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Unique closed-form solutions of portfolio selection subject to mean-skewness-normalization constraints

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ABSTRACT

This paper originally proposes two unique closed-form solutions, respectively to risky assets only and a risk-free asset existing situations, of the mean-variance-skewness (MVS) optimization model subject to mean-skewness-normalization constraints for portfolio selection. The efficient frontier and capital allocation surface (CAS) respectively derived from the two solutions are two hyperboloids, and tangent to each other at one hyperbola referred to as the market portfolio curve. Moreover, this curve intersects the mean-skewness plane of the portfolio return with zero-variance (zero-risk) at a line. Calculating the distance between a point on the coincident curve with the vertex of the CAS, we present a novel ratio to measure the performance of the risk-adjusted returns of market portfolio. The ratio is similar to the Sharpe ratio, moreover, under the more realistic assumption that portfolio returns follow a skew-normal distribution, the novel ratio can quantify the degree (or absence) of market portfolio exuberance.

1. Introduction

In the modern portfolio theory proposed by Markowitz [18], obtaining a mean-variance (MV) efficient portfolio denotes finding a weight vector (proportions) of assets in a portfolio, which makes the minimal variance of the portfolio return (gross-rate), and is subject to the two constraints: a certain and pre-specified mean of the portfolio return has to be achieved; the weights have to be summed up to 1. In order to clearly addressing the issue of this traditional theory, at first, some notations need to be reformulated here.

Definition 1. Suppose that there exist $p \in \mathbb{N}$ assets with their returns $\{X_i \mid i = 1, \dots, p\}$. Let $X_i \in \mathbb{R}$ be a random variable, and all of them can be vectorized as a random vector $X \triangleq [X_1 \dots X_p]^T \in \mathbb{R}^{p \times 1}$. Then, its mean vector $\mu \in \mathbb{R}^{p \times 1}$, covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$, coskewness matrix $\Delta \in \mathbb{R}^{p \times (p+p)}$ are defined as

$$\mu \triangleq E(X) \triangleq \mathbf{m}_1,$$

$$\Sigma \triangleq E\{[X - \mu][X - \mu]^T\} \triangleq \tilde{\mathbf{M}}_2,$$

$$\Delta \triangleq E\{[X - \mu][X - \mu]^T \otimes [X - \mu]^T\} \triangleq \tilde{\mathbf{M}}_3.$$

Here, μ , Σ , and Δ can also be considered as the first moment \mathbf{m}_1 , second central moment $\tilde{\mathbf{M}}_2$, third central moment $\tilde{\mathbf{M}}_3$, of X , respectively. \otimes denotes Kronecker product. When the investor proportionately selects these assets as

a portfolio, the portfolio return \bar{X} can be calculated by $\bar{X} = \mathbf{w}^T X$, where $\mathbf{w} \triangleq [w_1 \dots w_p]^T$ is the weight vector (proportions) subject to $\mathbf{w}^T \mathbf{1}_p = 1$, and $\mathbf{1}_p \triangleq [1 \dots 1]^T$ is a $p \times 1$ unitary vector. Then, the mean $\bar{\mu}$, variance $\bar{\sigma}^2$, skewness $\bar{\gamma}^3$ of \bar{X} can be defined as

$$\bar{\mu} \triangleq E(\bar{X}) = \mathbf{w}^T \mu,$$

$$\bar{\sigma}^2 \triangleq E\{[\bar{X} - \bar{\mu}]^2\} = \mathbf{w}^T \Sigma \mathbf{w},$$

$$\bar{\gamma}^3 \triangleq E\{[\bar{X} - \bar{\mu}]^3\} = \mathbf{w}^T \Delta (\mathbf{w} \otimes \mathbf{w}).$$

Then, based on these notations and above assumption, the MV optimization model is formulated as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T \Sigma \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{1}_p = 1, \quad \mathbf{w}^T \mu = \bar{\mu}. \end{aligned} \tag{1}$$

Here, note that the portfolio return is characterized by a normal distribution, its higher-order moments (> 2) are zero, and not be considered in the model. Hence, the solution of this model is so simple as to be widely used until now.

However, Peiro [20] with plenty of other empirical studies pointed out that the portfolio returns do not follow the normal distributions. Accordingly, a variety of recent researches suggested using the

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skewness or higher order moments to improve the MV optimization model in portfolio theory. The improved one via the skewness is referred to as mean-variance-skewness (MVS) optimization model. For example, Konno et al. [14] formulated a primal model maximizing the skewness subject to the fixed mean and variance of the portfolio return:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T \Delta (\mathbf{w} \otimes \mathbf{w}) \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{I}_p = 1, \quad \mathbf{w}^T \boldsymbol{\mu} = \bar{\mu}, \quad \mathbf{w}^T \Sigma \mathbf{w} = \bar{\sigma}^2. \end{aligned} \quad (2)$$

In addition, based on the seminal work by Lai [15], given a MVS optimization model with two or more goals, e.g., with the skewness and mean maximizations:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T \Delta (\mathbf{w} \otimes \mathbf{w}), \quad \max_{\mathbf{w}} \mathbf{w}^T \boldsymbol{\mu} \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{I}_p = 1, \quad \mathbf{w}^T \Sigma \mathbf{w} = \bar{\sigma}^2, \end{aligned} \quad (3)$$

[5,21] with some other researchers suggested using polynomial goal programming (PGP) approach to find a compromise between multiple goals. Furthermore, Li et al. [16] proposed a fuzzy method to solve this multiobjective optimization problem with more goals.

On the other hand, Pressacco and Stucchi [22], Maillet and Jurczenko [17], Jurczenko and Maillet [12], de Athayde and Flores [6], Mencia and Sentana [19], and Harvey et al. [10] stated that the skewness of portfolio return needs to be set as a constraint in the MVS optimization model:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T \Sigma \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{I}_p = 1, \quad \mathbf{w}^T \boldsymbol{\mu} = \bar{\mu}, \quad \mathbf{w}^T \Delta (\mathbf{w} \otimes \mathbf{w}) = \bar{\gamma}^3. \end{aligned} \quad (4)$$

Moreover, Dudzinska-Baryla et al. [7] deformed this MVS model as

$$\begin{aligned} \max_{\mathbf{w}} \quad & \lambda_1 \mathbf{w}^T \boldsymbol{\mu} - \mathbf{w}^T \Sigma \mathbf{w} + \lambda_2 \mathbf{w}^T \Delta (\mathbf{w} \otimes \mathbf{w}) \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{I}_p = 1, \quad w_i \geq 0, \quad i = 1, \dots, p, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0. \end{aligned} \quad (5)$$

If $w_i \geq 0$, $i = 1, \dots, p$ is left out, this deformation is similar with the MVS model (4).

However, for all of the models mentioned above, since the first-conditions of their Lagrangians with respect to \mathbf{w} are quadratic equations, they may have more than one solution.

In recent years, it has been noticed that if the random vector of the asset returns is characterized by another special distribution beyond the normal one, the solution of the MVS optimization model may be unique. During this period, Azzalini and Valle [3] introduced the multivariate skew-normal distribution, which includes the normal distribution, but skews it with one simple parameter. It can be used to solve the problem of the non-unique solution. Azzalini [2] summarized this distribution family, where a p -dimensional skew-normal random vector is defined as $X \sim SN_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha})$ by three parameters, i.e., the location vector $\boldsymbol{\xi}$, scale matrix $\boldsymbol{\Omega}$, and slant vector $\boldsymbol{\alpha}$. Then, its probability density function (PDF) is written as

$$\phi_p(\mathbf{x}; \boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\alpha}) \triangleq 2\phi_p(\mathbf{x}; \boldsymbol{\xi}, \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}^T \boldsymbol{\omega}^{-1}(\mathbf{x} - \boldsymbol{\xi})).$$

Here, $\phi_p(\mathbf{x}; \boldsymbol{\xi}, \boldsymbol{\Omega})$ is the PDF of the p -dimensional normal distribution $N_p(\boldsymbol{\xi}, \boldsymbol{\Omega})$; $\Phi(x)$ denotes the integral function of $\phi_1(x; 0, 1)$, and $\boldsymbol{\omega}$ is a matrix defined by setting the non-diagonal components of $\boldsymbol{\Omega}$ to zero and obtaining its square root as

$$\boldsymbol{\omega} \triangleq (\boldsymbol{\Omega} \diamond \mathbf{I}_p)^{\frac{1}{2}},$$

where \diamond denotes the Hadamard product. In order to accurately describe the skewness of SN_p , Azzalini and Valle [3] introduced a vector parameter $\boldsymbol{\delta}$ to regulate the skewness, which is similar to the coskewness matrix, but without correlated components. Every component value of $\boldsymbol{\delta}$ is limited to the interval $(-1, 1)$, and can be transformed from $\boldsymbol{\alpha}$ using

$$\boldsymbol{\delta} = (1 + \boldsymbol{\alpha}^T \boldsymbol{\Omega} \boldsymbol{\alpha})^{-\frac{1}{2}} \boldsymbol{\Omega} \boldsymbol{\alpha},$$

where $\boldsymbol{\Omega} = \boldsymbol{\omega} \boldsymbol{\Omega} \boldsymbol{\omega}$. Moreover, $\boldsymbol{\delta}$ can be directly calculated as the cube

roots of the skewnesses in the coskewness matrix Δ , or the cube roots of the uncorrelated components in the third central moment \bar{M}_3 .

Therefore, for realizing the unique solution of the MSV optimization model, Simaan [25] seminally supposed that the random vector is on a spherical distribution, which is similar to the skew-normal distribution and is proposed by Kelker [13], and can be abbreviated the coskewness matrix as a skewness vector without coskewnesses. Then, when the skewness vector is fixed as one constraint to replace the coskewness matrix, the MVS optimization model can be rewritten as follows.

Definition 2. Suppose there exist $p \in \mathbb{N}$ assets with returns $\{X_i \mid i = 1, \dots, p\}$. Let these returns be a set of random variables which can be vectorised as a random vector $X \triangleq [X_1 \dots X_p]^T \in \mathbb{R}^{p \times 1}$ satisfying $X \sim SN_p(\boldsymbol{\xi}, \boldsymbol{\Omega}, \boldsymbol{\delta})$. When the investor proportionately selects these assets as a portfolio, the portfolio return \bar{X} can be calculated using $\bar{X} = \mathbf{w}^T X$, where $\mathbf{w} \triangleq [w_1 \dots w_p]^T$ is the weight vector subjected to $\mathbf{w}^T \mathbf{I}_p = 1$, and $\mathbf{I}_p \triangleq [1 \dots 1]^T$ is a $p \times 1$ unitary vector. Then, the mean $\bar{\xi}$, variance $\bar{\omega}$, and skewness $\bar{\delta}$ of \bar{X} can be respectively defined as

$$\bar{\xi} \triangleq \mathbf{w}^T \boldsymbol{\xi}, \quad \bar{\omega}^2 \triangleq \mathbf{w}^T \boldsymbol{\Omega} \mathbf{w}, \quad \bar{\delta} \triangleq \mathbf{w}^T \boldsymbol{\delta}.$$

Then, when the skewness vector is fixed as one constraint to replace the coskewness matrix, the MVS optimization model can be written as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T \boldsymbol{\Omega} \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^T \mathbf{I}_p = 1, \quad \mathbf{w}^T \boldsymbol{\xi} = \bar{\xi}, \quad \mathbf{w}^T \boldsymbol{\delta} = \bar{\delta}. \end{aligned} \quad (6)$$

Here, since the first-condition of its Lagrangian with respect to \mathbf{w} is a linear equation, there just exists one solution of this model. Simaan [25] deduced two linear equations with three unknown coefficients to represent the solution. However, in our opinion, this solution is not the final one, which needs to be transformed in a closed form.

