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An economic order quantity model with reverse logistics program

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ABSTRACT

The economic order quantity (EOQ) model has evolved greatly over past decades on the strength of incorporating realistic factors. In recent times, interest has been geared towards studying the effect of reverse flow of products into inventory system. In practice, businesses may operate return policy (and may reuse products and material) in an effort to increase customer loyalty and recover assets. In this paper, a reverse logistics EOQ model is studied. The inventory problem of when to order and how much quantity to order when there is reverse flow of items into the system is addressed in the form of profit maximisation. To find the optimal solution of the model (a nonlinear maximisation problem), the Karush-Kuhn-Tucker (KKT) conditions for the objective function are presented and the square-root formulae for the firm's order size and price are derived. A simple algorithm is proposed for a search of the optimal solution. To illustrate the applicability of the model, some numerical examples are given. Finally, sensitivity analyses are carried out on several key parameters in order to gain some useful insight.

1. Introduction

The economic order quantity (EOQ) model is a powerful scientific inventory-management framework that helps businesses in determining the level of stock that should be held to minimise costs associated with maintaining inventories and meeting customer demand. One of the earliest papers on mathematical models of inventory management was Harris [1], who described the EOQ problem and presented a solution. This basic EOQ model depended in its derivation on the simplistic assumptions of fixed demand rate, instantaneous replenishment, no shortages and that items are homogenous, non-perishable and of perfect quality. When one or all of these assumptions is relaxed, the EOQ model no longer gives a minimum-cost solution. This paper focuses on developing a frame for replenishing an inventory system operating a reverse logistic program with an increasing rate of reverse logistic investment cost. The analysis is set out to answer the following questions:

Given that the investment cost in the reverse logistics scheme increases with a increase in return rate, what is the optimal choice for the return policy, order quantity, cycle length and unit selling price of item? These questions will guide the goal of the paper.

Several authors have extended the basic EOQ model, and it has evolved greatly over past decades on the strength of incorporating realistic factors (see for e.g. [2–5]). Conventional modelling of inventory systems often ignores the dynamic nature of reverse flow of items because of the complexity of developing a framework

accommodating a return structure. This leads to incorrect inventory decisions and inefficient use of inventory data. Articles on EOQ models with reverse-logistics schemes are relatively few in number, and the present literature only gives results for cases with no deterioration of items. Recent developments in inventory control allow us to incorporate reverse-flow of item to the regular inventory model. This makes it possible to take quantitative decision on the return rate, order quantity and cycle length. In additional, more information can be extracted from it than the conventional EOQ model. Reverse logistic EOQ model can be developed rapidly and cheaply thus helping to redefine the EOQ models as well as allowing the reuse of returned items. This approach, however, generate a new set of challenges. The most significant of these is the complexity of the procedures required to adequately characterise the optimal solution. In the present analysis, we proposed an efficient EOQ frame with reverse logistic program, whilst allowing flexibility in the model so that it can be used on a wide range of inventory problems, e.g. system containing item with expiry date.

As mentioned as above, traditional EOQ models ignored the impact of return products via a reverse logistics program on the inventory decision. However, there are many real-world scenarios in which businesses operate return policy (and reuse products and material) in a bid to increase customer loyalty and recover assets. Recently, there have been several attempts to incorporate reverse logistics scheme into the traditional inventory models (see e.g. [6–9]). For example, in some industries such as book publishing, catalogue retailing, and greeting

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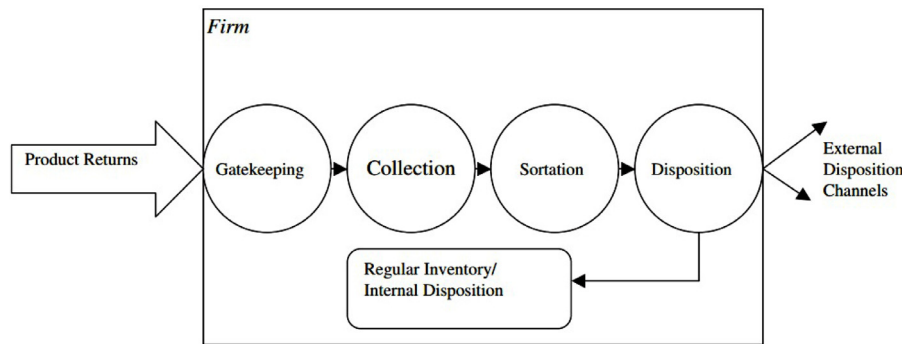


Fig. 1. A retail reverse logistics process; adapted from [12].

cards, over 20% of all products sold are eventually returned to the vendor [10]. Hawks [11] defined reverse logistics as the process of moving goods from the point of destination to the point of origin for the purpose of recapturing value or proper disposal. The reverse flow of goods affects inventory level and total inventory cost, and it should be taken into account when analysing inventory systems. According to [11], returns reduce the profitability of retailers by 4.3%, and he argued that retailers are more likely to sell returns to a broker or similar entity than to the manufacturers who are more inclined to re-manufacture or refurbish which would seem logical given that manufacturers are better at manufacturing than retailers. Gains due to reverse logistics include enhanced customer satisfaction, decreased inventory levels and reduced cost.

The retail-level reverse-chain process is shown in Fig. 1. As can be seen, the core components of reverse logistics are gate-keeping, collection, sortation and disposition [11–13], which can be collectively referred to as the central return centre. The individual unit of this central return centre are discussed in length by [12]. The expenses associated with the operation of reverse logistic have a direct effect on inventory decisions. Over the last few years, reverse logistics has attracted a great deal of attention from researchers in supply-chain management. The present analysis is motivated by the need to investigate the impact of returned products on inventory decisions. In this case, whether to investment or not in reverse logistics-program depends on inventory model or the different model values, and the resulting output is usually a function of order size, price and cycle time.

