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Operations Research Perspectives

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Elsevier

Suggested Citation: Herbon, Avi (2020) : An approximated solution to the constrained integrated manufacturer-buyer supply problem, Operations Research Perspectives, ISSN 2214-7160, Elsevier, Amsterdam, Vol. 7, pp. 1-9, https://doi.org/10.1016/j.orp.2020.100140

This Version is available at:
https://hdl.handle.net/10419/246410

Operations Research Perspectives 7 (2020) 100140

Contents lists available at ScienceDirect

Operations Research Perspectives

journal homepage: www.elsevier.com/locate/orp

An approximated solution to the constrained integrated manufacturer-buyer supply problem

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ARTICLE INFO

Keywords:
Integrated manufacturer-buyer problem
bounded production cycle length
nested formulation

ABSTRACT

In recent decades, the integrated manufacturer-buyer supply chain problem has been widely studied by many scholars, as it provides benefits to both parties in terms of planning flexibility, information sharing and joint costs. The manufacturer produces a production lot of size Q, with variable production rate $P \leq U$, where U denotes the maximal production rate (e.g., due to technological restrictions or due to limited machinery and/or manpower capacities). The manufacturer delivers the lot to the buyer in n (integer) smaller shipments, each of size q. An upper bound on the production cycle length is assumed (e.g., to enable the scheduling of maintenance periods or idle windows of time in which workers are not required to work.) In order to solve the problem mathematically, we suggest a sub-optimal nested formulation of the problem that utilizes existing formulas for $n^*(r)$ and $q^*(r)$ (where r denotes the demand-to-production rate ratio, $r = D/P$) for the unconstrained problem. The optimal solution of the accurate formulation is obtained through numerical optimization, utilizing the IBM CPLEX solver, and is compared with the proposed sub-optimal method. Among the advantages of the suggested approach are that the solution is analytically derived, it is very simple to implement, and it yields a minimal joint cost that is close to the accurate formulation.

1. Introduction

The joint optimization concept for buyer and vendor was first described by Goyal [1]. There is evidence to suggest that improved synchronization of the production and purchasing processes between the manufacturer and the buyer, as well as greater information sharing, are the most effective means of meeting demand and increasing profits along the supply chain. Cooperation and information sharing are the foundations of a good working relationship between the two players [2, 3], one that aims to decrease the joint costs of holding inventories, transportation, internal shipment and production set-up [4]. Synchronization between a manufacturer and a retailer who share information about the market demand and about production capacity is exemplified by the case of Toyota's automobile production system [5].

This study focuses on the production process, which is a particularly important area for synchronization. Schweitzer and Seidmann [6] point out that machine production rates can often easily be changed. Consider an approach in which the manufacturer has flexibility in determining the production rate, up to the limitations of the available production speed [7]. Increasing the production speed has several advantages. Among them are postponing the subsequent production lot, decreasing the possibility of shortage, enabling maintenance time-windows, and shifting to the production of other products – a distinct advantage in the current climate due to the growing competition among multiple products for a limited storage capacity. On the other hand, faster accumulation of inventory has a negative effect due to increased holding costs.

A further influential variable with regard to the production process is the length of the production cycle – in particular, whether the cycle length is bounded or unbounded. An unlimited cycle length reduces planning flexibility, whereas a bound on the production cycle allows utilization of the time when the production system is idle (i.e., the delay between production lots) for processing other products. Furthermore, such delays allow workers free time (e.g., for personal or career-development purposes). These delays may be end of weeks when the production floor might be not active. Limiting the production cycle also enables flexibility in reacting to changes in demand for specific requirements. According to Milgrom and Roberts [8], in 1988, General Motors' engineers were able, for the first time in company history, to use a regular producing facility to make pilots of the next year's model. The engineers set the equipment to make 1989 models after the workers had left the factory on Friday afternoon, ran the equipment to manufacture the new models over the weekend, and then reset the equipment to produce 1988 model cars so that regular production could be resumed on Monday morning.

We seek to simultaneously find the optimal values of key decisions along the supply chain such that the overall joint cost is minimized. In particular, the decisions are: (1) production rate, P; (2) the shipment size delivered from the vendor to the buyer, q; and (3) the shipment frequency, n (an integer). The suggested methodology is essentially

https://doi.org/10.1016/j.orp.2020.100140
Received 17 June 2019; Received in revised form 14 December 2019; Accepted 9 January 2020
Available online 16 January 2020

analytical, thus serving as a possible benchmark that could be used in future studies to develop methods for solving similar problems or more complex ones. The quality of the proposed approximation method is measured by comparing the obtained results with the optimal solution. The results show a relatively small deviation from the optimal objective value.

2. Literature review

2.1. Manufacturer-buyer coordination

Goyal and Gupta [4] were the first to review the literature relating to the integrated vendor-buyer paradigm. At least three additional reviews have followed since. A review focusing on the coordination mechanisms between vendor and buyer through quantity discount schemes is presented in Sarmah et al. [9]. A selected review of the literature on the joint economic lot sizing problem (JELP) is provided by Ben-Daya et al. [10]. A comprehensive review of lot-size models that focuses on coordinated inventory replenishment decisions between buyer and vendor, and their impact on the performance of the supply chain, is presented by Glock [2].

Wahab et al. [11] determine the optimal production–shipment policy by minimizing the total expected cost per unit time in an international coordinated vendor–buyer green supply chain with a return policy that consists of sending back the defective items to the vendor. Hoque [12] analyzes coordinated vendor-buyer models with an exponential distribution for the lead time in a two-stage supply chain. The vendor produces a product at a finite rate and delivers the lot to the buyer in a number of equal-sized batches (sub-lots) to meet the deterministic demand, where backlog is allowed. Sajadieh and Larsen [13] consider a manufacturer-retailer model for a two-stage supply chain. The retailer and the manufacturer face random demand and random yield, respectively. Marchi et al. [14] develop a financial collaboration model between a single manufacturer and a single buyer in which the vendor has the option to invest in increasing its production rate. Our model specializes in determining the production rate for the integrated manufacturer-buyer supply problem when constraints are present.