Furthermore, Adcock [1] used skew-Student-t distribution to define another MVS optimization model but without the normalization term $\mathbf{w}^T \mathbf{I}_p = 1$. Similarly, Gan [8] applied the skew-normal distribution to define a model as

$$\begin{aligned} \min_{\mathbf{w}} \quad & \mathbf{w}^T \boldsymbol{\Omega} \mathbf{w} \\ \text{s. t.} \quad & \mathbf{w}^T \boldsymbol{\xi} = \bar{\xi}, \quad \mathbf{w}^T \boldsymbol{\delta} = \bar{\delta}, \end{aligned}$$

without the normalization term too.

Therefore, our purpose of this paper is to derive the unique closed-form solution of the MVS optimization model (6) based on the skew-normal distribution and subject to mean-skewness-normalization constraints. Then we can derive the efficient frontier and capital allocation surface (CAS) from the solutions, which are tangent to each other at one partial hyperbola referred to as the curve of market portfolio. The concept of the efficient frontier will be extended from two-dimensional space to three-dimensional space. By measuring the distance from a point in the curve of market portfolio to the variance-axis through the vertex of CAS, a novel ratio of return versus risk will be presented here. It is similar to the Sharpe ratio, but is more flexible with additional information owing to the more realistic assumption that the portfolio returns follow the skew-normal distribution. The novel ratio can measure not only the performance of the risk-adjusted returns of market portfolio versus their risks, but also the degree (or absence) of their exuberance that expresses the skewnesses of the risk-adjusted returns.

2. Preliminaries

For the purposes of comparison, the famous solutions of the traditional MV optimization model (1), respectively to risky assets only and a risk-free asset existing situations, are reformulated here. Their detailed deriving processes can be referred to Back [4] or other literatures.

2.1. Only risky assets

At first, the weight vector \mathbf{w} can be derived from the following theorem when there only exist risky assets in the portfolio.

Theorem 1. Under Definition 1, supposing that only risky assets are selected by the investor, the Lagrangian \mathcal{L} of (1) can be defined as

$$\begin{aligned} \mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2) &\triangleq \mathbf{w}^T \Sigma \mathbf{w} + \lambda_1 (1 - \mathbf{w}^T \mathbf{1}_p) + \lambda_2 (\bar{\mu} - \mathbf{w}^T \boldsymbol{\mu}) \\ &= \mathbf{w}^T \Sigma \mathbf{w} + \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \left(\begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix} - \begin{bmatrix} \mathbf{1}_p \\ \boldsymbol{\mu}^T \end{bmatrix} \mathbf{w} \right). \end{aligned} \tag{7}$$

Here, given the following definitions:

$$\begin{aligned} A &\triangleq \mathbf{1}_p^T \Sigma^{-1} \mathbf{1}_p, \quad B \triangleq \mathbf{1}_p^T \Sigma^{-1} \boldsymbol{\mu} = \boldsymbol{\mu}^T \Sigma^{-1} \mathbf{1}_p, \\ C &\triangleq \boldsymbol{\mu}^T \Sigma^{-1} \boldsymbol{\mu}, \quad P \triangleq AC - B^2, \end{aligned} \tag{8}$$

we can arrive the solution of \mathbf{w} :

$$\mathbf{w} = \Sigma^{-1} \left[\frac{C\mathbf{1}_p - B\boldsymbol{\mu}}{P} + \frac{A\boldsymbol{\mu} - B\mathbf{1}_p}{P} \bar{\mu} \right], \tag{9}$$

through Appendix A.2.

Note that \mathbf{w} denotes global optimization and implies that an asset with a negative weight is not eliminated from the portfolio. This result differs from the findings of the traditional capital asset pricing model (CAPM) and will be further discussed in Section 4.3.

2.2. Allowing for a risk-free asset

If there exists a risk-free asset among the selected assets, we can also obtain the solution of the MS optimization model (1).

Definition 3. Let a random vector $X \triangleq [X_1 \dots X_p]^T$ be the return vector on p risky assets with a weight vector \mathbf{w}_X , and fulfill $X \sim \mathcal{N}(\boldsymbol{\mu}_X, \Sigma_{XX})$. Let a random variable R denote the return on a risk-free asset with a weight $w_R \triangleq 1 - \mathbf{1}_p^T \mathbf{w}_X$, and fulfill $R \sim \mathcal{N}(\mu_R, \sigma_{RR}^2)$. We merge X and R into a random vector \bar{X} to denote the return vector on these $p + 1$ assets, and fulfill $\bar{X} \sim \mathcal{N}(\bar{\boldsymbol{\mu}}, \bar{\Sigma})$, where

$$\bar{X} \triangleq \begin{bmatrix} X \\ R \end{bmatrix}, \quad \bar{\boldsymbol{\mu}} \triangleq \begin{bmatrix} \boldsymbol{\mu}_X \\ \mu_R \end{bmatrix}, \quad \bar{\Sigma} \triangleq \begin{bmatrix} \Sigma_{XX} & \Sigma_{XR} \\ \Sigma_{RX} & \sigma_{RR}^2 \end{bmatrix}, \quad \bar{\mathbf{w}} \triangleq \begin{bmatrix} \mathbf{w}_X \\ w_R \end{bmatrix}.$$

Here, these covariance matrices and variance are defined as

$$\begin{aligned} \Sigma_{XX} &\triangleq E[(X - \boldsymbol{\mu}_X)(X - \boldsymbol{\mu}_X)^T], \\ \Sigma_{XR} &\triangleq E[(X - \boldsymbol{\mu}_X)(R - \mu_R)] \triangleq \Sigma_{RX}^T, \\ \sigma_{RR}^2 &\triangleq E[(R - \mu_R)(R - \mu_R)]. \end{aligned}$$

Then, the MS optimization model (1) can be rewritten as

$$\begin{aligned} \min_{\bar{\mathbf{w}}} & \quad \bar{\mathbf{w}}^T \bar{\Sigma} \bar{\mathbf{w}} \\ \text{s. t.} & \quad \bar{\mathbf{w}}^T \mathbf{1}_{p+1} = 1, \quad \bar{\mathbf{w}}^T \bar{\boldsymbol{\mu}} = \bar{\mu}. \end{aligned} \tag{10}$$

Theorem 2. Under Definition 3, the Lagrangian \mathcal{L} of Eq. (10) is defined as

$$\mathcal{L}(\bar{\mathbf{w}}, \lambda_1, \lambda_2) \triangleq \bar{\mathbf{w}}^T \bar{\Sigma} \bar{\mathbf{w}} + \lambda_1 (1 - \bar{\mathbf{w}}^T \mathbf{1}_{p+1}) + \lambda_2 (\bar{\mu} - \bar{\mathbf{w}}^T \bar{\boldsymbol{\mu}}).$$

Since R is a constant and uncorrelated to X , we have $\sigma_{RR}^2 = 0$, $\Sigma_{XR} = \mathbf{0}_p$, and $\Sigma_{RX} = \mathbf{0}_p^T$. Given the notations:

$$\boldsymbol{\mu} \triangleq \boldsymbol{\mu}_X, \quad r \triangleq \mu_R, \quad \Sigma \triangleq \Sigma_{XX}, \quad \mathbf{w} \triangleq \mathbf{w}_X, \quad w_0 \triangleq w_R,$$

the constrained minimization problem (10) is equivalent to

$$\begin{aligned} \min_{\mathbf{w}} & \quad \mathbf{w}^T \Sigma \mathbf{w} \\ \text{s. t.} & \quad \mathbf{w}^T \boldsymbol{\mu} + w_0 r = \bar{\mu} + (1 - \mathbf{w}^T \mathbf{1}_p)r = \bar{\mu}. \end{aligned} \tag{11}$$

Then, the corresponding Lagrangian \mathcal{L} can be defined as

$$\begin{aligned} \mathcal{L}(\mathbf{w}, \lambda) &\triangleq \mathbf{w}^T \Sigma \mathbf{w} + \lambda [\bar{\mu} - \mathbf{w}^T \boldsymbol{\mu} - (1 - \mathbf{w}^T \mathbf{1}_p)r] \\ &= \mathbf{w}^T \Sigma \mathbf{w} + \lambda [(\bar{\mu} - r) - (\mathbf{u} - r\mathbf{1}_p)^T \mathbf{w}], \end{aligned}$$

with one Lagrange multiplier λ . Given the following definition:

$$\begin{aligned} G &= (\mathbf{u} - r\mathbf{1}_p)^T \Sigma^{-1} (\mathbf{u} - r\mathbf{1}_p) \\ &= C - 2Br + Ar^2, \end{aligned} \tag{12}$$

we can obtain the solution of \mathbf{w} :

$$\mathbf{w} = \Sigma^{-1} \left[\frac{\boldsymbol{\mu} - r\mathbf{1}_p}{G} (\bar{\mu} - r) \right], \tag{13}$$

through Appendix A.3.

2.3. Efficient frontier

If solutions (9) and (13) are deduced, the corresponding efficient frontier and market portfolio can be plotted.

Corollary 1. Under the solution (9) of \mathbf{w} , let $\bar{X} = \mathbf{w}^T \bar{X}$, based on Appendix A.2, its variance is

$$\bar{\sigma}^2 = \mathbf{w}^T \Sigma \mathbf{w} = \frac{1}{P} (A\bar{\mu}^2 - 2B\bar{\mu} + C). \tag{14}$$

The above equation can be considered as a bullet-shaped hyperbola, sometimes referred to as the Markowitz Bullet:

$$\mathcal{L}_{EF}: -P\bar{\sigma}^2 + A\bar{\mu}^2 - 2B\bar{\mu} + C = 0, \tag{15}$$

with its vertex $\mathcal{P}_{EF} \triangleq (\bar{\mu}_{EF}, \bar{\sigma}_{EF})$ at

$$\bar{\mu}_{EF} = \frac{B}{A}, \quad \bar{\sigma}_{EF} = \left(\frac{1}{A} \right)^{\frac{1}{2}}. \tag{16}$$

We call the bullet-shaped hyperbola efficient frontier of the MV optimization model.

Corollary 2. On the other hand, under the solution (13) of \mathbf{w} , based on Appendix A.3, the variance of the portfolio return \bar{X} on \mathbf{w} is

$$\bar{\sigma}^2 = \mathbf{w}^T \Sigma \mathbf{w} = \frac{1}{G} (\bar{\mu} - r)^2, \tag{17}$$

or

$$\bar{\sigma} = \begin{cases} (\bar{\mu} - r)/G^{\frac{1}{2}}, & \text{if } \bar{\mu} \geq r \\ (r - \bar{\mu})/G^{\frac{1}{2}}, & \text{otherwise} \end{cases}. \tag{18}$$

This can also be considered as double half-lines:

$$\mathcal{L}_{CAL}: -G\bar{\sigma}^2 + (\bar{\mu} - r)^2 = 0, \tag{19}$$

referred to as the best possible capital allocation line (CAL), also known as the capital market line (CML), where its vertex $\mathcal{P}_{CAL} \triangleq (\bar{\mu}_{CAL}, \bar{\sigma}_{CAL})$ is located at $(r, 0)$.

2.4. Market portfolio

Corollary 3. Merging Eqs. (14) and (17) according to Appendix A.6, the efficient frontier and CAL are tangent to each other at one tangency point $\mathcal{P}_{EAC} \triangleq (\bar{\mu}_{EAC}, \bar{\sigma}_{EAC})$, referred to as market portfolio, where

$$\begin{aligned} \bar{\mu}_{EAC} &= \begin{cases} \frac{P}{A(AG - P)^{\frac{1}{2}}} + \frac{B}{A} = \frac{G}{(AG - P)^{\frac{1}{2}}} + r, & \text{if } \bar{\mu}_{EF} \geq r \\ \frac{P - 2AG}{A(AG - P)^{\frac{1}{2}}} + \frac{B}{A} = -\frac{G}{(AG - P)^{\frac{1}{2}}} + r, & \text{otherwise} \end{cases} \\ \bar{\sigma}_{EAC} &= \left(\frac{G}{AG - P} \right)^{\frac{1}{2}}. \end{aligned} \tag{20}$$

Corollary 4. Furthermore, given $\mathcal{P}_{EAC} \triangleq (\bar{\mu}_{EAC}, \bar{\sigma}_{EAC})$, we can define Sharpe ratio $S(\mathcal{P}_{EAC})$ of return versus risk as

$$S(\mathcal{P}_{EAC}) \triangleq \begin{cases} \frac{\bar{\mu}_{EAC} - r}{\bar{\sigma}_{EAC}} = G^{\frac{1}{2}}, & \text{if } \bar{\mu}_{EF} \geq r \\ \frac{r - \bar{\mu}_{EAC}}{\bar{\sigma}_{EAC}} = -G^{\frac{1}{2}}, & \text{otherwise} \end{cases} \quad (21)$$

In the traditional CAPM, the vertex of the efficient frontier $\bar{\mu}_{EF}$ is not less than the risk-free ratio of return r given the assumption that investors always maintain long positions. By contrast, this study supposes that investors can have short positions, while $\bar{\mu}_{EF}$ is negative relative to r . This produces a relatively negative market portfolio, as described in Section 4.4. In Section 4.5, we propose asset picking rules that are applicable to both the MV and MVS optimization models.