The idea of incorporating a reverse-logistics program into an inventory model is new and critical to inventory modelling. Demand rate and deterioration, on the other hand, have long been recognised as essential aspects of an inventory model. Bose et al. [14] proposed a finite-horizon EOQ model for items with linear demand rate and constant deterioration. Taleizadeh [5,15,16] introduced a number of EOQ models with deterioration rate and prepayment options. Chen et al. [17] developed an EOQ model for an inventory system in which return of used product is allowed and in which items in the system do not deteriorate. Sanni et al. [18] extended the model to include items with constant rate of deterioration. Recently, O'Neill and Sanni [19] presented a general approach for analysing the EOQ problem.

One of the pioneer papers on reverse logistic EOQ control model was [20], who developed a deterministic EOQ-based model, assuming constant demand and return rates and fixed lead times for external orders and recovery. Ritcher [6] studied an EOQ model with constant disposal and used-product collection rates. Alinovi et al. [7] proposed a stochastic EOQ-control model with random demand, return fraction and lead time. They assumed a non-zero ordering and manufacturing lead time in which the lot sizes are fixed parameters coming from external constraints or a firm's estimation. Liao and Deng [8,9] developed an extended EOQ/SEIQ model for re-manufacturing system under uncertain market demand and uncertain acquisition quantity, in which production and environmental costs are considered. Every practical

inventory system experiences some forms of deterioration. Losses due to deterioration of items affect the total average inventory cost and the availability of items for consumer demand, and hence should be taken into account when analysing an inventory system. In these models for reverse logistics, the deterioration rate of item is relaxed and the impact of unit selling on the order size is ignored. These models are suitable in cases where items are non-perishable over the cycle length but they do not accurately represent cases where items deteriorate. In particular, Alinovi et al. [7]'s analysis focuses on describing the impacts of demand uncertainty, return fraction and manufacturing lead time on inventory decisions in which the order quantity and cycle length are fixed parameters assumed to be derived from external constraints or the firm's estimations. They ignored impact of deterioration on items while using the return rate as a decision variable.

In the literature, the focus has been primarily on methods for either determining appropriate order size to minimise system cost or to determine optimal return rate. In these models, little or no attention has been given to obtain a combined results when considering the reverse flow of items. Hence, the problem of including a reverse-logistics arrangement in inventory decisions remains largely an open research area.

In comparison with previous research, this work aims at modifying the EOQ model to be more representative of real contexts; with this purpose in mind, we have paid particular attention to:

- Formulating the EOQ model to include price-dependent demand and time-varying deterioration rate under reverse logistics program with increasing investment cost;
- Determining optimal solutions for the return policy, order quantity, cycle length and unit selling price; where return rate, order size, selling price are the decision variables.

Conversely, we do not concentrate on demand uncertainty, assuming the firm or business is able to predict demand with high degree of accuracy, taking only the selling price as the main driver of consumer demand.

Thus, in the work, we attempt to fill this gap and answer the question of how to replenish an inventory system under reverse logistics program with increasing rate of logistics investment cost. In particular, we study the EOQ model with a reverse-logistics program in a specific form allowing for linear demand rate and constant rate of deterioration and time-varying deterioration of items. We formulate an optimisation problem for the firm and obtain an optimal inventory policy. The results are demonstrated using numerical studies.

2. Mathematical development

In this section, we follow the model form set out in Chen et al. [17] using similar assumptions. The notation in the EOQ model component are as follows.

- a : Fixed ordering cost per replenishment
- b : Purchase cost per unit item
- c : Inventory holding per unit per unit time
- p : Unit selling price of product
- $D(p)$: Price-dependent demand rate
- $\theta(t)$: Deterioration rate as a function of time with θ denoting constant deterioration
- y : Order quantity per replenishment
- T : Cycle length.

The notation for modelling used product collection operation are as follows.

- b_1 : The unit price of used collected product sold by the firm to its supplier
- b_2 : The unit cost of collecting, holding, sorting and handing a returned item, which includes fees paid by the firm to consumers.
- ν : Return rate for collecting used product.
- $R(\nu)$: Sum of return rate for used product in the current replenishment cycle.

The following assumptions are made in the model development.

1. A single-item inventory system is considered which receives an order size of $y > 0$ units at the starting time $t = 0$.
2. No shortages is allowed.
3. The demand is a deterministic price dependent demand.
4. The possibility of item deteriorating inventory is assumed.
5. A reverse logistics program is set up to absorb returned item and reused.
6. A substantial amount, s is invested in the operation of reverse flow of item.
7. The investment cost of running a reverse-logistics scheme is a quadratic function of return rate coefficient, ν . This ensures (and follows from intuition) that investment cost increases at an increasing rate of returned items.
8. The inventory holding costs for collected items are ignored (see Chen et al. [17] for similar treatment).
9. We assumed a positive salvage value $b_1 - b_2$ of collected used product.

We consider the problem of profit maximisation for a firm with a reverse-logistics program. The demand is affected by the unit selling price the firm sets for its goods and a fraction of items deteriorates over the cycle time. Suppose that the firm wishes to maximise its profits for an appropriate choice of the order size y , the unit selling price p and the return rate parameter ν . For $\nu, T > 0$ and $p > 0$, we have $y > 0$, so that the profit consists of revenue from sales of new and returned items, and is reduced by purchase cost, holding cost and the cost of running the reverse-logistics system. Therefore, for all $T > 0$ the average profit-rate is given by

$$\pi(p, \nu, T) = \left[p + (b_1 - b_2)R(\nu) \right] D(p) - \frac{1}{T} \left[a + b \cdot y(p, T) + c \cdot h(p, T) + c_1(\nu) \right], \tag{1}$$

where b is the purchase cost per unit item, a is the fixed cost per order and c is the carrying cost of a item per unit per unit time. The parameter b_1 represents the unit cost of collecting, holding, sorting and handing a returned item, which includes the collection fees paid by the firm to consumers. The parameter b_2 is the unit price of a used returned product sold by the firm to its supplier; here we assume a positive salvage value $b_1 - b_2$ of collected used product. (See [17] for a similar treatment.)

