Lee, Chang-Yong; Lee, Dongju

Article

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An efficient method for solving a correlated multi-item inventory system

Chang-Yong Lee*, Dongju Lee

The Department of Industrial & Systems Engineering, Kongju National University, Cheonan 330–717, South Korea

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ABSTRACT

We propose an efficient method of finding an optimal solution for a multi-item continuous review inventory model in which a bivariate Gaussian probability distribution represents a correlation between the demands of different items. By utilizing appropriate normalizations of the demands, we show that the normalized demands are uncorrelated. Furthermore, the set of equations coupled with different items can be decoupled in such a way that the order quantity and reorder point for each item can be evaluated independently from those of the other. As a result, in contrast to conventional methods, the solution procedure for the proposed method can be much simpler and more accurate without any approximation. To demonstrate the advantage of the proposed method, we present a solution scheme for a multi-item continuous review inventory model in which the demand of optional components depend on that of a "vanilla box," representing the customer's stochastic demand, under stochastic payment and budget constraints. We also perform a sensitivity analysis to investigate the dependence of order quantities and reorder points on the correlation coefficient.

1. Introduction

In a competitive global market, a manufacturer usually needs to provide a wide variety of products with a short amount of lead time to improve customer satisfaction and increase market share. When an order from a customer arrives, differentiation to the tailored order is often postponed down the assembly line to achieve both product variety and short lead time. In general, the more diverse a product, the longer lead time it requires. Manufacturers often postpone final products and secure semi-finished products to meet customers various orders effectively by assembling the optional components to semi-finished products. Modularization and postponement (or delayed differentiation) can be an effective means to reduce the lead time while maintaining a wide variety of products. Many modularization and postponement studies have shown that these concepts offer an advantage in terms of reducing uncertainty and forecasting errors with regard to demand [3,9,10] in addition to creating product variety and customization at low cost [2]. Thus, modularization and postponement have become important concepts in the market to provide better service to customers and make the business process more efficient.

Saghiri and Barnes [7] investigated the link between supplier flexibility and postponement implementation through empirical data, and found that the supplier impacts on postponement implementation for the buying companies depend on the types of postponement and supplier flexibility. Simão et al. [8] evaluated the impact of postponement on supply chain performance considering green supply chain by using computer simulation. They showed that packing postponement and logistics can improve the performance of logistics and reduce CO_2 emissions. Zhou et al. [13] developed a two-stage queueing network for form postponement considering correlated demand characterized by Markovian arrival process, and found that the variance and correlation coefficient of the demand can increase total cost.

In a product line such as computer retailing or automobile assembly, concepts of modularization and postponement are realized by an assembly process that consists of a semi-finished product "vanilla box" and optional components that are directly used in the final assembly. The vanilla box consists of components, known as the commonality of parts, needed to assemble the final product with appropriate optional components, and the vanilla box approach has been shown to be effective under high variance [9].

Applications of modularization and postponement to an inventory system require a multi-item model in which the demand of each of the several optional components depends on the

* Corresponding author.
E-mail addresses: clee@kongju.ac.kr (C.-Y. Lee), djlee@kongju.ac.kr (D. Lee).

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demand or the presence of the vanilla box. The vanilla box represents the customer's stochastic demand, and optional components, in turn, depend on the demand of the vanilla box for final assembly. Thus, the demands of the vanilla box and optional components are stochastically correlated. As the inventory model becomes complicated because of the correlation, it is desirable to find an efficient and accurate method to solve the model system.

A multi-item inventory model was proposed to comply with the concept of modularization and postponement [11]. The model consists of a vanilla box and optional components in which the correlation between the two types of items is implemented as a bivariate Gaussian probability distribution whereas the optional components are independent of each other. This model handles a continuous review inventory system in which an order quantity Q is placed whenever an inventory level reaches a certain reorder point r under the presence of service level and budget constraints. Subsequently, a stochastic payment is also included in the model in such a manner that the total inventory cost does not exceed a predetermined budget [12].

There have been investigated two types of the budget constraint depending on when the purchasing cost is paid. The first type pays the purchasing cost at the time an order is placed, whereas the second type pays when an order arrives. The latter case is referred to as a *stochastic* budget constraint because the inventory level at placing an order is, in general, different from that at the arrival of the order. In this case, the maximum inventory investment is random due to the random nature of the inventory level. Thus, the stochastic budget constraint takes a form of a probability in that the maximum inventory investment to be within a budget should be greater than a certain level when the purchasing cost is paid at the time an order arrives.

In this paper, we propose an efficient method to solve a correlated multi-item continuous review inventory model in which the correlation between the vanilla box and an optional component is represented by a bivariate Gaussian probability distribution. In addition, when the variables are normally distributed, many results, such as conditional distribution, can be derived analytically in explicit form. By using appropriate normalizations of the demands of items, we show that the set of equations coupled with the vanilla box and optional components can be reduced to sets of decoupled equations for each item. Furthermore, each set of decoupled equations is simplified in a closed form and solved without any approximation. Thus, the equations for each item can be solved independently of each other.

The conventional method for solving such a model system is based on a heuristic of combining a Newton–Raphson method and a Hadley–Whitin iterative procedure [12]. At each iteration, a candidate solution is found by using the Newton–Raphson method in which numerical integrations are carried out where required. The iteration proceeds until both Q and r sufficiently converge. Briefly, the conventional method takes the set of simultaneous equations for Q's and r's as a whole and utilizes heuristic approximating procedures. Given that the conventional method uses a rather complicated approximation and iteration, it requires heavy computation time.

In contrast, the proposed method does not rely on any approximation or heuristics. As a result, the solution procedure for the proposed method is much simpler, more accurate, and offers shorter computing time than the conventional method. We apply the proposed method to a correlated multi-item continuous review inventory model to demonstrate its usefulness. We also perform a sensitivity analysis in terms of the correlation to further characterize the behavior of the order quantity and reorder point of optional components. In addition, the proposed scheme can be used as a dependable method for a more generalized multi-item continuous review inventory model with much more complicated correlations among items.

The rest of this paper is organized as follows. Section 2 discusses the normalization of the demands and introduces an illustrative model to show how the set of simultaneous equations can be decoupled and simplified. Section 3 describes the proposed method for solving the model system. Section 4 presents experimental results and discusses the sensitivity analysis, together with an application of the proposed method to an extended model. Section 5 summarizes the study and gives our conclusions.

2. Normalization and multi-item inventory model

2.1. Correlation and normalization

Consider a multi-item inventory model that includes the correlation between a vanilla box and an optional component. Furthermore, we allow multiple optional components and each optional component is dependent on the vanilla box through a bivariate Gaussian probability distribution, whereas optional components are independent of each other.

