:  $U_1(w) = 1 - \exp(1 - \alpha w)$ .



(a) Data Set 1



(b) Data Set 2

Fig. 4. Results for hyperbolic absolute risk aversion:  $U_2(w) = \frac{1-\alpha}{\alpha} \left( \frac{aw}{1-\alpha} + b \right)^\alpha$  where  $\alpha = 0.1$ .

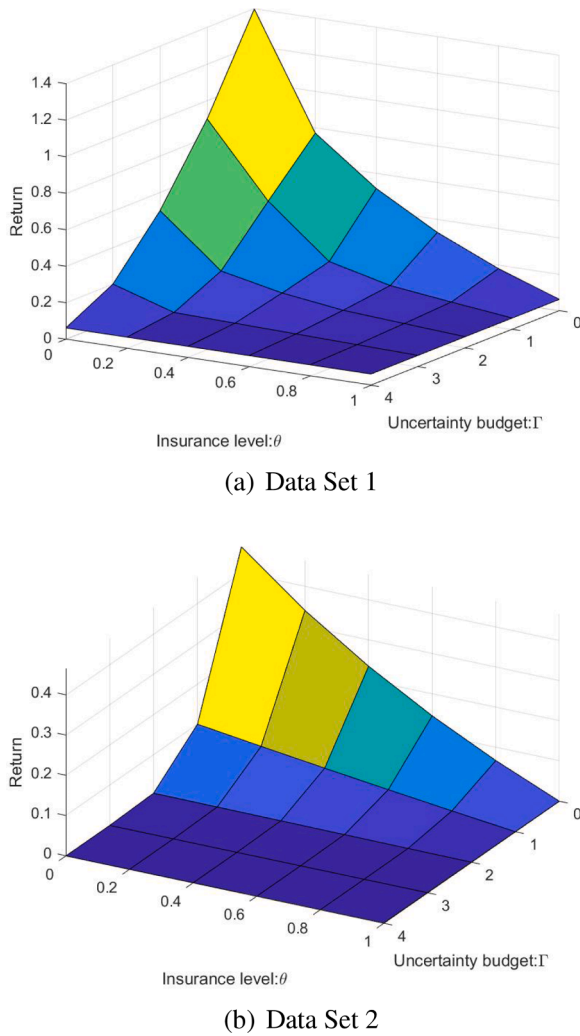


Fig. 5. Tradeoff of weak and strong guarantee.

value of  $\Gamma$ . The value of  $\Gamma$  decreases as the mean of wealth increases, i.e., the leftmost symbol is for  $\Gamma = 4$  and the rightmost one is  $\Gamma = 0$ . Different  $\alpha$  values lead to different models with the highest expected utility. As an example, for the blue circles corresponding to the utility function with  $\alpha = 0.5$ , the model with  $\Gamma = 0$  (the rightmost point) has the highest expected utility. For the green diamonds corresponding to the utility function with  $\alpha = 10$ , the model with  $\Gamma \geq 4$  (leftmost point) has the highest utility value. As the coefficient of Constant Absolute Risk Aversion  $\alpha$  increases, the value of  $\Gamma$  achieving the highest expected utility increases.

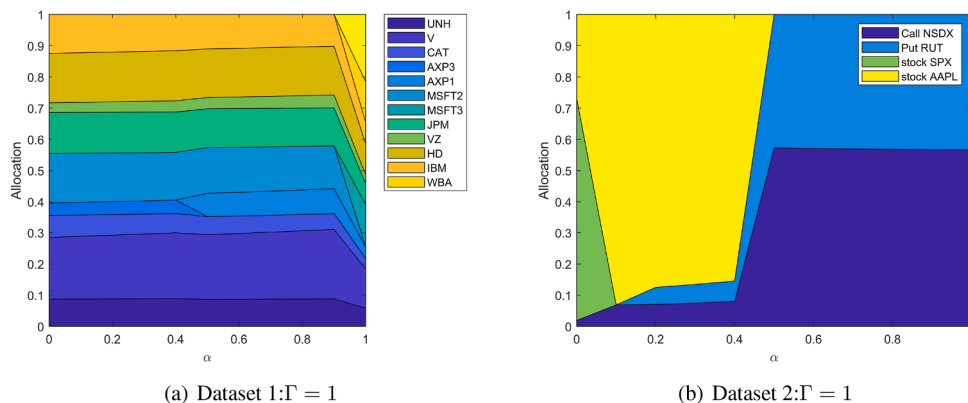


Fig. 6. Diversification reduction when we change the  $\rho$  parameter.