Some expenses are associated with running a reverse-logistics scheme. The investment cost of the firm operating a reverse-logistics program is assumed to be a quadratic function of the return-rate coefficient, $c_1(\nu) = s\nu^2$. This assumption ensures that the incremental rate of

the operational cost of the reverse-logistic program depends on the return rate. In line with [17], it is assumed that products sold in past replenishment cycles are collected in the current replenishment cycle, and that the return-rate for product sold in the past i th replenishment cycle follows a geometric series, with initial return-rate coefficient ν and common ratio μ , $0 < \mu < 1$. The return rate for the products sold in the past i th replenishment cycle is $\nu\mu^{i-1}$ and the sum of the return-rate for used products in the current replenishment cycle is represented by $R(\nu) = \sum_{i=1}^{\infty} \nu\mu^{i-1} = \nu/(1 - \mu)$. The parameter μ can be estimated, from past data, by taking the ratio of the return rates of any two consecutive replenishment cycles¹— say for e.g. $\mu = \nu^i/\nu^{i-1}$ while ν is determined in an optimal manner. It is only reasonable to assume the total number of collected used items be non-negative, and that this number should not exceed the number of items sold in each replenishment cycle. Thus, the scope of the firm’s profit maximisation is limited by the constraint $0 < R(\nu) \leq 1$.

To determine the inventory level and total item-holding time, let the inventory start at the order size y , which then depletes due to the simultaneous effects of demand and deterioration until it becomes empty at the cycle time $t = T$. At this time, it is replenished by a new order, and the process repeats. The inventory process $I(p, T) = \{I(t|p, T) | 0 \leq t \leq T\}$ over the cycle time T ; its variation is described by the inventory linear first-order differential equation. A simple version was used by [21],

$$\overbrace{\frac{dI}{dt}(t|p, T)}^{\text{Overall rate-of-change}} = \overbrace{-D(p)}^{\text{Inventory depletion}} - \overbrace{\theta(t)I(t|p, T)}^{\text{Deterioration}}, \tag{2}$$

for all $0 \leq t \leq T$,

where $I(t)$ is the inventory-level function, and $D(p)$ and $\theta(t)$ are the rate functions for demand and deterioration, respectively.

To facilitate the analysis, define the cumulative deterioration function Θ by

$$\Theta(t) = \int_0^t \theta(s) ds \quad \text{for all } 0 \leq t \leq T.$$

Solving Eq. (2), using the cumulative deterioration, the inventory-level function is:

$$I(t|p, T) = D(p) \int_t^T \exp\{\Theta(s) - \Theta(t)\} ds. \tag{3}$$

The order size at the beginning of the inventory system is given by:

$$y(p, T) = D(p) \int_0^T \exp\{\Theta(s)\} ds. \tag{4}$$

The total item-holding time is given by:

$$h(p, T) = \int_0^T I(t|p, T) dt = D(p) \int_0^T \int_t^T \exp\{\Theta(s) - \Theta(t)\} ds dt. \tag{5}$$

The above analysis gives the general form of the inventory functions—specific forms are determined by the features of the demand and deterioration rates. The general results can be applied to specific models to obtain the optimal policy for various kinds of inventory systems under a reverse-logistics program. We present the case in which a constant proportion of inventory items deteriorate per cycle, followed by the case which the deterioration rate is time varying.

Now, consider a constant deteriorating inventory item. The equation governing the inventory process $I(p, T) = \{I(t|p, T) | 0 \leq t \leq T\}$ is

$$I(0|p, T) = y(p, T), \quad \frac{dI}{dt}(t|p, T) + \theta I(t|p, T) = -D(p),$$

for all $0 \leq t \leq T$,

where θ is the constant proportion items deteriorating per cycle time.

¹ In the case where the business is just starting, μ can be estimated by soliciting expert opinion or by seeking advice from similar businesses already in existence.

The solution to the inventory equation is:

$$I(t|p, T) = D(p) \frac{\exp\{\theta(T-t)\} - 1}{\theta}, \quad \text{for all } 0 \leq t \leq T.$$

The order size $y(p, T)$ and the total item-holding time $h(p, T)$ are obtained using Eqs. (4) and (5):

$$y(p, T) = D(p) \frac{\exp\{\theta T\} - 1}{\theta}, \quad h(p, T) = D(p) \frac{\exp\{\theta T\} - \theta T - 1}{\theta^2}.$$

Substituting these expressions into Eq. (1), the firm's profit function can be written as

$$\pi(p, \nu, T) = \left(\left(p + \frac{c}{\theta} \right) + \frac{(b_1 - b_2)\nu}{1 - \mu} - \left(b + \frac{c}{\theta} \right) \frac{\exp\{\theta T\} - 1}{\theta T} \right) D(p) - \frac{a + s\nu^2}{T}. \tag{6}$$

There is no exact solution for the decision variables using the profit function in Eq. (6) because of the presence of $1/(\theta T)$ and $\exp\{\theta T\}$, which does not augur well for solving explicitly for T . The exact solution for the cycle length T can be described in terms of the Lambert W-function, also called the Product Log function. Although it is possible to obtain the exact solution using iterative schemes, we adopt a quadratic approximation of the exponential term for the sake of analytic and computational convenience. Some authors have favoured this approach (see for example [5,22]). To obtain the specific structure of the rate of withdrawal of items from the inventory system, we assume a linear demand rate $D(p) = m - kp$ where m is the fixed demand and k is the elasticity coefficient. The average profit-rate for this model is then:

$$\pi(p, \nu, T) = \left(\left(p + \frac{c}{\theta} \right) + \frac{(b_1 - b_2)\nu}{1 - \mu} - \left(b + \frac{c}{\theta} \right) \frac{\exp\{\theta T\} - 1}{\theta T} \right) (m - kp) - \frac{a + s\nu^2}{T}. \tag{7}$$

Therefore, the optimisation problem is $\max_{p, T, \nu > 0} \{\pi(p, \nu, T)\}$, subject to $0 < R(\nu) \leq 1$.

