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## Article

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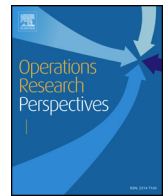
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# Analysis of a multi-component system with failure dependency, $N$ -policy and vacations

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## ABSTRACT

This paper studies a repairable  $k$ -out-of- $n$ : $G$  system with failure dependencies,  $N$ -policy and repairman's multiple vacations. Whenever there are no broken components, the repairman leaves for a vacation which obeys a phase type distribution. Upon returning from his/her vacation, he/she takes another vacation if broken components waiting less than  $N$ . This pattern continues until at least  $N$  broken components are waiting. By using the Markov process theory, the matrix analytical method and the probabilistic properties of the phase type distribution, the availability and the rate of occurrence of failures of the system are derived in transient and stationary regimes. Further, numerical examples discuss the behavior of the system reliability measures.

## 1. Introduction

In the area of reliability theory, the  $k$ -out-of- $n$ : $G$  system is often encountered in industrial systems. Such a system consists of  $n$  components in which all the  $n$  components are operating initially even though only  $k$  out of  $n$  are required for the system to be normal. Its applications include the power transmission and distribution systems, the communication systems with multiple transmitters, the design of electronic circuits, the multi-engine system in an aircraft, the multi-pump system in a hydraulic control system and the multi-display system in a cockpit. For instance, in a data processing system with six video displays, a minimum of three video displays are in good condition can be possible to display the full data. Thus the display system works as a 3-out-of-6: $G$  system. Due to the importance of  $k$ -out-of- $n$ : $G$  system in industry systems, a lot of authors have studied their availability and reliability. Comprehensive discussion of the multi-component system is provided in the bibliographies by Kuo and Zuo [1], and Cao and Cheng [2].

When  $k$ -out-of- $n$ : $G$  systems are considered for modelling, random times involved are usually assumed to be exponential distributions. For example, Byun et al. [3], Wu et al. [4], Jain and Gupta [5], Yuan [6], Moghaddass et al. [7], Zhang and Wu [8], Khatab et al. [9], Tang and Zhang [10], Ushakumari and Krishnamoorthy [11]. They studied such a multi-component system with different assumptions. Wu et al. [4] investigated a repairable  $k$ -out-of- $n$ : $G$  system with vacations and  $N$ -policy, in which the repairman is activated as soon as the total number of broken components accumulates to value  $N$  ( $1 \leq N \leq n - k + 1$ ). Various

steady-state system performance measures such as the availability, the rate of occurrence of failures are derived. Khatab et al. [9] provided an algorithm for automatic construction system state transition diagram to study availability of the system with non-identical components and repair priority rule. Subsequently, Moghaddass et al. [7] generalized their work to  $k$ -out-of- $n$ : $G$  system with non-identical components, similar or different repair priorities and shut-off rules. Besides, employing the semi-Markov process theory and the Laplace (Laplace-Stieltjes) transform, Wu et al. [12,13] studied the  $k$ -out-of- $n$ : $G$  repairable system with general repair times. The mean time to the first failure, the steady-state system availability and rate of occurrence of failures of the system are derived.

In traditional  $k$ -out-of- $n$ : $G$  system, it is usually supposed that the repairman remains idle until broken components are presented or some repair control policies are met. However in many real-world systems, the repairman may be assigned to perform some extra jobs such as additional work, preventive maintenance in his/her idle time. Here the time spent by the repairman to take additional tasks is called vacation time. Such type of repair models with vacation policy have been attracted attention in the literature. Yuan [6] studied a  $k$ -out-of- $n$ : $G$  repairable system with repairmen's multiple vacations where the operating times and repair times, the vacation times all follow exponential distributions. Later, Wu et al. [14] discussed the stationary reliability measures for such a system with single vacation by using the supplementary variables technique. For more detail on this topic the reader are referred to the Chakravorthy et al. [15], Zhang et al. [16], and Wu

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et al. [17] and therein. With this knowledge, the case of repairman's multiple vacations is taken into account in this paper.

In multi-component systems, another factor is the failure dependency which describes the interactions among system components. The dependency is the most common phenomenon in real world situations. For example, the failure/adding one component will change system reliability both by loss/gain the component's reliability and by reconfiguration of system loading. In 2007, Yu et al. [18] introduced a specific failure dependency called redundant dependency where any component can be viewed as a redundancy for another components. They studied the reliability optimization problem of a parallel system with  $n$  identical components. In their work, the dependence function is originally defined to quantify the redundant dependency. According to the dependence function, the redundant dependencies are further classified into independence, weak, linear and strong dependency. Recently, Yu et al. [19] proposed a constructive approach to optimize the system availability of an  $n$  components parallel repairable system through modeling the dependency of components. The optimization problem is provided and the resolution procedure is progressively developed.

This paper dealt with a  $k$ -out-of- $n$ : $G$  repairable system with repairman's multiple vacations and redundant dependency. In order to make the model more reasonable and flexible, the repair is according to  $N$ -policy. Utilizing the Markov process theory, the matrix analytical method and the probabilistic properties of the phase type (PH) distribution, the system availability, the rate of occurrence of failures along with other system reliability measures are derived in transient and stationary regimes.

This paper is arranged as follows. Section 2 gives some notations and the assumptions of the model. Section 3 provides the infinitesimal generator of the Markov process. System reliability measures are derived in Section 4. Section 5 gives numerical results. Finally, conclusions are drawn in Section 6.