From Fig. 4, we see that lower values of  $a$  and higher values of  $b$  make the optimal value of  $\Gamma$  increases.

3.3.2. Effect of insurance constraints

Fig. 5 depicts the effects of the insurance constraint on the optimal objective value (worst-case return over the uncertainty set) as a function of  $\Gamma \in [0, 4]$  and  $\theta \in [0, 1]$ . For any fixed  $\Gamma$ , the optimal worst-case return monotonically decreases with  $\theta$ . When the uncertainty set is small (small  $\Gamma$  value), the optimal worst-case return over the uncertainty set is relatively high. Therefore, the inclusion of the insurance guarantee has a significant impact due to the high insurance costs that are introduced. When the budget of uncertainty  $\Gamma$  increases, which increases the size of the uncertainty set, the value of the optimal worst-case return over the uncertainty set drops when  $\theta$  remains constant, and  $\theta$  needs to decrease for the optimal objective value to remain constant.

3.3.3. Effect of stock-factor dependency

While studying the impact of correlation between pairs of assets would be cumbersome due to the number of pairs involved, we study the dependency between stock and factor by changing the value of the off-diagonal elements in  $\hat{R}$  as follows:  $\hat{R}_{new} = (1 - \rho) * diag(\hat{R}) + \rho * \hat{R}$  where  $\rho \in [0, 1]$ . (In our model, there are as many factors as there are stocks.) When  $\rho = 0$  the  $\hat{R}$  matrix is diagonal and the asset returns are independent, with each factor affecting only one stock return. When  $\rho = 1$ , we have the original  $\hat{R}$  matrix. In both Data Sets 1 and 2, we observe that, when the group of options invested in (non-zero allocation) stays the same, the allocation exhibits little change (Fig. 6.) The main change comes from having new calls or puts options or stocks enter the portfolio, which leads to significant change in the allocation (Tables 8 and 9).

3.4. Extension to multiple periods

Tables 10 and 11 show the optimal solutions of the multi-period model where the uncertainty budget function is chosen to be (1,2,3,4) (e.g.,  $\Gamma_3 = 3$ ). For both data sets, the portfolio is the most diversified at time 1 as the allocation for Data Set 1 uses 9 underlying assets and the allocation for Data Set 2 uses 3. In subsequent time periods, the allocation for Data Set 1 uses only 1 underlying asset and the allocation for Data Set 2 uses 2. Recall that the decision variables are monetary values and not fractions for  $t \geq 2$ .

Multi-period (open-loop) robust portfolio optimization with options therefore seems a promising area of research that we hope to combine with other types of options, such as American or Asian options, in future work. Of particular interest would be to study the impact of various families of uncertainty budgets on the optimal allocation. Further, because of the increasing difficulty in estimating parameters such as nominal stock returns and the stock-factor dependence matrix, it would