The firm seeks to maximise the average profit rate by appropriate choice of the price p , return rate ν and cycle length T , which from the expression of $y(p, T)$ determines the order size. We solve the optimisation problem by considering four possible scenarios. Firstly, we consider the price level and the return rate to be fixed exogenous values, and determine the optimal cycle length for a given price and return rate. Secondly, we determine the optimal price level for a fixed cycle length and return rate. Thirdly, we consider the price level to be a fixed exogenous value, and determine the optimal cycle length and optimal return-rate. Finally, we consider the case in which the price level, the return-rate and the cycle length are all decision variables limited only by the return-rate constraint.

Proposition 2.1. *If the price level, the return rate and the cycle length are decision variables and are endogenously determined by the firm, then the Karush-Kuhn-Tucker (KKT) conditions are necessary conditions for the optimal solutions to the inventory problem with Lagrangian:*

$$\mathcal{L}(p, \nu, T, \lambda) = \pi(p, \nu, T) - \lambda \left(1 - \frac{\nu}{1 - \mu} \right), \tag{8}$$

where λ is the Lagrange multiplier (see Appendix for proof).

The following results are some consequences of Proposition 2.1.

Result 2.1. For any given price level p and fixed return rate ν , the optimal cycle length for the firm is given by

$$\hat{T} = \sqrt{\frac{2(a + s\nu)}{c + b\theta} \cdot \frac{1}{m - kp}}.$$

Result 2.2. For any fixed cycle length and return rate, an endogenous price level is given by

$$\hat{p} = \frac{1}{2k} \left[m - k \left(\frac{(b_1 - b_2)\nu}{1 - \mu} + \frac{c}{\theta} + \left(b + \frac{c}{\theta} \right) \left(1 + \frac{1}{2}\theta T \right) \right) \right].$$

Result 2.3. For a fixed exogenous price level, the optimal cycle length \hat{T} and the optimum return rate $\hat{\nu}$ are given respectively by

$$\hat{T} = \sqrt{\frac{2a}{(m - kp)[(c + b\theta) - \psi]}}, \quad \psi = \frac{(b_1 - b_2)^2(m - kp)}{2s(1 - \mu)^2},$$

$$\hat{\nu} = \frac{(b_1 - b_2)(m - kp)\hat{T}}{2s(1 - \mu)}.$$

If the optimal return rate $\hat{\nu}$ falls outside the interval $(0, 1 - \mu]$, violating the scope of optimisation, it is set as $\hat{\nu} = 1 - \mu$ and a new optimal cycle length is obtained.

(The proof of Results 2.1, and–2.3 stem from Proposition 2.1)

Remark: The use of quadratic approximation of exponential terms allows finding closed-form equations for the decision variables in the propositions.

In this problem, the KKT procedure yields pseudo closed-form equations for the optimal values of the decision variables. In this case, these equations are used iteratively in the search for a solution until they are all satisfied at a given threshold. To achieve this, we consider a simple iterative procedure that shuttles between p and T using Result 2.1 and Result 2.2. The procedure starts with the basic (EOQ) cycle length, uses Result 2.2 to determine a price level, then uses this price level in Result 2.1 to compute an updated value of the cycle length, and so forth until a threshold is achieved. The value of price level at the point where the threshold is satisfied can then be used in Result 2.3 to determine \hat{T} and $\hat{\nu}$. The steps are outlined in the algorithm below.

Another feasible starting value for the price level is $p_0 = \frac{m}{2k}$.

Having analysed a reverse-logistics inventory system with constant proportion of deteriorating items, we now present the development for systems with time-varying deterioration rate. Let the time-varying deterioration rate be described by a power function $\theta(t) = \theta\beta(t - \gamma)^{\beta-1}$, so that the cumulative deterioration $\Theta(t) = \theta(t - \gamma)^\beta$ for all $t \geq \gamma$. The inventory-level function is:

$$I(t|p, T) = \begin{cases} D(p)(\gamma - t) + D(p) \int_\gamma^T \exp\{\theta(s - \gamma)^\beta\} ds, & \text{for } t \leq \gamma, \\ D(p) \int_t^T \exp\{\theta[(s - \gamma)^\beta - (t - \gamma)^\beta]\} ds, & \text{for } t > \gamma. \end{cases}$$

The order quantity and the total item-holding time are given respectively by:

$$y(p, \alpha) = D(p)\gamma \left(1 + \phi_0(\alpha) \right),$$

$$h(p, \alpha) = D(p)\gamma^2 \left(\frac{1}{2} + \phi_0(\alpha) + \frac{1}{2}\phi_1(\alpha) \right), \tag{9}$$

where the cycle length is parameterised, $T = \alpha\gamma$, so that all times are now expressed as multiples of the non-deterioration time. The quantities ϕ_0 and ϕ_1 are given by:

$$\phi_0(\alpha) \equiv \int_0^{\alpha-1} \exp\{\theta\gamma^\beta t^\beta\} dt,$$

$$\phi_1(\alpha) \equiv 2 \int_0^{\alpha-1} \int_t^{\alpha-1} \exp\{\theta\gamma^\beta (s^\beta - t^\beta)\} ds dt.$$

