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**Article**

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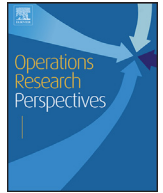
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# Robust optimization: Sensitivity to uncertainty in scalar and vector cases, with applications

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## ABSTRACT

The question we address is how robust solutions react to changes in the uncertainty set. We prove the location of robust solutions with respect to the magnitude of a possible decrease in uncertainty, namely when the uncertainty set shrinks, and convergence of the sequence of robust solutions.

In decision making, uncertainty may arise from incomplete information about people's (stakeholders, voters, opinion leaders, etc.) perception about a specific issue. Whether the decision maker (DM) has to look for the approval of a board or pass an act, they might need to define the strategy that displeases the minority. In such a problem, the feasible region is likely to be unchanged, while uncertainty affects the objective function. Hence the paper studies only this framework.

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## 1. Introduction

*Robust optimization* is proving to be a fruitful tool in the study of problems with uncertain data. Since the seminal paper by Ben-Tal and Nemirovski [1], several authors have studied the problem both in scalar and vector optimization settings (see e.g. [2–6]). More recently, a detailed monograph has been devoted to the topic [7] and survey papers [8,9] have collected the major issues and applications of robust optimization.

The need for such a tool arises when a constrained optimization problem (scalar or vector) depends upon uncertain parameters that may affect both the objective function and the constraints. This occurs in many real-world applications of optimization in industries, energy markets, finance, to quote some fields (see e.g. [10–14] and the references therein), due to unknown future developments, measurement or manufacturing errors, incomplete information in model development, and so on. In such circumstances, stochastic programming is often applied, but this approach requires the choice of a probability distribution that can hardly be motivated but for the technical capability of solving the problem.

In robust optimization, no arbitrary assumption on the distribution of parameters is required. Instead, a robust solution is defined introducing a different optimization problem known as a *robust counterpart* that allows to find a “worst-case-oriented” optimal solution. Since the seminal papers by Ben-Tal et al., robust optimization has also been extended to multiobjective problems, see e.g. [15,16] and several theoretical issues have been investigated. A major problem in optimization theory that has been addressed only recently for robust optimization is the stability of solutions with respect to the uncertainty set  $\mathcal{U}$ , where parameters are assumed to take values. In [16–18], using set-optimization techniques, some well-posedness for robust optimization problems has been proven under quasiconvexity assumptions.

In this paper, we investigate the behaviour of scalar and vector robust optimization problems upon variation of the uncertainty set  $\mathcal{U}$ . Indeed, in several applications of robust optimization, the decision maker (DM) can act to reduce uncertainty, e.g. by improving the technology of measurement or manufacturing or acting on stakeholder perception on some issues, to steer the uncertainty toward a desired level and eventually a desired robust solution. Mathematically, we model this possibility allowing  $\mathcal{U}$  to change. We provide some estimates to locate the optimal values and the optimal solutions of a robust optimization problem with respect to the solution of a problem without uncertainty. When multiobjective optimization problems are considered, the results make use of set orderings introduced by Kuroiwa [19].

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The remaining sections of the paper are organized as follows. In Section 2 we present the basic notation and develop our results in the scalar case. Section 3 presents a decision-making issue that can be addressed through our results. Not only does robust optimization provide DMs with a worst-case solution concepts, it also allows the “safe” solution to be steered toward a more desirable solution, provided uncertainty, interpreted as stakeholders perception, can be influenced[A2]. An academic numerical example is also provided. Further, in Section 4 the results are extended to a vector optimization problem with uncertainty and then applied to a portfolio selection model with two risky assets.

**2. Scalar case**

Throughout this paper  $\mathbb{R}^n$  denotes the Euclidean space of dimension  $n$ ,  $\langle x, y \rangle$  denotes the inner product of  $x$  and  $y$ , and  $\|x\| = \sqrt{\langle x, x \rangle}$  denotes the Euclidean norm of  $x$ . Given a lower bounded function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  and a closed convex set  $X \subseteq \mathbb{R}^n$ , we consider the problem

$$\inf_{x \in X} g(x). \tag{P}$$

A point  $x^0 \in X$  is a solution to (P) when  $g(x^0) = \inf_{x \in X} g(x)$ . Given an arbitrary  $\varepsilon > 0$ , we define  $x^0 \in X$  an  $\varepsilon$ -solution to (P) when  $g(x^0) \leq \inf_{x \in X} g(x) + \varepsilon$ .

We introduce the uncertain optimization problem by

$$\inf_{x \in X} f(x, u), \tag{UP}$$

where  $X \subseteq \mathbb{R}^n$  is a closed convex set,  $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ ,  $u$  is an uncertain parameter, with  $u \in \mathcal{U}$  for some convex compact subset  $\mathcal{U} \subseteq \mathbb{R}^p$ . We assume  $f$  is continuous with respect to  $u$  and  $f(\cdot, u)$  is lower bounded on  $X$  for every  $u \in \mathcal{U}$ .

Problem (UP) has been studied extensively in the literature (see e.g. [7] and the references therein) through its associated *robust counterpart*

$$\inf_{x \in X} \max_{u \in \mathcal{U}} f(x, u). \tag{RP}$$

Various degrees of uncertainty can occur for the same objective function. Here we also introduce a compact, convex subset  $\mathcal{U}^0 \subseteq \mathcal{U}$  to represent the *nominal instance* of our robust optimization problem, as the least achievable uncertainty. Moreover, we consider sets

$$\mathcal{W}^\lambda := (1 - \lambda)\mathcal{U}^0 + \lambda\mathcal{U}, \quad \lambda \in [0, 1],$$

that provide average level of uncertainty. Clearly,

$$\begin{aligned} \mathcal{W}^1 &= \mathcal{U} && \text{(high uncertainty),} \\ \mathcal{W}^0 &= \mathcal{U}^0 && \text{(low uncertainty).} \end{aligned}$$

An extreme, yet meaningful situation is represented by  $\mathcal{U}^0 = \{u^0\}$ , i.e. absence of uncertainty, that also would coincide with  $\lambda = 0$ .