**Table 8**  
Impact on allocation of changing  $\rho$  for Data Set 1.

$\rho$ \stock	call UNH	call V	call CAT	call3 AXP	call1 AXP	call2 MSFT	call3 MSFT	call JPM	call VZ	call HD	call BA	put IBM	put WBA
0	0.087	0.198	0.071	0.040	0	0.158	0	0.132	0.031	0.158	0.125	0	0
0.1	0.088	0.201	0.069	0.041	0	0.157	0	0.131	0.032	0.159	0.123	0	0
0.2	0.088	0.204	0.067	0.042	0	0.156	0	0.131	0.033	0.159	0.121	0	0
0.3	0.089	0.207	0.065	0.042	0	0.154	0	0.130	0.035	0.160	0.118	0	0
0.4	0.089	0.211	0.063	0.043	0	0.153	0	0.129	0.036	0.160	0.116	0	0
0.5	0.087	0.207	0.059	0	0.075	0.146	0	0.124	0.036	0.156	0.110	0	0
0.6	0.087	0.211	0.057	0	0.076	0.144	0	0.124	0.037	0.156	0.108	0	0
0.7	0.087	0.214	0.055	0	0.077	0.142	0	0.123	0.038	0.156	0.106	0	0
0.8	0.088	0.218	0.053	0	0.079	0.139	0	0.123	0.040	0.156	0.104	0	0
0.9	0.088	0.223	0.052	0	0.080	0.136	0	0.123	0.041	0.156	0.102	0	0
1	0.058	0.125	0.033	0	0.037	0.000	0.137	0.067	0.028	0.099	0.069	0.130	0.216

**Table 9**  
Impact on allocation of changing  $\rho$  for Data Set 2.

$\rho$ \stock	call NDXP	put RUT	stock SPX	stock APPL
0	0.019	0	0.713	0.268
0.1	0.069	0	0	0.931
0.2	0.070	0.055	0	0.875
0.3	0.075	0.060	0	0.866
0.4	0.080	0.065	0	0.855
0.5	0.572	0.428	0	0
0.6	0.571	0.429	0	0
0.7	0.5694	0.431	0	0
0.8	0.568	0.432	0	0
0.9	0.567	0.433	0	0
1	0.566	0.434	0	0

**Table 10**  
Optimal allocation for Data Set 1.

$t$	Solution
1	call1 V: 0.125, call2 CAT: 0.033, call1 JNJ: 0.037, call3 MSFT, call3 JPM:0.067, call1 VZ: 0,028, call3 BA: 0.069, put3 IBM: 0.13, put3 WBA: 0.216
2	put3 WBA: 0.428, call3 WBA: 0.751
3	put3 WBA: 0.451, call3 WBA: 0.791
4	put3 WBA: 0.482, call3 WBA: 0.840

**Table 11**  
Optimal allocation for Data Set 2.

$t$	Solution
1	c1 NDXP: 0.565, p15 RUT: 0.434, p15 AAPL: 0.001
2	c1 NDXP: 0.091, p15 RUT: 0.093, p15 AAPL: 0.001, s RUT: 0.880
3	c1 NDXP: 0.092, p15 RUT: 0.093, p15 AAPL: 0.001, s RUT: 0.906
4	c1 NDXP: 0.11, p15 RUT: 0.094, p15 AAPL: 0.001, s RUT: 0.920

be valuable to investigate uncertainty budget functions that are convex with time for this application. This is, however, beyond the scope of the present paper.

3.5. Summary of results

We summarize the insights gained in the numerical results and investment insights as follows. First, a key focus of our paper was to compare the robust portfolios obtained with polyhedral uncertainty sets with those obtained with ellipsoidal uncertainty sets in Zymler et al. [24]. We observe that both types of uncertainty sets lead to portfolios that perform very well in practice, as measured by the Sharpe ratio. Our models behave slightly better than the benchmarks on the two data sets considered. Further, in practice, the allocations for both types of uncertainty sets show similar diversification properties; in particular, we invest in at most one call and at most one put with the same underlying asset. However, we are only able to prove this property for polyhedral uncertainty sets.

The corresponding investment insight is that it is suboptimal to invest in more than one call or more than one put of the same underlying asset when uncertainty is modeled using a polyhedral uncertainty set and the investor maximizes his worst-case return over the uncertainty set, as in Bertsimas and Sim [5]. While the investor may find diversification across multiple calls and multiples puts on the same underlying asset reassuring, this is actually counterproductive because the options

are on the same asset. The investor would be better served selecting the “best” call and/or the “best” put as determined by the optimization problem as having the best combination of price and payoff. Another investment insight is that the use of options reduces the need for diversification across multiple assets, which we explain intuitively by the fact that exercising the option in essence improves the price of the stock and so improves its return enough that the investor does not need to hedge his bets as much by investing in multiple assets.