Note that we have assumed that the non-deterioration parameter γ is positive and $T > \gamma$. This restriction seems reasonable since it excludes cases in which items start deteriorating before arriving in the system (negative γ indicates deterioration of items in-transit) and cases in which there is no deterioration at all during the inventory cycle ($T \leq \gamma$). Given that a reverse-logistics program is incorporated into the inventory model, a negative γ may precipitate a very high return rate of used products and quite often render the solution of the problem unobtainable. For the case under consideration, we have the profit-rate function for the reverse-logistics inventory model as:

- 1: Input the problem data, set index $j = 0$, specify a terminal parameter $\epsilon > 0$, set the initial value of the cycle time to the basic cycle length $T_0 = \sqrt{\frac{2a}{c_0} \cdot \frac{1}{m}}$ and let $p_0 = 0$.
- 2: Setting the index j to increment by 1 and using the current value of T_j , compute:
 - (i) the $(j + 1)$ th return rate

$$v_{j+1} = \frac{(b_1 - b_2)(m - kp_j)}{2s(1 - \mu)} T_j \quad j = 0, 1, \dots;$$

if v_{j+1} falls outside the interval $(0, 1 - \mu]$, it is set as $v_{j+1} = 1 - \mu$.

- (ii) the $(j + 1)$ th price level

$$p_{j+1} = \frac{1}{2k} \left[m - k \left(\frac{(b_1 - b_2)v_{j+1}}{1 - \mu} + \frac{c}{\theta} + \left(b + \frac{c}{\theta} \right) \left(1 + \frac{1}{2} \theta T_j \right) \right) \right] \quad j = 0, 1, \dots$$

- 3: Calculate T_{j+1} , given the price level p_{j+1} and v_{j+1} , such that

$$T_{j+1} = \left(\frac{2(a + sv_{j+1})}{c + b\theta} \cdot \frac{1}{m - kp_{j+1}} \right)^{1/2} \quad j = 0, 1, \dots$$

- 4: If $|T_{j+1} - T_j| \leq \epsilon$, stop; the optimal solution is $\hat{T} = T_{j+1}$, $\hat{p} = p_{j+1}$; evaluate the average profit and the order size. Otherwise, replace T_j with the new cycle length, T_{j+1} and go to Step 2.

Algorithm 1. Simultaneous computation of price level and optimal cycle length.

$$\begin{aligned} \pi(p, v, \alpha) &= \frac{D(p)}{\alpha} \left[\alpha p + \frac{(b_1 - b_2)\alpha v}{1 - \mu} - \left(b + \frac{c\gamma}{2} \right) - (b + c\gamma)\phi_0(\alpha) - \frac{c\gamma\phi_1(\alpha)}{2} \right] \\ &\quad - \frac{a + sv^2}{\alpha\gamma}. \end{aligned} \tag{10}$$

Therefore, the objective of the problem is to maximise Eq. (10) subject to $0 \leq R(\nu) \leq 1$. The KKT setup in Proposition 2.1 can be used to find the optimal policy by substituting Eq. (10) in Eq. (8) and differentiating with respect to the desired variables. For computational convenience, the special functions ϕ_0 and ϕ_1 can be represented as infinite sums of vanishing terms (truncated at an appropriate level of numerical accuracy) so that:

$$\begin{aligned} \phi_0(\alpha) &= (\alpha - 1) \sum_{k=0}^{\infty} \frac{(\theta\gamma^\beta(\alpha - 1))^k}{k!(\beta k + 1)}, \\ \phi_1(\alpha) &= (\alpha - 1)^2 \sum_{k=0}^{\infty} \frac{(\theta^2\gamma^{2\beta}(\alpha - 1)^{2\beta})^k}{(k!)^2(\beta k + 1)^2}. \end{aligned}$$

When $k \gg \theta\gamma^\beta(\alpha - 1)^\beta$, the terms in these infinite sums rapidly become small, and so the sums can be truncated at only a few terms and high precision levels still obtained. The derivatives of $\phi_0(\alpha)$ and $\phi_1(\alpha)$ can be obtained using Leibniz's integral rule [see e.g. 19,23, for treatment].

3. Numerical example

Example 3.1 Inventory management with reverse logistics. An inventory problem with prepayment option, reverse logistics program and constant decay is studied. We consider the distributor of a certain electronic product reviewing its inventory policy for one particular model. The distributor's supplier requires it to prepay 40% of the purchase cost by making a series of three fixed prepayments at equal intervals of 0.2 year. The demand for the product is 800 units per month with elasticity coefficient 0.65. The other relevant inventory data are: $a = \$60$ per order; $b = \$12$ per unit; $b_1 = \$3.5$ per unit; $b_2 = \$1$ per unit; $c = \$0.05$ per unit per year; $s = 125$; $\mu = 0.2$; $\theta = 0.1$; $i = \$0.35$ per year.