2.2. Variable production rate and constraints over the production cycle and the production rate

Khouja and Mehrez [15] and Khouja (1995) extend the classical economic production lot size (EPL) model to cases where the production rate is a decision variable and the unit production cost becomes a function of the production rate. Khouja (1995) assumes a deterministic setting while Khouja and Mehrez [15] assume a random setting where the quality of the products produced by the system is related to the speed of production. Eiamkanchanalai and Banerjee (1999) develop a model for simultaneously determining the optimal run length and the rate of production for a single item. They assume that the production cost per unit is a quadratic function of the production rate. Note, however, that the above studies did do not consider an integrated supply chain but rather a single player. Their problem settings include fuzzy demand, inspection errors and an adjustable production rate.

Hajji et al. [16] address the joint supplier selection, replenishment and manufacturing control problem in a dynamic, stochastic context. The production rate is bounded and the decision variable is the production rate as a function of time, where the production system is subject to machine failures. A dynamic stochastic model is proposed, and the resulting optimality conditions are solved numerically. Unlike our problem, the system under optimization consists only of the manufacturer. Jauhari et al. [17] investigate an integrated production-inventory system consisting of a single vendor and a single buyer. They extend the model of Lin [18] by addressing additional considerations, including fuzzy annual demand, stochastic demand, inspection errors, partial backorders, and an adjustable production rate. Jauhari and Saga [19] extend the work of Jauhari et al. [17] by including fuzzy ordering cost, setup cost reduction, and service-level constraint.

Lee and Rosenblatt [20] suggest limiting the production cycle length in order to allow scheduled preventive maintenance and machine calibration. McCormick et al. [21] suggest a heuristic for maximizing throughput in a multi-product assembly line. The scheduling problem is to minimize the cycle time. Ben-Daya and Hariga develop a model under which the total lead-time, which consists of the production cycle time and the transportation time from the manufacturer to the buyer, is limited.

2.3. Closely related work and contribution of present study

Jauhari and Pujawan [22] present a synchronized vendor-buyer model in which they employ iterative solution procedures to determine the delivery lot size, delivery frequency, production lot, raw material lot size, and production rate – with the objective of minimizing the total cost. Aidurgam et al. [23] utilize the EPQ (Economic Production Quantity) model to analyze the joint problem of the two players. They assume that the demand is stochastic and that the production rate is a decision variable.

We follow the framework proposed by David and Eben-Chaime [24] in the sense that we analytically address a similar problem and we assume that each of the two players participates in the internal shipment costs. The buyer operates according to the known EOQ (Economic Order Quantity) model and the manufacturer follows the EPQ model (see Fig. 1). David and Eben-Chaime [24] show that the lot-for-lot regime (i.e., a single shipment delivered for each production lot) is not necessarily optimal. In another paper (see [25]), these authors show that under specified conditions, the gap between the optimal cost when assuming an integer shipment frequency and the optimal cost under continuous shipment frequency does not exceed 1.5%. Yang and Pan [26] and Glock [27] also assume an integer shipment frequency. In these studies, the authors carry out convex optimization of the continuous cost as a function of shipment frequency.

The present study contributes to the relatively modest literature that combines an integrated inventory and a variable production rate (see Table 1 in [17]). In contrast to the aforementioned algorithmic solutions, we analytically address the manufacturer-buyer problem, given a bound on the production cycle length, and we provide explicit solutions. We simultaneously seek the integer shipment frequency, shipment size and production rate. We introduce a simplified nested formulation that is solved analytically and is numerically compared with the optimal solution. The simplified formulation utilizes known optimal solutions by David and Eben-Chaime [25] for the reduced problem (which does not consider production rate as a decision variable, nor does it assume a bound on the production cycle length).

3. Modeling the problem

3.1. Supply-chain description and assumptions

Consider a supply-chain (see Fig. 1) in which a single manufacturer produces continuously a production lot of size Q items with constant production rate $P \geq D$, where D is the given demand rate posed by the buyer (e.g., the market). We define the demand-to-production rate ratio $r = D/P$.

We assume that the two players cooperate and seek to minimize their joint total cost associated with inventory holding, production setup and internal shipping. Shortages are not planned in either warehouse. In order to prevent endless production (i.e., $r = 1$), we define an upper bound $r_{\max}$ on the production rate ratio r. We define also an upper bound U on the production rate. This bound is associated with technological restrictions or limited machinery and/or manpower capacities. To prevent long production cycles and exploit the advantages

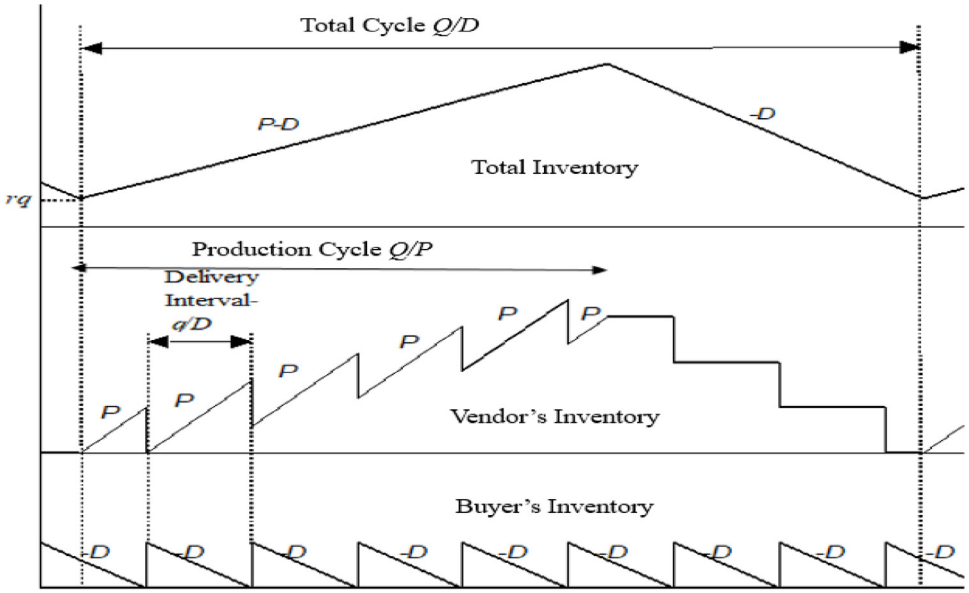

Fig. 1. Inventory level at the buyer, inventory level at the manufacturer, and total inventory (adapted from David, I., & Eben-Chaime, M., 2008)