We are now interested in the family of robust counterparts to (UP)

$$\inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w). \tag{RP}^\lambda$$

When  $\lambda = 0$  the robust counterpart shows the lowest level of uncertainty achievable (possibly none at all) and accordingly we denote the special instance as

$$\inf_{x \in X} \max_{u \in \mathcal{U}^0} f(x, u). \tag{RP0}$$

The following notation will also be used throughout the paper

$$E_f(x) := \max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}^0} f(x, u);$$

$$\bar{E}_f := \sup_{x \in X} E_f(x);$$

and

$$\underline{E}_f := \inf_{x \in X} E_f(x).$$

Clearly, since  $\mathcal{U}^0 \subseteq \mathcal{U}$ ,  $E_f(x) \geq 0$ , for all  $x \in X$ , but we need to assume  $\bar{E}_f$  is finite.

**Remark 1.** We can provide some characterization of  $E_f(x)$  and  $\bar{E}_f$  as follows:

(i) If  $f(x, u)$  is continuous in  $x$  for every  $u \in \mathcal{U}$ , then  $E_f(x)$  is also continuous. Hence  $\bar{E}_f$  is finite whenever  $X$  is compact.

(ii) Assume that  $\mathcal{U}^0 + \mathcal{V} \supseteq \mathcal{U}$  for some compact convex subset  $\mathcal{V} \subseteq \mathbb{R}^p$ .

(a) If  $f(x, u)$  is subadditive with respect to  $u$ , the following holds:

$$\begin{aligned} E_f(x) &\leq \max_{v \in \mathcal{V}, u' \in \mathcal{U}^0} f(x, v + u') - \max_{u'' \in \mathcal{U}^0} f(x, u'') \\ &\leq \max_{v \in \mathcal{V}, u' \in \mathcal{U}^0} [f(x, v) + f(x, u')] - \max_{u'' \in \mathcal{U}^0} f(x, u'') = \max_{v \in \mathcal{V}} f(x, v). \end{aligned}$$

(b) If  $f(x, u)$  is affine with respect to  $u$ , i.e.  $f(x, u) = \langle g(x), u \rangle + h(x)$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , the following holds:

$$E_f(x) \leq \max_{v \in \mathcal{V}} \langle g(x), v \rangle.$$

In addition, if  $\mathcal{U}^0 + \mathcal{V} = \mathcal{U}$ , then

$$E_f(x) = \max_{v \in \mathcal{V}} \langle g(x), v \rangle,$$

which, for  $\mathcal{U}^0 = \{u^0\}$  and  $\mathcal{U} = \mathcal{B}(u^0)$ , entails  $E_f(x) = \|g(x)\|$ , where  $\mathcal{B}(u)$  is the closed unit ball in  $\mathbb{R}^p$  centred in  $u \in \mathbb{R}^p$ .

(iii)  $\bar{E}_f$  is finite without compactness of  $X$  in the following case:  $n = p$ ,  $f(x, u) = \langle Ax, u \rangle + h(x)$  where  $A$  is an  $n \times n$  orthogonal matrix and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\{b_1, b_2, \dots, b_n\}$  is an orthonormal basis in  $\mathbb{R}^n$ ,  $X = \{\alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n \mid \frac{1}{2}(\alpha_1^2 + \dots + \alpha_n^2) - \frac{1}{2} \leq \alpha_1\}$ , and  $\{u^0\} \subseteq \mathcal{U}^0 \subseteq \mathcal{U} = \mathcal{B}(u^0) - Ab_1$ . Let  $\mathcal{V} = \mathcal{B}(0) - Ab_1$ , then  $\mathcal{U}^0 + \mathcal{V} \supseteq \mathcal{U}$  holds. Then, for any  $x \in X$ ,

$$E_f(x) \leq \max_{v \in \mathcal{V}} \langle Ax, v \rangle = \|Ax\| - \langle Ax, Ab_1 \rangle = \|x\| - \langle x, b_1 \rangle \leq 1.$$

The final inequality follows from the fact:  $\frac{1}{2}(\alpha_1^2 + \dots + \alpha_n^2) - \frac{1}{2} \leq \alpha_1$  implies  $\sqrt{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2} - \alpha_1 \leq 1$ . Therefore  $\bar{E}_f$  is finite, while  $X$  is unbounded.

The following results provide an answer to how the level of uncertainty affects the robust optimal value (and possibly the robust solution) of (UP). We first provide the “location” of the optimal value, depending on the average uncertainty described by some  $\lambda$ .

**Proposition 1 (Location of optimal values of (RP) $^\lambda$ ).**

(i) Let  $f(x, u)$  be convex with respect to  $u$ , then the following holds:

$$\inf_{x \in X} \max_{u \in \mathcal{U}^0} f(x, u) \leq \inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w) \leq \inf_{x \in X} \max_{u \in \mathcal{U}} f(x, u) + \lambda \bar{E}_f. \tag{1}$$

(ii) Let  $f(x, u)$  be concave with respect to  $u$ , then the following holds:

$$\inf_{x \in X} \max_{u \in \mathcal{U}^0} f(x, u) + \lambda \underline{E}_f \leq \inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w). \tag{2}$$

**Proof.**

(i) Let  $w^\lambda \in \mathcal{W}^\lambda$ , that is  $w^\lambda = (1 - \lambda)u^0 + \lambda u$ , for some  $u^0 \in \mathcal{U}^0$  and  $u \in \mathcal{U}$ . From convexity with respect to  $u$  it follows that

$$f(x, w^\lambda) \leq (1 - \lambda)f(x, u^0) + \lambda f(x, u), \quad \text{for all } x \in X.$$

Hence, recalling that  $\mathcal{U}^0 \subseteq \mathcal{W}^\lambda$  for every  $\lambda \in [0, 1]$ , we have

$$\begin{aligned} \max_{u \in \mathcal{U}^0} f(x, u) &\leq \max_{w \in \mathcal{W}^\lambda} f(x, w) \\ &= \max_{u^0 \in \mathcal{U}^0, u \in \mathcal{U}} f(x, (1 - \lambda)u^0 + \lambda u) \\ &\leq (1 - \lambda) \max_{u^0 \in \mathcal{U}^0} f(x, u^0) + \lambda \max_{u \in \mathcal{U}} f(x, u) \\ &= \max_{u \in \mathcal{U}^0} f(x, u) + \lambda E_f(x), \quad \text{for all } x \in X. \end{aligned} \tag{3}$$

It follows that

$$\inf_{x \in X} \max_{u \in \mathcal{U}^0} f(x, u) \leq \inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w) \leq \inf_{x \in X} \max_{u \in \mathcal{U}} f(x, u) + \lambda \bar{E}_f.$$

(ii) The proof follows along the same lines.