A bivariate Gaussian probability distribution function (PDF) of the random variables X_v and X_j of the demand during the lead time for the vanilla box and the jth optional component, respectively, is given by

$$f(x_v, x_j) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1}(\vec{x}-\vec{\mu})\right\}. \tag{1}$$

Here, the variable vector $\vec{x}$, the mean vector $\vec{\mu}$, the covariance matrix Σ, and the correlation coefficient ρ_j between X_v and X_j are expressed, respectively, as

$$\vec{x} = \begin{bmatrix} x_v \\ x_j \end{bmatrix}, \quad \vec{\mu} = \begin{bmatrix} \mu_v \\ \mu_j \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_v^2 & \sigma_{vj} \\ \sigma_{vj} & \sigma_j^2 \end{bmatrix}, \quad \text{and} \quad \rho_j \equiv \frac{\sigma_{vj}}{\sigma_v \sigma_j} \tag{2}$$

with $|\Sigma|$ being the determinant of the 2×2 matrix of Σ. Given that X_j depends on X_v, we express the bivariate PDF as a product of the marginal PDF of X_v and the conditional PDF of X_j given $X_v = x_v$. In this way, the bivariate PDF of Eq. (1) can be written as

$$\begin{aligned} f(x_v, x_j) &= \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left\{-\frac{(x_v-\mu_v)^2}{2\sigma_v^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_j\sqrt{1-\rho_j^2}} \\ &\quad \times \exp\left\{-\frac{\left[x_j - \left(\mu_j + \rho_j \frac{\sigma_j}{\sigma_v}(x_v-\mu_v)\right)\right]^2}{2\sigma_j^2(1-\rho_j^2)}\right\} \\ &\equiv f_{X_v}(x_v)\ f_{X_j|X_v}(x_j|x_v). \end{aligned} \tag{3}$$

In general, the demand for the vanilla box is equal to the customer's demand, whereas the demand for each optional component depends on the safety stock of the vanilla box. Note that optional components, by definition, cannot be assembled by themselves. Rather, they should be combined with the vanilla box to form a final product. Therefore, the demand X_j of the jth optional component during the lead time depends on the current level r_v of the vanilla box. That is, the demand of an optional component depends on how many vanilla boxes needed to be finalized during the lead time (i.e., the current level r_v) in the assembly line. In this sense, the PDF of the demand X_j of the jth optional component during the lead time should be conditioned for a given $X_v = r_v$. Given that the safety stock of the vanilla box is $r_v - \mu_v$, the demand for each optional component depends on the reorder point of the vanilla box r_v. This implies that the conditional PDF of X_j is evaluated at $X_v = r_v$.

Motivated by this characteristic, we define the normalized random variables as

$$Z_v \equiv \frac{X_v - \mu_v}{\sigma_v} \quad \text{and} \quad Z_j \equiv \frac{X_j - \mu_{oj}}{\sigma_{oj}}, \tag{4}$$

where

$$\sigma_{oj} \equiv \sigma_j \sqrt{1-\rho_j^2} \quad \text{and} \quad \mu_{oj} \equiv \mu_j + \rho_j \frac{\sigma_j}{\sigma_v}(r_v - \mu_v). \tag{5}$$

Note that the random variable Z_j contains not only the demand X_j of the jth optional component but also the reorder point r_v of the vanilla box. With the normalization, the bivariate PDF at $X_v = r_v$ can be rewritten as

$$f_{X_v}(x_v)\ f_{X_j|X_v}(x_j|r_v) = \frac{1}{\sqrt{2\pi}} e^{-z_v^2/2} \frac{1}{\sqrt{2\pi}} e^{-z_j^2/2}. \tag{6}$$

We see from Eq. (6) that the normalization decomposes the bivariate PDF into a product of two PDFs of Z_v and Z_j, both of which can be regarded as independent standard *univariate* PDFs. This implies that the notion of the conditional PDF does not exist and there is no distinction between dependent and independent items. In what follows, we take a correlated multi-item continuous review inventory system as an illustration of the advantage of normalization.

2.2. Model formulation and set of equations

The illustrative model we consider in this paper is a correlated multi-item continuous review inventory system that includes two types of items: a vanilla box and many optional components. Furthermore, as stated in Section 2.1, each optional component depends on the vanilla box through a bivariate Gaussian probability distribution. We use the following notations to formulate the model:

- A : fixed procuring cost,
- C : unit variable procurement cost,
- D : expected annual demand,
- h : carrying cost,
- p : unit shortage cost,
- κ : service cost rate, and
- β : available budget limit.

Note that each of these terms can be used for both the vanilla box and optional components whenever possible. We distinguish the vanilla box from the jth optional component by the subscripts v and oj, respectively. For instance, A_v and A_{oj} represent the fixed procuring costs of the vanilla component and the jth optional component, respectively. The model is composed of the sum of the expected average annual cost (EAC) of the two types of items under budgetary constraint. The budget constraint can be expressed in terms of the total inventory investment, which consists of the purchase cost and the service cost. The service cost for the vanilla box is assumed to be proportional to the service level, which is the cumulative distribution of demand during the lead time. Thus, the service cost for the vanilla box is $\kappa_v F_{X_v}(r_v)$, where κ_v is the proportionality. Again, similar argument can be applied to the case of the optional components. As the stochastic budget constraint assumes that the purchase cost is paid at the time an order arrives, the inventory level has a stochastic nature in that the probability of total inventory investment to be within budget should be greater than a certain value. A detailed account of the model development can be found in [11,12].

The objective of the model is to minimize the sum of EAC for the vanilla box and m optional components under a stochastic budgetary constraint. That is, we would like to find (Q_v, r_v) and $(\vec{Q_o}, \vec{r_o})$, where $\vec{Q_o} = (Q_{o1}, Q_{o2}, \ldots, Q_{om})$ and $\vec{r_o} = (r_{o1}, r_{o2}, \ldots, r_{om})$, that minimize

$$EAC(\vec{Q}, \vec{r}) = EAC_1(Q_v, r_v) + EAC_2(\vec{Q_o}, \vec{r_o}) \tag{7}$$

subject to

$$Prob\left\{ C_v(Q_v + r_v - X_v) + \sum_{j=1}^{m} C_{oj}(Q_{oj} + r_{oj} - X_j) + \kappa_v F_{X_v}(r_v) + \sum_{j=1}^{m} \kappa_{oj} F_{X_j|X_v}(r_{oj}|r_v) \le \beta \right\} \ge \eta \tag{8}$$

for

$$Q_v \ge 0, \quad r_v \ge 0 \quad \text{and} \quad Q_{oj} \ge 0, \quad r_{oj} \ge 0 \quad \text{for} \quad j = 1, 2, \ldots, m. \tag{9}$$

In addition, $F_{X_v}(r_v)$ and $F_{X_j|X_v}(r_{oj}|r_v)$ are the cumulative density functions (CDFs) of $f_{X_v}(x_v)$ and $f_{X_j|X_v}(x_j|x_v)$, respectively.