To solve this problem, we must first adjust the unit cost b to accommodate or absorb the prepayment cost. A formula for augmenting the unit cost with prepayment is given in [5] and [19]:

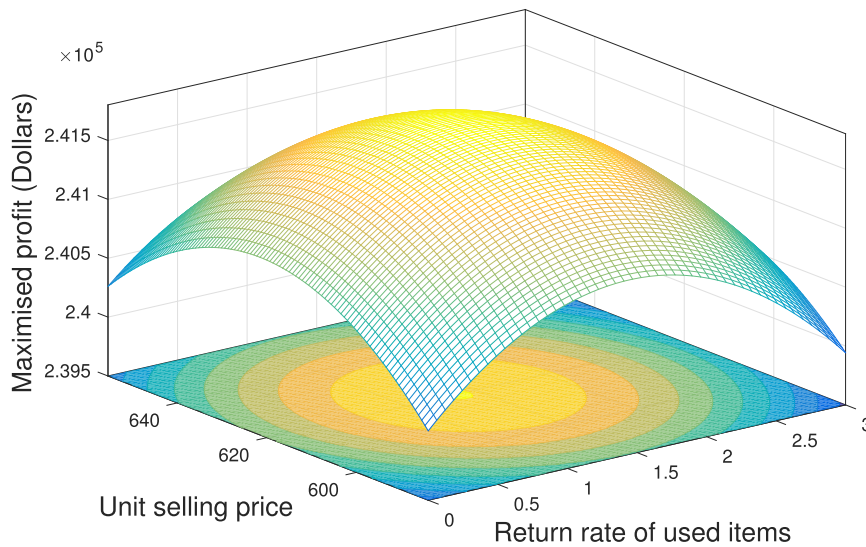
$$b_* = b \left(1 + \frac{\delta(1+i)^{u(n+1)} - n(1+i)^u + (n-1)}{n(1+i)^u - 1} \right), \tag{11}$$

and the new total payment b_*y . In this problem, $n = 3$, $u = 0.2$, $\delta = 0.4$, $b = 12$, $i = 0.35$. By adjusting the cost-per-item to absorb prepayment, the new inflated cost-per-item becomes $b_* = 14.219$. Note that the new cost-per-item parameter b_* does not depend on the control variables, and so the profit function has the same structure as before and the optimisation problem is unchanged. However, the inclusion of the timing of payment in the problem significantly reduces the system total profit. Using Eq. (7), Proposition 2.1 and the algorithm, we obtain the optimal return rate $\hat{\nu} = 0.8$; and the optimal price $p = \$620.56$; other outcomes are:

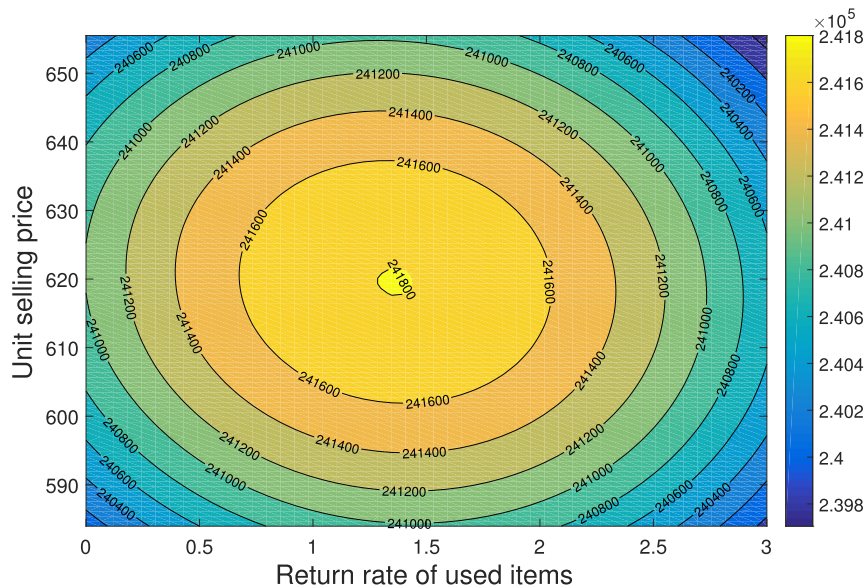
$$\hat{\nu} = 299.082 \text{ units}, \hat{T} = 0.7270 \text{ year}, \hat{\pi} = \$241, 840.67 \text{ per annum.}$$

Fig. 2 shows the firm's profit function for various prices and return rates. Fig. 2(a) reveals that a unique optimal policy exists for the problem, since the graph indicates concavity of the profit function with respect to the unit selling price and the return rate. The contour plot in Fig. 2(b) shows that the surface is symmetric and peaks in the centre at around 241800. Though the return rate is plotted over a range of 0 to 3 (this means considering an unconstrained problem for the plot), the optimisation process is restricted to the region 0 to 1. Fig. 2(a) and (b) also show increase in the maximised profit for which the optimal return rate $\hat{\nu}$ falls outside the interval $(0, 1 - \mu]$, validating the restriction $0 < R(\nu) \leq 1$ on the cumulative return rate. In accordance with Result 2.3, the optimal return rate is set as $\hat{\nu} = 1 - \mu$ (that is $\hat{\nu} = 0.8$) and a new policy is obtained.

Example 3.2 Deteriorating item inventory with reverse logistics. An inventory problem with a logistics program, constant deterioration and no prepayment option is studied. Consider the inventory data from Example 3 of [17]: $b = 10$; $b_1 = 3$; $b_2 = 1$; $s = 10000$; $a = 3000$; $c = 0.8$;



(a) A concavity plot of profit function with respect to return rate and price.



(b) A contour plot showing profit rate as a function of return rate and price.

Fig. 2. Maximum profit function with respect to return rate and price.

$\mu = 0.2$; and demand function $D(p) = 1000 - 15p$. In addition, let a constant fraction of the on-hand inventory $\theta = 0.3$ deteriorate per unit time.

Using the solution in Proposition 2.1 and the algorithm, the optimal solution is:

$$\hat{v} = 0.0843 \quad \hat{p} = 40.15 \quad \hat{y} = 879.09 \text{ units}$$

$$\hat{T} = 1.6955 \quad \hat{\tau} = 8,735.41.$$

In comparison, the model in [17], which ignores the impact of deterioration of items would overestimate the optimal return rate and the optimal order size, and underestimate the optimum unit selling price if deterioration were present.