## 2. Some notations and model assumptions

### 2.1. Some notations

**Definition 1.** A distribution  $H(x)$  on  $[0, +\infty)$  is of phase type with representation  $(\delta, S)$ , if it is the distribution of the time until absorption in a Markov process on the states  $\{1, 2, \dots, m+1\}$  with infinitesimal generator

$$\begin{pmatrix} S_{m \times m} & S^0_{m \times 1} \\ \mathbf{0}_{1 \times m} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

and initial probability vector  $(\delta_1 \times m, \mathbf{0}_{1 \times 1})$ . Assuming that the states  $\{1, 2, \dots, m\}$  are all transient, and the state  $m+1$  is absorbing. Hence the matrix  $S$  can be interpreted as the rate transient matrix among the transient states, while  $S^0$  represents the column vector of absorption rates. The matrix  $S$  is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies  $-S\mathbf{e}_{m \times 1} = S^0$ . The probability distribution  $H(x) = 1 - \delta_1 \times m \exp(Sx)\mathbf{e}_{m \times 1}$ ,  $x \geq 0$ , and its mean  $E(\chi) = -\delta_1 \times m S^{-1}\mathbf{e}_{m \times 1}$ .

For easy reference, let  $I_l$  be an identity matrix of order  $l$ , and  $\mathbf{e}_l$  be a column vector of order  $l$  of 1's. Denote by  $\mathbf{0}_{l \times j}$  a zero matrix of dimension  $l \times j$ , and by  $f^*(s)$  the Laplace transform of an arbitrary function  $f(t)$ ,  $t \geq 0$ , namely,  $f^*(s) = \int_0^\infty e^{-st}f(t)dt$ , by  $\omega_1 \times \kappa(i)$  is a row vector of dimension  $\kappa$  with 1 in the  $i$ th position and 0 others.

### 2.2. Model assumptions

The detailed assumptions of the system are provided as follows.

**A1.** The system consists of  $n$  identical components where all  $n$  components are operating initially even though only  $k$  out of  $n$  are required for the system to be good. The system is down as long as the total

number of components in the operating state goes down to  $k-1$ . When it fails, no failure will occur for other operating components.

**A2.** The operating time  $X$  of component is an exponential distribution with nominal failure rate  $\lambda(\lambda > 0)$ . Once an operating component breaks, it is repaired by a repairman. The repair time  $Y$  of broken component is an  $m$ th order PH distribution with the irreducible representation  $(\alpha, T)$  and mean repair time  $1/\mu = -\alpha T^{-1}\mathbf{e}_m$ . A repaired component is as good as new.

**A3.** The repairman leaves for a vacation whenever there are no broken components. Upon returning from his/her vacation, if there are at least  $N$  ( $1 \leq N \leq n-k+1$ ) broken components waiting, the repairman starts to repair. Otherwise, he/she leaves for another vacation. This pattern continues until there are at least  $N$  broken components waiting in the system. Further, the vacation time  $V$  follows a PH distribution with representation  $(\beta, H)$  of order  $r$  and mean vacation time  $1/\theta = -\beta H^{-1}\mathbf{e}_r$ .

**A4.** The redundant dependency which was firstly defined in [18] is taken into account. As the components are identical and redundant to each other in the system, the dependency is symmetric to these components. To combine the redundant dependency into the system failure, assume that the failure rate of components is given by its nominal failure rate  $\lambda$  and the dependence function which provided below

$$\frac{\lambda}{g(i)}, \quad i \geq 2, \quad g(1) \equiv 1, \quad (1)$$

where  $g(i)$  is the dependence function,  $i$  is the total number of operating components in the system. When one component breaks, the failure rate of the operating components should update upon the dependence function  $g(i)$ .

**A5.** Random variables  $X$ ,  $Y$ , and  $V$  are mutually independent.

## 3. The infinitesimal generator matrix

This section constructs the infinitesimal generator matrix which describes the behavior of the repairable  $k$ -out-of- $n$ : $G$  system. According to the model assumptions, the studied system can be viewed as a block-structured continuous-time Markov chain. The details of the Markov process theory could be seen in Stroock [20], Stewart [21]. To achieve this, denote by  $L(t)$  the number of broken components (either waiting or being repaired) at time  $t$  in the system,  $L(t) = i$  ( $i = 0, 1, \dots, n-k+1$ ). Define  $J(t)$  be the state of the repairman at time  $t$ , and

$$J(t) = \begin{cases} 1, & \text{the repairman is repairing broken components at time } t \\ 2, & \text{the repairman is on vacation at time } t. \end{cases}$$

Further, we define

- $\omega_1(t)$ : the phase of the repair process at time  $t$ ,  $\omega_1(t) = 1, 2, \dots, m$ .
- $\omega_2(t)$ : the phase of the vacation process at time  $t$ ,  $\omega_2(t) = 1, 2, \dots, r$ .

Due to the system is a redundant system with dependent components, the failure rate  $\lambda_i$  of the system is given below

$$\lambda_i = (n-i) \frac{\lambda}{g(n-i)}, \quad 0 \leq i \leq n-k.$$

Based on the model assumptions, the stochastic process  $\mathcal{X}(t) = \{L(t), J(t): t \geq 0\}$  is a continuous-time Markov chain with state space given by

$$\Omega = \Delta_0^2 \cup \left( \bigcup_{i=1}^{n-k+1} (\Delta_i^1 \cup \Delta_i^2) \right),$$

where

$\Delta_i^2 = \{L(t) = i, J(t) = 2, \omega_2(t) = l_2 | i = 0, 1, \dots, n-k+1, l_2 = 1, 2, \dots, r\}$  describes that there are  $i$  broken components in the system, and the repairman takes his/her vacation with phase  $l_2$ ,  $\Delta_i^1 = \{L(t) = i, J(t) = 1, \omega_1(t) = l_1 | i = 1, 2, \dots, n-k+1, l_1 = 1, 2, \dots, m\}$

describes that there are  $i$  broken components in the system, and the repairman is repairing component with phase  $l_1$ .