The expected average annual total cost consists of the ordering, purchasing, holding, and shortage costs. The ordering, purchasing, and holding costs for the vanilla box are given, in order of precedence, as

$$\frac{A_v D_v}{Q_v} + C_v D_v + h_v\left(\frac{Q_v}{2} + r_v - \mu_v\right). \tag{10}$$

The shortage for the vanilla box occurs when the demand X_v during the lead time exceeds the current level r_v. Thus, the expected average shortage (or penalty) cost can be expressed as the expectation value of the cost due to the shortage $(x - r_v)$ for $x > r_v$:

$$\frac{p_v D_v}{Q_v} \int_{r_v}^{\infty} (x - r_v) f_{X_v}(x) dx, \tag{11}$$

where $f_{X_v}(x)$ is PDF of X_v. The similar argument can be applied to the optional components taking into account the conditional PDF. Thus, EAC_1 and EAC_2 are given as

$$EAC_1(Q_v, r_v) = \frac{A_v D_v}{Q_v} + C_v D_v + h_v\left(\frac{Q_v}{2} + r_v - \mu_v\right) + \frac{p_v D_v}{Q_v} \int_{r_v}^{\infty} (x - r_v) f_{X_v}(x) dx, \tag{12}$$

$$EAC_2(\vec{Q_o}, \vec{r_o}) = \sum_{j=1}^{m} \left[\frac{A_{oj} D_{oj}}{Q_{oj}} + C_{oj} D_{oj} + h_{oj}\left(\frac{Q_{oj}}{2} + r_{oj} - \mu_{oj}\right) + \frac{p_{oj} D_{oj}}{Q_{oj}} \int_{r_{oj}}^{\infty} (x - r_{oj}) f_{X_j|X_v}(x|r_v) dx \right], \tag{13}$$

where μ_{oj} is defined as in Eq. (5). It is readily shown that the constraint of Eq. (8) can be rewritten as

$$C_v(Q_v + r_v) + \sum_{j=1}^{m} C_{oj}(Q_{oj} + r_{oj}) + \kappa_v F_{X_v}(r_v) + \sum_{j=1}^{m} \kappa_{oj} F_{X_j|X_v}(r_{oj}|r_v) \le \beta + \mu_Y + z_{1-\eta}\sigma_Y, \tag{14}$$

where

$$\mu_Y \equiv C_v \mu_v + \sum_{j=1}^{m} C_{oj} \mu_{oj}, \quad \sigma_Y^2 \equiv C_v^2 \sigma_v^2 + \sum_{j=1}^{m} C_{oj}^2 \sigma_{oj}^2, \tag{15}$$

and $z_{1-\eta} = F^{-1}(1-\eta)$ with $F^{-1}(1-\eta)$ being the inverse of the standard Gaussian CDF of the probability η. The detailed derivation of Eq. (14) can be found in Appendix C.

With the model, the Lagrangian function J using the Lagrangian relaxation, with $\lambda \ge 0$, can be written as

$$J = EAC_1(Q_v, r_v) + EAC_2(\vec{Q_o}, \vec{r_o}) + \lambda \left\{ C_v(Q_v + r_v) + \sum_{j=1}^{m} C_{oj}(Q_{oj} + r_{oj}) + \kappa_v F_{X_v}(r_v) + \sum_{j=1}^{m} \kappa_{oj} F_{X_j|X_v}(r_{oj}|r_v) - \left(\beta + \mu_Y + z_{1-\eta}\sigma_Y\right) \right\}. \tag{16}$$

$$\int_{r_v}^{\infty}(x-r_v)f_{X_v}(x)\,dx=\sigma_v\left\{\frac{1}{\sqrt{2\pi}}e^{-z_v^2/2}-z_vG(z_v)\right\} \qquad \frac{d}{dr_v}\int_{r_v}^{\infty}(x-r_v)f_{X_v}(x)\,dx=-G(z_v)$$

$$\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)\,dx=\sigma_v\left\{\frac{1}{\sqrt{2\pi}}e^{-z_{oj}^2/2}-z_{oj}G(z_{oj})\right\} \qquad \frac{\partial}{\partial r_{oj}}\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)\,dx=-G(z_{oj})$$

$$\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{\partial}{\partial r_v}f_{X_j|X_v}(x|r_v)\,dx=\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)G(z_{oj}) \qquad \frac{d}{dr_v}F_X(r_v)=\frac{1}{\sqrt{2\pi}\sigma_{oj}}e^{-z_v^2/2}$$

$$\frac{\partial}{\partial r_v}F_{X_j|X_v}(r_{oj}|r_v)=-\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)\frac{1}{\sqrt{2\pi}\sigma_{oj}}e^{-z_{oj}^2/2} \qquad \frac{\partial}{\partial r_{oj}}F_{X_j|X_v}(r_{oj}|r_v)=\frac{1}{\sqrt{2\pi}\sigma_{oj}}e^{-z_{oj}^2/2}$$

Fig. 1. The list of expressions that are used to simplify the first order necessary condition.

The first order necessary conditions can be achieved by differentiating J with respect to Q_v, r_v, Q_{oj}, r_{oj}, and λ:

$$\frac{\partial J}{\partial Q_{oj}}=-\frac{A_{oj}D_{oj}}{Q_{oj}^2}+\frac{h_{oj}}{2}-\frac{p_{oj}D_{oj}}{Q_{oj}^2}\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx+\lambda C_{oj}=0 \tag{17}$$

$$\frac{\partial J}{\partial r_{oj}}=h_{oj}+\frac{p_{oj}D_{oj}}{Q_{oj}}\frac{\partial}{\partial r_{oj}}\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx+\lambda C_{oj}+\lambda\kappa_{oj}\frac{\partial}{\partial r_{oj}}F_{X_j|X_v}(r_{oj}|r_v)=0 \tag{18}$$

$$\frac{\partial J}{\partial Q_v}=-\frac{A_vD_v}{Q_v^2}+\frac{h_v}{2}-\frac{p_vD_v}{Q_v^2}\int_{r_v}^{\infty}(x-r_v)f_{X_v}(x)dx+\lambda C_v=0 \tag{19}$$

$$\begin{aligned}\frac{\partial J}{\partial r_v}=&h_v+\frac{p_vD_v}{Q_v}\frac{\partial}{\partial r_v}\int_{r_v}^{\infty}(x-r_v)f_{X_v}(x)dx\\&+\sum_{j=1}^{m}\frac{p_{oj}D_{oj}}{Q_{oj}}\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{\partial}{\partial r_v}f_{X_j|X_v}(x|r_v)dx\\&-\sum_{j=1}^{m}\left(h_{oj}+\lambda C_{oj}\right)\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)+\lambda C_v+\lambda\kappa_v\frac{d}{dr_v}F_{X_v}(r_v)\\&+\sum_{j=1}^{m}\lambda\kappa_{oj}\frac{\partial}{\partial r_v}F_{X_j|X_v}(r_{oj}|r_v)=0\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial J}{\partial\lambda}=&C_v(Q_v+r_v)+\sum_{j=1}^{m}\left[C_{oj}(Q_{oj}+r_{oj})\right]+\kappa_vF_{X_v}(r_v)\\&+\sum_{j=1}^{m}\kappa_{oj}F_{X_j|X_v}(r_{oj}|r_v)-\left(\beta+\mu_Y+z_{1-\eta}\sigma_Y\right)=0.\end{aligned} \tag{21}$$

Note that Eqs. (17)–(20) are simultaneous equations for Q_{oj}, r_{oj}, Q_v, and r_v.