□

Under further assumptions, we can obtain a better result.

**Corollary 1.** *If  $f(x, u)$  is affine with respect to  $u$ , then both (1) and (2) hold. If, in particular,  $\mathcal{U}^0 = \{u^0\}$ ,  $\mathcal{U} = \mathcal{B}(u^0)$  and  $f(x, u) = \langle g(x), u \rangle + h(x)$ , with  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , the following holds:*

$$\inf_{x \in X} f(x, u^0) + \lambda \inf_{x \in X} \|g(x)\| \leq \inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w) \leq \inf_{x \in X} f(x, u^0) + \lambda \sup_{x \in X} \|g(x)\|.$$

Moreover if  $n = p$ ,  $g(x) = Ax$ , and  $A$  is an  $n \times n$  orthogonal matrix, then the following holds:

$$\inf_{x \in X} f(x, u^0) + \lambda \inf_{x \in X} \|x\| \leq \inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w) \leq \inf_{x \in X} f(x, u^0) + \lambda \sup_{x \in X} \|x\|.$$

The following proposition is an immediate consequence of Proposition 1.

**Proposition 2.** *Let  $f(x, u)$  be convex with respect to  $u$ . Then every solution of problem  $(RP^\lambda)$  is an  $\varepsilon$ -solution of problem  $(RP^0)$  with  $\varepsilon = \lambda \bar{E}_f$ .*

Moreover, the robust optimal values of the robust counterparts  $(RMP^\lambda)$  converge to those of the nominal instance  $(RP^0)$ , as uncertainty decreases.

**Proposition 3 (Convergence of the optimal values of  $(RP^\lambda)$ ).** *Assume  $f(x, u)$  is convex with respect to  $u$ . Hence,*

$$\inf_{x \in X} \max_{w \in \mathcal{W}^\lambda} f(x, w) \rightarrow \inf_{x \in X} \max_{u \in \mathcal{U}^0} f(x, u) \tag{4}$$

as  $\lambda \downarrow 0$ .

**Proof.** The proof is immediate from Proposition 1. □

**Remark 2.** (4) holds in the following cases:

- (i)  $f(x, u) = \langle g(x), u \rangle + h(x)$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is continuous,  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , and  $X$  is bounded (including a linear problem with bounded feasible region);
- (ii)  $f(x, u) = \langle g(x), u \rangle + h(x)$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is bounded on  $X$ , and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  (including a problem with unbounded feasible region, see (iii) of Remark 1).

**Remark 3.** Observe that Proposition 3 requires no convexity assumption with respect to  $x$ .

Finally, some convergence of robust solutions to  $(RMP^\lambda)$  toward the robust solution with lowest uncertainty to  $(RP^0)$  can also be proved.

**Proposition 4** (Convergence of the optimal solutions of  $(RP^\lambda)$ ). *Assume that for every  $\lambda \in [0, 1]$  functions  $\max_{w \in \mathcal{W}^\lambda} f(x, w)$  are convex and l.s.c. with respect to  $x$  on the convex closed set  $X$ . Assume further that the set of solutions of  $(RP^0)$  is nonempty and compact. Let  $\lambda_n \downarrow 0$ , set  $\mathcal{W}^n = \mathcal{W}^{\lambda_n}$  and assume  $x_n = x_{\lambda_n} \in X$  are solutions of problem  $(RP^\lambda)$ .*

*Then there exists a subsequence  $x_{n_k}$  of  $x_n$  such that  $x_{n_k} \rightarrow \bar{x} \in X$  and  $\bar{x}$  is a solution of  $(RP^0)$ .*

**Proof.** From Proposition 2 we know that for every  $n$ ,  $x_n$  is an  $\varepsilon$ -solution of  $(RP^0)$  with  $\varepsilon = \lambda_n \bar{E}_f$ . The assumptions guarantee that there exists a subsequence of  $x_n$  converging to a solution of  $(RP^0)$  (see e.g. [20]). □

### 3. An application

A recent contribution by Hu and Mehrotra [21] applied a robust framework, so-called McRow (multicriteria robust optimization with weight set), to address a multiobjective optimization problem through (linear) scalarization. Within the same line, other papers have used similar techniques to model a decision-making problem involving the satisfaction of several stakeholders. Similar arguments apply to management science, politics, as well as R&D project selection [11,22,23].

Basically, any decision-making process implies the need to face conflicts among stakeholders who, ultimately, must approve a certain action. Whether it is a management strategy that needs to be accepted by a board, an environmental policy or an act to be passed by a government, stakeholder disagreement, due to different opinions or perceptions, must be addressed. We assume that a decision is represented by some vector  $x \in X \subseteq \mathbb{R}^n$ , where  $X$  is the feasible region, modelling budget, cultural, technical, etc. constraints. A group of  $r$  stakeholders shares the same criteria  $l_k(x)$ ,  $k = 1, \dots, p$ , to evaluate the decision. As an example, we can think of the decision to build some logistic infrastructure that must be evaluated according to the revenues, environmental consequences, security, etc. Without loss of generality, we can assume that each criteria has to be minimized, hence we define a vector-valued loss function  $l : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p$ . Each party involved (e.g. environmental activists, entrepreneur, residents, trade unions, etc.) has a different opinion about the relative importance of each criteria and, hence, defines a different vector of nonnegative weights  $u^s = (u^s_1, \dots, u^s_p)$ ,  $s = 1, \dots, r$ , for each criterion. Hence, each stakeholder faces the individual optimization problem that is the minimization of the scalarized loss function  $L(x, u^s) = \langle l(x), u^s \rangle$

$$\inf_{x \in X} L(x, u^s) = L^*(u^s), \tag{5}$$

where  $L^*(u^s)$  provides the  $s$ th stakeholder with the most desirable outcome, even if it may not be granted by the strategy proposed by the DM. Hence, given a course of action  $x \in X$ , chosen by the DM, we can measure the dissatisfaction of each stakeholder as the distance between the scalarized value of the loss function and  $L^*(u^s)$ , namely

$$d_s(x, u^s) = L(x, u^s) - L^*(u^s). \tag{6}$$

The DM must minimize the weighted sum of dissatisfaction among the stakeholders

$$\inf_{x \in X} \sum_{s=1}^r \mu_s d_s(x, u^s), \tag{7}$$

where  $\mu_s \geq 0$  represents the relative importance or influence each stakeholder has and  $\sum_{s=1}^r \mu_s = 1$ .