By partitioning the state space into levels with respect to the number of broken components and applying lexicographical sequence for these states, the corresponding infinitesimal generator matrix  $Q$  of  $\mathcal{X}(t)$  is of dimension  $(n - k + 1)(m + r) + r$ , exhibiting the following block-structured form

$$Q = \begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & B_2 & A_2 & C_2 & \\ & & \ddots & \ddots & \ddots \\ & & & B_{n-k} & A_{n-k} & C_{n-k} \\ & & & & B_{n-k+1} & A_{n-k+1} \end{pmatrix},$$

where

$$B_i = \begin{pmatrix} T^0 \beta \\ 0_{r \times r} \end{pmatrix}, \quad B_i = \begin{pmatrix} T^0 \alpha & 0_{m \times r} \\ 0_{r \times m} & 0_{r \times r} \end{pmatrix}, \quad i = 2, 3, \dots, n - k + 1,$$

$$A_0 = -\lambda_0 I_r + H + H^0 \beta,$$

$$A_i = \begin{pmatrix} -\lambda_i I_m + T & 0_{m \times r} \\ 0_{r \times m} & -\lambda_i I_r + H + H^0 \beta \end{pmatrix}, \quad i = 1, 2, \dots, N - 1,$$

$$A_i = \begin{pmatrix} -\lambda_i I_m + T & 0_{m \times r} \\ H^0 \alpha & -\lambda_i I_r + H \end{pmatrix}, \quad i = N, N + 1, \dots, n - k,$$

$$A_{n-k+1} = \begin{pmatrix} T & 0_{m \times r} \\ H^0 \alpha & H \end{pmatrix},$$

$$C_0 = (0_{r \times m}, \lambda_0 I_r), \quad C_i = \lambda_i I_{m+r}, \quad i = 1, 2, \dots, n - k.$$

Here we give an explanation on how to derive these elements of the infinitesimal generator matrix  $Q$ .

$(i, 2) \rightarrow (i, 2)$ ,  $i = 0, 1, \dots, N - 1$ : The block  $-\lambda_i I_r + H + H^0 \beta$  means that all the  $n - i$  operating components are good following  $-\lambda_i$ , a change among the phase of the repairman's vacation following  $H$ , or the repairman returns from his/her vacation following  $H^0$  and there are less than  $N$  broken components in the system, then he/she restarts another vacation immediately following  $\beta$ .

$(i, 2) \rightarrow (i, 2)$ ,  $i = N, N + 1, \dots, n - k$ : The block  $-\lambda_i I_r + H$  corresponding to all the  $n - i$  operating components are good following  $-\lambda_i$ , and a change among the phase of the repairman's vacation following  $H$ .

$(n - k + 1, 2) \rightarrow (n - k + 1, 2)$ : The block  $H$  indicates that a change among the phase of the repairman's vacation following  $H$ .

$(i, 1) \rightarrow (i, 1)$ ,  $i = 1, 2, \dots, n - k$ : The block  $-\lambda_i I_m + T$  means that all the  $n - i$  operating components are good following  $-\lambda_i$ , and a change among the phase of the repair following  $T$ .

$(n - k + 1, 1) \rightarrow (n - k + 1, 1)$ : The block  $T$  indicates that a change among the phase of the repair following  $T$ .

$(i, 2) \rightarrow (i, 1)$ ,  $i = N, N + 1, \dots, n - k + 1$ : The block  $H^0 \alpha$  means that the repairman comes from vacation following vector  $H^0$ , and he/she starts to repair broken components immediately following  $\alpha$ .

$(0, 2) \rightarrow (1, 2)$ : The block  $\lambda_0 I_r$  corresponds to one of the  $n$  operating components breaks down following  $\lambda_0$ , while the repairman's vacation phases do not change.

$i \rightarrow i + 1$ ,  $i = 1, 2, \dots, n - k$ : The block  $\lambda_i I_{m+r}$  indicates that one of the  $n - i$  operating components breaks down following  $\lambda_i$ , while the repair phases do not change, and the repairman's vacation phases do not change.

$(1, 1) \rightarrow (0, 2)$ : The block  $T^0 \beta$  means that the repair is completed following vector  $T^0$ , then the repairman takes another vacation immediately following vector  $\beta$ .

$(i, 1) \rightarrow (i - 1, 1)$ ,  $i = 2, 3, \dots, n - k + 1$ : The block  $T^0 \alpha$  indicates that the repair is completed following vector  $T^0$ , and then the repairman begins to repair other broken components immediately following vector  $\alpha$ .

**Table 1**

Classification of redundant dependencies.

dependence types	dependence function
independence	$g(i) = 1, i \geq 1$
weak dependence	$1 < g(i) < i, i \geq 2, g(1) = 1$
linear dependence	$g(i) = i, i \geq 1$
strong dependence	$g(i) > i, i \geq 2, g(1) = 1$

## 4. System reliability measures

### 4.1. Transient reliability measures

This subsection studies the transient behavior of system reliability measures by employing the Laplace transform method. First, supposed that all the components are new and the repairman is on vacation initially, and define the following probabilities

$$P_i((i, 1, l_1), t) = P\{\mathcal{X}(t) = (i, 1, l_1) | \mathcal{X}(0) = (0, 2, \hat{l})\},$$

$$\hat{l} = 1, 2, \dots, r, \quad l_1 = 1, 2, \dots, m, \quad i = 1, 2, \dots, n - k + 1.$$

$$P_i((i, 2, l_2), t) = P\{\mathcal{X}(t) = (i, 2, l_2) | \mathcal{X}(0) = (0, 2, \hat{l})\},$$

$$\hat{l}, l_2 = 1, 2, \dots, r, \quad i = 0, 1, \dots, n - k + 1.$$

Moreover, the above transition probabilities could be written compactly in matrix form as follows

$$P_0(t) = (P_i((0, 2, l_2), t))_{r \times r}, \quad \hat{l}, l_2 = 1, 2, \dots, r,$$

$$P_i(t) = (P_i((i, 1, l_1), t), P_i((i, 2, l_2), t))_{r \times (m+r)}, \quad l_1 = 1, 2, \dots, m,$$

$$\hat{l}, l_2 = 1, 2, \dots, r, \quad i = 1, 2, \dots, n - k + 1.$$

By a straightforward analysis, the Kolmogorov forward equation of the Markov process  $\{\mathcal{X}(t), t \geq 0\}$  is formed in matrix format as

$$\frac{d}{dt} P(t) = P(t) Q, \quad (2)$$

where  $P(t) = (P_0(t), P_1(t), \dots, P_{n-k+1}(t))$ .