2.3. Simplification of equations using normalization

Given that Q_{oj} and r_{oj} are coupled with Q_v and r_v from Eqs. (17)–(20), the equations are intractable to solve directly. For example, two equations Eqs. (17) and (18) contain three variables Q_{oj}, r_{oj}, and r_v. It turns out, however, that the normalizations discussed in Section 2.1 can not only simplify the various expressions in Eqs. (17)–(20), but also, more importantly, decouple the equations for the optional components Eqs. (17) and (18) from the equations for the vanilla box Eqs. (19) and (20).

Similarly to the normalization of Eq. (4), we further define the normalized reorder points as

$$z_v\equiv\frac{r_v-\mu_v}{\sigma_v}\quad\text{and}\quad z_{oj}\equiv\frac{r_{oj}-\mu_{oj}}{\sigma_{oj}}=\frac{(r_{oj}-\mu_j)-\rho_j\frac{\sigma_j}{\sigma_v}(r_v-\mu_v)}{\sigma_j\sqrt{1-\rho_j^2}}, \tag{22}$$

where we have used the definition of μ_{oj} of Eq. (5). Note that the normalized reorder point z_{oj} is a function of r_{oj} and r_v.

With the normalization defined in Eqs. (4) and (22), the various expressions in Eqs. (17)–(20) can be simplified as shown in Fig. 1. We derive the simplified expressions in detail in Appendix A. With the normalization, Eqs. (17)–(21) can be re-expressed as

$$\frac{\partial J}{\partial Q_{oj}}=-\frac{A_{oj}D_{oj}}{Q_{oj}^2}+\frac{h_{oj}}{2}-\frac{p_{oj}D_{oj}}{Q_{oj}^2}\sigma_{oj}L(z_{oj})+\lambda C_{oj}=0 \tag{23}$$

$$\frac{\partial J}{\partial r_{oj}}=h_{oj}-\frac{p_{oj}D_{oj}}{Q_{oj}}G(z_{oj})+\lambda C_{oj}+\lambda\frac{\kappa_{oj}}{\sigma_{oj}}f(z_{oj})=0 \tag{24}$$

$$\frac{\partial J}{\partial Q_v}=-\frac{A_vD_v}{Q_v^2}+\frac{h_v}{2}-\frac{p_vD_v}{Q_v^2}\sigma_vL(z_v)+\lambda C_v=0 \tag{25}$$

$$\frac{\partial J}{\partial r_v}=h_v-\frac{p_vD_v}{Q_v}G(z_v)+\lambda C_v+\lambda\frac{\kappa_v}{\sigma_v}f(z_v)=0 \tag{26}$$

$$\begin{aligned}\frac{\partial J}{\partial\lambda}=&C_v(Q_v+\sigma_vz_v)+\sum_{j=0}^{m}C_{oj}\left(Q_{oj}+\sigma_{oj}z_{oj}\right)+\kappa_vF(z_v)+\sum_{j=0}^{m}\kappa_{oj}F(z_{oj})\\&-\left(\beta+\mu_Y+z_{1-\eta}\sigma_Y\right)=0.\end{aligned} \tag{27}$$

Here, we define

$$f(z)\equiv\frac{1}{\sqrt{2\pi}}e^{-z^2/2},\quad G(z)\equiv\int_z^{\infty}f(t)\,dt,\quad F(z)\equiv1-G(z),\quad\text{and}\quad L(z)\equiv f(z)-zG(z). \tag{28}$$

Note that, owing to the normalization, Eqs. (17)–(20) are decoupled into two sets of equations: Eqs. (23) and (24) and Eqs. (25) and (26). Furthermore, each set of equations is identical and differs from the other only by the subscript. This implies that Eqs. (23) and (24) can be solved independently from Eqs. (25) and (26). This is expected because the PDF of the bivariate Gaussian distribution [Eq. (3)] can be expressed as the product of the PDF of the normalized variables Z_v and Z_j [Eq. (6)]. Thus, Eqs. (23) and (24) can be solved for Q_{oj} and z_{oj} directly. Similarly, Eqs. (25) and (26) can be solved for Q_v and z_v. Once z_{oj} and z_v are found, we can use Eqs. (4) and (22) to get r_{oj} and r_v. In the next section, we describe how to solve the set of Eqs. (23)–(26) under the constraint of Eq. (27).

3. Proposed method to solve the model system

Similar to the approach by [4], the proposed method to solve the model system consists of two parts. First, we regard Eqs. (23)–(26) as a subproblem for a given λ. Second, we repeatedly solve the subproblem until we find the solution λ to Eq. (27).

3.1. Procedures for solving the subproblem

For a given λ, we regard Eqs. (23)–(26) as a subproblem and solve them for Q_{oj}, z_{oj}, Q_v, and z_v. The crucial point is that solving Eqs. (23) and (24) for z_{oj} and Q_{oj} is independent of solving Eqs. (25) and (26) for z_v and Q_v, respectively. Given that $G(z_{oj})$ can be numerically evaluated for any z_{oj}, we note that Eqs. (23) and (24) are functions of Q_{oj} and z_{oj} only. By eliminating Q_{oj} from Eqs. (23) and (24), we obtain an equation for z_{oj}. In particular, Eqs. (23) and (24) can be rewritten, respectively, in terms of Q_{oj} as

$$Q_{oj}^2 = \frac{A_{oj}D_{oj} + p_{oj}D_{oj}\sigma_{oj}L(z_{oj})}{h_{oj}/2 + \lambda C_{oj}} \quad \text{and}$$

$$Q_{oj} = \frac{p_{oj}D_{oj}G(z_{oj})}{h_{oj} + \lambda C_{oj} + \lambda\left(\kappa_{oj}/\sigma_{oj}\right)f(z_{oj})}. \tag{29}$$

We can eliminate Q_{oj} from Eq. (29), resulting in an equation for $g_{oj}(z_{oj})$ in terms of z_{oj} only:

$$g_{oj}(z_{oj}) \equiv \frac{p_{oj}D_{oj}G(z_{oj})}{h_{oj} + \lambda C_{oj} + \lambda\left(\kappa_{oj}/\sigma_{oj}\right)f(z_{oj})} - \left\{\frac{A_{oj}D_{oj} + p_{oj}D_{oj}\sigma_{oj}L(z_{oj})}{h_{oj}/2 + \lambda C_{oj}}\right\}^{1/2} = 0. \tag{30}$$

There exists a unique solution $z_{oj} = z_{oj}^*$ of Eq. (30) if the following three conditions are satisfied:

(a) $g_{oj}(z_{oj})$ is a continuous function in z_{oj}.
(b) $g_{oj}(z_{oj})$ is strictly monotonic in z_{oj}.
(c) There exist two distinct values z_1 and z_2 such that $g_{oj}(z_1)\,g_{oj}(z_2) < 0$.