In order to gain further insight, we perform a sensitivity analysis on some input parameters of Example 3.2 by generating a table that shows what the optimal solutions would be if the true inputs were as given in Table 1- changing one parametric value at a time while holding other

input values constant.

In Table 1, when the deterioration rate $\theta = 0.00001$, a negligible rate, the solutions are similar to the results in [17], in which the impact of deterioration was completely ignored. Further, it can be seen that the optimal profit rate decreases as the deterioration rate increases. An increase in the reverse-logistics (investment) scaling parameter leads to a decrease in profit. Thus, as s increases, management needs to order less quantity, reduce cycle length and slightly increase price to cushion the affects of reverse-logistics cost; maximum profit decreases.

It can also be seen from Table 1 that a reduction in the reverse-logistics program cost to \$1000 brought a 4.54% increase in the profit rate. Conversely, a 91.67% increase in the reverse-logistics operational cost decreased the profit by 4.26%, which confirms the assertion that returns reduce the profitability of retailers by 4.3%, as reported in [11].

Before concluding the numerical analysis, note that the solutions have allowed for the order size to be any real number of items, and so the optimal profit-maximising solution is characterised by the exact

Table 1
Sensitivity analysis on deterioration rate θ and reverse logistics-scaling parameter for s Example 3.2.

Parameter	Value	Optimal values					
		\hat{T}	\hat{p}	$\hat{\nu}$	$\hat{\lambda}$	$\hat{\pi}$	$\% \Delta \hat{\pi}$
θ	1×10^{-5}	4.6416	1929.057	38.96	0.2411	10742.99	+ 22.98%
	0.1	2.6922	1259.395	39.49	0.1372	9894.50	+ 13.27%
	0.2	2.0446	1015.941	39.85	0.1028	9265.76	+ 6.07%
	2	0.5965	405.823	43.11	0.0263	3281.16	-62.44%
	1000	1.8005	976.555	39.40	0.8000	9131.80	+ 4.54%
s	4000	1.7181	898.570	40.02	0.2146	8799.01	+ 0.73%
	12000	1.6931	876.998	43.11	0.0701	8728.43	-0.08%

critical-point conditions. In practice, it is usual that the items are indivisible, and so an integer number of items in the order size is required. In fact, in many cases the items may only come in denominations of a certain integer size, so that there is a further restriction to order sizes that are a multiple of the allowable denomination. In any case, as long as the functions involved are smooth continuous functions, we are able to obtain restricted optima of this kind simply by searching allowable discrete values near the maximising real value. In the above examples, we might round the optimum order size to the nearest integer or we might check the profit at each of the nearby integers in order to determine an optimum discrete order size. This would then change the other quantities slightly, and would lead to slightly less profit than in the continuous optimisation solution.

4. Conclusion

We introduced an EOQ model with a reverse logistics strategy for deteriorating products. We examined the impacts of deterioration of items and the reverse logistics program on inventory management decisions. By optimising the reverse-logistics processes, businesses can experience long-term benefits and reduce revenue spent on returns. Our work complements the EOQ literature by showing that investment in a reverse-logistics program and deterioration of items affect the firm's optimal choices (as evident in the sensitivity analysis in Table 1).

This work can be extended in several ways. The authors are presently developing a reorder-point inventory with a reverse logistics program. Another fruitful extension of the model is to consider an

Appendix A

We set out to establish the KKT conditions for the firm's constrained maximisation problem for an inventory system with a reverse-logistics program.

We begin with the Lagrangian $\mathcal{L}(p, \nu, T, \lambda)$ of Eq. (8); taking partial derivatives with respect to the relevant decision variables, we have the first-order necessary conditions (KKT) for the nonlinear optimisation:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial T} &= -\frac{1}{2}(c + b\theta)(m - kp) + \frac{a + s\nu}{T^2} \leq 0 \Rightarrow T \left(-\frac{1}{2}(c + b\theta)(m - kp) + \frac{a + s\nu}{T^2} \right) = 0, \\ \frac{\partial \mathcal{L}}{\partial p} &= m - 2k \left(p + \frac{1}{2}\Lambda \right) \leq 0 \Rightarrow p \left(m - 2k \left(p + \frac{1}{2}\Lambda \right) \right) = 0, \\ \frac{\partial \mathcal{L}}{\partial \nu} &= \frac{(b_1 - b_2)(m - kp)}{1 - \mu} - \frac{2s\nu}{T} - \frac{\lambda}{1 - \mu} \leq 0 \Rightarrow \nu \left(\frac{(b_1 - b_2)(m - kp)}{1 - \mu} - \frac{2s\nu}{T} - \frac{\lambda}{1 - \mu} \right) = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= \frac{\nu}{1 - \mu} - 1 \leq 0 \Rightarrow \lambda \left(\frac{\nu}{1 - \mu} - 1 \right) = 0, \end{aligned}$$

where $T \geq 0, p \geq 0, \nu \geq 0, \lambda \geq 0$, with $\Lambda = \left(\frac{(b_1 - b_2)\nu}{1 - \mu} + \frac{c}{\theta} - \left(b + \frac{c}{\theta} \right) \left(1 + \frac{1}{2}\theta T \right) \right)$ and λ is the Lagrange multiplier. The resulting systems are solved to find the optimal values, from which the proposition follow. O'Neill and Sanni [19] have shown in their Theorem 3 and Theorem 4 (and Corollary 3), for a general class of inventory optimisation problems with no reverse-logistics consideration, that the profit-rate function is quasi-concave with respect to p and T . Since in this development, the reverse-logistics component is a quadratic function of investment in return cost, the profit-rate function is now the sum of two concave functions. Thus, the KKT necessary conditions are sufficient to guarantee a global maximum policy for the firm.

inventory system with multiple items.