Table 1
Optimal solution r^* for all feasible cases of the relationship between the critical points r_{min}, r_0, r_{up}, r_{max}.

no.	Case	Condition	r^*
1	$r_0 < r_{min} < r_{up} < r_{max}$	-	r_{up}
2	$r_0 < r_{min} < r_{max} < r_{up}$	-	r_{max}
3a	$r_{min} < r_0 < r_{up} < r_{max}$	$C(r_{min}) < C(r_{up})$	r_{min}
3b	$r_{min} < r_0 < r_{up} < r_{max}$	$C(r_{min}) \geq C(r_{up})$	r_{up}
4a	$r_{min} < r_0 < r_{max} < r_{up}$	$C(r_{min}) < C(r_{max})$	r_{min}
4b	$r_{min} < r_0 < r_{max} < r_{up}$	$C(r_{min}) \geq C(r_{max})$	r_{max}
5	$r_{min} < r_{up} < r_0 < r_{max}$	-	r_{min}
6	$r_{min} < r_{max} < r_0 < r_{up}$	-	r_{min}
7	$r_{min} < r_{up} < r_{max} < r_0$	-	r_{min}
8	$r_{min} < r_{max} < r_{up} < r_0$	-	r_{min}

of a limited production cycle length, we define a bound on the production cycle length, T_p. Very long production cycles hinder the scheduling of maintenance periods (which usually take place at the end of the week) or idle windows of time in which workers are not required to work. Very long production cycles also result in the accumulation of large quantities of the product in storage, which might exceed the vendor's storage capacity. The key decisions are: the demand-to-production rate ratio r, the shipment size q, and the shipment frequency n (integer).

3.2. Notations and assumptions

The following notations will be used to develop the model:
Parameters:

U - upper bound on the production rate
D - demand rate
$r_{\max}$ - upper bound on the production rate ratio
T_p - upper bound on the production cycle length
k - total delivery cost of a single shipment from the manufacturer to the buyer
c_V - unit cost to the manufacturer
K - production set-up cost
h_V - unit holding cost per unit time in the manufacturer's storage facility
h_B - unit holding cost per unit time in the buyer's storage facility

Decision variables:

r - production rate ratio ($r = D/P$)
Q - production lot size
q - shipment size
n - number of shipments in each production lot (integer)

The following assumptions are used in our model:

(1) all parameters are fixed and known
(2) the unit cost to the manufacturer is independent of the production rate
(3) both players share information and have an agreement regarding sharing the joint profit
(4) the production rate is bounded from above
(5) the production rate is greater than the demand rate
(6) the production cycle length is bounded
(7) shipment frequency is an integer
(8) each production lot incurs a production set-up cost
(9) each shipment incurs a delivery cost.

3.3. Problem formulation

The problem we address in this paper is formally defined as

$$\begin{aligned} &\min_{r,q,n} C(r, q, n) \\ &s.\ t \\ &\quad (1.1) D/U \leq r \leq r_{\max} \\ &\quad (1.2) rnq/D \leq T_p \\ &\qquad q > 0 \\ &\quad n \geq 1, \textit{Integer} \end{aligned} \tag{1}$$

where the joint total cost per unit time $C(r, q, n)$is computed from Fig. 1 and is given by

$$C(r, q, n) = Dc_V + \frac{DK}{nq} + \frac{kD}{q} + h_V\left(rq + \frac{nq}{2}(1 - r) - \frac{q}{2}\right) + h_B\frac{q}{2} \tag{2}$$

In (2), c_V is the unit cost to the manufacturer, k denotes the total delivery cost of a single shipment from the manufacturer to the buyer, K denotes the production set-up cost, and h_V and h_B are the unit holding costs per unit time in the manufacturer's storage facility and in the

buyer's storage facility, respectively. In order to solve problem (P1) we use the common IBM CPLEX optimization solver and numerically obtain the optimal solution.

4. Mathematical analysis

We compare the optimal solution of formulation (P1) with the optimal solution (with rounding to obtain an integer value of n) of the proposed sub-optimal nested formulation (P2), which is described in the following.

4.1. Nested formulation

We present below a sub-optimal formulation of problem (P1) and subsequently assess its effectiveness.

Problem (P2):

$$\min_{r,q,intn} C(r, q, n) = \min_{\substack{\frac{D}{U} \le r \le r_{\max} \\ r \le \frac{DT_p}{nq}}} \left\{ \min_{q,intn} C(r, q, n) \right\} \tag{3}$$

The continuous-n version of the internal optimization problem, which does not consider any constraints, is analyzed by David and Even Chaime [25]). The solution is as follows:

Proposition 1. *If $h_B > h_V(1-2r)$, n is continuous, and $r < 1$ is given, then*

$$q^*(r) = \sqrt{\frac{2Dk}{h_B - h_V(1-2r)}}, \quad n^*(r) = \sqrt{\frac{K(h_B - h_V(1-2r))}{kh_V(1-r)}} \tag{4}$$

are minimal and unique solutions to the sub-problem $\min_{q,\, n} C(r, q, n)$.