Condition (a) is immediately satisfied because $f(z_{oj})$, $G(z_{oj})$, and $L(z_{oj})$ are continuous in z_{oj}. The following theorem satisfies condition (b).

Theorem 1. *$g_{oj}(z_{oj})$ is strictly decreasing function in z_{oj}.*

The proof is given in Appendix B. Finally, condition (c) imposes a restriction on the values of the parameters. Given that $g_{oj}(z_{oj})$ is a continuous and strictly decreasing function in z_{oj} from conditions (a) and (b), by assuming that $z_{oj} > 0$, the following two inequalities should be satisfied to meet condition (c): $g_{oj}(0) > 0$ and $g_{oj}(+\infty) < 0$. It is easy to see that the condition $g_{oj}(+\infty) < 0$ is satisfied by noting that $G(+\infty) = 0$ and $0 < L(+\infty) < \infty$. Therefore, the values of the parameters have to satisfy $g_{oj}(0) > 0$. That is,

$$g_{oj}(0) = \frac{1}{2}\frac{p_{oj}D_{oj}}{h_{oj} + \lambda C_{oj} + \lambda\kappa_{oj}/(\sqrt{2\pi}\sigma_{oj})} - \left\{\frac{A_{oj}D_{oj} + p_{oj}D_{oj}\sigma_{oj}/\sqrt{2\pi}}{h_{oj}/2 + \lambda C_{oj}}\right\}^{1/2} > 0. \tag{31}$$

The values of the parameters should satisfy this inequality for Eq. (30) to have a unique solution.

Given that Eq. (30) is a function of z_{oj} only, one can use a simple search technique, such as a bisection method (see, for instance, [5]), to solve it at least numerically if not analytically. Once the solution z_{oj}^* of $g_{oj}(z_{oj}^*) = 0$ is obtained, we can substitute it back into Eq. (29) to get Q_{oj}^*, the solution of Q_{oj}. We repeat the same procedure for $j = 1, 2, \ldots, m$ to obtain the normalized reorder point and order quantity for the optional components.

The normalized reorder point z_v and order quantity Q_v for the vanilla box can be obtained by applying a method similar to that used for the optional components. That is, Eqs. (25) and (26) can be rewritten respectively in terms of Q_v as

$$Q_v^2 = \frac{A_vD_v + p_vD_v\sigma_vL(z_v)}{h_v/2 + \lambda C_v} \quad \text{and} \quad Q_v = \frac{p_vD_vG(z_v)}{h_v + \lambda C_v + \lambda(\kappa_v/\sigma_v)f(z_v)}. \tag{32}$$

The existence of a unique solution z_v^* of

$$g_v(z_v) \equiv \frac{p_vD_vG(z_v)}{h_v + \lambda C_v + \lambda(\kappa_v/\sigma_v)f(z_v)} - \left\{\frac{A_vD_v + p_vD_v\sigma_vL(z_v)}{h_v/2 + \lambda C_v}\right\}^{1/2} = 0 \tag{33}$$

can be proven in a similar fashion to the case for z_{oj}^* of Eq. (30). Thus, we can solve numerically for $z_v = z_v^*$; subsequently, we can solve for $Q_v = Q_v^*$ by substituting z_v^* into either Eq. (25) or (26).

3.2. Algorithm for solving the model system

The solution scheme for the subproblem discussed in Section 3.1 reduces the set of Eqs. (23)–(27) to one equation [Eq. (34)] with one unknown λ, which is implicitly dependent on the variables:

$$g(\lambda) = C_v(Q_v + \sigma_v z_v) + \sum_{j=0}^{m} C_{oj}\left(Q_{oj} + \sigma_{oj}z_{oj}\right) + \kappa_v F(z_v) + t\sum_{j=0}^{m}\kappa_{oj}F(z_{oj}) - \left(\beta + \mu_Y + z_{1-\eta}\sigma_Y\right). \tag{34}$$

If $g(\lambda) > 0$, then the constraint is violated; otherwise (that is, $g(\lambda) \le 0$), the constraint is satisfied. By using the Lagrangian relaxation [4], the proposed algorithm to obtain the optimal reorder points and order quantities for the optional components and vanilla box [r_{oj}, Q_{oj}, r_v, Q_v] is as follows:

Step 1: Find λ_1 and λ_2 such that $g(\lambda_1) > 0$ and $g(\lambda_2) < 0$.
Step 2: For each λ_1 and λ_2, solve the subproblem as follows:
Step 2(a): For $j = 1$ to m, numerically solve Eq. (30) for z_{oj} to obtain z_{oj}^*, and substitute z_{oj}^* into Eq. (29) to get Q_{oj}^*.
Step 2(b): Numerically solve Eq. (33) for z_v to obtain z_v^*, and substitute z_v^* into Eq. (32) to get Q_v^*.
Step 3: Let $\lambda_{new} = (\lambda_1 + \lambda_2)/2$ and find Q_{oj}^*, z_{oj}^*, Q_v^*, and z_v^* from Steps 2(a) and 2(b). If $g(\lambda_{new}) > 0$, then let $\lambda_1 = \lambda_{new}$; otherwise let $\lambda_2 = \lambda_{new}$.
Step 4: Repeat Steps 2 and 3 until $|g(\lambda_1)| < \epsilon$ or $|g(\lambda_2)| < \epsilon$, where ϵ is a predetermined error.
Step 5: Use Eq. (22) to get r_v^* and r_{oj}^* from z_v^* and z_{oj}^*, respectively.

We implemented the above algorithm with C language by using the libraries, such as “trapzd”, “polint”, “qromb”, and “rtbis” in Ref. [6].

4. Experimental results and discussion

We illustrate the performance of the proposed method by an experiment that consists of one vanilla box and two optional components (i.e., $m = 2$). The parameters for the vanilla box and two optional components are listed in Table 1. They are the same as the parameters used in [12] and $\epsilon = 10^{-7}$. Table 2 lists the solution to the model for a given input from Table 1 together with the results of [12].