3. Main theorems

Based on the premises mention above, the MVS optimization model (6) can be completely solved here in two closed forms.

3.1. Only risky assets

Theorem 3. Under Definition 2, supposing that the investor only selects risky assets, the Lagrangian \mathcal{L} of (6) can then be defined as

$$\begin{aligned} \mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2, \lambda_3) &\triangleq \mathbf{w}^T \Omega \mathbf{w} + \lambda_1(1 - \mathbf{w}^T \mathbf{I}_p) + \lambda_2(\bar{\xi} - \mathbf{w}^T \bar{\xi}) + \lambda_3(\bar{\delta} - \mathbf{w}^T \bar{\delta}) \\ &= \mathbf{w}^T \Omega \mathbf{w} + \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}^T \left(\begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_p^T \\ \bar{\xi}^T \\ \bar{\delta}^T \end{bmatrix} \mathbf{w} \right) \end{aligned} \quad (22)$$

Here, given the following definitions:

$$\begin{aligned} P &\triangleq AC - B^2, & Q &\triangleq AE - D^2, & R &\triangleq CE - F^2, \\ U &\triangleq BD - AF, & V &\triangleq BF - CD, & W &\triangleq DF - BE, \\ \Xi &\triangleq AR + BW + DV = BW + CQ + FU = DV + FU + EP \\ &= ACE - AF^2 + 2BDF - B^2E - CD^2, \end{aligned} \quad (23)$$

which are calculated by

$$\begin{aligned} A &\triangleq \mathbf{I}_p^T \Omega^{-1} \mathbf{I}_p, & B &\triangleq \mathbf{I}_p^T \Omega^{-1} \bar{\xi} = \bar{\xi}^T \Omega^{-1} \mathbf{I}_p, \\ C &\triangleq \bar{\xi}^T \Omega^{-1} \bar{\xi}, & D &\triangleq \mathbf{I}_p^T \Omega^{-1} \bar{\delta} = \bar{\delta}^T \Omega^{-1} \mathbf{I}_p, \\ E &\triangleq \bar{\delta}^T \Omega^{-1} \bar{\delta}, & F &\triangleq \bar{\xi}^T \Omega^{-1} \bar{\delta} = \bar{\delta}^T \Omega^{-1} \bar{\xi}, \end{aligned} \quad (24)$$

we can obtain the solution of \mathbf{w} :

$$\mathbf{w} = \Omega^{-1} \left[\frac{R\mathbf{I}_p + W\bar{\xi} + V\bar{\delta}}{\Xi} + \frac{W\mathbf{I}_p + Q\bar{\xi} + U\bar{\delta}}{\Xi} \bar{\xi} + \frac{V\mathbf{I}_p + U\bar{\xi} + P\bar{\delta}}{\Xi} \bar{\delta} \right], \quad (25)$$

through Appendix A.4.

3.2. Allowing for a risk-free asset

When investors add a risk-free asset to their selected risky assets, the MVS optimization model (6) can be solved using the following approach.

Definition 4. Let a random vector $X \triangleq [X_1 \dots X_p]^T$ be the return vector on p risky assets with a weight vector \mathbf{w}_X , satisfying $X \sim SN_p(\xi_X, \Omega_{XX}, \delta_X)$. Let a random variable R denote the return on a risk-free asset with a weight $w_R \triangleq 1 - \mathbf{I}_p^T \mathbf{w}_X$, satisfying $R \sim SN_1(\xi_R, \omega_{RR}^2, \delta_R)$. We will merge X and R into a random vector \bar{X} to denote the return vector of these $p + 1$ assets where

$$\begin{aligned} \bar{X} &\triangleq \begin{bmatrix} X \\ R \end{bmatrix}, & \bar{\xi} &\triangleq \begin{bmatrix} \xi_X \\ \xi_R \end{bmatrix}, & \bar{\Omega} &\triangleq \begin{bmatrix} \Omega_{XX} & \Omega_{XR} \\ \Omega_{RX} & \omega_{RR}^2 \end{bmatrix}, \\ \bar{\delta} &\triangleq \begin{bmatrix} \delta_X \\ \delta_R \end{bmatrix}, & \bar{\mathbf{w}} &\triangleq \begin{bmatrix} \mathbf{w}_X \\ w_R \end{bmatrix}. \end{aligned}$$

Then, the MVS optimization model (6) can be re-written as

$$\begin{aligned} \min_{\bar{\mathbf{w}}} & \quad \bar{\mathbf{w}}^T \bar{\Omega} \bar{\mathbf{w}} \\ \text{s. t.} & \quad \bar{\mathbf{w}}^T \mathbf{I}_{p+1} = 1, \quad \bar{\mathbf{w}}^T \bar{\xi} = \bar{\xi}, \quad \bar{\mathbf{w}}^T \bar{\delta} = \bar{\delta}. \end{aligned} \quad (26)$$

Theorem 4. Under Definition 4, The associated Lagrangian \mathcal{L} is

$$\begin{aligned} \mathcal{L}(\bar{\mathbf{w}}, \lambda_1, \lambda_2, \lambda_3) &\triangleq \bar{\mathbf{w}}^T \bar{\Omega} \bar{\mathbf{w}} + \lambda_1(1 - \bar{\mathbf{w}}^T \mathbf{I}_{p+1}) + \lambda_2(\bar{\xi} - \bar{\mathbf{w}}^T \bar{\xi}) \\ &\quad + \lambda_3(\bar{\delta} - \bar{\mathbf{w}}^T \bar{\delta}). \end{aligned}$$

Since R is a constant and uncorrelated to X , we have $\omega_{RR}^2 = 0$, $\Omega_{XR} = \mathbf{0}_p$, $\Omega_{RX} = \mathbf{0}_p^T$, and $\delta_R = 0$. Given the notations:

$$\begin{aligned} \xi &\triangleq \xi_X, & r &\triangleq \xi_R, & \Omega &\triangleq \Omega_{XX}, \\ \delta &\triangleq \delta_X, & \mathbf{w} &\triangleq \mathbf{w}_X, & w_0 &\triangleq w_R, \end{aligned}$$

the constrained minimisation problem (26) is equivalent to

$$\begin{aligned} \min_{\mathbf{w}} & \quad \mathbf{w}^T \Omega \mathbf{w} \\ \text{s. t.} & \quad \mathbf{w}^T \bar{\xi} + w_0 r = \mathbf{w}^T \bar{\xi} + (1 - \mathbf{w}^T \mathbf{I}_p)r = \bar{\xi}, \\ & \quad \mathbf{w}^T \bar{\delta} = \bar{\delta}. \end{aligned} \quad (27)$$

¹Then, the corresponding Lagrangian \mathcal{L} can be defined as

$$\begin{aligned} \mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2) &= \mathbf{w}^T \Omega \mathbf{w} + \lambda_1[\bar{\xi} - \mathbf{w}^T \bar{\xi} - (1 - \mathbf{w}^T \mathbf{I}_p)r] + \lambda_2(\bar{\delta} - \mathbf{w}^T \bar{\delta}) \\ &= \mathbf{w}^T \Omega \mathbf{w} + \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \left(\begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix} - \begin{bmatrix} (\bar{\xi} - r\mathbf{I}_p)^T \\ \bar{\delta}^T \end{bmatrix} \mathbf{w} \right), \end{aligned}$$

with two Lagrange multipliers λ_1 and λ_2 . Here, given the following definitions:

$$\begin{aligned} G &\triangleq (\bar{\xi} - r\mathbf{I}_p)^T \Omega^{-1} (\bar{\xi} - r\mathbf{I}_p) = C - 2Br + Ar^2 \\ H &\triangleq \bar{\delta}^T \Omega^{-1} (\bar{\xi} - r\mathbf{I}_p) = F - Dr \\ \Pi &\triangleq EG - H^2 = (CE - F^2) + 2(DF - BE)r + (AE - D^2)r^2 \\ &= R + 2Wr + Qr^2, \end{aligned} \quad (28)$$

we can obtain the solution of \mathbf{w} :

$$\mathbf{w} = \Omega^{-1} \left[\frac{E(\bar{\xi} - r\mathbf{I}_p) - H\bar{\delta}}{\Pi} (\bar{\xi} - r) + \frac{G\bar{\delta} - H(\bar{\xi} - r\mathbf{I}_p)}{\Pi} \bar{\delta} \right], \quad (29)$$

through Appendix A.4.

3.3. Efficient frontier

Similar to the MV optimization model, if solutions (25) and (29) are deduced, the corresponding efficient frontier and tangency portfolio can be plotted.

Corollary 5. Under the solution (25) of \mathbf{w} , let $\bar{X} = \mathbf{w}X$ be the portfolio return, based on Appendix A.4, its variance is

$$\bar{\omega}^2 = \mathbf{w}^T \Omega \mathbf{w} = \frac{1}{\Xi} (R + Q\bar{\xi}^2 + P\bar{\delta}^2 + 2W\bar{\xi} + 2V\bar{\delta} + 2U\bar{\xi}\bar{\delta}). \quad (30)$$

Similar to the hyperbola mentioned in the mean-variance analysis, sometimes referred to as the Markowitz Bullet, the above equation can be

¹ Jiang et al. [11] proposed a MVS model including a risk-free asset like (27), and used its solution of \mathbf{w} to describe an efficient frontier. Here, we use the solution of \mathbf{w} for the MVS model (27) to generate a best possible allocation surface (CAS). Our proposed efficient frontier is formed by the solution of \mathbf{w} derived from the MVS model (6). These two surfaces are described in Section 3.3.

considered as a bullet-shaped hyperboloid:

$$\mathcal{H}_{EF}: -\Xi\bar{\omega}^2 + R + Q\bar{\xi}^2 + P\bar{\delta}^2 + 2W\bar{\xi} + 2V\bar{\delta} + 2U\bar{\xi}\bar{\delta} = 0. \quad (31)$$

with its vertex $\mathcal{P}_{EF} \triangleq (\bar{\xi}_{EF}, \bar{\delta}_{EF}, \bar{\omega}_{EF})$ at

$$\bar{\xi}_{EF} = \frac{PW - UV}{U^2 - PQ}, \quad \bar{\delta}_{EF} = \frac{QV - UW}{U^2 - PQ}. \quad (32)$$

We call the bullet-shaped hyperboloid a novel efficient frontier of the MSV optimization model.