We approximate the optimal solution of problem (P1) by inserting solution (4) into $C(r, q, n)$, thus reducing the objective to a function of a single variable r:

$$C(r, q, n) = \sqrt{2DKh_V(1-r)} + \sqrt{2Dk(h_B - h_V(1-2r))}. \tag{5}$$

4.2. The feasible domain

Problem (P2) includes two constraints, $D/U \le r \le r_{max}$ and $rnq/D \le T_p$. Combining (4) with the constraint $rnq/D \le T_p$, we obtain

$$\sqrt{\frac{2r^2K}{Dh_V(1-r)}} \le T_p \tag{6}$$

Alternatively,

$$r^2 + r\frac{T_p^2 Dh_V}{2K} - \frac{T_p^2 Dh_V}{2K} \le 0 \tag{7}$$

The solution of inequality (7) is:

$$r_{up} = \frac{-\frac{T_p^2 Dh_V}{2K} + \sqrt{\frac{T_p^2 Dh_V}{2K}\left(\frac{T_p^2 Dh_V}{2K} + 4\right)}}{2} \tag{8}$$

The following result connects the original problem (P1) to the approximated one (P2). In particular, Theorem 1 specifies the conditions under which approximating the solution is not required since the obtained solution coincides with the optimal solution.

Theorem 1. *If $T_p \ge T_p^{min}$, where T_p^{min} is the root of $r_{up}(T_p) = r_{max}$ (see (8)), then the optimal solution of the sub-optimal nested formulation (P2) is also optimal for the general formulation (P1).*

Proof. When $r_{up} \ge r_{max}$, constraint (1.2) is redundant. T_p^{min} is obtained by equating $r_{up}(T_p) = r_{max}$. ∎

4.3. The optimal solution of problem (P2)

To construct the optimal solution of problem (P2), we utilize the observation (from the first-order condition) that $C(r)$ has an interior unique maximal point, r_0, when $0 < r_0 < 1$. Otherwise, $C(r)$ is strictly decreasing in r, where

$$r_0 = \begin{cases} \frac{4kh_V - K(h_B - h_V)}{2h_V(K+2k)} & K < \frac{4kh_V}{h_B - h_V} \\ 0 & otherwise \end{cases}. \tag{9}$$

Critical point r_0, together with r_{max}, r_{min} and r_{up}, assist in establishing the approximation developed in our paper, i.e., the optimal solution of problem (P2).

Theorem 2. *Let us define $r_{min} = D/U$. Problem (P2) has a unique global optimal solution, r^*, as presented in Table 1 below.*

$$r_0 < r_{min} < r_{up} < r_{max}$$

Proof. Based on the unimodality of $C(r)$ with r, and on Eqs. (8) and (9), problem (P2) reduces to a single-variable, convex optimization search: $\min_{r,q,\text{int}n} C(r, q, n) = \min_{\frac{D}{U} \le r \le (r_{\max}, \frac{DT_p}{nq})} C(r)$.∎

Below we prove a technical lemma supporting the unimodality of the objective with n when r is given, and as a result make the search for the optimal integer shipment frequency, n, significantly simpler.

Lemma 1. *Given a demand-to-production rate ratio r, if $h_B > h_V(1-2r)$, the cost function $C(r, q^*(n), n)$ is unimodal with variable n.*

Proof. See Appendix A.

Table 1 shows 10 feasible cases, two of which (1 and 3b) have the optimal solution r_{up}, i.e., a faster production rate than the optimal rate under problem (P1), r_{max}. In order to overcome the difficulties associated with a continuous shipment size n, we use Lemma 1 referring to the unimodality of $C(n)$, and replace n^* by the better of the two (if feasible) closest integer neighbors, i.e., $n^*_{\text{int}} = \text{argmin}_{n=n^*, n^*+1} C(r^*, q^*(r^*), n^*)$.

5. Numerical illustration

In Section 4 we optimally solved problem (P2). Yet, problem (P2) has two flaws. The first is that the shipment size n is allowed to be continuous; however, this has been dealt with by replacing n^* with n^*_{int}. The second is that the nested formulation (3) may be too restricting with respect to the original formulation (1), in which case it could result in a non-optimal solution under problem (P1). In this section we present a numerical example illustrating the applicability of the approximated formulation (P2).

5.1. Comparison of the optimal solutions of problems (P1) and (P2)

Consider the data presented in Table 2.

Given the above settings, $r_0 = 0.038$ and $r_{min} = 0.4$. Table 3 presents the optimal solution of problem (P1) and the optimal solution of problem (P2) for different values of the bound on the production cycle length, T_p. We define $\rho_{T_p}(\%)$ as the percentage difference between the value of rnq/D obtained by the optimal solution of problem (P2) and the

Table 2
Parameter values chosen for the numerical example

Parameter	k	K	D	U	h_V	h_B	r_{max}	c_V
	100$	5000$	200 units per day	500 units per day	10$ per unit per unit time	10$ per unit per unit time	0.8	100$

Table 3
Comparison of the optimal solutions of problems (P1) and (P2)

T_p (days)	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
r^*(P1)	r_{min}	0.419	0.619	r_{max}	r_{max}	r_{max}	r_{max}	r_{max}	r_{max}	r_{max}
r^*(P2)	non feasible	non feasible	0.482	0.580	0.656	0.715	0.762	r_{max}	r_{max}	r_{max}
n^*(P1)	4	7	9	10	12	15	18	20	20	20
n^*(P2)	–	–	9	11	13	15	17	20	20	20
q^*(P1)	62.5	68.19	53.85	50.00	52.08	50.00	48.61	50.00	50.00	50.00
q^*(P2)	–	–	64.41	58.72	55.21	52.89	51.23	50.00	50.00	50.00
C^*(P1)	25320.00	24060.64	23691.33	23300.00	23025.73	22883.33	22818.24	22800.00	22800.00	22800.00
C^*(P2)	–	–	23847.42	23585.78	23352.19	23147.31	22965.37	22800.00	22800.00	22800.00
T_p^*(P1)	0.5	1	1.5	2	2.5	3	3.5	4	4	4
T_p^*(P2)	–	–	1.397	1.873	2.354	2.836	3.318	4	4	4
ρ_{T_p}(%)	–	–	–6.87	–6.35	–5.84	–5.46	–5.20	0	0	0
ρ_C(%)	–	–	0.66	1.22	1.42	1.15	0.64	0	0	0

optimal production cycle length *rnq/D* obtained by numerically solving problem (P1). We define ρ_C(%) as the percentage difference (always positive) between the optimal cost according to the optimal solution of problem (P2) and the optimal cost obtained by numerically solving problem (P1).