Table 1
Parameters for the vanilla box and the two optional components with $\eta = 0.9031$ so that $z_{1-\eta} = -1.3$.

	A_v	C_v	D_v	h_v	p_v	κ_v	μ_v	σ_v	β
Vanilla box	700	150	10,000	6	8	4000	300	40	150,000
Optional component	A_{oj}	C_{oj}	D_{oj}	h_{oj}	p_{oj}	κ_{oj}	μ_j	σ_j	ρ_j
1	40	3	4000	0.7	1.0	200	100	15	0.5
2	20	2	6000	0.4	0.7	150	170	20	0.8

Table 2
Solutions for Q_v, r_v, Q_{oj}, and r_{oj}, together with λ and the total cost.

	Q_v	r_v	Q_{o1}	r_{o1}	Q_{o2}	r_{o2}	λ	$EAV(\vec{Q}, \vec{r})$
Proposed method	860.8246	341.6691	580.8890	121.5989	648.4425	202.7676	0.045190	1,536,070
[12]	862.3301	340.3125	579.6005	122.7817	647.4532	203.3750	0.045044	1,536,061

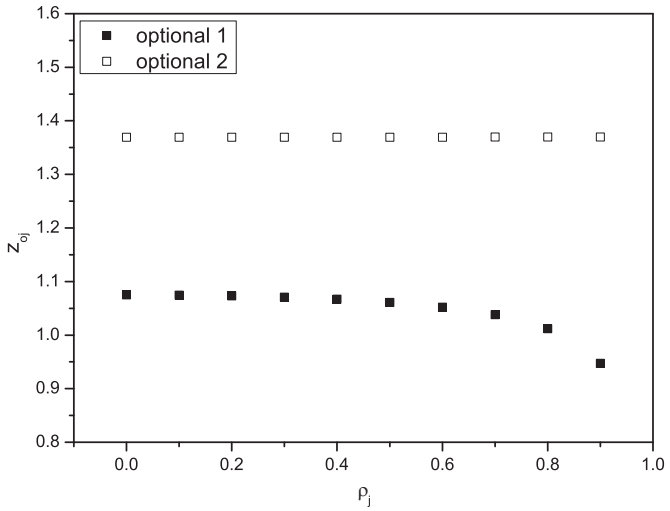

Fig. 2. Plots of the normalized reorder points z_{oj} for the optional components versus the correlation coefficient ρ_j.

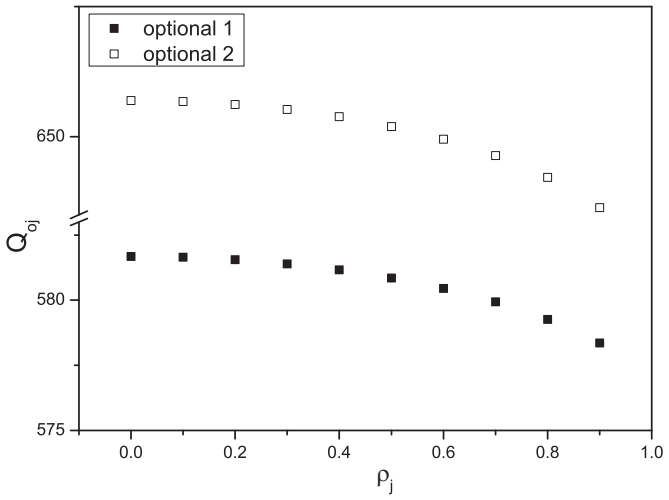

Fig. 3. Plots of the order quantities Q_{oj} for the optional components versus the correlation coefficient ρ_j.

As we have stated in Introduction [11,12] solved the simultaneous equations for the inventory system as a whole by utilizing rather complicated heuristical approximations and iteration. Thus, their method requires relatively a large amount of computational time and the solutions are not exact. In contrast, the proposed method is a direct method in the sense that it does not require any approximation and iteration. Thus, the proposed method is more accurate because it does not use any heuristical approximation. Furthermore, the proposed method is also fast since it finds optimal values directly without any iteration. Therefore, the proposed method is much simpler and more accurate than the methods by [11,12]. Based on these, although we cannot numerically compare the performance due to unavailability of the codes for [11,12], we believe the proposed method has advantages in finding optimal solution.

We also perform a sensitivity analysis of the order quantities and reorder points for the optional components with respect to the correlation coefficient. It should be noted that Q_v and r_v are independent of ρ_j from Eqs. (32) and (33). For the sensitivity analysis, we first need to find the behavior of z_{oj} as ρ_j varies. Fig. 2 shows that z_{oj} is almost constant with respect to ρ_j although z_{o1} decreases slightly for large value of ρ_1. This implies that the normalized reorder points are insensitive to the correlation coefficient.

Fig. 3 shows the behavior of order quantities Q_{oj} of the optional components as the correlation coefficient ρ_j varies. From the first equation in Eq. (29), we see that the dependence of Q_{oj} on ρ_j stems from $\sigma_{oj}L(z_{oj})$. Because z_{oj} is more or less insensitive to ρ_j Fig. 2, so is $L(z_{oj})$. Thus, considering ρ_j dependence only, we have

$$Q_{oj} \propto \sigma_{oj} \approx \sigma_j \sqrt{1-\rho_j^2}. \tag{35}$$

This implies that Q_{oj} decreases as the absolute value of ρ_j increases and Q_{oj} reaches its maximum when $\rho_j = 0$ as shown in Fig. 3. The maximum of Q_{oj} at $\rho_j = 0$ can also be proved as follows. From the first equation in Eq. (29), it can be readily shown that

$$\frac{dQ_{oj}}{d\rho_j} = -\frac{p_{oj}D_{oj}}{\{h_{oj} + 2\lambda C_{oj}\}} \left\{ \frac{L(z_{oj}) + z_{oj}G(z_{oj})}{Q_{oj}} \right\} \frac{\sigma_j^2}{\sigma_{oj}} \rho_j. \tag{36}$$

Thus, Q_{oj} is extreme when $\rho_j = 0$. Furthermore,

$$\left. \frac{d^2Q_{oj}}{d\rho_j^2} \right|_{\rho_j=0} = -\frac{p_{oj}D_{oj}}{\{h_{oj} + 2\lambda C_{oj}\}} \left\{ \frac{L(z_{oj}) + z_{oj}G(z_{oj})}{Q_{oj}} \right\} \sigma_j < 0 \tag{37}$$

implies that Q_{oj} has its maximum when $\rho_j = 0$.