4. Conclusions

In this work, we have explored the impact of options to mitigate risk in static robust portfolio allocation, when the uncertain returns are described using polyhedral uncertainty sets parametrized by a budget of uncertainty. We have derived tractable reformulations for settings involving strong guarantees, weak guarantees and a combination of both, and explored theoretical properties of those models. In numerical experiments, the optimal portfolio with options exhibits less diversification but superior performance to the robust portfolio without options. We have also compared the performance of our proposed portfolios with those presented in Zymler et al. [24] for ellipsoidal sets, and identified values of the budget of uncertainty for which our models outperform the models obtained with ellipsoidal sets, as measured by the Sharpe ratio. We have also evaluated how the optimal choice of the budget of uncertainty varies with parameters defining the investor’s utility functions, considered the trade-off between weak and strong guarantees and

**Table 12**  
Strike prices of options.

Stock (i)	Strike price of call option j ( $K_j^+$ )					Strike price of put option k ( $K_k^-$ )				
	call1	call2	call3	call4	call5	put1	put2	put3	put4	put5
AAPL	95	155	187.5	75	85	230	227.5	232.5	235	285
AXP	107	105	108	100	95	109	108	110	111	112
BA	350	160	352.5	220	150	355	357.5	362.5	360	367.5
CAT	145	144	140	141	130	147	146	148	175	149
CSCO	47	45	40.5	46	46.5	47	45.5	40	46	44.5
CVX	116	114	115	70	55	130	117	120	119	118
DIS	110	111	105	60	70	115	116	120	114	111
DWDP	67.5	68.5	66.5	69	60	74	70	71	72	72.5
GS	195	210	145	185	260	230	212.5	242.5	245	235
HD	155	207.5	200	205	140	210	212.5	225	217.5	220
IBM	148	147	145	90	146	150	180	149	175	170
INTC	45	44.5	44	43.5	42.5	47.5	45.5	48	47	46
JNJ	139	132	135	105	100	146	140	150	155	160
JPM	113	111	112	110	105	115	114	116	160	117
KO	40	43	45	44	41	60	47	46.5	49	48.5
MCD	120	160	85	135	125	200	195	190	162.5	165
MMM	207.5	140	205	220	145	217.5	210	230	240	220
MRK	65	68	67.5	68.5	62.5	72.5	71.5	70.5	70	71
MSFT	110	112	111	105	106	114	113	115	118	125
NKE	83	82	82.5	81	80	83.5	86	84.5	84	87
PFE	42.5	41	41.5	42	40	43	44	43.5	47	35
PG	80	77.5	83	62.5	81	95	83.5	84	100	105
TRV	120	125	100	110	115	135	140	130	125	115
TUX	135	133	130	95	115	160	137	138	136	140
UNH	100	195	210	260	262.5	280	267.5	265	270	275
V	146	147	140	145	144	148	149	155	150	150.5
VZ	54.5	49	50	52.5	53.5	55	55.5	56	56.5	61
WBA	70	67	68.5	64.5	65.5	70.5	71	71.5	90	72
WMT	95	100	92	92.5	90	140	130	96	95.5	97
XOM	80	82	81	81.5	80.5	83	83.5	82.5	84	90

**Table 13**  
Stocks' prices and expected return.

Stock (i)	Stock's price ( $S_i$ )	mean of return ( $\bar{R}_i$ )
AAPL	225.84	1.007093
AXP	108.765	1.012155
BA	356.0768	1.01088
CAT	145	1.006576
CSCO	47.16	1.005035
CVX	115.48	1.002632
DIS	110.14	1.002187
DWDP	69.28	1.003539
GS	227.83	1.003822
HD	209.75	1.00572
IBM	147.90	1.000628
INTC	45.22	1.002778
JNJ	139.42	1.002311
JPM	112.88	1.006081
KO	45.96	1.001686
MCD	163.01	1.004198
MMM	206.87	1.002437
MRK	69.95	1.002023
MSFT	112.84	1.007093
NKE	83.32	1.004592
PFE	42.57	1.00317
PG	83.091	1.000251
TRV	128.29	1.001682
TUX	134.83	1.003496
UNH	264.23	1.006791
V	147.46	1.005917
VZ	54.64	1.001852
WBA	70.47	0.999437
WMT	95.46	1.003536
XOM	82.57	1.000859

investigated the stock-factor dependency as a proxy for portfolio-wide correlation. Finally, we have considered an extension to a multi-period robust setting. Our numerical experiments suggest that our robust portfolio performs very well in practice.