The results in Table 3 support Theorem 1, indicating that the two formulations reach identical optimal solutions for all bounds $T_p \geq T_p^{min}$, where $T_p^{min} \in (3.5, 4]$. For small values of T_p, the decision variables (i.e., *r, n* and *q*) obtained by the suggested approximation may be distant from those of the optimal solution. Nevertheless, values of the decision variables that are distant from the optimal ones do not necessarily imply poorer decisions, provided no constraints are violated and the difference between both costs remains relatively small. For the case where $T_p = 2$, for example, the shipment frequency differs only by 1 (11 instead of 10). At the same time, the production rate ratio *r* is significantly smaller (0.580 instead of 0.8). However, these changes are "compensated for" by the larger shipment size *q* (58.72 instead of 50), such that there is a relatively small gap (1.22%) in the optimal joint cost. In this example, our approach suggests a faster production rate and, accordingly, a shorter production cycle. This outcome could be considered as advantageous over the optimal policy in the sense that it frees up more time that the decision-makers can then dedicate to other activities (e.g., maintenance, processing other products). In general, when the bound T_p is less than T_p^{min}, the gap between the optimal cost and the cost obtained by the suggested model does not exceed 1.5%, and is even less on average.

In all instances, the optimal production rate and shipment size associated with problem (P2) are larger than those obtained for the general problem (P1). When the bound on the production cycle length is active (for the original problem), the optimal cycle length obtained by problem (P2) is smaller (by approximately 6%) than the bound T_p. This is a direct outcome of constraint (6). Interestingly, for very small values of T_p, the resulting solution of problem (P2) is infeasible. This result is explained by reaching an infeasible production rate $r_{up} < D/U$. Keeping in mind that there is always a feasible solution to the original problem (P1), the manufacturer can always, in such scenarios, set $r^* = D/U$ while assigning values for the couple set (*q, n*) that comply with $nq \leq UT_p$, i.e., that comply with (1.2).

5.2. Sensitivity analysis of the joint cost and production cycle length

Figs. 2 and 3 present, respectively, the sensitivity of the gaps in the optimal cost, ρ_C(%), and the optimal production cycle length, ρ_{T_p}(%), to changes in the values of the key model parameters. The parameter values correspond to the data presented in Table 2, and the baseline value for the bound on the production cycle length was set at $T_p = 2.5$. Considering the baseline values $\rho_{T_p}(\%) = -5.84\%$, and $\rho_C(\%) = 1.42\%$, (see Table 3), it is evident from the results presented in Figs. 2 and 3 that even large changes in each of the key parameters have a negligible influence on the effectiveness of the suggested approach. In particular, the cost parameters *k*, h_V and h_B barely affect the gap (in terms of joint cost per unit time) between the models. The largest value of ρ_C(%) due to deviations of between -50% and 50% in one of the cost parameter values is obtained when the unit cost c_V decreases by half (i.e., $\rho_C(\%) = 1.7624 \cdot 1.42 = 2.5\%$).

Increasing each of the parameters *D, k*, and c_V generally decreases the gap ρ_C(%); thus, for these scenarios the joint cost per unit time in the two models becomes more alike. Increasing each of the parameters *D, k*, and h_B generally decreases the gap ρ_{T_p}(%); i.e., the production cycle length in the two models becomes more alike. Decreasing the production set-up cost *K* decreases the gap between the models' optimal unit-time cost *C*, while decreasing the total delivery cost of a single shipment, *k*, decreases the gap between the models' optimal production cycle length *rnq/D*. Decreasing the unit holding cost h_B by half results in the highest gap of $\rho_{T_p}(\%) = -1.4762 \cdot 5.84 = -8.62\%$, meaning that utilizing the suggested method results in a production cycle length that is nearly 9% shorter than the bound.

Unlike a gap between the models in terms of the unit-time cost *C*, a gap in terms of the production cycle length *rnq/D* (e.g., a shorter production cycle in our model) does not imply an inferior solution, since the additional available time may be utilized. Interestingly, the percentage change in ρ_C(%) with demand rate, *D,* and with the unit holding cost per unit time in the manufacturer's storage facility, h_V, shows a concave curve (see Fig. 2); i.e., the deviations from the optimal unit-time cost *C* are bounded.

In contrast to the relatively simple curves obtained for the deviation in ρ_C(%) (see Fig. 2), the deviation in ρ_{T_p}(%) with changes in the model parameters (see Fig. 3) exhibits oscillated curves. We assign the results in Fig. 3 to the rounding of *n* after obtaining the continuous optimization. These oscillations may indicate that even for very large deviations (%) from the original parameter values, the percentage change in the gap between cycle lengths remains limited.