Expanding the above matrix equation gives below

$$\frac{d}{dt} P_0(t) = P_0(t) A_0 + P_1(t) B_1, \quad (3)$$

$$\frac{d}{dt} P_1(t) = P_0(t) C_0 + P_1(t) A_1 + P_2(t) B_2, \quad (4)$$

$$\frac{d}{dt} P_i(t) = P_{i-1}(t) C_{i-1} + P_i(t) A_i + P_{i+1}(t) B_{i+1}, \quad i = 2, 3, \dots, n - k, \quad (5)$$

$$\frac{d}{dt} P_{n-k+1}(t) = P_{n-k}(t) C_{n-k} + P_{n-k+1}(t) B_{n-k+1}. \quad (6)$$

By taking the Laplace transform on Eqs. (3)–(6), we obtain the following algebraic system with the help of initial condition  $P_0(0) = I_r$ ,

$$sP_0^*(s) - I_r = P_0^*(s) A_0 + P_1^*(s) B_1, \quad (7)$$

$$sP_1^*(s) = P_0^*(s) C_0 + P_1^*(s) A_1 + P_2^*(s) B_2, \quad (8)$$

$$sP_i^*(s) = P_{i-1}^*(s) C_{i-1} + P_i^*(s) A_i + P_{i+1}^*(s) B_{i+1}, \quad i = 2, 3, \dots, n - k, \quad (9)$$

$$sP_{n-k+1}^*(s) = P_{n-k}^*(s) C_{n-k} + P_{n-k+1}^*(s) B_{n-k+1}. \quad (10)$$

Eqs. (7)–(10) is an algebraic system that can be solved recursively. After some calculation, the expressions for the Laplace transform of the transient probability is given in terms of  $P_0^*(s)$

$$P_i^*(s) = P_0^*(s) \prod_{k=1}^i \xi_k, \quad i = 1, 2, \dots, n - k + 1, \quad (11)$$

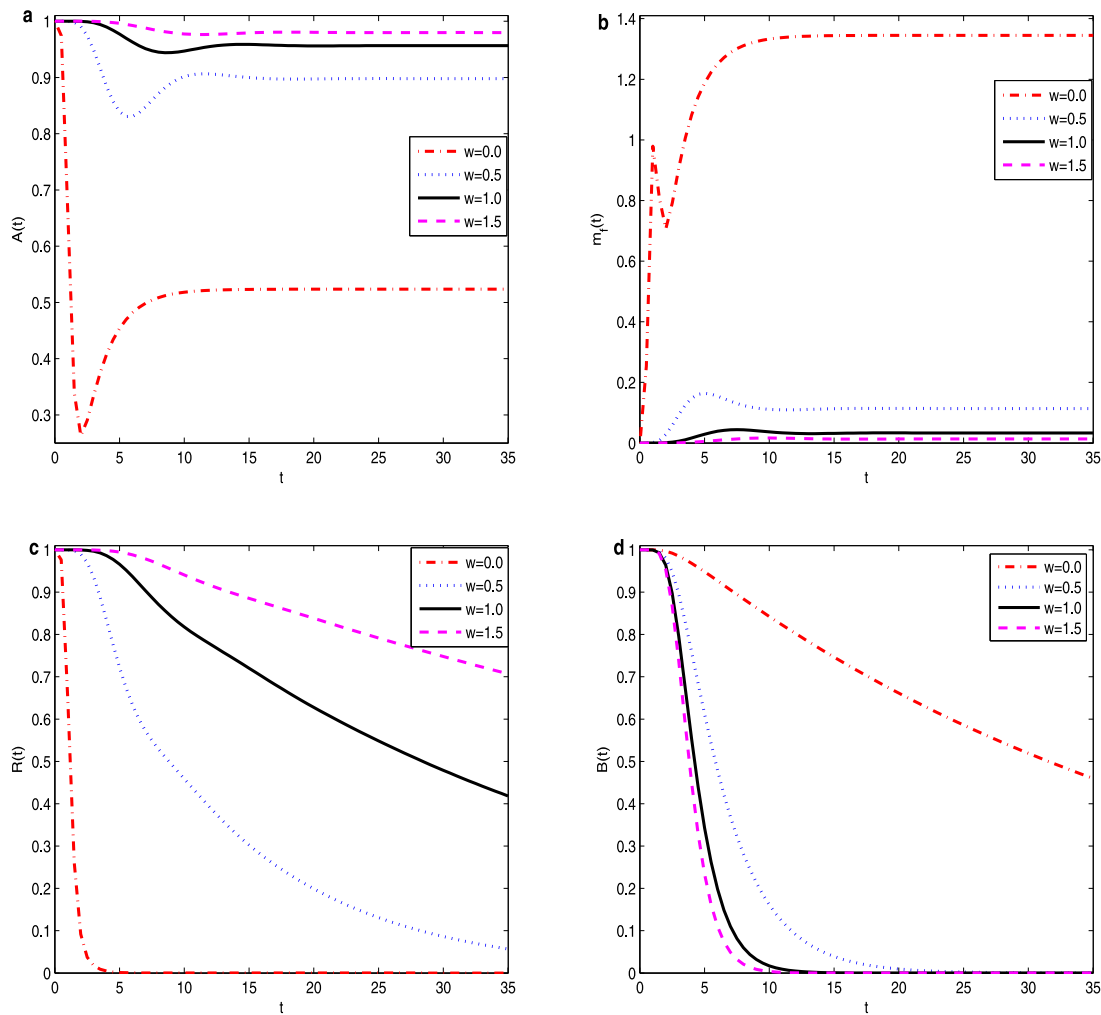
where  $\xi_i = C_{i-1}(sI_{m+r} - A_i - \xi_{i+1} B_{i+1})^{-1}$ ,  $i = 1, 2, \dots, n - k$ ,  $\xi_{n-k+1} =$