In real-world applications, precise values of  $u^s$  are not likely known in advance, leading to uncertainty in the problem. We assume that for all  $s = 1, \dots, r$ ,  $u^s \in \mathcal{U}^s \subseteq \mathbb{R}^p$ , with  $\mathcal{U}^s$  convex and compact and we set  $\mathcal{U} := \mathcal{U}^1 \times \dots \times \mathcal{U}^r$ ,  $u = (u^1, \dots, u^r) \in \mathcal{U}$ . An overly conservative DM would like to minimize the maximum dissatisfaction originated by the decision, leading to the following robust counterpart

$$\inf_{x \in X} \sup_{u \in \mathcal{U}} f(x, u), \tag{8}$$

where  $f(x, u) = \sum_{s=1}^r \mu_s d_s(x, u^s)$ .

We assume now that  $\mathcal{U}$  models the highest uncertainty, while the DM may wish to limit it to a smaller subset  $\mathcal{U}^0 \subseteq \mathcal{U}$ . We assume  $\mathcal{U}^0 = \{u^0\}$ , with  $u^0 = (u^{0,1}, \dots, u^{0,r})$ , that is the DM's opinion or perception, although the case where  $\mathcal{U}^0$  is not a singleton might also be of some interest. When projects or policies require approval by a board of stakeholders, it is common knowledge that

the DM seeks consensus by not just trying to please as many stakeholders as possible, but also by steering the stakeholders' viewpoints towards the DM's own, i.e. toward  $u^0$ . Any activity (advertising, campaigns, lectures, lobbying, etc.) to change the attitude of the stakeholders will possibly result in a reduction of uncertainty. According to the notation of the previous section, we shall consider the set  $\mathcal{W}^\lambda$  as the uncertainty set reachable by consensus, where  $\lambda = 0$  is the largest consensus reachable and  $\lambda = 1$  means no action has been taken to influence the stakeholders. Hence, we have a desired optimization problem, the nominal instance,

$$\inf_{x \in X} f(x, u^0), \tag{9}$$

where  $f(x, u^0) = \sum_{s=1}^r \mu_s d_s(x, u^{0,s})$  and a robust optimization problem

$$\inf_{x \in X} \sup_{u \in \mathcal{W}^\lambda} f(x, u). \tag{10}$$

According to the results proved in the previous section, we have the following result to provide the location of the solution depending on the strength of the campaign enforced.

**Corollary 2.** Assume that  $\mathcal{U}^0 = \{u^0\}$ . Then

(i) we have

$$\begin{aligned} \inf_{x \in X} f(x, u^0) &\leq \inf_{x \in X} \sup_{u \in \mathcal{W}^\lambda} f(x, u) \\ &\leq \inf_{x \in X} f(x, u^0) + \lambda \sum_{s=1}^r \mu_s \sup_{x \in X} \sup_{u^s \in \mathcal{U}^s - u^{0,s}} d_s(x, u^s); \end{aligned}$$

(ii) when  $\lambda \downarrow 0$ , we have

$$\inf_{x \in X} \sup_{u \in \mathcal{W}^\lambda} f(x, u) \rightarrow \inf_{x \in X} f(x, u^0).$$

**Proof.** Since  $L^*(u^s)$  is a super-linear function as the infimum of linear functions, it follows that  $d_s(x, u^s)$  are sub-linear functions as the sum of a linear function and a sub-linear function. Hence, the result follows from Propositions 1 and 3 and Remark 1.  $\square$

**Example 1.** Let  $r = 2$ ,  $n = 1$  and  $p = 2$ . Assume the loss function is  $l(x) = (x, -x)$  and  $X = [1, 2]$ . Let  $u^1 = (u_1^1, u_2^1) \in \mathcal{U}_1 = \{(\alpha, 1 - \alpha) \mid \alpha \in [0.2, 0.8]\}$  and  $u^2 = (u_1^2, u_2^2) \in \mathcal{U}_2 = \{(\alpha, 1 - \alpha) \mid \alpha \in [0.3, 0.7]\}$ , so that problem (5) is reduced to

$$\inf_{x \in [1, 2]} (2u_1^s - 1)x$$

for each  $s = 1, 2$ . Therefore,

$$\operatorname{argmin}_{x \in X} f(x, u^s) = \begin{cases} \{1\} & \text{if } u_1^s > 0.5 \\ [1, 2] & \text{if } u_1^s = 0.5 \\ \{2\} & \text{if } u_1^s < 0.5 \end{cases}$$

and

$$L^*(u^s) = \begin{cases} 2u_1^s - 1 & \text{if } u_1^s \geq 0.5, \\ 2(2u_1^s - 1) & \text{if } u_1^s < 0.5, \end{cases} \tag{11}$$

while

$$d_s(x, u^s) = \begin{cases} (2u_1^s - 1)x - 2u_1^s + 1 & \text{if } u_1^s \geq 0.5 \\ (2u_1^s - 1)x - 2(2u_1^s - 1) & \text{if } u_1^s < 0.5 \end{cases} \tag{12}$$

for each  $s = 1, 2$ .

Assume  $\mu_1 = \mu_2 = \frac{1}{2}$ , i.e. the DM equally weights the dissatisfaction of each stakeholder. Hence, problem (7) is reduced to

$$\inf_{x \in [1, 2]} \frac{1}{2} d_1(x, u^1) + \frac{1}{2} d_2(x, u^2).$$

In view of (12), we can compute

$$\sup_{(u^1, u^2) \in \mathcal{U}} \frac{1}{2} d_1(x, u^1) + \frac{1}{2} d_2(x, u^2) = \begin{cases} -0.5x + 1 & \text{if } x \in [1, 1.5], \\ 0.5x - 0.5 & \text{if } x \in (1.5, 2]. \end{cases}$$

Therefore, the robust solution is  $x^* = 1.5$ .

When  $\mathcal{U}^0 = \mathcal{U}_1^0 \times \mathcal{U}_2^0 = \{(0.5, 0.5)\} \times \{(0.5, 0.5)\}$ , we have

$$\inf_{x \in X} f(x, u^0) = 0 \quad \text{and} \quad \sum_{s=1}^r \mu_s \sup_{x \in X} \sup_{u^s \in \mathcal{U}^s - u^{0,s}} d_s(x, u^s) = 1.5,$$

by direct calculation, and then we have

$$0 \leq \inf_{x \in X} \sup_{u \in \mathcal{W}^\lambda} f(x, u) \leq 1.5\lambda$$

from Corollary 2.