Declaration of Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

CRedit authorship contribution statement

S. Sanni: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Visualization, Validation. **Z. Jovanoski:** Supervision, Project administration, Funding acquisition, Writing - review & editing, Resources, Validation. **H.S. Sidhu:** Supervision, Project administration, Funding acquisition, Writing - review & editing, Resources, Validation.

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Supplementary material

Supplementary material associated with this article can be found, in the online version, at [10.1016/j.orp.2019.100133](https://doi.org/10.1016/j.orp.2019.100133).

References

- [1] Harris FW. How many parts to make at once. *Factory: Mag Manage* 1913;10(2):135–6. 152.
- [2] Sebatjane M, Adetunji O. Economic order quantity model for growing items with imperfect quality. *Operations Research Perspectives* 2019;6:100088. <https://doi.org/10.1016/j.orp.2018.11.004>. <http://www.sciencedirect.com/science/article/pii/S2214716018301866>
- [3] Mosca A, Vidyarthi N, Satir A. Integrated transportation inventory models: A review. *Operations Research Perspectives* 2019;6:100101. <https://doi.org/10.1016/j.orp.2019.100101>. <http://www.sciencedirect.com/science/article/pii/S2214716018302562>
- [4] Chakrabarty T, Giri B, Chaudhuri K. An eoq model for items with weibull distribution deterioration, shortages and trended demand: An extension of philip's model. *Computers & Operations Research* 1998;25(7):649–57. [https://doi.org/10.1016/S0305-0548\(97\)00081-6](https://doi.org/10.1016/S0305-0548(97)00081-6). <http://www.sciencedirect.com/science/article/pii/S0305054897000816>
- [5] Taleizadeh AA. An economic order quantity model for deteriorating item in a purchasing system with multiple prepayments. *Appl Math Model* 2014;38(23):5357–66.
- [6] Richter K. The extended eoq repair and waste disposal model. *International Journal of Production Economics* 1996;45(1):443–7. [https://doi.org/10.1016/0925-5273\(95\)00143-3](https://doi.org/10.1016/0925-5273(95)00143-3). <http://www.sciencedirect.com/science/article/pii/S0925527395001433>
- [7] Alinovi A, Bottani E, Montanari R. Reverse logistics: a stochastic eoq-based inventory control model for mixed manufacturing/remanufacturing systems with return policies. *Int J Prod Res* 2012;50(5):1243–64. <https://doi.org/10.1080/00207543.2011.571921>.
- [8] Liao H, Deng Q. A carbon-constrained eoq model with uncertain demand for re-manufactured products. *Journal of Cleaner Production* 2018;199:334–47. <https://doi.org/10.1016/j.jclepro.2018.07.108>. <http://www.sciencedirect.com/science/article/pii/S0959652618320870>
- [9] Liao H, Deng Q. Ees-eoq model with uncertain acquisition quantity and market demand in dedicated or combined remanufacturing systems. *Applied Mathematical Modelling* 2018;64:135–67. <https://doi.org/10.1016/j.apm.2018.07.026>. <http://www.sciencedirect.com/science/article/pii/S0307904X18303421>
- [10] Robinson A.. What is reverse logistics and how is it different than traditional logistics?2014. <https://cerasis.com/2014/02/19/what-is-reverse-logistics/>.
- [11] Hawks K. What is reverse logistics? *Reverse Logistics Magazine* 2006;Winter/Spring. <http://www.rlmagazine.com/edition01p12.php>
- [12] Wilcox W, Horvath PA, Griffis SE, Autry CW. A markov model of liquidity effects in reverse logistics processes: The effects of random volume and passage. *International Journal of Production Economics* 2011;129(1):86–101. <http://www.sciencedirect.com/science/article/pii/S0925527310003270>
- [13] Horvath PA, Autry CW, Wilcox WE. Liquidity implications of reverse logistics for retailers: a markov chain approach. *J Retail* 2005;81(3):191–203.
- [14] Bose S, Goswami A, Chaudhuri K. An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting. *J Oper Res Soc* 1995;46(6):771–82.
- [15] Taleizadeh AA. Lot-sizing model with advance payment pricing and disruption in supply under planned partial backordering. *Int Trans Oper Res* 2017;24(4):783–800.
- [16] Taleizadeh AA. An eoq model with partial backordering and advance payments for an evaporating item. *Int J Prod Econ* 2014;155:185–93.
- [17] Chen C-K, Weng T-C, Lo C-C. Optimal replenishment quantity for new products and return rate of used products for a retailer. *Appl Math Model* 2016;40(23):9754–66.
- [18] Sanni S, O'Neill B, Jovanoski Z, Sidhu H. Optimisation of perishable inventory items with geometric return rate of used product. 22nd International Congress on Modelling and Simulation. 2017.
- [19] O'Neill B, Sanni S. Profit optimisation for deterministic inventory systems with linear cost. *Comput Ind Eng* 2018;122:303–17.
- [20] Schrady DA. A deterministic inventory model for repairable items. *Naval Res Logist Q* 1967;14(3):391–8.
- [21] Ghare PM, Schrader GF. A model for exponentially decaying inventory. *J Ind Eng* 1963;14(5):238–43.
- [22] Sanni S, Chigbu P. Optimal replenishment policy for items with three-parameter weibull distribution deterioration, stock-level dependent demand and partial backlogging. *Yugoslav J Oper Res* 2017;28(1):107–21.
- [23] Sanni S, O'Neill B. Inventory optimisation in a three-parameter weibull model under a prepayment system. *Comput Ind Eng* 2019;128:298–304.