**Table 2**  
Transient availability for different values of  $w$ .

$w$	$t = 0.05$ transient availability $A(t)$	$t = 1.5$	$t = 6.0$	$t = 15$	$t = 35$	$t = \infty$ $A^a$
0	0.97660904	0.34201890	0.48163202	0.52319329	0.52361451	0.52361452
0.5	0.99999869	0.99804110	0.83077065	0.89982153	0.89797169	0.89797446
1.0	0.99999999	0.99995324	0.96285746	0.95881161	0.95652641	0.95652964
1.5	1.00000000	0.99999638	0.99246097	0.97931742	0.97989364	0.97989035

**Table 3**  
Transient rate of occurrence of failures for different values of  $w$ .

$w$	$t = 0.05$ transient rate of occurrence of failures $m_f(t)$	$t = 1.5$	$t = 6.0$	$t = 15$	$t = 35$	$t = \infty$ $m_f$
0	0.23710169	0.81292025	1.25042557	1.34395920	1.34490638	1.34490641
0.5	0.00001926	0.00831939	0.15455548	0.11306537	0.11391032	0.11390898
1.0	0.00000027	0.00022341	0.03894288	0.03191271	0.03317650	0.03317663
1.5	0.00000002	0.00001807	0.00894089	0.01307950	0.01346656	0.01346320



**Fig. 1.** Illustration of reliability measures under different values of  $w$ .

$$C_{n-k}(sI_{m+r} - A_{n-k+1})^{-1}.$$

Furthermore,  $P_0^*(s)$  could be solved by Eq. (7) and is given by

$$P_0^*(s) = (sI_r - A_0 - \xi_1 B_1)^{-1}. \quad (12)$$

Thus the transient probability functions at time  $t$  can be computed by using the numerical algorithm for the inverse Laplace transform. In this paper, we use the EULER algorithm proposed by Abate and Whitt

[22], which is a variant of Fourier-series method based on the Euler summation and the Bromwich contour inversion integral. The aim is to compute values of a real-valued function  $f(t)$  for various  $t$  from the Laplace transform  $f^*(s) = \int_0^\infty e^{-st}f(t)dt$ , where  $s$  is a complex variable with nonnegative real part. The numerical inversion is mainly based on the following important formula (see Eq. (2) in Abate and Whitt [22])

**Table 4**  
State probabilities for 7-out-of-13:G system with Coxian distribution.

$(i, j, l)$	$P_{i,j,l}(t)$	$(i, j, l)$	$P_{i,j,l}(t)$	$(i, j, l)$	$P_{i,j,l}(t)$
(0,2,1)	0.01643583	(3,1,1)	0.05235657	(5,2,1)	0.00534730
(0,2,2)	0.00344131	(3,1,2)	0.03183399	(5,2,2)	0.00395434
(0,2,3)	0.00171559	(3,1,3)	0.02552045	(5,2,3)	0.00378766
(1,1,1)	0.01541181	(3,2,1)	0.00835334	(6,1,1)	0.08010165
(1,1,2)	0.00901992	(3,2,2)	0.00514342	(6,1,2)	0.05011607
(1,1,3)	0.00630686	(3,2,3)	0.00420980	(6,1,3)	0.04431219
(1,2,1)	0.01421884	(4,1,1)	0.06800305	(6,2,1)	0.00434747
(1,2,2)	0.00513300	(4,1,2)	0.04179587	(6,2,2)	0.00334552
(1,2,3)	0.00315037	(4,1,3)	0.03479272	(6,2,3)	0.00335833
(2,1,1)	0.03398735	(4,2,1)	0.00663369	(7,1,1)	0.05878872
(2,1,2)	0.02035795	(4,2,2)	0.00458207	(7,1,2)	0.04015415
(2,1,3)	0.01547772	(4,2,3)	0.00410801	(7,1,3)	0.04550793
(2,2,1)	0.01074854	(5,1,1)	0.07804666	(7,2,1)	0.01385791
(2,2,2)	0.00545314	(5,1,2)	0.04840692	(7,2,2)	0.01106343
(2,2,3)	0.00394552	(5,1,3)	0.04157042	(7,2,3)	0.01179652

$$f(t) = \frac{2 \exp(at)}{\pi} \int_0^\infty \Re(f^*(a + iu)) \cos(ut) du, \quad (13)$$

where  $i = \sqrt{-1}$ ,  $\Re(s)$  is the real part of  $s$ , and  $a$  is a number such that  $f^*(s)$  has no singularities on or to the right of the vertical line  $s = a$ . We can numerically evaluate the integral (13) by means of the trapezoidal rule to obtain an approximate alternating series.

Once the transient probabilities are calculated, some important reliability measures in transient regime are easily derived.