5.3. Sensitivity analysis of the decision variables r, q and n

Figs. 2 and 3 addressed the deviations in terms of the main measures, i.e., the unit-time cost *C* and the production cycle length *rnq/D*. The following tables address the deviations in terms of the values of the decision variables, i.e., the production rate ratio *r*, the shipment size *q*, and the shipment frequency, *n*. The baseline differences in the values obtained by solving problem (P2) and those obtained by optimally solving problem (P1), for the case where the bound on the production cycle length is $T_p = 2.5$, are $d_r = -0.144$, $d_q = 3.13$ and $d_n = 1$ (calculated from Table 3). Tables 4a and b show the changes in these differences as a result of assuming negative and positive deviations (respectively) in the original model parameter values. For example, a change of 1 in the shipment frequency, such as occurs for a -30% deviation in the demand rate (see Tables 4a), implies a difference of 2

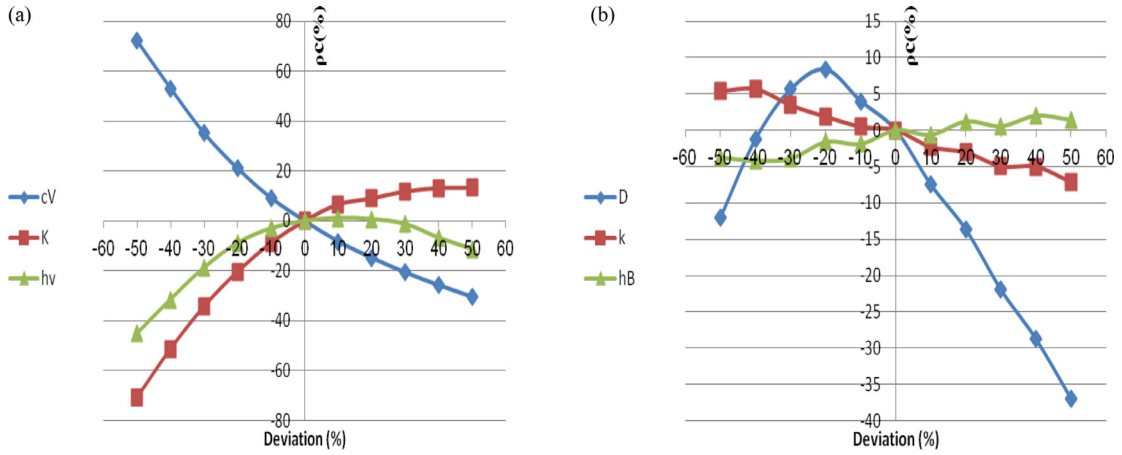

Fig. 2. a. Percentage change in $\rho_C(\%)$ with deviation (%) from original parameter value (c_V, K, h_V). Figure 2b. Percentage change in $\rho_C(\%)$ with deviation (%) from original parameter value (D, k, h_B)

relative to the optimal value of n obtained by solving (P1).

Since the analyzed problem is modeled as a non-linear integer formulation, one would expect decisions to alter even under moderate changes in the parameter values. The performance of the suggested approach is mainly tested against in terms of the objective (i.e., joint cost). As illustrated in Fig. 2, the suggested approach achieves relatively good performance according to this measure; in fact, for a wide range of parameters, it achieves the optimal solution. Thus, the decision maker can achieve relatively good performance even under substantial deviations in the original parameters. However, as shown in Tables 4a and b, at the same time, the values of the decision variables may alter quite substantially. As argued above, in principle, there is nothing wrong with deviations in the decision variables from the optimal values, as long as no constraints are violated and the performance remains near optimal. From the results presented in Tables 4, it can be seen that the value of the shipment frequency n does not notably increase from the baseline value (i.e., the maximal deviation is 1). The maximal deviation from the optimal production rate ratio (-0.226 = -0.144-0.082) is obtained due to a change in parameter h_V (when its value is decreased by 40%). This of course indicates a significantly higher production rate than the optimal value. In order to maintain a near-optimal joint cost, as shown in Fig. 2a (for the parameter h_V), the suggested approach chooses also a significant increase in the shipment size (of 8.38). We conclude that, relative to the optimal method, the suggested approach may choose higher production rates (and accordingly a shorter production cycle), as well as bigger shipment sizes; yet it simultaneously keeps the main objective (i.e., joint cost) close to the optimal value.

5.4. Comparison of the optimal solutions of problems (P1) and (P2) for the case where the unit cost depends on the production rate

The assumption used in our model, whereby the unit cost is independent of the production rate, simplified the mathematical analysis and assisted in deriving explicit solutions for the constrained integrated supply chain under discussion. Assuming that the unit cost is independent of the production rate does not mean, as some may assume, that the production costs are fixed when the production rate changes. When the production rate increases by 25%, for example, the quantity produced per unit time also increases by 25%, and accordingly, the production cost per unit time increases. Yet, in numerous scenarios this "linearity" does not accurately represent real production floors and the unit cost may alter as is disputed in existing literature. In this subsection we follow Khouja and Mehrez [15], who extend the economic production lot size (EPL) problem and assume the unit cost $c_V(P)$ depends on the production rate P according to

$$c_V(P) = \frac{a}{P} + bP \tag{10}$$

Thus, the joint objective (2) becomes

$$C(r, q, n) = D\left(\frac{ar}{D} + \frac{bD}{r}\right) + \frac{DK}{nq} + \frac{kD}{q} + h_V\left(rq + \frac{nq}{2}(1-r) - \frac{q}{2}\right) + h_B\frac{q}{2} \tag{11}$$

We numerically evaluate the effect of using the above dependency

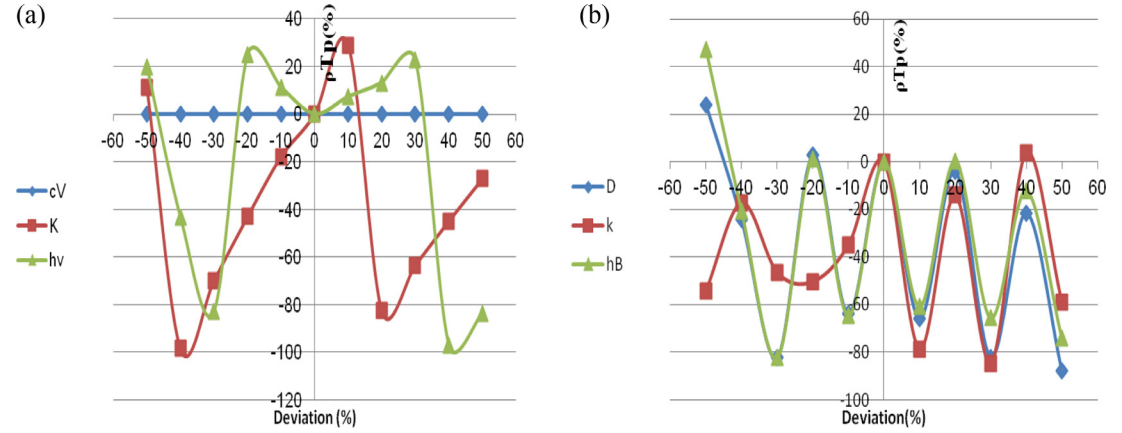

Fig. 3. . a. Percentage change in $\rho_{T_p}(\%)$ with deviation (%) from original parameter value (c_V, K, h_V). Figure 3b. Percentage change in $\rho_{T_p}(\%)$ with deviation (%) from original parameter value (D, k, h_B)