Fig. 4 shows the behavior of reorder points r_{oj} of the optional components as the correlation coefficient ρ_j varies. Unlike the order quantity, r_{oj} reaches its maximum at a positive value of ρ_{oj}. For the behavior of r_{oj}, we can rewrite Eq. (22) as follows:

$$r_{oj} = \mu_j + \rho_j \frac{\sigma_j}{\sigma_v}(r_v - \mu_v) + \sigma_j \sqrt{1-\rho_j^2}\, z_{oj}. \tag{38}$$

Because z_{oj} is almost independent of ρ_j, as ρ_j increases, the second term on the right-hand side of Eq. (38) also increases while

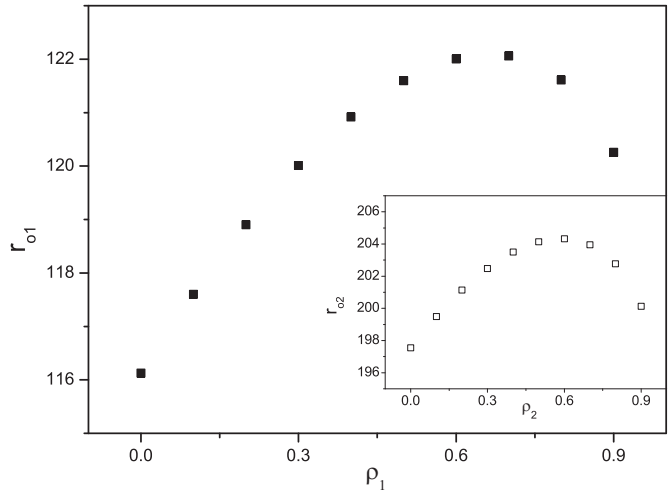

Fig. 4. Plots of the reorder points r_{o1} for optional component 1 versus the correlation coefficient ρ_1 while $\rho_2 = 0.5$. Inset: Same plot for optional component 2.

the third term decreases. Thus, there is a trade-off between the second and the third terms, resulting in an optimum value of r_{oj}. Furthermore, the maximum r_{oj} occurs when

$$\rho_j^{\max} \approx \frac{r_v - \mu_v}{\sqrt{\sigma_v^2\ z_{oj}^2 + \sigma_j^2\ (r_v - \mu_v)^2}}. \tag{39}$$

This implies that $\rho_j^{\max}$ depends on the safety stock $r_v - \mu_v$ of the vanilla box. Given that the safety stock is a positive quantity, we have $0 < \rho_j^{\max} < 1$ unlike the maximum of Q_{oj}.

5. Summary and conclusion

In this paper, we presented an efficient method for finding an optimal solution (Q, r) to a correlated multi-item continuous review inventory model in which a bivariate Gaussian probability distribution is used as a correlation between the vanilla box and an optional component. By normalizations of the random variables for the demands, we showed that the bivariate Gaussian PDF can be expressed as a product of two independent Gaussian PDFs, which implies that the normalized random variables are uncorrelated.

To demonstrate the usefulness of the normalization, we solved a multi-item continuous review inventory (Q, r) model in which the vanilla box and optional components are correlated under stochastic payment and budget constraints. With normalization, we showed that the set of equations coupled with the vanilla box and optional components was decoupled into sets of equations for the normalized quantities. Furthermore, each set of decoupled equations was reduced to a closed form and could be solved numerically without any approximation.

We also performed the sensitivity analysis in terms of the correlation. We found that the order quantity and the reorder point of optional components depended on the strength of the correlation as we expected. In particular, we showed that the order quantity of an optional component reached its maximum when there was no correlation between the vanilla box and the optional component. In addition, the reorder point of an optional component reached a maximum that depended on the safety stock of the vanilla box.

The decoupling can be possible for other type of distributions depending on the characteristics of the distribution. The crucial point is whether a multivariate PDF can be decomposed into the product of the marginal and conditional probability distribution functions. Thus, a multivariate PDF that has an exponential form, such as a bivariate exponential PDF, can be decoupled. In contrast, a multivariate Cauchy PDF, as an example, cannot be decoupled.

The proposed method is a direct method of solving the model in the sense that it does not require any approximations and iterations that were adopted in the previous studies [11,12]. In this sense, the proposed method is much faster in computation time and more accurate in numerical results than the methods proposed early. In addition, the proposed method can be applied to the case where different optional components are correlated, besides the correlation between the vanilla box and each optional component, by taking advantage of the decomposition property of the multivariate Gaussian distribution. Based on these, we believe that the proposed method has advantages in finding optimal solution over the early studies.

The proposed method can be used as a dependable method for a generalized multi-item continuous review inventory model with complicated interactions among items. It would be interesting to investigate how far the proposed method can be applied to other types of correlation, such as a multivariate Gaussian or other multivariate distributions.

Acknowledgment

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Appendix A. Simplification of various functions

In this appendix, we show how to simplify the expressions in Eqs. (17)–(20). To this end, we define various normalized variables as follows:

$$z \equiv \frac{x - \mu_v}{\sigma_v} \quad \text{and} \quad z_v \equiv \frac{r_v - \mu_v}{\sigma_v}, \tag{A-1}$$

$$z_j \equiv \frac{x - \mu_{oj}}{\sigma_{oj}} \quad \text{and} \quad z_{oj} \equiv \frac{r_{oj} - \mu_{oj}}{\sigma_{oj}}, \tag{A-2}$$

where

$$\sigma_{oj} \equiv \sigma_j\sqrt{1-\rho_j^2} \quad \text{and} \quad \mu_{oj} \equiv \mu_j + \rho_j\frac{\sigma_j}{\sigma_v}(r_v - \mu_v). \tag{A-3}$$

In addition, from Eq. (3), we have

$$f_{X_v}(x) = \frac{1}{\sqrt{2\pi}\sigma_v}\exp\left\{-\frac{(x-\mu_v)^2}{2\sigma_v^2}\right\} \quad \text{and} \quad f_{X_j|X_v}(x|r_v) = \frac{1}{\sqrt{2\pi}\sigma_{oj}}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}. \tag{A-4}$$

Finally, the G-function is defined as

$$G(z) \equiv \int_z^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz. \tag{A-5}$$

A1. Evaluation of $\int_{r_v}^\infty (x - r_v) f_{X_v}(x)dx$ *and* $\frac{d}{dr_v}\int_{r_v}^\infty (x - r_v) f_{X_v}(x)dx$

From Eqs. (A-1) and (A-4), we have

$$\int_{r_v}^\infty (x - r_v) f_{X_v}(x)dx = \frac{\sigma_v}{\sqrt{2\pi}}\int_{z_v}^\infty (z - z_v)e^{-z^2/2}dz = \sigma_v\left\{\frac{1}{\sqrt{2\pi}}e^{-z_v^2/2} - z_v G(z_v)\right\}. \tag{A-6}$$