Corollary 6. On the other hand, under the solution (29) of \mathbf{w} , based on Appendix A.5, the variance of the portfolio return \bar{X} on \mathbf{w} is

$$\begin{aligned} \bar{\omega}^2 &= \mathbf{w}^T \mathbf{\Omega} \mathbf{w} = \frac{1}{\Pi} [E(\bar{\xi} - r)^2 + G\bar{\delta}^2 - 2H(\bar{\xi} - r)\bar{\delta}] \\ &= \frac{1}{\Pi} (Er^2 + E\bar{\xi}^2 + G\bar{\delta}^2 + 2Hr\bar{\delta} - 2Er\bar{\xi} - 2H\bar{\xi}\bar{\delta}), \end{aligned} \quad (33)$$

or

$$\bar{\omega} = \begin{cases} [E^{\frac{1}{2}}(\bar{\xi} - r) - G^{\frac{1}{2}}\bar{\delta}]/\Pi^{\frac{1}{2}}, & \text{if } E^{\frac{1}{2}}(\bar{\xi} - r) \geq G^{\frac{1}{2}}\bar{\delta} \\ [G^{\frac{1}{2}}\bar{\delta} - E^{\frac{1}{2}}(\bar{\xi} - r)]/\Pi^{\frac{1}{2}}, & \text{otherwise} \end{cases}. \quad (34)$$

This can also be considered as a hyperboloid:

$$\mathcal{H}_{CAS}: -\Pi\bar{\omega}^2 + E(\bar{\xi} - r)^2 + G\bar{\delta}^2 - 2H(\bar{\xi} - r)\bar{\delta} = 0, \quad (35)$$

referred to as the best possible capital allocation surface (CAS), where its vertex $\mathcal{P}_{CAS} \triangleq (\bar{\xi}_{CAS}, \bar{\delta}_{CAS}, \bar{\omega}_{CAS})$ is located at $(r, 0, 0)$.

3.4. Market portfolio

Corollary 7. Merging Eqs. (30) and (33), the efficient frontier and CAS are tangent to each other at one partial quadratic polynomial C_{EAC} . It is referred to as the curve of tangency portfolio. Projecting it on the mean-skewness plane, we have

$$\begin{aligned} C'_{EAC}: (Q\Pi - E\Xi)\bar{\xi}^2 + (P\Pi - G\Xi)\bar{\delta}^2 + 2(U\Pi + H\Xi)\bar{\xi}\bar{\delta} \\ + 2(W\Pi + E\Xi r)\bar{\xi} + 2(V\Pi - H\Xi r)\bar{\delta} + (R\Pi - E\Xi r^2) = 0. \end{aligned}$$

Given the definitions:

$$\begin{aligned} a &\triangleq Q\Pi - E\Xi, & b &\triangleq P\Pi - G\Xi, & c &\triangleq U\Pi + H\Xi, \\ d &\triangleq W\Pi + E\Xi r, & e &\triangleq V\Pi - H\Xi r, & f &\triangleq R\Pi - E\Xi r^2. \end{aligned} \quad (36)$$

Eq. (36) can be written as

$$C'_{EAC}: a\bar{\xi}^2 + 2b\bar{\xi}\bar{\delta} + c\bar{\delta}^2 + 2d\bar{\xi} + 2e\bar{\delta} + f = 0.$$

From Appendix A.7, we obtain

$$ac = b^2 \geq 0, \quad af = d^2 \geq 0, \quad cf = e^2 \geq 0. \quad (37)$$

Thus, C'_{EAC} is a coincident line:

$$C'_{EAC}: \bar{\delta} = \alpha\bar{\xi} + \beta \quad (38)$$

on the mean-skewness plane, where

$$\alpha \triangleq -\left(\frac{a}{c}\right)^{\frac{1}{2}}, \quad \beta \triangleq -\left(\frac{f}{c}\right)^{\frac{1}{2}}. \quad (39)$$

Its projections on the mean-variance and skewness-variance planes are two hyperbolas, respectively.

Corollary 8. Furthermore, given a point $\mathcal{P}_a \triangleq (\bar{\xi}_a, \bar{\delta}_a, \bar{\omega}_a)$ on C_{EAC} , by calculating the distance from \mathcal{P}_a to the variance-axis through \mathcal{P}_{CAS} , we can define a novel ratio $R(\mathcal{P}_a)$ as

$$R(\mathcal{P}_a) \triangleq \frac{[(\bar{\xi}_a - r)^2 + \bar{\delta}_a^2]^{\frac{1}{2}}}{\bar{\omega}_a}. \quad (40)$$

This is similar to the Sharpe ratio (Sharpe [23], [24]), and can be used to measure the historical risk-adjusted return.

Corollary 9. There exists a point $\hat{\mathcal{P}}_a$ maximizing $R(\mathcal{P}_a)$ on C_{EAC} . This point is referred to as market portfolio $\mathcal{P}_{EAC} \triangleq (\bar{\xi}_{EAC}, \bar{\delta}_{EAC}, \bar{\omega}_{EAC})$, and can be derived from Appendix A.8.

Corollary 10. Furthermore, given the market portfolio $\mathcal{P}_{EAC} \triangleq (\bar{\xi}_{EAC}, \bar{\delta}_{EAC}, \bar{\omega}_{EAC})$, we can define the novel ratio $R(\mathcal{P}_{EAC})$ as

$$R(\mathcal{P}_{EAC}) \triangleq \frac{[(\bar{\xi}_{EAC} - r)^2 + \bar{\delta}_{EAC}^2]^{\frac{1}{2}}}{\bar{\omega}_{EAC}}, \quad (41)$$

to quantify the degree of market portfolio exuberance.

4. Discussion

4.1. Skewness computation

Azzalini and Valle [3] proposed a moment-generating function of the p -dimensional skew-normal distribution. Based on this, Genton et al. [9] gave its central third moment:

$$\bar{\mathbf{M}}_3 = \frac{4 - \pi}{\pi} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \boldsymbol{\delta} \otimes \boldsymbol{\delta}^T \otimes \boldsymbol{\delta}. \quad (42)$$

Thus, $\boldsymbol{\delta}$ can also be calculated by inverting $\bar{\mathbf{M}}_3$. To simplify this calculation, since $\bar{\mathbf{M}}_3$ is expressed by the Kronecker cube of $\boldsymbol{\delta}$, and $\boldsymbol{\delta}$ is a vector, $\bar{\mathbf{M}}_3$ can be split into scalar expressions as

$$m_3(n(p+1) + 1, n+1) = C[\boldsymbol{\delta}(n+1)]^3, \quad n = 0, \dots, p-1,$$

except for the correlated components, where

$$\bar{\mathbf{M}}_3 \triangleq \begin{bmatrix} m_3(1, 1) & m_3(1, 2) & \dots & m_3(1, p) \\ m_3(2, 1) & m_3(2, 2) & \dots & m_3(2, p) \\ \vdots & \vdots & \ddots & \vdots \\ m_3(p^2, 1) & m_3(p^2, 2) & \dots & m_3(p^2, p) \end{bmatrix}, \quad C \triangleq \frac{4 - \pi}{\pi} \left(\frac{2}{\pi}\right)^{\frac{1}{2}},$$

$$\boldsymbol{\delta} \triangleq [\boldsymbol{\delta}(1)\boldsymbol{\delta}(2)\dots\boldsymbol{\delta}(p)]^T.$$

Every expression here can be inverted to a closed-form expression to estimate the corresponding component of $\boldsymbol{\delta}$ as

$$\boldsymbol{\delta}(n+1) = [C^{-1}m_3(n(p+1) + 1, n+1)]^{\frac{1}{3}}, \quad n = 0, \dots, p-1.$$

Here, since the cube roots in all of the closed-form expressions are vectorized as a 1/3 Hadamard power, the above equations can be written as

$$\begin{aligned} \boldsymbol{\delta} &= (C^{-1}\mathbf{m}_3)^{\diamond \frac{1}{3}}, \quad n = 0, \dots, p-1, \\ \mathbf{m}_3 &\triangleq \left[\underbrace{m_3(1, 1) \dots m_3(n(p+1) + 1, n+1) \dots m_3(p^2, p)}_p \right]^T. \end{aligned} \quad (43)$$

Therefore, $\bar{\boldsymbol{\delta}} = \mathbf{w}^T \boldsymbol{\delta}$ contains only the independent elements of the central third moment $\bar{\mathbf{M}}_3$, whereas $\bar{\boldsymbol{\gamma}} = [\mathbf{w}^T \bar{\mathbf{M}}_3 (\mathbf{w} \otimes \mathbf{w})]^{\frac{1}{3}}$ includes all elements of $\bar{\mathbf{M}}_3$. We consider $\bar{\boldsymbol{\delta}}$ to be the simplified version of $\bar{\boldsymbol{\gamma}}$ and let the optimization model (6) approximate the origin (4) to obtain the unique closed-form solutions subject to the mean-skewness-normalization constraints.

4.2. Unified form

Theorem 5. Let a standard quadratic form with equality constraints be

$$\begin{aligned} \min_{\mathbf{w}} & \quad \mathbf{w}^T \mathbf{S} \mathbf{w} \\ \text{s. t.} & \quad \mathbf{T}^T \mathbf{w} = \mathbf{t}, \end{aligned} \quad (44)$$

where $\mathbf{S} \in \mathbb{R}^{n \times n}$ is a positive definite and invertible matrix, $\mathbf{T} \in \mathbb{R}^{m \times n}$ with $m < n$ and $\mathbf{t} \in \mathbb{R}^m$. Since $\mathbf{w}^T \mathbf{S} \mathbf{w}$ is convex, the Karush–Kuhn–Tucker (KKT)

Table 1
All of the optimization models in Theorems 1–4 summarized as a standard and unified quadratic form with their corresponding global optimal solutions of \mathbf{w} .

Theorem	S	T	t	w
Theorem 1	Σ	$\begin{bmatrix} \mathbf{I}_p^T \\ \mu^T \end{bmatrix}$	$\begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix}$	Eq. (9)
Theorem 2	Σ	$(\mu - r\mathbf{I}_p)^T$	$\bar{\mu} - r$	Eq. (13)
Theorem 3	Ω	$\begin{bmatrix} \mathbf{I}_p^T \\ \xi^T \\ \delta^T \end{bmatrix}$	$\begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix}$	Eq. (25)
Theorem 4	Ω	$\begin{bmatrix} (\xi - r\mathbf{I}_p)^T \\ \delta^T \end{bmatrix}$	$\begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix}$	Eq. (29)

points are the global optimal solution in a closed form. With the Lagrangian function:

$$\mathcal{L}(\mathbf{w}, \lambda) \triangleq \mathbf{w}^T \mathbf{S} \mathbf{w} + \lambda^T (\mathbf{t} - \mathbf{T} \mathbf{w}), \tag{45}$$

the first-order conditions for \mathbf{w} and λ are

$$\begin{aligned} \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \lambda) &= 2\mathbf{S} \mathbf{w} - \mathbf{T}^T \lambda = 0, \\ \nabla_{\lambda} \mathcal{L}(\mathbf{w}, \lambda) &= \mathbf{t} - \mathbf{T} \mathbf{w} = 0. \end{aligned}$$

In the first equation, $\mathbf{w} = \frac{1}{2} \mathbf{S}^{-1} \mathbf{T}^T \lambda$. Substituting \mathbf{w} into the second equation, we get $\lambda = 2(\mathbf{T} \mathbf{S}^{-1} \mathbf{T}^T)^{-1} \mathbf{t}$, and

$$\mathbf{w} = \mathbf{S}^{-1} \mathbf{T}^T (\mathbf{T} \mathbf{S}^{-1} \mathbf{T}^T)^{-1} \mathbf{t}. \tag{46}$$

Corollary 11. Given Eq. (46), the quadratic form $\mathbf{w}^T \mathbf{S} \mathbf{w}$ can be expressed as

$$\begin{aligned} \mathbf{w}^T \mathbf{S} \mathbf{w} &= [\mathbf{t}^T (\mathbf{T} \mathbf{S}^{-1} \mathbf{T}^T)^{-1} \mathbf{T} \mathbf{S}^{-1}] \mathbf{S} [\mathbf{S}^{-1} \mathbf{T}^T (\mathbf{T} \mathbf{S}^{-1} \mathbf{T}^T)^{-1} \mathbf{t}] \\ &= \mathbf{t}^T (\mathbf{T} \mathbf{S}^{-1} \mathbf{T}^T)^{-1} \mathbf{t}. \end{aligned} \tag{47}$$

As shown in the Table 1, the optimization models in Theorems 1–4 can be summarized as a standard and unified quadratic form with equality constraints in Theorem 5. It is easy to derive their corresponding global optimal solutions for \mathbf{w} using the proofs in Appendixes A.2, A.3, A.4, and A.5.