Corollary 2 guarantees that enforcing lobbying strategies that spread consensus leads to a robust optimal value close to the target of the DM. Moreover, it also provides a suitable location of the robust optimal value in terms of distance from the desired one. Finally, applying also Proposition 2, we have that the robust solution to problem (10) is an  $\varepsilon$ -solution of problem (9).

**Proposition 5.** Let  $l_k$  be convex for all  $k = 1, \dots, p$  and l.s.c. Assume that the set of solutions to problem (9) is nonempty and compact and  $\lambda_n \downarrow 0$ . Let  $x_{\lambda_n} \in X$  be a solution to problem (10) for each  $\mathcal{W}^{\lambda_n}$ . Then there exists a subsequence of  $x_{\lambda_n}$  converging to a solution  $\bar{x}$  of problem (9).

#### 4. Multiobjective case

We now consider the multiobjective optimization problem

$$\min_{x \in X} g(x), \tag{MP}$$

where  $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $X \subseteq \mathbb{R}^n$  is a closed convex set. In vector optimization, the notion of a solution to (MP) is not unique (see e.g. [24]). A point  $x^0 \in X$  is said to be an efficient solution to (MP) when

$$(\operatorname{Im}(g) - g(x^0)) \cap (-\mathbb{R}_+^m) = \{0\},$$

where  $\operatorname{Im}(g)$  is the range of  $g$  or, equivalently, there does not exist  $x \in X$  such that  $g(x) \leq g(x^0)$  and  $g(x) \neq g(x^0)$ , where  $a \leq b$  if  $a \in b - \mathbb{R}_+^m$  whenever  $a, b \in \mathbb{R}^m$ .

A point  $x^0$  is said to be a weakly efficient solution to (MP) when

$$(\operatorname{Im}(g) - g(x^0)) \cap (-\operatorname{int} \mathbb{R}_+^m) = \emptyset$$

or, equivalently, there does not exist  $x \in X$  such that  $g(x) < g(x^0)$ , where  $a < b$  if  $a \in b - \operatorname{int} \mathbb{R}_+^m$ .

Also the set of efficient (respectively, weakly efficient) solutions of problem (MP) is denoted by  $\operatorname{Eff}(\operatorname{MP})$  (respectively,  $\operatorname{WEff}(\operatorname{MP})$ ). We set  $\operatorname{Min}(\operatorname{MP}) = \{g(x) : x \in \operatorname{Eff}(\operatorname{MP})\}$  and  $\operatorname{WMin}(\operatorname{MP}) = \{g(x) : x \in \operatorname{WEff}(\operatorname{MP})\}$  to denote the set of optimal values corresponding to efficient and weak efficient solutions, respectively.

We now consider an uncertain multiobjective optimization problem:

$$\min_{x \in X} (f_1(x, u_1), \dots, f_m(x, u_m)), \tag{UMP}$$

where  $f_i: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , are continuous functions with respect to  $u_i$ , with  $u_i \in \mathcal{U}_i$  for some convex compact subset  $\mathcal{U}_i \in \mathbb{R}^p$ .

When  $m = 1$ , problem (UMP) becomes a scalar uncertain minimization problem. Recently, Kuroiwa and Lee [15] introduced the robust counterpart of (UMP), defined as

$$\min_{x \in X} \left( \sup_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \dots, \sup_{u_m \in \mathcal{U}_m} f_m(x, u_m) \right). \tag{RMP}$$

A robust efficient solution to (UMP) is defined as a vector  $x^0 \in X$  that is an efficient solution to (RMP). A robust weakly efficient solution to (UMP) is defined as a weakly efficient solution to (RMP).



Similarly to the previous section, consider convex compact sets  $\mathcal{U}_i^0 \subseteq \mathcal{U}_i \subseteq \mathbb{R}^p$ ,  $i = 1, \dots, m$ , to represent the lowest degree of uncertainty achievable on each criterion. Likewise, the sets

$$\mathcal{W}_i^\lambda := (1 - \lambda)\mathcal{U}_i^0 + \lambda\mathcal{U}_i \quad \lambda \in [0, 1]$$

represent intermediate degrees of uncertainty on each criterion  $i = 1, \dots, p$ . As in the scalar case, we assume  $\max_{u \in \mathcal{U}_i^0} f_i(\cdot, u)$  is lower bounded on  $X$  for all  $i = 1, \dots, m$ , and we set for all  $i = 1, \dots, m$

$$E_{f_i}(x) := \max_{u \in \mathcal{U}_i} f_i(x, u) - \max_{u \in \mathcal{U}_i^0} f_i(x, u);$$

$$\bar{E}_{f_i} := \sup_{x \in X} E_{f_i}(x);$$

and

$$\underline{E}_{f_i} := \inf_{x \in X} E_{f_i}(x);$$

assuming  $\bar{E}_{f_i}$  is finite, see Remark 1, for all  $i = 1, \dots, m$ .

The robust multiobjective counterpart associated to  $\mathcal{W}_i^\lambda$  is

$$\min_{x \in X} \left( \max_{w_1 \in \mathcal{W}_1^\lambda} f_1(x, w_1), \dots, \max_{w_m \in \mathcal{W}_m^\lambda} f_m(x, w_m) \right). \quad \text{RMP}^\lambda$$

We now study the behaviour of the solutions of (RMP $^\lambda$ ). For notational simplicity, we set:

$$(i) \ f^\lambda(x) := \left( \max_{w_1 \in \mathcal{W}_1^\lambda} f_1(x, w_1), \dots, \max_{w_m \in \mathcal{W}_m^\lambda} f_m(x, w_m) \right);$$

$$(ii) \ \bar{\mathbf{E}}_f := (\bar{E}_{f_1}, \dots, \bar{E}_{f_m});$$

$$(iii) \ \underline{\mathbf{E}}_f := (\underline{E}_{f_1}, \dots, \underline{E}_{f_m}).$$

By definition,  $f^0 \leq f^\lambda$  and  $0 \leq \underline{\mathbf{E}}_f \leq \bar{\mathbf{E}}_f$ .