**CRedit authorship contribution statement**

**Hedieh Ashrafi:** Data curation, Investigation, Methodology, Formal analysis, Software, Visualization, Writing - original draft. **Aurélie C. Thiele:** Conceptualization, Project administration, Supervision, Validation, Writing - review & editing.

**Declaration of Competing Interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgments**

The authors would like to thank three anonymous reviewers and the Associate Editor, whose thoughtful comments have significantly improved the paper.

**Appendix A. Option parameters for Data Set 1**

**References**

- [1] Markowitz H. Portfolio selection. *J Finance* 1952;7(1):77–91.
- [2] Consigli G, Kuhn D, Brandimarte P. Optimal financial decision making under uncertainty. *Optimal financial decision making under uncertainty*. Springer; 2017. p. 255–90.
- [3] Hull JC. *Options, futures, and other derivatives*. 10th ed. Pearson; 2017.
- [4] Ben-Tal A, Nemirovski A. Robust solutions of uncertain linear programs. *Oper Res Lett* 1999;25(1):1–13.
- [5] Bertsimas D, Sim M. The price of robustness. *Oper Res* 2004;52(1):35–53.
- [6] Lim GJ, Kardar L, Ebrahimi S, Cao W. A risk-based modeling approach for radiation therapy treatment planning under tumor shrinkage uncertainty. *Eur J Oper Res* 2020;280(1):266–78.