Table 4
a. Changes in the difference in the values of the decision variables between (P2) and (P1) with negative deviations (%) from the original parameter values. b. Difference ((P2) and (P1)) in d_r, d_q and d_n with positive deviations (%) from the original parameter values

Change in % in parameterValueParameter	−50	−40	−30	−20	−10	
D	−0.044	−0.080	−0.064	−0.038	−0.016	d_r
	1.74	4.97	1.76	1.46	3.12	d_q
	0	0	1	0	0	d_n
	0.109	0.082	0.058	0.037	0.017	d_r
K	−0.088	0.846	1.715	−1.48	−0.715	d_q
	0	1−	−1	0	0	d_n
	0	0	0	0	0	d_r
k	1.19	0.58	1.41	1.62	1.187	d_q
	0	0	0	0	0	d_n
h_V	−0.059	−0.082	−0.061	−0.038	−0.018	d_r
	2.729	5.248	4.2	2.79	5.401	d_q
	1	1	1	0	0	d_n
h_B	0	0	0	0	0	d_r
	4.567	6.296	7.667	4.773	2.245	d_q
	−1	−1	0	1−	0	d_n
c_V	0	0	0	0	0	d_r
	0	0	0	0	0	d_q
	0	0	0	0	0	d_n

Change in % in parameter Value parameter	10	20	30	40	50	
D	0.016	0.030	0.04	0.055	0.066	d_r
	1.216	−1.68	0.359	1.29	−1.17	d_q
	0	0	0	−1	0	d_n
	0.016−	0.031−	0.045−	−0.057	−0.069	d_r
K	0.709	1.358	1.98	2.42	3.171	d_q
	0	1	1	1	1	d_n
	0	0	0	0	0	d_r
k	2.706	0.552	3.022	5.40	7.69	d_q
	0	0	0	−1	0	d_n
h_V	0.016	0.030	0.043	0.055	0.066	d_r
	2.586	1.381	0.13	−0.79	−2.2	d_q
	−1	−1	−1	0	0	d_n
h_B	0	0	0	0	0	d_r
	2.029	0.239	2.047	0.571	−0.783	d_q
	0	0	0	0	1	d_n
c_V	0	0	0	0	0	d_r
	0	0	0	0	0	d_q
	0	0	0	0	0	d_n

of the unit cost on production rate with regard to the performance of our simplified approach. Specifically, the solution approach presented previously is not altered and it is compared with the optimal solution (numerical search) of problem (P'1) where $c_V(P)$ is given by (10). We replace (P1) with (P'1) and (P2) with (P2′) to denote the two generalized problems. In order for the comparison to be "fair", we find initial values for the parameters a and b that result in a function $c_V(P)$ that crosses the former fixed line $c_V(P) = c_V = 100$ (see Table 2) twice (i.e., the first time when it is decreasing and the second time when it is increasing). Accordingly, the chosen initial values are $a=17142$ and $b=0.142857$. Similar to Sections 5.2 and 5.3, the baseline value for the bound on the production cycle length is set at T_p=2.5.

Tables 5 shows that the three decision variables are assigned almost identical values by the two models. The suggested approach obtains a joint cost that is only 0.04% higher than that obtained by the optimal policy. In comparison to the case where the unit cost is independent of the production rate, the gap between the production rates in the two approaches is significantly decreased (see Table 3 for $T_p = 2.5$). One might imagine that the selected parameters (i.e., a and b) are the main cause of the high-quality result. Thus, in the following two figures we present the effect of selecting other values of these two parameters on the performance of our approach. In particular, we present the values of ρ_{T_p}(%)and ρ_C(%) for several values of a and b, plotted in terms of their deviation (%) from their original values.

Fig. 4 shows that the suggested approach results in an increase of no more than 5.5% in the joint cost due to deviations of between -50% and 50% in one of the parameters that determines the shape of the unit cost function $c_V(P)$. This indicates that the quality of the suggested approximation remains high even when it is applied to a more complex problem design than that initially assumed. For the production cycle

Table 5
Comparison of the optimal solutions of problems (P'1) and (P'2)

r^* (P'1)	r^* (P'2)	n^* (P'1)	n^* (P'2)	q^* (P'1)	q^* (P'2)
0.639	0.656	13	13	56.98	55.22
C^* (P'1)	C^* (P'2)	T_p^* (P'1)	T_p^*(P'2)	ρ_{T_p}(%)	ρ_C(%)
23298.41	23308.01	2.367	2.354	−0.55	0.04

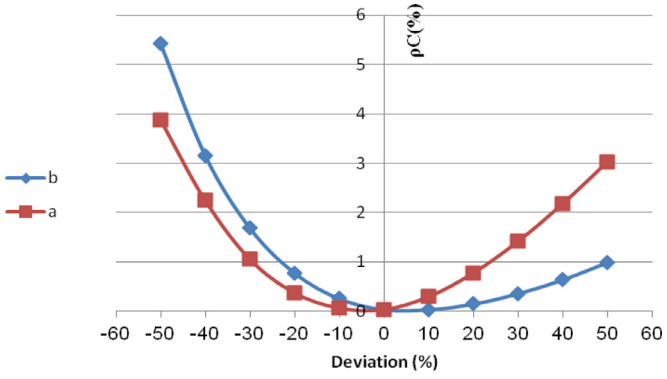

Fig. 4. ρ_C(%) with deviation (%) from original parameter value (a, b)