Remember that  $A \subseteq \mathbb{R}^m$  is said to be  $\mathbb{R}_+^m$ -closed if  $A + \mathbb{R}_+^m$  is closed and  $A$  is said to be minorized if there exists  $\underline{a} \in A$  such that  $\underline{a} + \mathbb{R}_+^m \supseteq A$ . When  $\text{Im}(f^\lambda)$  is  $\mathbb{R}_+^m$ -closed for  $\lambda \in [0, 1]$ , we can see that  $\text{Min}(\text{RMP}^\lambda)$  is nonempty and the domination property for  $\text{Im}(f^\lambda)$ , that is,

$$\text{Im}(f^\lambda) \subseteq \text{Min}(\text{RMP}^\lambda) + \mathbb{R}_+^m \quad (13)$$

holds, since  $\text{Im}(f^\lambda)$  is minorized by the lower boundedness assumption of  $\max_{u \in \mathcal{U}_i^0} f_i(\cdot, u)$  on  $X$ , see for example Lemma 3.5 and Theorems 3.3 and 4.3 in [24]. The domination property is important in the theory of decision making. We need the following relation between sets introduced in [19]. For  $A, B \subseteq \mathbb{R}^m$ , we use the notation

$$A \leq^l B \quad \text{if } A + \mathbb{R}_+^m \supseteq B,$$

or, equivalently, if for every  $b \in B$ , there exists  $a \in A$  such that  $a \leq b$  holds. The relation is reflexive and transitive, but not antisymmetric. By using the notation, the inclusion in the domination property (13) can be written as follows:

$$\text{Min}(\text{RMP}^\lambda) \leq^l \text{Im}(f^\lambda).$$

We now discuss the locations of the robust efficient or weakly efficient optimal values.

**Proposition 6.** Assume that  $\text{Im}(f^0)$  is  $\mathbb{R}_+^m$ -closed. Then the set relation

$$\text{Min}(\text{RMP}^0) \leq^l \text{Im}(f^\lambda)$$

holds for every  $\lambda \in [0, 1]$  and, hence, in particular

$$\text{Min}(\text{RMP}^0) \leq^l \text{Min}(\text{RMP}^\lambda) \quad \text{and}$$

$$\text{WMin}(\text{RMP}^0) \leq^l \text{WMin}(\text{RMP}^\lambda).$$

**Proof.** For every  $x \in X$ , by using the domination property for  $\text{Im}(f^0)$ , there exists  $\hat{x} \in X$  such that  $f^0(\hat{x}) \in \text{Min}(\text{RMP}^0)$  and  $f^0(\hat{x}) \leq f^0(x)$ . From  $f^0(x) \leq f^\lambda(x)$ , then we have

$$f^\lambda(x) = f^0(\hat{x}) + (f^\lambda(x) - f^0(x)) + (f^0(x) - f^0(\hat{x})) \\ \in \text{Min}(\text{RMP}^0) + \mathbb{R}_+^m.$$

This shows  $\text{Min}(\text{RMP}^0) \leq^l \text{Im}(f^\lambda)$ .  $\square$

**Proposition 7.** Assume that for all  $i = 1, \dots, m$ ,  $f_i(x, u_i)$  is convex in both variables  $x \in X$  and  $u_i \in \mathcal{U}_i$ , and  $\text{Im}(f^\lambda)$ ,  $\lambda \in [0, 1]$ , is  $\mathbb{R}_+^m$ -closed. Then the set relation

$$\text{WMin}(\text{RMP}^\lambda) \leq^l \text{Im}(f^0) + \lambda \bar{\mathbf{E}}_f$$

holds and, hence, in particular

$$\text{WMin}(\text{RMP}^\lambda) \leq^l \text{WMin}(\text{RMP}^0) + \lambda \bar{\mathbf{E}}_f.$$

**Proof.** By contradiction, if the set relation  $\text{WMin}(\text{RMP}^\lambda) \leq^l \text{Im}(f^0) + \lambda \bar{\mathbf{E}}_f$  does not hold, there exists  $\bar{x} \in X$  such that  $f^0(\bar{x}) + \lambda \bar{\mathbf{E}}_f \notin \text{WMin}(\text{RMP}^\lambda) + \mathbb{R}_+^m$ . From the convexity of  $f_i(\cdot, u_i)$  and the domination property for  $\text{Im}(f^\lambda)$ ,  $\text{WMin}(\text{RMP}^\lambda) + \mathbb{R}_+^m$  is closed convex. Indeed, by using the domination property for  $\text{Im}(f^\lambda)$ ,

$$\text{Im}(f^\lambda) \subseteq \text{Min}(\text{RMP}^\lambda) + \mathbb{R}_+^m \subseteq \text{WMin}(\text{RMP}^\lambda) + \mathbb{R}_+^m \subseteq \text{Im}(f^\lambda) + \mathbb{R}_+^m,$$

this shows  $\text{WMin}(\text{RMP}^\lambda) + \mathbb{R}_+^m = \text{Im}(f^\lambda) + \mathbb{R}_+^m$ . From the convexity of  $f_i(\cdot, u_i)$ , we have that  $f^\lambda$  is  $\mathbb{R}_+^m$ -convex, therefore  $\text{Im}(f^\lambda) + \mathbb{R}_+^m$  is convex. Also from the  $\mathbb{R}_+^m$ -closedness of  $\text{Im}(f^\lambda)$ ,  $\text{Im}(f^\lambda) + \mathbb{R}_+^m$  is closed. By using the strong separation theorem, there exist  $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_m) \in \mathbb{R}^m \setminus \{0\}$  and  $\beta \in \mathbb{R}$  such that

$$\langle \bar{\alpha}, f^0(\bar{x}) + \lambda \bar{\mathbf{E}}_f \rangle < \beta \leq \langle \bar{\alpha}, y \rangle$$

hold for every  $y \in \text{WMin}(\text{RMP}^\lambda) + \mathbb{R}_+^m$ . It is easy to show that  $\bar{\alpha} \in \mathbb{R}_+^m$ . Also by using the domination property for  $\text{Im}(f^\lambda)$ ,  $\text{Im}(f^\lambda) \subseteq \text{Min}(\text{RMP}^\lambda) + \mathbb{R}_+^m$  holds. By putting  $y = f^\lambda(\bar{x})$ ,