4.3. Negative weight

A remarkable outcome of the traditional CAPM is the market portfolio, which is a theoretical bundle of investments comprising every type of asset available in the global financial market. It lies on the mean-variance-efficient frontier; in other words, the set of portfolios with mean-variance characteristics cannot be improved upon. Since the weight of each asset in a frontier portfolio is proportional to its total market capitalization, the CAPM is inconsistent with the efficient frontiers, wherein every frontier portfolio has at least one negative weight or short position.

By contrast, contingent on whether an individual is allowed to short sell, this study assumes that an individual’s frontier portfolio has at least one asset with a negative weight in a submarket, although the market as a whole has non-negative weights. Therefore, portfolio asset weight (positive or negative) is used to proportionally measure exposure to a given asset class (long or short).

In this case, a short position on an uncertain asset means borrowing the asset, selling it now, buying it back later, and then returning it to the original owner. This means that the asset will have a negative weight in the portfolio. However, this only holds for an individual if we add up all promises on a single security, and as a result, certain individuals will eventually hold that security, causing the market as a whole to have non-negative weights.

4.4. Relatively negative market portfolio

Under the above-mentioned assumption, the CAL comprises two tangent lines (upper and lower) drawn from the risk-free rate of return r to a feasible region for risky assets. If the vertex of the efficient frontier \mathcal{P}_{EF} is greater than the risk-free rate of return r , the upper CAL will be tangent to the efficient frontier \mathcal{L}_{EF} with a tangency point \mathcal{P}_{EAC} . Here, the tangency point denotes an optimal market portfolio that offers the lowest risk for a fixed level of expected returns, irrespective of whether the portfolio includes the risk-free asset having zero weight ($w_0 = 1 - \mathbf{w}^T \mathbf{I}_p$). By contrast, if the vertex of the efficient frontier \mathcal{P}_{EF} is less than the risk-free rate of return r , the lower CAL will be tangent to the efficient frontier \mathcal{L}_{EF} at another tangency point \mathcal{P}_{EAC} . The two market portfolios on the upper and lower CALs are considered the positive and negative market portfolios relative to the risk-free asset and do not occur simultaneously. However, the traditional CAPM only accepts the upper CAL with the relatively positive market portfolio.

4.5. Asset picking rule

Given the two assumptions for negative weight and a relatively negative market portfolio, the following can be considered rules of thumb for asset picking:

- If there is a relatively positive market portfolio, sell risk-free asset and buy risky assets aligned with the market portfolio, that is, buy and sell the risky assets on the basis of their positive (for buying) and negative (for selling) weights.
- In the case of a relatively negative market portfolio, sell risky assets in the market portfolio, that is, sell and buy the risky assets on the basis of their positive (for buying) and negative (for selling) weights and buy risk-free asset.

The two cases starkly contrast each other. Here, as mentioned above, selling assets (maintaining a short position) requires security, particularly if investors do not own them.

It should be noticed that since the risk-free asset in the market portfolio has zero weight, its quantity for selling or buying is undetermined. Sometimes investors tend to set the total price of the risk-free asset for selling and buying equal to that of the risky assets inversely for buying and selling.

5. Empirical study

The data set tested here contains daily returns for all stocks except Visa Inc. included in the Dow Jones Industrial Average index, from January, 2001 through December, 2014. For comparing the differences between the MV and MVS models, we extract three subsets from it, whose time spans are set from January, 2003 through December, 2007, from January, 2008 through December, 2012, and from January, 2010 through December, 2014, respectively. The effective daily federal funds rate is considered as the risk-free rate of return, which is fluctuant for every subset.

First, we present the two solutions of the MV optimization model for every subsets, and severally plot the results by MATLAB in Figs. 1 (a)–3 (a). The Efficient frontier \mathcal{L}_{EF} and CAL \mathcal{L}_{CAL} are plotted in red and blue with their vertexes \mathcal{P}_{EF} and \mathcal{P}_{CAL} , respectively. Since \mathcal{P}_{EF} is lower than \mathcal{P}_{CAS} for every subset, the corresponding market portfolio \mathcal{P}_{EAC} is relatively negative to the risk-free rate of return, and instructs the investors to enter into short position. Moreover, since the Sharpe ratios $S(\mathcal{P}_{EAC})$ of the market portfolios for these subsets are 1.9264, 0.2124 and 0.1188, the selling pressure in the first subset is stronger than others.

On the other hand, by calculating the multivariate skew-normal distribution of every datasets with R Package ‘sn’, we also give the two solutions of the MVS optimization model, and plot the results in

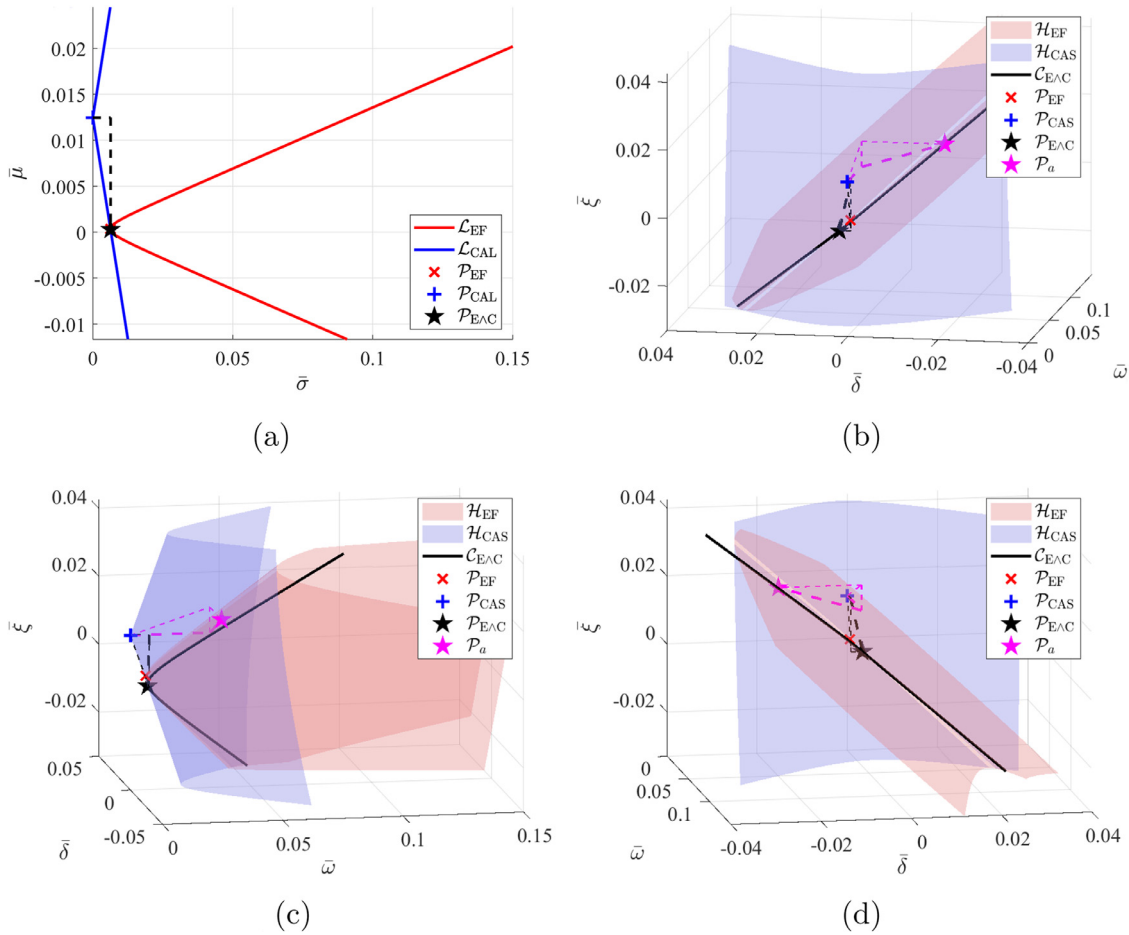


Fig. 1. Solutions of MV and MVS optimization models calculated from the first subset (from January, 2003 through December, 2007). (a): Efficient frontier \mathcal{L}_{EF} in red and CAL \mathcal{L}_{CAL} in blue with their vertices \mathcal{P}_{EF} in red and \mathcal{P}_{CAL} in blue are tangent at market portfolio \mathcal{P}_{EAC} in black; (b), (c) and (d): efficient frontier \mathcal{H}_{EF} in red and CAS \mathcal{H}_{CAS} in blue with their vertices \mathcal{P}_{EF} in red and \mathcal{P}_{CAS} in blue are tangent at the curve of market portfolio C_{EAC} in black in three views (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.).

figures (b), (c) and (d) of Figs. 1–3. The efficient frontier \mathcal{H}_{EF} and CAS \mathcal{H}_{CAS} are plotted in red and blue with their vertices \mathcal{P}_{EF} and \mathcal{P}_{CAS} , respectively. Then, their tangent curve (the curve of market portfolio) C_{EAC} is drawn in black. Given a portfolio $\mathcal{P}_a(\bar{\xi}_a = 0.02)$ marked in magenta in C_{EAC} , we use the magenta line to indicate the distance $[(\bar{\xi}_a - r)^2 + \bar{\delta}_a^2]^{\frac{1}{2}}$ from \mathcal{P}_a to the variance-axis through \mathcal{P}_{CAS} , and use another magenta line to denotes $\bar{\omega}_a$. Their ratio $R(\mathcal{P}_a)$ defined by Corollary 8 is considered as the novel ratio to measure the historical risk-adjusted return of \mathcal{P}_a . Moreover, we can obtain the market portfolio \mathcal{P}_{EAC} with the greatest novel ratio by Corollary 9, which is marked in black in C_{EAC} . The novel ratios $R(\mathcal{P}_{EAC})$ of the market portfolios for these subsets are 2.0001, 0.7825 and 0.8382, In addition, the pressures in the first and second subsets are selling pressures according to their relatively negative market portfolios. Therefore, the selling pressure in the first subset is stronger than that of the second subset. But since the market portfolio of the third subset is positive relative to the risk-free rate of return, its novel ratio bring the investors a buying pressure. Different from the pressure measured by the Sharpe ratio, this pressure measured by our proposed ratio is generated by not only the risk-adjusted return of the market portfolio with its risk, but also its exuberance. Here, the exuberance reflects the positive or negative skewness of the risk-adjusted return.

The solutions of weight for MV and MVS models calculated from the three subsets are shown in Figs. 4–6, respectively. We find that the overall shapes of weight for MV and MVS models of the first subset in Fig. 4 are similar to each other, hence both the models suggest the investors entering into similar short positions. Here, there exist a few

differences between the two solutions of weight, and only DIS, GE, JPM, MRK, MSFT TRV and UNH are inverse. It denotes that the portfolio exuberance, determined by the skewness of the risk-adjusted return, is not outstanding in this MVS model.

For the MV and MVS models of the second subset in Fig. 5, a few same elements of the two solutions of weight keep the similar and great values, such as JNJ, KO, MCD and T, but the others have diverse values with different volumes and inverse sign patterns. Therefore, we consider that the portfolio exuberance is playing an important role in the optimization of the MVS model for the second subset.

As shown in Fig. 6 for the third subset, except CSCO, GS and IBM, all of the same elements of the two solutions of weight have significant differences from each other. It causes the negative portfolio risk-adjusted return, thus the MV model suggests the investors entering into a short position. On the contrary, the MVS model implies that the investors need to take a long position because the market portfolio has become exuberant, although its portfolio risk-adjusted return is not remarkably positive.