- [7] Yuan W, Wang J, Qiu F, Chen C, Kang C, Zeng B. Robust optimization-based resilient distribution network planning against natural disasters. *IEEE Trans Smart Grid* 2016;7(6):2817–26.
- [8] Hu J, Li J, Mehrotra S. A data driven functionally robust approach for coordinating pricing and order quantity decisions with unknown demand function. *Oper Res* 2019.
- [9] Gabrel V, Murat C, Thiele A. Recent advances in robust optimization: an overview. *Eur J Oper Res* 2014;235(3):471–83.
- [10] Goldfarb D, Iyengar G. Robust portfolio selection problems. *Math Oper Res* 2003;28(1):1–38.
- [11] Tütüncü RH, Koenig M. Robust asset allocation. *Ann Oper Res* 2004;132(1-4):157–87.
- [12] Garlappi L, Uppal R, Wang T. Portfolio selection with parameter and model uncertainty: a multi-prior approach. *Rev Financ Stud* 2007;41–81.
- [13] Cornuejols G, Tütüncü R. *Optimization methods in finance*. Cambridge University Press; 2006.
- [14] Pachamanova DA. Handling parameter uncertainty in portfolio risk minimization. *J Portf Manag* 2006;32(4):70–8.
- [15] Kim JH, Kim WC, Fabozzi FJ. Recent advancements in robust optimization for investment management. *Ann Oper Res* 2017:1–16.
- [16] Fabozzi FJ, Kolm PN, Pachamanova DA, Focardi SM. *Robust portfolio optimization and management*. John Wiley & Sons; 2007.
- [17] Bertsimas D, Pachamanova D. Robust multiperiod portfolio management in the presence of transaction costs. *Comput Oper Res* 2008;35(1):3–17.
- [18] Zhu S, Fukushima M. Worst-case conditional value-at-risk with application to robust portfolio management. *Oper Res* 2009;57(5):1155–68.
- [19] Fabozzi FJ, Huang D, Zhou G. Robust portfolios: contributions from operations research and finance. *Ann Oper Res* 2010;176(1):191–220.
- [20] Kawas B, Thiele A. A log-robust optimization approach to portfolio management. *OR Spectrum* 2011;33(1):207–33.
- [21] Kim JH, Kim WC, Fabozzi FJ. Recent developments in robust portfolios with a worst-case approach. *J Optim Theory Appl* 2014;161(1):103–21.
- [22] Rujeerapaiboon N, Kuhn D, Wiesemann W. Robust growth-optimal portfolios. *Manag Sci* 2016;62(7):2090–109.
- [23] Lutgens F, Sturm J, Kolen A. Robust one-period option hedging. *Oper Res* 2006;54(6):1051–62.
- [24] Zymler S, Rustem B, Kuhn D. Robust portfolio optimization with derivative insurance guarantees. *Eur J Oper Res* 2011;210(2):410–24.
- [25] Zymler S, Kuhn D, Rustem B. Distributionally robust joint chance constraints with second-order moment information. *Math Program* 2013;137(1-2):167–98.
- [26] Zymler S, Rustem B, Kuhn D. Worst-case value at risk of nonlinear portfolios. *Eur J Oper Res* 2013;59(1):172–88.
- [27] Matmoura Y, Penev S. Multistage optimization of option portfolio using higher order coherent risk measures. *Eur J Oper Res* 2013;227(1):190–8.
- [28] Davari-Ardakani H, Aminnayeri M, Seifi A. Multistage portfolio optimization with stocks and options. *Int Trans Oper Res* 2016;23(3):593–622.
- [29] Zhao L, Palomar DP. A Markowitz portfolio approach to options trading. *IEEE Trans Signal Process* 2018;66(16):4223–38.
- [30] Chopra VK, Ziemba WT. The effect of errors in means, variances, and covariances on optimal portfolio choice. *Handbook of the fundamentals of financial decision making: part I*. World Scientific; 2013. p. 365–73.
- [31] Leung P-L, Ng H-Y, Wong W-K. An improved estimation to make Markowitz's portfolio optimization theory users friendly and estimation accurate with application on the us stock market investment. *Eur J Oper Res* 2012;222(1):85–95.
- [32] Bertsimas D, Iancu DA, Parrilo PA. Optimality of affine policies in multistage robust optimization. *Math Oper Res* 2010;35(2):363–94.
- [33] Bandi C, Bertsimas D. Robust option pricing. *Eur J Oper Res* 2014;239(3):842–53.
- [34] Frey R, Sin CA. Bounds on European option prices under stochastic volatility. *Math Finance* 1999;9(2):97–116.
- [35] DeMarzo PM, Kremer I, Mansour Y. Robust option pricing: Hannan and Blackwell meet Black and Scholes. *J Econ Theory* 2016;163:410–34.
- [36] Shen R, Zhang S. Robust portfolio selection based on a multi-stage scenario tree. *Eur J Oper Res* 2008;191(3):864–87.
- [37] Marzban S, Mahootchi M, Khamseh AA. Developing a multi-period robust optimization model considering american style options. *Ann Oper Res* 2015;233(1):305–20.
- [38] Ling A, Sun J, Wang M. Robust multi-period portfolio selection based on downside risk with asymmetrically distributed uncertainty set. *Eur J Oper Res* 2020;285(1):81–95.
- [39] Yin L, Han L. International assets allocation with risk management via multi-stage stochastic programming. *Comput Econ* 2020;55(2):383–405.
- [40] Cornuejols G, na Javier P, Tütüncü RH. *Optimization methods in finance*. Cambridge University Press; 2018.
- [41] Bertsimas D, Tsitsiklis J. *Introduction to linear optimization*. Athena Scientific; 1997.
- [42] Bertsimas D, Thiele A. A robust optimization approach to inventory theory. *Oper Res* 2006;54(1):150–68.
- [43] Longstaff FA. Are negative option prices possible? The callable us treasury-bond puzzle. *J Bus* 1992;571–92.
- [44] Iyengar SS, Lepper MR. When choice is demotivating: Can one desire too much of a good thing? *J Personal Soc Psychol* 2000;79(6):995–1006.
- [45] von Neumann J, Morgenstern O. *Theory of games and economic behavior*. Princeton University Press; 1944.