$$\beta \leq \langle \bar{\alpha}, f^\lambda(\bar{x}) \rangle \\ = \sum_{i=1}^m \max_{u_i^0 \in \mathcal{U}_i^0, u_i \in \mathcal{U}_i} \bar{\alpha}_i f_i(\bar{x}, (1 - \lambda)u_i^0 + \lambda u_i) \\ \leq \sum_{i=1}^m \max_{u_i^0 \in \mathcal{U}_i^0, u_i \in \mathcal{U}_i} \bar{\alpha}_i ((1 - \lambda)f_i(\bar{x}, u_i^0) + \lambda f_i(\bar{x}, u_i)) \\ = \sum_{i=1}^m \bar{\alpha}_i \left( \max_{u_i^0 \in \mathcal{U}_i^0} f_i(\bar{x}, u_i^0) + \lambda E_{f_i}(\bar{x}) \right) \\ \leq \sum_{i=1}^m \bar{\alpha}_i \left( \max_{u_i^0 \in \mathcal{U}_i^0} f_i(\bar{x}, u_i^0) + \lambda \bar{E}_{f_i} \right) \\ = \langle \bar{\alpha}, f^0(\bar{x}) + \lambda \bar{\mathbf{E}}_f \rangle \\ < \beta.$$

This is a contradiction. Then the set relation  $\text{WMin}(\text{RMP}^\lambda) \leq \text{Im}(f^0) + \lambda \bar{\mathbf{E}}_f$  holds. The latter set relation is clear.  $\square$

**Remark 4.** The convexity assumption on  $f_i(\cdot, u_i)$  in Proposition 7 can be weakened by the  $\mathbb{R}_+^m$ -convex-likeness of  $f^\lambda$  or, equivalently,  $\mathbb{R}_+^m$ -convexity of  $\text{Im}(f^\lambda)$ , that is  $\text{Im}(f^\lambda) + \mathbb{R}_+^m$  is a convex set.

**Corollary 3.** Assume that for all  $i = 1, \dots, m$ ,  $f_i(x, u_i)$  is convex in both variables  $x \in X$  and  $u_i \in \mathcal{U}_i$ , and  $\text{Im}(f^\lambda)$ ,  $\lambda \in [0, 1]$ , are  $\mathbb{R}_+^m$ -closed. Then the set relations

$$\text{WMin}(\text{RMP}^0) \leq^l \text{WMin}(\text{RMP}^\lambda) \leq^l \text{WMin}(\text{RMP}^0) + \lambda \bar{\mathbf{E}}_f \quad (14)$$

hold.

Let  $\mathcal{C}$  be the family of all  $\mathbb{R}_+^m$ -closed-convex minorized nonempty subsets of  $\mathbb{R}^m$ , that is,  $A \in \mathcal{C}$  if and only if  $A + \mathbb{R}_+^m$  is a nonempty closed convex set, and there exists  $\underline{a} \in \mathbb{R}^m$  such that  $\underline{a} + \mathbb{R}_+^m \supseteq A$ . We define a binary relation  $\equiv$  on  $\mathcal{C}$  by  $A \equiv B$  if  $A + \mathbb{R}_+^m = B + \mathbb{R}_+^m$  for any  $A, B \in \mathcal{C}$ . Then  $\equiv$  is an equivalence relation and we can define the equivalence class  $[A] = \{B \in \mathcal{C} \mid A \equiv B\}$  and the quotient set  $\mathcal{C}/\equiv = \{[A] \mid A \in \mathcal{C}\}$ , for further details see [25]. For

a bounded base of  $\mathbb{R}_+^m$ , for example  $D = \{d \in \mathbb{R}_+^m \mid \|d\| = 1\}$ , function  $H : (C/\cong)^2 \rightarrow \mathbb{R}$ , defined as

$$H(A, B) := H([A], [B]) := \sup_{d \in D} \left| \inf_{a \in A} \langle d, a \rangle - \inf_{b \in B} \langle d, b \rangle \right|,$$

is a metric, see [26] for details. Then, from (14), we have the following result.

**Corollary 4.** Under the assumptions of Corollary 3, we have

$$H(\text{WMin}(\text{RMP}^\lambda), \text{WMin}(\text{RMP}^0)) \leq \lambda \|\bar{\mathbf{E}}_f\|. \tag{15}$$

and

$$[\text{WMin}(\text{RMP}^\lambda)] \rightarrow [\text{WMin}(\text{RMP}^0)] \text{ as } \lambda \downarrow 0$$

in the metric  $H$ .

**Proof.** Put  $A = \text{WMin}(\text{RMP}^0)$  and  $B = \text{WMin}(\text{RMP}^\lambda)$ . The relations (14) imply

$$\inf_{a \in A} \langle d, a \rangle \leq \inf_{b \in B} \langle d, b \rangle \leq \inf_{a \in A} \langle d, a \rangle + \lambda \|\bar{\mathbf{E}}_f\|$$

for all  $d \in D$ . Hence we have (15).  $\square$

**Proposition 8.** Assume that for all  $i = 1, \dots, m$ ,  $f_i(x, u_i)$  is convex in both variables  $x \in X$  and concave in  $u_i \in \mathcal{U}_i$ , and  $\text{Im}(f^0)$  is  $\mathbb{R}_+^m$ -closed. Then the set relation

$$\text{WMin}(\text{RMP}^0) \leq^l \text{Im}(f^\lambda) - \lambda \mathbf{E}_f$$

holds and, hence, in particular

$$\text{WMin}(\text{RMP}^0) \leq^l \text{WMin}(\text{RMP}^\lambda) - \lambda \mathbf{E}_f.$$

**Proof.** The proof follows along the same lines as the proof of Proposition 7, by using the separation theorem for a closed convex set  $\text{WMin}(\text{RMP}^0) + \mathbb{R}_+^m$  and the domination property for  $\text{Im}(f^0)$ .  $\square$

**Corollary 5.** Assume that  $f_i(x, u_i) = \langle g_i(x), u_i \rangle + h_i(x)$ , where  $g_i : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are convex,  $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , and  $\text{Im}(f^\lambda)$  is  $\mathbb{R}_+^m$ -closed. Then the set relations

$$\text{WMin}(\text{RMP}^0) + \lambda \mathbf{E}_f \leq^l \text{WMin}(\text{RMP}^\lambda) \leq^l \text{WMin}(\text{RMP}^0) + \lambda \bar{\mathbf{E}}_f \tag{16}$$

hold. Moreover, if  $\mathcal{U}_i^0 = \{u_i^0\}$  and  $\mathcal{U}_i = \mathcal{B}(u_i^0)$ ,  $i = 1, \dots, m$ , then the following holds:

$$\begin{aligned} & \text{WMin}(\text{RMP}^0) + \lambda \left( \inf_{x \in X} \|g_1(x)\|, \dots, \inf_{x \in X} \|g_m(x)\| \right) \\ & \leq^l \text{WMin}(\text{RMP}^\lambda) \leq^l \text{WMin}(\text{RMP}^0) \\ & + \lambda \left( \sup_{x \in X} \|g_1(x)\|, \dots, \sup_{x \in X} \|g_m(x)\| \right). \end{aligned}$$

From (16), we also have the following result.