6. Concluding remarks

By ignoring the distribution used to characterise the asset and portfolio returns, the mean-variance-skewness (MVS) optimization model can be constructed only using the moments from the first to third or a high-order one. However, its solution is too complicated to be popularly applied. Hence, finding a probability distribution with a simple skewness becomes important. The skew-normal distribution

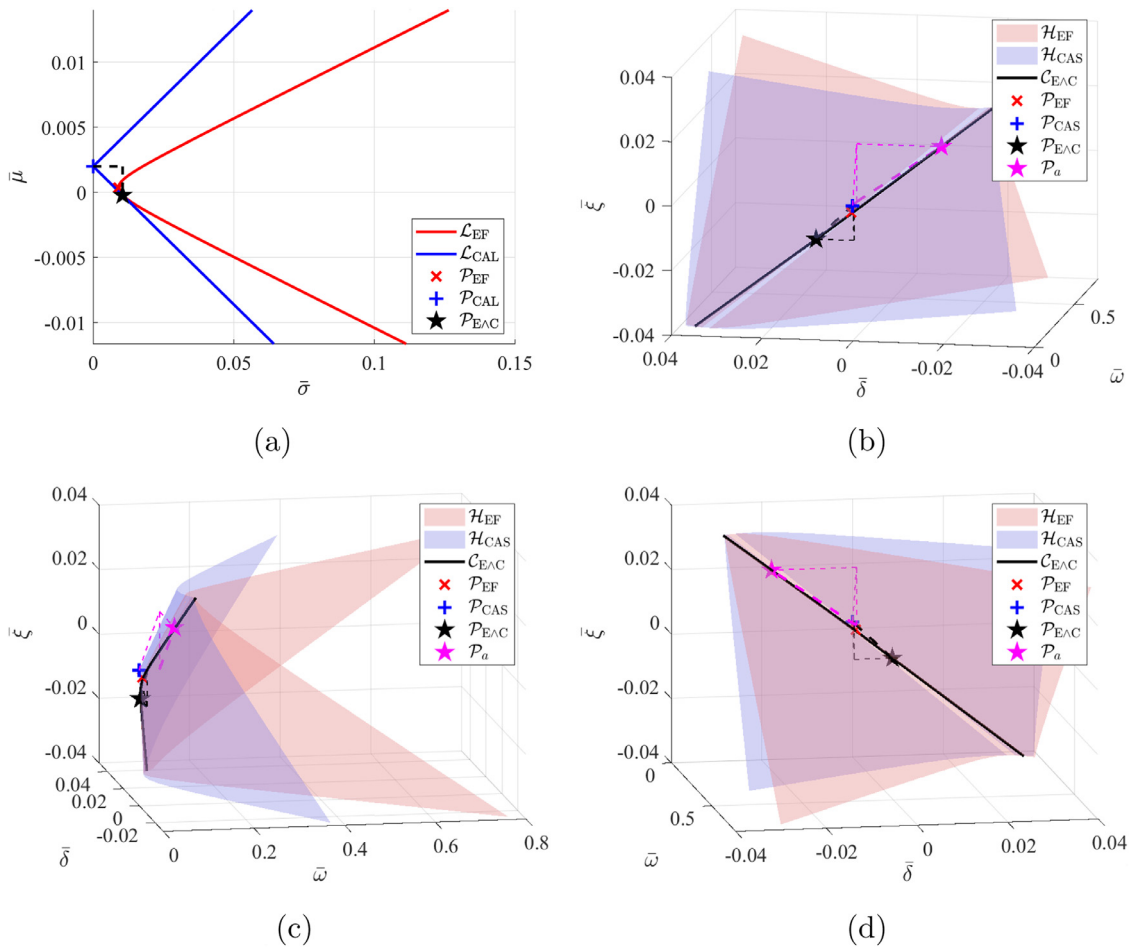


Fig. 2. Solutions of MV and MVS optimization models calculated from the second subset (from January, 2008 through December, 2012). (a): Efficient frontier \mathcal{L}_{EF} in red and CAL \mathcal{L}_{CAL} in blue with their vertexes \mathcal{P}_{EF} in red and \mathcal{P}_{CAL} in blue are tangent at market portfolio \mathcal{P}_{EAC} in black; (b), (c) and (d): efficient frontier \mathcal{H}_{EF} in red and CAS \mathcal{H}_{CAS} in blue with their vertexes \mathcal{P}_{EF} in red and \mathcal{P}_{CAS} in blue are tangent at the curve of market portfolio \mathcal{C}_{EAC} in black in three views (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.).

with its related families gives the opportunity to build a solvable model (6) due to its simple skewness vector. Based on this, we provided two unique closed-form solutions of (6) by adapting the model with three constraints, i.e., the skewness, mean, and normalization. Although these solutions are more complicated than those of the mean-variance (MV) optimization model, they are more easily calculated using a computer than the traditional non-unique or unclosed-form solutions of the various MVS optimization models.

The efficient frontier and capital allocation surface (CAS) derived from the solutions are both partial hyperboloids. They are tangent to each other at one partial hyperbola referred to as the curve of market portfolio. The concept of the efficient frontier is extended from two-dimensional space to three-dimensional space.

By measuring the distance from a point in the curve of tangency portfolio to the variance-axis through the vertex of CAS, a novel ratio of return versus risk is presented here. It is similar to the Sharpe ratio, but

is more flexible with additional information owing to the more realistic assumption that the portfolio returns follow the skew-normal distribution. The novel ratio can measure not only the performance of the risk-adjusted returns of market portfolio versus their risks, but also the degree (or absence) of their exuberance, special for the case that the expected ratio of return is close to zero. Here, the great exuberance quantified by the great skewness implies the high probability of potential profit for portfolio selection in short time,

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Appendix A

A1. Lemma 1

Lemma 1. An $(m \times m)$ matrix A^{-1} is the inverse of the $(m \times m)$ matrix A , if $AA^T = A^{-1}A = I_m$. For instance, for $m = 2$, given

$$A \triangleq \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

the inverse of A is

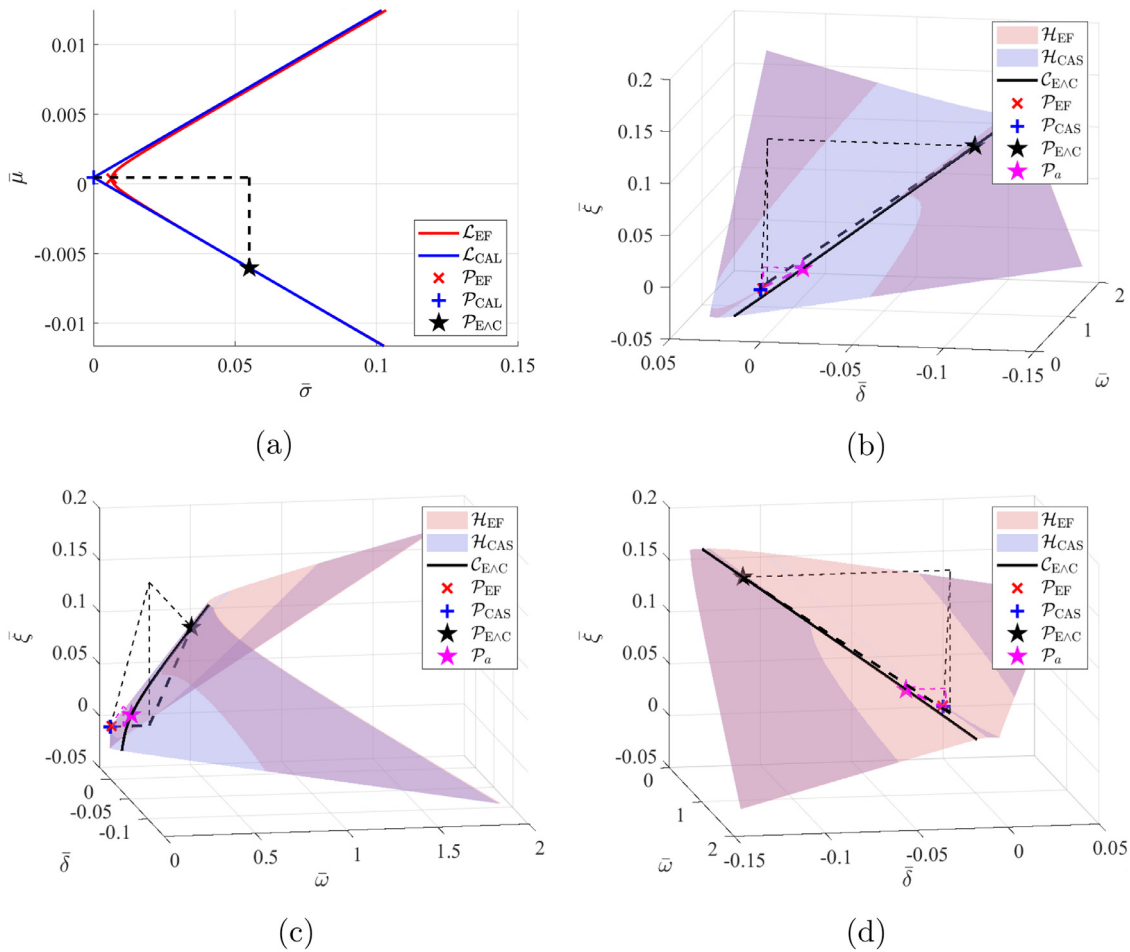


Fig. 3. Solutions of MV and MVS optimization models calculated from the third subset (from January, 2010 through December, 2014). (a): Efficient frontier \mathcal{L}_{EF} in red and CAL \mathcal{L}_{CAL} in blue with their vertices \mathcal{P}_{EF} in red and \mathcal{P}_{CAL} in blue are tangent at market portfolio \mathcal{P}_{EAC} in black; (b), (c) and (d): efficient frontier \mathcal{H}_{EF} in red and CAS \mathcal{H}_{CAS} in blue with their vertices \mathcal{P}_{EF} in red and \mathcal{P}_{CAS} in blue are tangent at the curve of market portfolio \mathcal{C}_{EAC} in black in three views (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.).

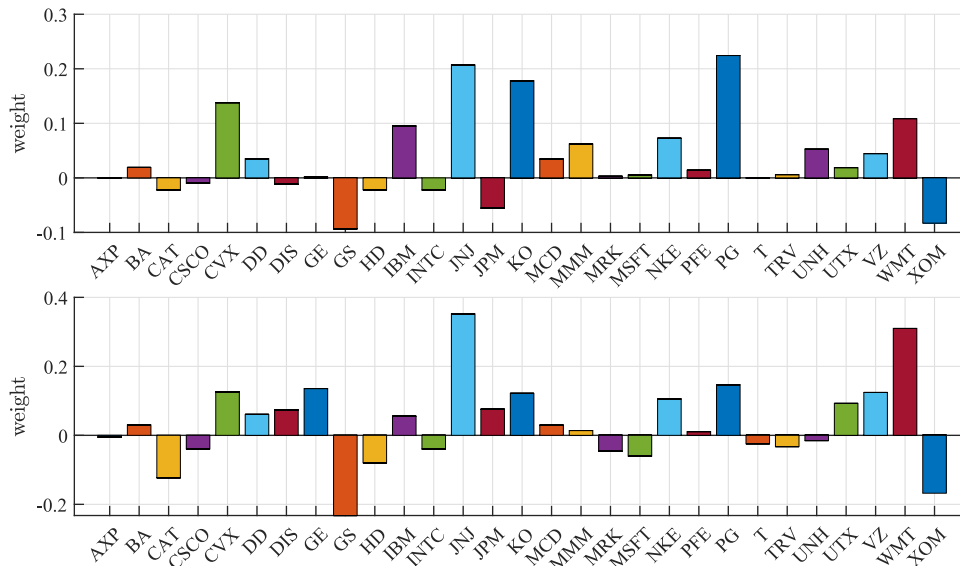


Fig. 4. Optimized weights for MV and MVS models calculated from the first subset (from January, 2003 through December, 2007). Upper: optimized weight for MV model; lower: optimized weight for MVS model.