- System availability

$$A(t) = \beta P_0(t) e_r + \sum_{i=1}^{n-k} \beta P_i(t) e_{m+r}. \quad (14)$$

- The rate of occurrence of failures at time  $t$

$$m_f(t) = \beta P_{n-k}(t) C_{n-k} e_{m+r}. \quad (15)$$

#### 4.2. Steady-state reliability measures

Let  $\pi$  be the steady-state probability vector of matrix  $Q$ . We partition  $\pi$  according to the number of broken components as  $\pi = (\pi_0, \pi_1, \dots, \pi_{n-k+1})$  where  $\pi_0 = (\pi_{0,2,1}, \pi_{0,2,2}, \dots, \pi_{0,2,r})$ ,  $\pi_i = (\pi_{i,1,1}, \dots, \pi_{i,1,m}, \pi_{i,2,1}, \dots, \pi_{i,2,r})$ ,  $i = 1, 2, \dots, n-k+1$ . Then the steady-state equations expressed in matrix form as  $\pi Q = 0$  are given by

$$\pi_0 A_0 + \pi_1 B_1 = 0_{1 \times r}, \quad (16)$$

$$\pi_0 C_0 + \pi_1 A_1 + \pi_2 B_2 = 0_{1 \times (m+r)}, \quad (17)$$

$$\pi_{i-1} C_{i-1} + \pi_i A_i + \pi_{i+1} B_{i+1} = 0_{1 \times (m+r)}, \quad i = 2, 3, \dots, n-k, \quad (18)$$

$$\pi_{n-k} C_{n-k} + \pi_{n-k+1} A_{n-k+1} = 0_{1 \times (m+r)}, \quad (19)$$

and the following normalizing equation must be satisfied

$$\pi_0 e_r + \sum_{i=1}^{n-k+1} \pi_i e_{m+r} = 1. \quad (20)$$

It follows from Eq. (19) that

$$\pi_{n-k+1} = \pi_{n-k} C_{n-k} (-A_{n-k+1})^{-1} = \pi_{n-k} \phi_{n-k+1}. \quad (21)$$

Substituting the result to Eq. (18) and performing some routine manipulations, we obtain the following results

$$\pi_i = \pi_{i-1} \phi_i, \quad i = 2, 3, \dots, n-k, \quad (22)$$

where  $\phi_i = -C_{i-1}(A_i + \phi_{i+1} B_{i+1})^{-1}$ ,  $i = 2, 3, \dots, n-k$ .

Similarly, we obtain from Eq. (17) that

$$\pi_1 = -\pi_0 C_0 (A_1 + \phi_2 B_2)^{-1} = \pi_0 \phi_1. \quad (23)$$

Consequently,  $\pi_i$  ( $i = 1, 2, \dots, n-k+1$ ) can be written in terms of  $\pi_0$  as  $\pi_i = \pi_0 \Phi_i$  where  $\Phi_i = \phi_1 \phi_2 \dots \phi_i$ ,  $i = 1, 2, \dots, n-k+1$ . Once the steady-state probability  $\pi_0$  being obtained, the steady-state solutions  $\pi = (\pi_0, \pi_1, \dots, \pi_{n-k+1})$  are then determined.  $\pi_0$  could be satisfies the following equations:

$$\pi_0 (A_0 + \Phi_1 B_1) = 0_{1 \times r}, \quad (24)$$

$$\pi_0 \left( e_r + \sum_{i=1}^{n-k+1} \Phi_i e_{m+r} \right) = 1. \quad (25)$$

With the stationary probability vector, we can compute some useful performance measures straightforwardly.

- The steady-state system availability

$$A = \pi_0 e_r + \sum_{i=1}^{n-k} \pi_i e_{m+r} = \pi_0 \left( e_r + \sum_{i=1}^{n-k} \Phi_i e_{m+r} \right). \quad (26)$$

- The steady-state rate of occurrence of failures

$$m_f = \pi_{n-k} C_{n-k} e_{m+r} = \lambda_{n-k} \pi_0 \Phi_{n-k} e_{m+r}. \quad (27)$$

- The expected number of failed components in the system

$$E[\zeta] = \sum_{i=1}^{n-k+1} i \pi_i e_{m+r} = \pi_0 \sum_{i=1}^{n-k+1} i \Phi_i e_{m+r}. \quad (28)$$

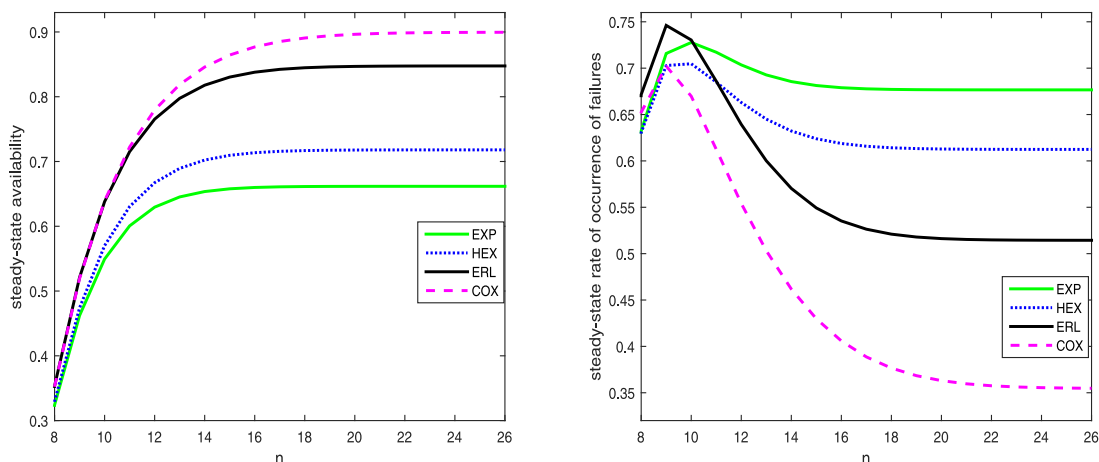


Fig. 2. Different service time distributions on performance measures.