**Corollary 6.** Under the assumptions of Corollary 5, we have

$$\lambda \|\mathbf{E}_f\| \leq H(\text{WMin}(\text{RMP}^\lambda), \text{WMin}(\text{RMP}^0)) \leq \lambda \|\bar{\mathbf{E}}_f\|.$$

**Example 2.** Fliege and Werner [27] proposed an application to portfolio selection of robust optimization. They considered a financial market with  $n$  risky assets defined on a suitable probability space in a single period setting. We assume their multivariate distribution has parameters  $\mu$  and  $\Sigma$  representing the vector of expected returns and the variance–covariance matrix, respectively. We also assume  $X \neq \emptyset$  is the convex and compact set of feasible portfolios and that portfolio weights add up to 1 (i.e.  $X \subseteq \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1\}$ ).

The efficient frontier in portfolio optimization is obtained as the set of solutions of the following problem:

$$\min_{x \in X} (f_1(x), f_2(x)), \tag{17}$$

where

$$\begin{aligned} f_1(x) &= x^T \Sigma x, \\ f_2(x) &= -\langle \mu, x \rangle. \end{aligned}$$

However, the nominal values of  $\mu$  and  $\Sigma$  are not known before the optimal portfolio is selected, although their realization will affect the payoff. The DM, therefore, faces an uncertainty problem that we can model by assuming  $\mathcal{U}_1 \subseteq \mathbb{R}^n$  and  $\mathcal{U}_2 \subseteq \mathbb{R}^{n^2}$  are the uncertainty sets of  $\mu \in \mathbb{R}^n$  and  $\Sigma \in \mathbb{R}^{n^2}$ , respectively, where we represent the matrix  $\Sigma$  as a vector  $\Sigma$  with  $n^2$  components.

Following the approach in [27], the multiobjective robust counterpart is

$$\min_{x \in X} \left( \max_{\Sigma \in \mathcal{U}_2} f_1(x), \max_{\mu \in \mathcal{U}_1} f_2(x) \right) \tag{18}$$

Assuming  $(\mu^0, \Sigma^0) \in \mathcal{U}_2 \times \mathcal{U}_1$  is a nominal instance (e.g. the one that will be realized or the one can be expected under some arbitrary distribution assumption), Fliege and Werner [27] defined the loss of efficiency due to uncertainty, i.e. choosing the robust solution by Crespi et al. (18).

Through results proved in Section 4, we can provide an estimation of the loss of efficiency under weak efficiency. Indeed, both  $f_1$  and  $f_2$  satisfy the assumptions in Propositions 7 and 8 and the subsequent corollaries. For the sake of simplicity, we assume  $n = 2$  and

$$\begin{aligned} \Sigma &= (\sigma_1^2, \sigma_2^2, \sigma_{1,2}, \sigma_{1,2}) \in \mathbb{R}^4, \\ \mu &= (\mu_1, \mu_2) \in \mathbb{R}^2, \end{aligned}$$

while uncertainty sets are

$$\begin{aligned} \mathcal{W}_1^\lambda &= (1 - \lambda)\mu^0 + \lambda\mathcal{U}_1, \\ \mathcal{W}_2^\lambda &= (1 - \lambda)\Sigma^0 + \lambda\mathcal{U}_2, \end{aligned}$$

where  $\mathcal{U}_1 = \mathcal{B}(\mu^0) \subseteq \mathbb{R}^2$  and  $\mathcal{U}_2 = \mathcal{B}(\Sigma^0) \subseteq \mathbb{R}^4$  are neighbourhoods of the nominal instances. Feasible portfolios are defined in  $X := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$ , i.e. no short sales are allowed. By direct calculations we obtain

$$\begin{aligned} E_{f_1}(x_1, x_2) &= x_1^2 + x_2^2 = \|(x_1, x_2)\|^2 \\ \text{and} \\ E_{f_2}(x_1, x_2) &= \|(x_1, x_2)\|. \end{aligned}$$

Therefore, we have  $\bar{E}_{f_1} = \bar{E}_{f_2} = 1$  and  $E_{f_1} = \frac{1}{2}$ ,  $E_{f_2} = \frac{\sqrt{2}}{2}$ . Hence, combining Propositions 7 and 8, we obtain

$$\begin{aligned} & \text{WMin}(\text{RMP}^0) + \lambda \left[ \frac{1}{\frac{\sqrt{2}}{2}} \right] \leq^l \text{WMin}(\text{RMP}^\lambda) \leq^l \text{WMin}(\text{RMP}^0) \\ & + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned}$$

while Corollary 6 gives

$$\frac{\sqrt{3}}{2} \lambda \leq H(\text{WMin}(\text{RMP}^\lambda), \text{WMin}(\text{RMP}^0)) \leq \sqrt{2} \lambda.$$

### Concluding remarks

In this paper we investigate the stability of robust solutions to changes in the uncertainty set. Scalar results are proved and then extended to multiobjective optimization. In both settings, under mild assumptions we manage to define upper and lower bounds for the robust optimal values. We identify possible areas of applications of the obtained results to decision making and to portfolio selection by linking our results to the models developed in [21,27].

Further investigations will address the generalization of the obtained results and other possible areas of application.

Among lines of future research, a possible topic is the case when the level of uncertainty differs on each component of the objective function. However a first rough approach to this instance can be obtained by setting in the framework of this paper,  $\lambda$  to be the maximum of all possible  $\lambda_i$  (assuming a different uncertainty on each component). Therefore the upper and lower bounds proved in this paper provide also a first approximation of bounds in the more general setting.

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