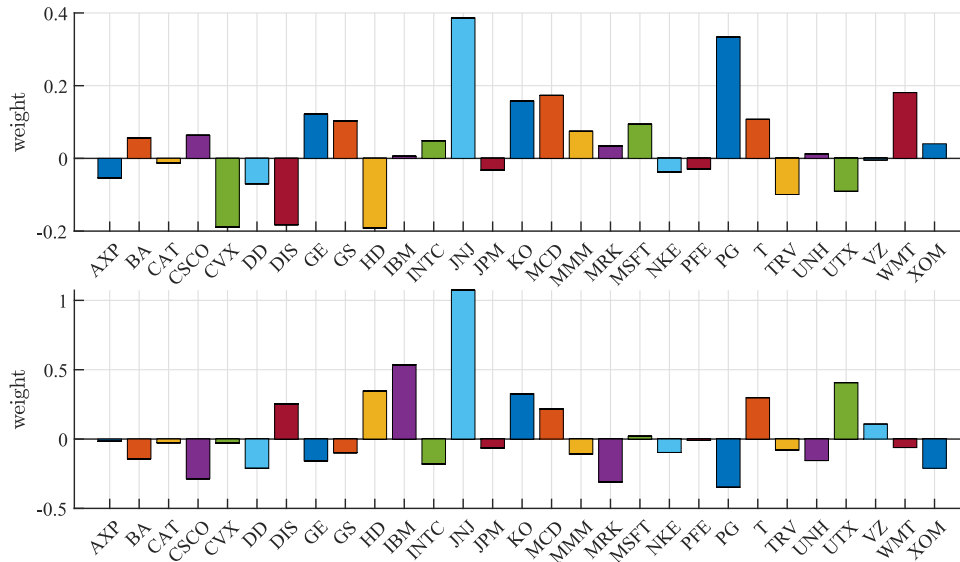


Fig. 5. Optimized weights for MV and MVS models calculated from the second subset (from January, 2008 through December, 2012). Upper: optimized weight for MV model; lower: optimized weight for MVS model.

$$\begin{aligned}
 \mathbf{A}^{-1} &= \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} \\
 &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix},
 \end{aligned}
 \tag{48}$$

where $\det(\mathbf{A})$ is the determinant of \mathbf{A} . For $m = 3$, given

$$\mathbf{A} \triangleq \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the inverse of \mathbf{A} is

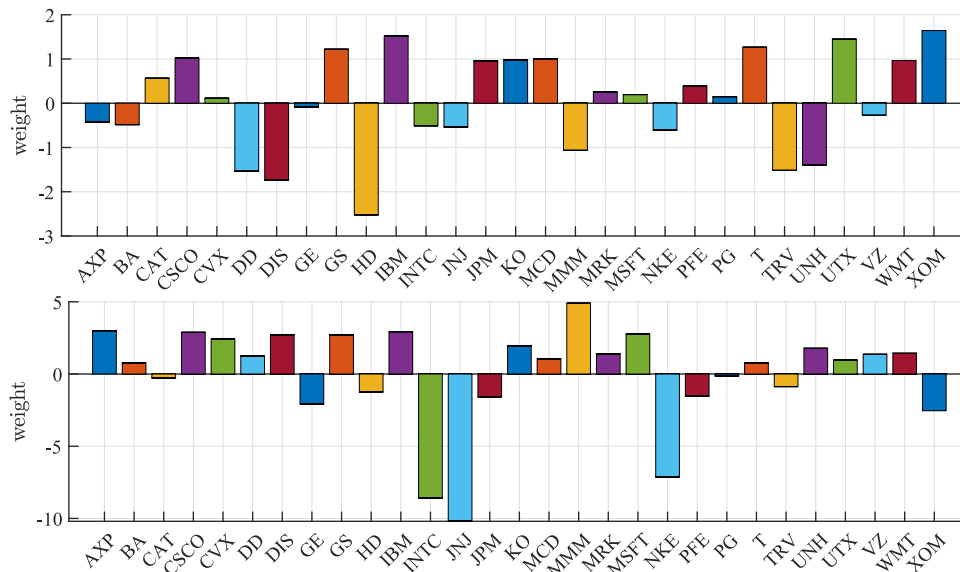


Fig. 6. Optimized weight for MV and MVS models calculated from the third subset (from January, 2010 through December, 2014). Upper: optimized weight for MV model; lower: optimized weight for MVS model.

$$\begin{aligned}
 \mathbf{A}^{-1} &= \frac{1}{\det(\mathbf{A})} \begin{bmatrix} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} & -\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} & \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
 -\det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & -\det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \\
 \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} & -\det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{bmatrix} \\
 &= \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\
 a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{11}a_{22} - a_{12}a_{21} \\
 a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}, \tag{49}
 \end{aligned}$$

where the determinant of \mathbf{A} is

$$\begin{aligned}
 \det(\mathbf{A}) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
 &= a_{21}(a_{12}a_{33} - a_{13}a_{32}) - a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{11}a_{32} - a_{12}a_{31}) \\
 &= a_{31}(a_{12}a_{23} - a_{13}a_{22}) - a_{32}(a_{11}a_{23} - a_{13}a_{21}) + a_{33}(a_{11}a_{22} - a_{12}a_{21}). \tag{50}
 \end{aligned}$$

A2. Proof of theorem 1

The Lagrangian function Eq. (7) in Theorem 1 can be considered as a case of Eq. (45) in Theorem 5. Therefore, by assuming

$$\mathbf{S} = \Sigma, \mathbf{T} = \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix}, \mathbf{t} = \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix}, \tag{51}$$

and substituting them into Eq. (46), we obtain the solution of \mathbf{w} in Theorem 1 as

$$\begin{aligned}
 \mathbf{w} &= \Sigma^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix} \left(\begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix} \\
 &= \Sigma^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix} \\
 &= \Sigma^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\mu}^\top \end{bmatrix} \frac{1}{P} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix} \tag{52}
 \end{aligned}$$

with Eq. (48) in Lemma (1). Moreover, by substituting (51) into Eq. (47) of Corollary 11, we have the corresponding quadratic form in Corollary 1 as

$$\mathbf{w}^\top \mathbf{S} \mathbf{w} = \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix}^\top \frac{1}{P} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\mu} \end{bmatrix}. \tag{53}$$

A3. Proof of Theorem 2

The Lagrangian function Eq. (12) in Theorem 2 can be considered as a case of Eq. (45) in Theorem 5. Therefore, by assuming

$$\mathbf{S} = \Sigma, \mathbf{T} = (\boldsymbol{\mu} - r\mathbf{I}_p)^\top, \mathbf{t} = \bar{\mu} - r, \tag{54}$$

and substituting them into Eq. (46), we obtain the solution of \mathbf{w} in Theorem 2 as

$$\begin{aligned}
 \mathbf{w} &= \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{I}_p) [(\boldsymbol{\mu} - r\mathbf{I}_p)^\top \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{I}_p)]^{-1}(\bar{\mu} - r) \\
 &= \Sigma^{-1}(\boldsymbol{\mu} - r\mathbf{I}_p) G^{-1}(\bar{\mu} - r). \tag{55}
 \end{aligned}$$

Moreover, by substituting (54) into Eq. (47) of Corollary 11, we have the corresponding quadratic form in Corollary 2 as

$$\mathbf{w}^\top \mathbf{S} \mathbf{w} = (\bar{\mu} - r) G^{-1}(\bar{\mu} - r). \tag{56}$$

A4. Proof of Theorem 3

The Lagrangian function Eq. (22) in Theorem 3 can be considered as a case of Eq. (45) in Theorem 5. Therefore, by assuming

$$\mathbf{S} = \Omega, \mathbf{T} = \begin{bmatrix} \mathbf{I}_p^\top \\ \boldsymbol{\xi}^\top \\ \boldsymbol{\delta}^\top \end{bmatrix}, \mathbf{t} = \begin{bmatrix} 1 \\ \xi \\ \delta \end{bmatrix}, \tag{57}$$

and substituting them into Eq. (46), we obtain the solution of \mathbf{w} in Theorem 3 as

$$\begin{aligned}
 \mathbf{w} &= \Omega^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \left(\begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \Omega^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \begin{bmatrix} A & B & D \\ B & C & F \\ D & F & E \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \frac{1}{\Xi} \begin{bmatrix} CE - F^2 & DF - BE & BF - CD \\ DF - BE & AE - D^2 & BD - AF \\ BF - CD & BD - AF & AC - B^2 \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} \mathbf{I}_p^\top \\ \xi^\top \\ \delta^\top \end{bmatrix} \frac{1}{\Xi} \begin{bmatrix} R & W & V \\ W & Q & U \\ V & U & P \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix}
 \end{aligned} \tag{58}$$

with Eq. (49) in Lemma (1). Moreover, by substituting (57) into Eq. (47) of Corollary 11, we have the corresponding quadratic form in Corollary 5 as

$$\mathbf{w}^\top \mathbf{S} \mathbf{w} = \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix}^\top \frac{1}{\Xi} \begin{bmatrix} R & W & V \\ W & Q & U \\ V & U & P \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\xi} \\ \bar{\delta} \end{bmatrix}. \tag{59}$$

A5. Proof of Theorem 4

The Lagrangian function Eq. (28) in Theorem 4 can be considered as a case of Eq. (45) in Theorem 5. Therefore, by assuming

$$\mathbf{S} = \Omega, \mathbf{T} = \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix}, \mathbf{t} = \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix}, \tag{60}$$

and substituting them into Eq. (46), we obtain the solution of \mathbf{w} in Theorem 4 as

$$\begin{aligned}
 \mathbf{w} &= \Omega^{-1} \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \left(\begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \Omega^{-1} \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \begin{bmatrix} C - 2Br + Ar^2 & F - Dr \\ F - Dr & E \end{bmatrix}^{-1} \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \begin{bmatrix} G & H \\ H & E \end{bmatrix}^{-1} \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix} \\
 &= \Omega^{-1} \begin{bmatrix} (\xi - r\mathbf{I}_p)^\top \\ \delta^\top \end{bmatrix} \frac{1}{\Pi} \begin{bmatrix} E & -H \\ -H & G \end{bmatrix} \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix}
 \end{aligned} \tag{61}$$

with Eq. (48) in Lemma (1). Moreover, by substituting (60) into Eq. (47) of Corollary 11, we have the corresponding quadratic form in Corollary 6 as

$$\mathbf{w}^\top \mathbf{S} \mathbf{w} = \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix}^\top \frac{1}{\Pi} \begin{bmatrix} E & -H \\ -H & G \end{bmatrix} \begin{bmatrix} \bar{\xi} - r \\ \bar{\delta} \end{bmatrix}. \tag{62}$$

A6. Proof of Corollary 3

By Eqs. (8) and (12), we have

$$\begin{aligned}
 AG - P &= A(Ar^2 - 2Br + C) - (AC - B^2) \\
 &= A^2r^2 - 2ABr + B^2 \\
 &= (Ar - B)^2,
 \end{aligned}$$

which can be written as

$$Ar - B = -(AG - P)^{\frac{1}{2}}, \tag{63}$$

or

$$r = \frac{B}{A} - \frac{(AG - P)^{\frac{1}{2}}}{A}. \tag{64}$$

Then reforming

$$\begin{aligned}
 (A\bar{\mu} - B)^2 &= (A\bar{\mu} - Ar + Ar - B)^2 \\
 &= A^2(\bar{\mu} - r)^2 + 2A(\bar{\mu} - r)(Ar - B) + (Ar - B)^2 \\
 &= A^2G\bar{\sigma}^2 - 2AG^{\frac{1}{2}}(AG - P)^{\frac{1}{2}}\bar{\sigma} + AG - P,
 \end{aligned} \tag{65}$$

by Eqs. (17) and (63), we obtain

$$\begin{aligned}
 AP\bar{\sigma}^2 - (A^2\bar{\mu}^2 - 2AB\bar{\mu} + AC) &= 0 \\
 AP\bar{\sigma}^2 - (A^2\bar{\mu}^2 - 2AB\bar{\mu} + B^2) &= AC - B^2 \\
 AP\bar{\sigma}^2 - (A\bar{\mu} - B)^2 &= P \quad (\text{by Eq.(8)}) \\
 AP\bar{\sigma}^2 - A^2G\bar{\sigma}^2 + 2AG\frac{1}{2}(AG - P)\frac{1}{2}\bar{\sigma} - AG + P &= P \quad (\text{by Eq.(65)}) \\
 (AG - P)\bar{\sigma}^2 - 2G\frac{1}{2}(AG - P)\frac{1}{2}\bar{\sigma} + G &= 0.
 \end{aligned} \tag{66}$$