**Table 5**  
Transient availability for different values of  $n$ .

$n$	$t = 0.05$ transient availability $A(t)$	$t = 1.5$	$t = 6.0$	$t = 15$	$t = 35$	$t = \infty$ $A^a$
6	0.99946428	0.81692931	0.80378442	0.80378380	0.80378380	0.80380133
7	0.99997044	0.86598103	0.83601414	0.83599315	0.83599315	0.83600257
8	0.99999851	0.90076712	0.85246966	0.85231186	0.85231185	0.85231699
9	0.99999993	0.92647078	0.86116034	0.86061029	0.86061026	0.86061300
10	1.00000000	0.94571852	0.86594645	0.86470421	0.86470398	0.86470536
11	1.00000000	0.96015295	0.86878225	0.86662239	0.86662159	0.86662225
12	1.00000000	0.97093442	0.87065542	0.86746632	0.86746456	0.86746485
13	1.00000000	0.97893594	0.87205774	0.86781355	0.86781063	0.86781074
14	1.00000000	0.98483116	0.87323091	0.86794736	0.86794326	0.86794330

**Table 6**  
The rate of occurrence of failures for different values of  $n$ .

$n$	$t = 0.05$ the rate of occurrence of failures $m_f(t)$	$t = 0.8$	$t = 2.0$	$t = 5.0$	$t = 15$	$t = \infty$ $m_f^a$
6	0.03140669	0.66981394	0.68675591	0.68675671	0.68675671	0.68669534
7	0.00230697	0.51693278	0.57398444	0.57402399	0.57402399	0.57399101
8	0.00014616	0.40647855	0.51655782	0.51690852	0.51690853	0.51689053
9	0.00000836	0.32069192	0.48653056	0.48786401	0.48786410	0.48785449
10	0.00000044	0.25216336	0.47037098	0.47353551	0.47353611	0.47353124
11	0.00000002	0.19699781	0.46115833	0.46682235	0.46682444	0.46682213
12	0.00000000	0.15269614	0.45536326	0.46386938	0.46387406	0.46387303
13	0.00000000	0.11737534	0.45122308	0.46265496	0.46266282	0.46266239
14	0.00000000	0.08947619	0.44787504	0.46218754	0.46219861	0.46219845

#### 4.3. System reliability and the mean time to system failure

Now we use the theory of PH distribution to derive the formulas of the system reliability and the mean time to system failure. To achieve this, we lump all failure states  $(n - k + 1, 1, 1), (n - k + 1, 1, 2), \dots, (n - k + 1, 1, m), (n - k + 1, 2, 1), (n - k + 1, 2, 2), \dots, (n - k + 1, 2, r)$  together to make one absorbing state, say “ $\otimes$ ”. Then consider an absorbing Markov process with state space

$$\Omega = \Delta_0^2 \cup \left( \bigcup_{i=1}^{n-k} (\Delta_i^1 \cup \Delta_i^2) \right) \cup \otimes,$$

where

$\Delta_i^1 = \{L(t) = i, J(t) = 1, \omega_1(t) = l_1 | l_1 = 1, 2, \dots, n - k, l_1 = 1, 2, \dots, m\}$ ,  
 $\Delta_i^2 = \{L(t) = i, J(t) = 2, \omega_2(t) = l_2 | l_2 = 0, 1, \dots, n - k, l_2 = 1, 2, \dots, r\}$  are transient, and the state “ $\otimes$ ” is absorbing.

The infinitesimal generator of this absorbing Markov process is given by

$$\begin{pmatrix} S_{[(n-k)(m+r)+r] \times [(n-k)(m+r)+r]} & S_{[(n-k)(m+r)+r] \times 1}^0 \\ \mathbf{0}_{1 \times [(n-k)(m+r)+r]} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

where

$$S = \begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-k-1} & A_{n-k-1} & C_{n-k-1} \\ & & & B_{n-k} & A_{n-k} \end{pmatrix},$$

and initial probability vector is  $(\delta_{1 \times [(n-k)(m+r)+r]}, \mathbf{0}_{1 \times 1})$  with  $\delta_{1 \times [(n-k)(m+r)+r]} = \omega_{(n-k)(m+r)+r}(t)$ ,  $t = 1, 2, \dots, (n - k)(m + r) + r$  is determined by initial state of the system.

Based on the definition of the PH distribution, we obtain the system reliability is

$$R(t) = \delta \exp(S t) \mathbf{e}. \quad (29)$$

The mean time to system failure (MTTF) is

$$\text{MTTF} = -\delta S^{-1} \mathbf{e}. \quad (30)$$

#### 4.4. The busy period of the repairman

The repairman's busy period of the  $k$ -out-of- $n$ :G system is the time interval from the repairman starts repairing broken components until there is no broken component in the system. Next, we will use the properties of the PH distribution to derive the expressions of the distribution and the expected duration of the busy period of the repairman. Consider an newly absorbing Markov process with state space  $\{(i, 1, l_1) | l_1 = 1, 2, \dots, n - k + 1, l_1 = 1, 2, \dots, m\} \cup \{(0, 2, l_2) | l_2 = 1, 2, \dots, r\}$  where the state  $(0, 2, l_2)$  are absorbing and all other states are transient. Similarly, the infinitesimal generator is given by

$$\begin{pmatrix} \Gamma_{(n-k+1)m \times (n-k+1)m} & \Gamma_{(n-k+1)m \times 1}^0 \\ \mathbf{0}_{1 \times (n-k+1)m} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

where

$$\Gamma = \begin{pmatrix} T & T^0 \alpha & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \lambda_{n-k} \mathbf{I} & -\lambda_{n-k} \mathbf{I} + T & T^0 \alpha & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_{n-k-1} \mathbf{I} & -\lambda_{n-k-1} \mathbf{I} + T & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\lambda_2 \mathbf{I} + T & T^0 \alpha \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \lambda_1 \mathbf{I} & -\lambda_1 \mathbf{I} + T \end{pmatrix},$$

and initial probability vector  $(\varphi_{1 \times (n-k+1)m}, \mathbf{0}_{1 \times 1})$ . Here  $\varphi = \omega_{1 \times (n-k+1)m}(t)$ ,  $t = 1, 2, \dots, (n - k + 1)m$  is determined by the initial state of the system.