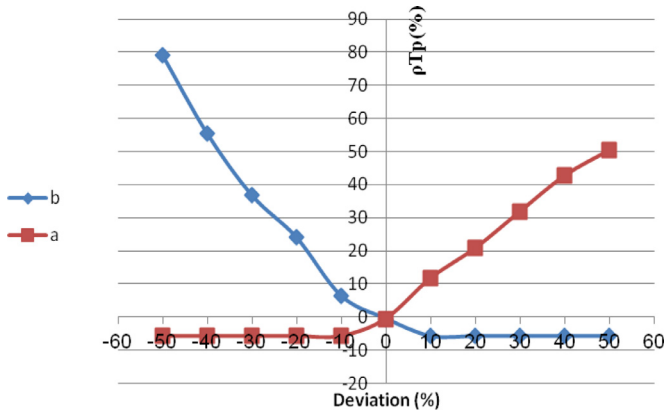

Fig. 5. $\rho_{T_p}(\%)$ with deviation (%) from original parameter value (*a, b*)

length, more substantial changes are observed (Fig. 5). In the independent scenario, the production cycle length was only ever shorter for (P2) than for (P1). The specific set (*a, b*) determines whether a relative increase or decrease in production cycle length is realized. Interestingly, when b exceeds some minimal threshold and when a is lower than some maximal threshold, the production cycle length is unaffected. This can be explained by the fact that an optimal value of the cycle length is active (i.e., $T_p = 2.5$). Small values of b or large values of a result in a cycle length that is larger for the suggested approach than for the optimal policy, i.e., a positive value of $\rho_{T_p}(\%)$. Yet, as argued above, as long as the joint cost remains alike and no constraints are violated, a difference in the value of the cycle length does not imply inferiority of the suggested approach.

6. Conclusions

6.1. Summary

In this paper we analytically analyze an important and well-known problem regarding the operational relations between the manufacturer and the buyer – two key players in many supply chains. A variable production rate has several advantages. Among them are postponing the subsequent production lot, decreasing the possibility of shortage, enabling maintenance time-windows, and shifting to the production of other products. We add to the existing formulations of this problem the consideration of a bound on the cycle length. Such a limitation may be imposed due to scheduled maintenance, end-of-week shut downs, or demand for other products requiring the processing machinery. Extensive numerical work is carried out to show the applicability and quality of the suggested approach.

6.2. Contribution and model advantages

We use existing optimal solutions for shipment frequency $n^*(r)$ and shipment size $q^*(r)$ obtained for the unconstrained problem as a basis for developing a simplified optimal solution to a nested formulation of the investigated problem. We use the critical points r_{min}, r_0, r_{up}, r_{max} and obtain the optimal demand-to-production rate ratio r^* for all feasible cases. We derive the theoretical relationship between the two problems through the existence of T_p^{min}, which is the root of $r_{up}(T_p) = r_{max}$; when $T_p \geq T_p^{min}$, the optimal solution of the sub-optimal nested formulation (P2) is also optimal for the general formulation (P1). The suggested approach is not optimal; yet it achieves solutions that are close approximations to the optimal one in terms of joint cost. When the bound is active, the production cycle length that is obtained is always smaller (due to a higher production rate) than the bound itself. This outcome may be utilized when there are machine breakdowns or when unexpected events arise that mean additional time is required for completing new assignments. The suggested approach is also compared with the optimal problem for the case where the unit cost function $c_V(P)$ depends on the production rate P. The numerical results support the conclusions drawn for the simplified case (where $c_V(P) = c_V$) regarding the quality of the suggested approximation.

6.3. Drawbacks and future research

The current approach of comparing the two models involves the use of numerical optimization of problem (P1). Although the sensitivity analysis of the key parameters reported in this study further supported the conclusions, and illustrated the effectiveness of the suggested approach, future research should adopt a more theoretical approach – one that analytically develops the optimal solution under the assumption of a discrete shipment frequency and that takes into consideration the discussed constraints. Furthermore, the developed model assumes a unit production cost that does not depend on the production rate. As this assumption is violated in numerous real-life production floors, we numerically evaluated the quality of the suggested model also for the case where the unit production cost depends on production rate. A possible future research direction, therefore, is to develop theoretical solutions for a specification of the model in which the unit production cost depends on the production rate. Such a research avenue could improve the quality of the suggested approach.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The author sincerely thanks MA student Amit Shaked for his advice and assistance.

Appendix A. Proof of Lemma 1

Let us begin by computing the Hessian matrix of $C(r, q, n)$, which results in

$$H(r, q, n) = \begin{bmatrix} \frac{2DK}{nq^3} + \frac{2kD}{q^3} & \frac{DK}{n^2q^2} + h_V(1/2(1-r)) \\ \frac{DK}{n^2q^2} + h_V(1/2(1-r)) & \frac{2DK}{n^3q} \end{bmatrix} \tag{A.1}$$

Since $\frac{2DK}{nq^3} + \frac{2kD}{q^3} > 0$, it remains to show that $det(H(r, q, n)) > 0$ is positive, i.e., that

$$\left(\frac{2DK}{nq^3} + \frac{2kD}{q^3}\right)\frac{2DK}{n^3q} - \left(\frac{DK}{n^2q^2} + h_V(1/2(1-r))\right)^2 > 0$$

Considering that, $q^*(r)n^*(r) = \sqrt{\frac{2DK}{h_V(1-r)}}$, assigning the Hessian matrix, FOC (4) results in

$$\left(\frac{3(h_V(1-r))^2}{4}+\frac{4D^2kK}{n^3q^4}\right)-\frac{3}{4(h_V(1-r))}^2>0.$$

Thus, the Hessian is positive definite. Since each point complying with the FOC is a local minimum, and since there is only one such point (e.g., (4)), this point is the global minimum and we conclude that objective $C(r, q, n)$ is unimodal with variable n. ∎

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