Since $\bar{\sigma} \geq 0$, we get

$$\bar{\sigma} = \left(\frac{G}{AG - P} \right)^{\frac{1}{2}}. \tag{67}$$

By substituting it into Eq. (17), we have

$$\bar{\mu} = \begin{cases} \frac{G}{(AG - P)^{\frac{1}{2}}} + r, & \text{if } \bar{\mu}_{EF} \geq r \\ -\frac{G}{(AG - P)^{\frac{1}{2}}} + r, & \text{otherwise} \end{cases}. \tag{68}$$

Moreover, by substituting Eq. (64) into the above equation, we obtain

$$\bar{\mu} = \begin{cases} \frac{P}{A(AG - P)^{\frac{1}{2}}} + \frac{B}{A}, & \text{if } \bar{\mu}_{EF} \geq r \\ \frac{P - 2AG}{A(AG - P)^{\frac{1}{2}}} + \frac{B}{A}, & \text{otherwise} \end{cases}. \tag{69}$$

A7. Proof of Eq. (37) in Corollary 7

$$\begin{aligned}
 b^2 - ac &= (U\Pi + H\Xi)^2 - (Q\Pi - E\Xi)(P\Pi - G\Xi) \\
 &= U^2\Pi^2 - 2HU\Xi\Pi + (H^2 - EG)\Xi^2 - Q\Pi\Pi^2 + GQ\Xi\Pi + EP\Xi\Pi \\
 &= \Pi[U^2\Pi - Q\Pi\Pi + GQ\Xi + 2HU\Xi - \Xi^2 + EP\Xi] \\
 &= \Pi[U^2\Pi - Q\Pi\Pi + (C - 2Br + Ar^2)Q\Xi \\
 &\quad + 2(F - Dr)U\Xi r - \Xi^2 + EP\Xi] \\
 &= \Pi[U^2\Pi - Q\Pi\Pi - (EP - \Xi + 2FU + CQ)\Xi - 2(BQ + DU)\Xi r + AQ\Xi r^2] \\
 &= \Pi[U^2\Pi - Q\Pi\Pi + AR\Xi + 2AW\Xi r + AQ\Xi r^2] \quad (\text{based on Eqs. (70) and (72)}) \\
 &= \Pi[U^2\Pi - Q\Pi\Pi + A\Xi(R + 2Wr + Qr^2)] \\
 &= \Pi^2[U^2 - QP + A\Xi] \\
 &= 0 \quad (\text{based on Eq. (74)})
 \end{aligned}$$

$$\begin{aligned}
 d^2 - af &= (W\Pi + E\Xi r)^2 - (Q\Pi - E\Xi)(R\Pi - E\Xi r^2) \\
 &= W^2\Pi^2 + 2EW\Xi\Pi r + E^2\Xi^2 r^2 - QR\Pi^2 + ER\Xi\Pi + EQ\Xi\Pi r^2 - E^2\Xi^2 r^2 \\
 &= \Pi[W^2\Pi - QR\Pi + ER\Xi + 2EW\Xi r + EQ\Xi r^2] \\
 &= \Pi[W^2\Pi - QR\Pi + E\Xi(R + 2Wr + Qr^2)] \\
 &= \Pi^2[W^2 - QR + E\Xi] \\
 &= 0 \quad (\text{based on Eq. (75)})
 \end{aligned}$$

$$\begin{aligned}
 e^2 - cf &= (V\Pi - H\Xi r)^2 - (P\Pi - G\Xi)(R\Pi - E\Xi r^2) \\
 &= V^2\Pi^2 - 2HV\Xi\Pi r + (H^2 - CG)\Xi^2 r^2 - PR\Pi^2 + GR\Xi\Pi + EP\Xi\Pi r^2 \\
 &= \Pi[V^2\Pi - PR\Pi + GR\Xi - 2HV\Xi r - \Xi^2 r^2 + EP\Xi r^2] \\
 &= \Pi[V^2\Pi - PR\Pi - (C + 2Br - Ar^2)R\Xi \\
 &\quad - 2(F - Dr)W\Xi r - \Xi^2 r^2 + EP\Xi r^2] \\
 &= \Pi[V^2\Pi - PR\Pi + CR\Xi - 2(BR + FV)\Xi r + (EP - \Xi + 2DV + AR)\Xi r^2] \\
 &= \Pi[V^2\Pi - PR\Pi + CR\Xi + 2CW\Xi r + CQ\Xi r^2] \quad (\text{based on Eqs. (71) and (73)}) \\
 &= \Pi[V^2\Pi - PR\Pi + C\Xi(R + 2Wr + Qr^2)] \\
 &= \Pi^2[V^2 - PR + C\Xi] \\
 &= 0 \quad (\text{based on Eq. (76)})
 \end{aligned}$$

$$\begin{aligned}
 BQ + DU &= B(AE - D^2) + D(BD - AF) \\
 &= A(BE - DF) \\
 &= -AW
 \end{aligned} \tag{70}$$

$$\begin{aligned}
 BR + FV &= B(CE - F^2) + F(BF - CD) \\
 &= C(BE - DF) \\
 &= -CW
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 EP - \Xi + 2FU + CQ &= EP - (DV + FU + EP) + 2FU + CQ \\
 &= -DV + FU + CQ \\
 &= -D(BF - CD) + F(BD - AF) + C(AE - D^2) \\
 &= ACE - AF^2 \\
 &= AR
 \end{aligned} \tag{72}$$

$$\begin{aligned}
 EP - \Xi + 2DV + AR &= EP - (DV + FU + EP) + 2DV + AR \\
 &= DV - FU + AR \\
 &= D(BF - CD) - F(BD - AF) + A(CE - F^2) \\
 &= ACE - CD^2 \\
 &= CQ
 \end{aligned} \tag{73}$$

$$\begin{aligned}
 QP - U^2 &= (AE - D^2)(AC - B^2) - (BD - AF)^2 \\
 &= A^2CE - ACD^2 - AB^2E + 2ABDF - A^2F^2 \\
 &= A(ACE - CD^2 - B^2E + 2BDF - AF^2) \\
 &= A\Xi
 \end{aligned} \tag{74}$$

$$\begin{aligned}
 QR - W^2 &= (AE - D^2)(CE - F^2) - (DF - BE)^2 \\
 &= ACE^2 - CD^2E - AEF^2 + 2BDEF - B^2E^2 \\
 &= E(ACE - CD^2 - AF^2 + 2BDF - B^2E) \\
 &= E\Xi
 \end{aligned} \tag{75}$$

$$\begin{aligned}
 PR - V^2 &= (AC - B^2)(CE - F^2) - (BF - AD)^2 \\
 &= AC^2E - B^2CE - ACF^2 + 2BCDF + C^2D^2 \\
 &= C(ACE - B^2E - AF^2 + 2BDF + CD^2) \\
 &= C\Xi
 \end{aligned} \tag{76}$$

A8. Proof of Corollary 7

The numerator and denominator of Eq. (40) are squared as

$$\begin{aligned}
 (\bar{\xi} - r)^2 + \delta^2 &= (\bar{\xi}^2 - 2r\bar{\xi} + r^2) + \delta^2 \\
 &= (\alpha^2 + 1)\bar{\xi}^2 + 2(\alpha\beta - r)\bar{\xi} + (\beta^2 + r^2),
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{\omega}^2 &= \frac{1}{\Xi}(R + Q\bar{\xi}^2 + P\delta^2 + 2V\delta + 2W\bar{\xi} + 2U\bar{\xi}\delta) \\
 &= (P\alpha^2 + 2U\alpha + Q)\bar{\xi}^2 + 2(P\alpha\beta + V\alpha + U\beta + W)\bar{\xi} \\
 &\quad + (P\alpha^2 + 2V\alpha + R),
 \end{aligned}$$

by Eq. (39), respectively. Thus squared Eq. (40) can be matrixed by

$$\frac{(\bar{\xi} - r)^2 + \delta^2}{\bar{\omega}^2} = \frac{\mathbf{t}^T \Phi \mathbf{t}}{\mathbf{t}^T \Psi \mathbf{t}}, \tag{77}$$

where

$$\Phi \triangleq \begin{bmatrix} \alpha^2 + 1 & \alpha\beta - r \\ \alpha\beta - r & \beta^2 + r^2 \end{bmatrix}, \quad \mathbf{t} \triangleq \begin{bmatrix} \bar{\xi} \\ 1 \end{bmatrix}, \tag{78}$$

and

$$\Psi \triangleq \begin{bmatrix} P\alpha^2 + 2U\alpha + Q & P\alpha\beta + V\alpha + U\beta + W \\ P\alpha\beta + V\alpha + U\beta + W & P\alpha^2 + 2V\alpha + R \end{bmatrix}. \tag{79}$$

Then Eq. (77) can be simplified as

$$\frac{\mathbf{t}^T \Phi \mathbf{t}}{\mathbf{t}^T \Psi \mathbf{t}} = \frac{\mathbf{t}^T \Psi^{-\frac{1}{2}} \Phi \Psi^{-\frac{1}{2}} \mathbf{t}}{\mathbf{t}^T \Psi^{-\frac{1}{2}} \mathbf{t}} = \frac{\tilde{\mathbf{t}}^T \tilde{\Phi} \tilde{\mathbf{t}}}{\tilde{\mathbf{t}}^T \tilde{\mathbf{t}}}, \tag{80}$$

where

$$\tilde{\Phi} \triangleq \Psi^{-\frac{1}{2}} \Phi \Psi^{-\frac{1}{2}}, \quad \tilde{\mathbf{t}} \triangleq \Psi^{-\frac{1}{2}} \mathbf{t}. \tag{81}$$

Here, Eq. (80) can be considered as a Rayleigh quotient:

$$RQ(\tilde{\Phi}, \tilde{t}) \triangleq \frac{\tilde{t}^T \tilde{\Phi} \tilde{t}}{\tilde{t}^T \tilde{t}}. \quad (82)$$

Suppose that symmetric matrix $\tilde{\Phi}$ is diagonalizable with only real eigenvalues, we obtain

$$RQ(\tilde{\Phi}, \tilde{t}) \geq \lambda_{\min}, \quad RQ(\tilde{\Phi}, \mathbf{v}_{\min}) = \lambda_{\min}, \quad (83)$$

which means $RQ(\tilde{\Phi}, \tilde{t})$ reaches its minimum λ_{\min} (the smallest eigenvalue of $\tilde{\Phi}$) when \tilde{t} is \mathbf{v}_{\min} (the corresponding eigenvector). Similarly, we have

$$RQ(\tilde{\Phi}, \tilde{t}) \leq \lambda_{\max}, \quad RQ(\tilde{\Phi}, \mathbf{v}_{\max}) = \lambda_{\max}. \quad (84)$$

Therefore, after eigenvalue decomposition of $\tilde{\Phi}$, by Eq. (84), we obtain the maximum of $RQ(\tilde{\Phi}, \tilde{t})$ as $RQ(\tilde{\Phi}, \mathbf{v}_{\max}) = \lambda_{\max}$, which is squared of the novel ratio of the market portfolio $R(\mathcal{P}_{E\wedge C})^2$. Here, let

$$\mathbf{v}_{\max} = \begin{bmatrix} \eta \bar{\xi}_{E\wedge C} \\ \eta \end{bmatrix}, \quad (85)$$

where η is a constant, we have $\bar{\xi}_{E\wedge C} = \eta \bar{\xi}_{E\wedge C} / \eta$. Then we calculate the corresponding $\bar{\delta}_{E\wedge C}$ by Eq. (38), and then get $\bar{\omega}_{E\wedge C}$ by Eq. (30) or (33).

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