Based on the phase type distribution, the distribution of the busy period of the repairman is

$$B(t) = 1 - \omega_{1 \times (n-k+1)m}(t) \exp(\Gamma t) \mathbf{e}_{(n-k+1)m \times 1},$$

and expected duration time of the repairman is continuously busy is

$$E[B] = -\omega_{1 \times (n-k+1)m}(t) \Gamma^{-1} \mathbf{e}_{(n-k+1)m \times 1}.$$



## 5. Numerical illustrations

### 5.1. Influence of redundant dependency on system reliability measures

The dependence function  $g(i)$  in Eq. (1) describes redundant dependency of components. When the number of operating components is given, the bigger  $g(i)$  is, the stronger the dependency is. Based on the reference [18], four types of redundant dependencies are classified by the value of  $g(i)$ .

According to Table 1, the redundant dependencies are independence, weak, linear and strong dependence. Consider the influence of four types of redundant dependencies on system reliability measures by taking the  $g(i) = wi + 1 - w$ ,  $w \geq 0$  as one specific form, and selecting  $k = 5$ ,  $n = 12$ ,  $\lambda = 0.85$ ,  $N = 4$ .

The service time is a PH distribution with 3 stages and mean 0.4025 where

$$\alpha = (1, 0, 0), \quad T = \begin{pmatrix} -7 & 4 & 2 \\ 2 & -9 & 5 \\ 1 & 3 & -10 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}.$$

The vacation time is a PH distribution with 3 stages and mean 2.2650 where

$$\beta = (1, 0, 0), \quad H = \begin{pmatrix} -1 & 0.6 & 0.2 \\ 0.7 & -2 & 1 \\ 0.5 & 1 & -4 \end{pmatrix}, \quad H^0 = \begin{pmatrix} 0.3 \\ 0.3 \\ 2.5 \end{pmatrix}.$$

The numerical example performs the above specific parameters under different values of  $w = 0, 0.5, 1.0$  and  $1.5$  (corresponding to independence, weak dependence, linear dependence and strong dependence, respectively). The computational results are displayed in Tables 2, 3 and Fig. 1.

It can be seen from Fig. 1 that the transient availability  $A(t)$  is a monotonically increasing function of  $w$ , while the rate of occurrence of failures decreases with  $w$  increase. This means that the system availability and the rate of occurrence of failure are sensitive to the dependency. Moreover, Tables 2, 3 and Fig. 1 show that the curves of the system availability  $A(t)$  and the rate of occurrence of failures  $m_f(t)$  exhibit violent fluctuations at the early stage, and after some time this fluctuations tend to disappear. The measure  $R(t)$  is an increasing function of  $w$ , while the  $B(t)$  is a decreasing function of  $w$ . They are the same pattern as the measures  $A(t)$  and  $m_f(t)$ .

### 5.2. Influence of service times on system reliability measures

We first choose  $k = 7$ ,  $N = 2$ ,  $\lambda = 0.55$ , and  $w = 0.07$ , and vacation time is a PH distribution with 3 stages and mean 1.5125 where

$$\beta = (1, 0, 0), \quad H = \begin{pmatrix} -2 & 1.4 & 0.3 \\ 0.8 & -3 & 2 \\ 0.6 & 0.4 & -3 \end{pmatrix}, \quad H^0 = \begin{pmatrix} 0.3 \\ 0.2 \\ 2 \end{pmatrix}.$$

Further, consider the following four cases of PH service time

- Exponential distribution (EXP)

$$\alpha = (1), \quad T = (-2), \quad T^0 = (2).$$

- Hyper-exponential distribution (HEX)

$$\alpha = (0.7, 0.3), \quad T = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- Erlangian distribution (ERL)

$$\alpha = (0.5, 0.2, 0.3), \quad T = \begin{pmatrix} -6 & 6 & 0 \\ 0 & -6 & 6 \\ 0 & 0 & -6 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}.$$

- Coxian distribution (COX)

$$\alpha = (0.6, 0.3, 0.1), \quad T = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -8 & 6 \\ 0 & 0 & -8 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}.$$

The expected values of the four PH distributions are 0.5, 0.45, 0.3667 and 0.3266, respectively. The results for state probabilities with Coxian distribution are provided in Table 4. Fig. 2 indicates the influence of the service time on the system reliability measures. It follows from Fig. 2 that the system availability increases with decreasing the mean of service time, while the rate of occurrence of failure goes on increasing with increasing the mean value. Similarly, performance measures are sensitive to the service time.

### 5.3. Special case

Set  $N = 1$ ,  $g(i) = 1$  and the mean vacation time tend to zero, our model reduce to the classical  $k$ -out-of- $n$ :G Markovian repairable system. Applying the Markov analysis method, Cao and Cheng [2] deduced the steady-state availability  $A^a$  and the rate of occurrence of failures  $m_f^a$  which are provided below.

$$A^a = \frac{\sum_{i=k}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i}{\sum_{i=k-1}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i}, \quad m_f^a = \frac{\frac{\mu}{(k-1)!} \left(\frac{\mu}{\lambda}\right)^{k-1}}{\sum_{i=k-1}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i}. \quad (31)$$

To illustrate the correctness of the formulae in the paper, we select  $k = 4$ ,  $\lambda = 0.65$ ,  $\mu = 3.5$ ,  $\beta = (1)$  and  $H = (-10^4)$ . Computation results are tabulated in Tables 5 and 6, respectively, which show that the formulae obtained in the present paper exactly agree with that given in Cao and Cheng [2].

## 6. Conclusions

This paper considered the  $k$ -out-of- $n$ :G system with  $N$ -policy and repairman's multiple vacations, in which the failure dependency is introduced to describe the interactions among the failures of components. System performance measures are derived in transient and stationary regimes by using the matrix analytical method and the properties of the phase type distribution.

In the future work, an interesting extension is to consider such a system with repairman's vacation policy and non-identical components subject to repair priorities. Once an operating components fails, it is immediately replaced and exchanged with a good component taken from the spare parts inventory. The failed component are sent to a central for repair considering priority rule.

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