By using the above result, its derivative becomes

$$\frac{d}{dr_v}\int_{r_v}^\infty (x - r_v) f_{X_v}(x)dx = \sigma_v\frac{dz_v}{dr_v}\frac{d}{dz_v}\left\{\frac{1}{\sqrt{2\pi}}e^{-z_v^2/2} - z_v G(z_v)\right\} = -G(z_v). \tag{A-7}$$

A2. Evaluation of $\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx$ *and* $\frac{\partial}{\partial r_{oj}}\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx$

From Eqs. (A-2)–(A-4), we have

$$\begin{aligned}&\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx\\&=\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{1}{\sqrt{2\pi}\sigma_{oj}}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}dx\\&=\sigma_j\sqrt{1-\rho_j^2}\int_{z_{oj}}^{\infty}(z_j-z_{oj})\frac{1}{\sqrt{2\pi}}e^{-z_j^2/2}dz_j\\&=\sigma_{oj}\left\{\frac{1}{\sqrt{2\pi}}e^{-z_{oj}^2/2}-z_{oj}G(z_{oj})\right\}.\end{aligned}\tag{A-8}$$

By using the above result, its derivative becomes

$$\begin{aligned}&\frac{\partial}{\partial r_{oj}}\int_{r_{oj}}^{\infty}(x-r_{oj})f_{X_j|X_v}(x|r_v)dx\\&=\sigma_{oj}\frac{dz_{oj}}{dr_{oj}}\frac{\partial}{\partial z_{oj}}\left\{\frac{1}{\sqrt{2\pi}}e^{-z_{oj}^2/2}-z_{oj}G(z_{oj})\right\}=-G(z_{oj}).\end{aligned}\tag{A-9}$$

A3. Evaluation of $\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{\partial}{\partial r_v}f_{X_j|X_v}(x|r_v)dx$

By using Eq. (A-4), the partial derivative becomes

$$\begin{aligned}\frac{\partial}{\partial r_v}f_{X_j|X_v}(x|r_v)&=\frac{d\mu_{oj}}{dr_v}\frac{\partial}{\partial\mu_{oj}}\frac{1}{\sqrt{2\pi}\sigma_{oj}}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}\\&=\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)\frac{(x-\mu_{oj})}{\sqrt{2\pi}\sigma_{oj}^3}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}.\end{aligned}\tag{A-10}$$

Thus, by using Eqs. (A-2) and (A-4), we have

$$\begin{aligned}&\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{\partial}{\partial r_v}f_{X_j|X_v}(x|r_v)dx\\&=\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)\int_{r_{oj}}^{\infty}(x-r_{oj})\frac{(x-\mu_{oj})}{\sqrt{2\pi}\sigma_{oj}^3}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}dx\\&=\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)\int_{z_{oj}}^{\infty}(z_j-z_{oj})\;z_j\frac{1}{\sqrt{2\pi}}e^{-z_j^2/2}dz_j=\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)G(z_{oj}).\end{aligned}\tag{A-11}$$

A4. Evaluation of $\frac{d}{dr_v}F_{X_v}(r_v)$, $\frac{\partial}{\partial r_v}F_{X_j|X_v}(r_{oj}|r_v)$, *and* $\frac{\partial}{\partial r_{oj}}F_{X_j|X_v}(r_{oj}|r_v)$

By using Eqs. (A-1) and (A-4), we have

$$F_{X_v}(r_v)=\int_{-\infty}^{r_v}\frac{1}{\sqrt{2\pi}\sigma_v}\exp\left\{-\frac{(x-\mu_v)^2}{2\sigma_v^2}\right\}dx=\int_{-\infty}^{z_v}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz.\tag{A-12}$$

Thus,

$$\frac{d}{dr_v}F_X(r_v)=\frac{dz_v}{dr_v}\frac{d}{dz_v}\int_{-\infty}^{z_v}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz=\frac{1}{\sqrt{2\pi}\sigma_v}e^{-z_v^2/2}.\tag{A-13}$$

Similarly, by using Eqs. (A-2)–(A-4), we have

$$\begin{aligned}F_{X_j|X_v}(r_{oj}|r_v)&=\int_{-\infty}^{r_{oj}}\frac{1}{\sqrt{2\pi}\sigma_{oj}}\exp\left\{-\frac{(x-\mu_{oj})^2}{2\sigma_{oj}^2}\right\}dx\\&=\int_{-\infty}^{z_{oj}}\frac{1}{\sqrt{2\pi}}e^{-z_j^2/2}dz_j.\end{aligned}\tag{A-14}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial r_v}F_{X_j|X_v}(r_{oj}|r_v)&=\frac{dz_{oj}}{dr_v}\frac{\partial}{\partial z_{oj}}\int_{-\infty}^{z_{oj}}\frac{1}{\sqrt{2\pi}}e^{-z_j^2/2}\;dz_j\\&=-\left(\rho_j\frac{\sigma_j}{\sigma_v}\right)\frac{1}{\sqrt{2\pi}\sigma_{oj}}e^{-z_{oj}^2/2}.\end{aligned}\tag{A-15}$$

Finally, by using Eqs. (A-2) and (A-4), we have

$$\frac{\partial}{\partial r_{oj}}F_{X_j|X_v}(r_{oj}|r_v)=\frac{dz_{oj}}{dr_{oj}}\frac{\partial}{\partial z_{oj}}\int_{-\infty}^{z_{oj}}\frac{1}{\sqrt{2\pi}}e^{-z_j^2/2}\;dz_j=\frac{1}{\sqrt{2\pi}\sigma_{oj}}e^{-z_{oj}^2/2}.\tag{A-16}$$

Appendix B. Proof of Theorem 1

For brevity, we drop the subscript *oj* for the optional component and assume $\lambda>0$. To prove that $g(z)$ of Eq. (30) is monotonically decreasing in z, it suffices to show that for any two values of z such that $z_1<z_2$, $g(z)$ satisfies $g(z_1)>g(z_2)$. We start with

$$g(z)\equiv\frac{pDG(z)}{h+\lambda C+\lambda\kappa/\sigma f(z)}-\left\{\frac{AD+pD\sigma L(z)}{h/2+\lambda C}\right\}^{1/2},\tag{B-1}$$

where the loss function $L(z)$, the standard Gaussian PDF $f(z)$, and $G(z)$ are defined, respectively, as

$$\begin{aligned}&L(z)\equiv\sigma\left\{\frac{1}{\sqrt{2\pi}}e^{-z^2/2}-zG(z)\right\},\quad f(z)\equiv\frac{1}{\sqrt{2\pi}}e^{-z^2/2},\\&\text{and}\quad G(z)\equiv\frac{1}{\sqrt{2\pi}}\int_z^{\infty}e^{-t^2/2}\;dt.\end{aligned}\tag{B-2}$$

From Eq. (B-1), we have

$$\begin{aligned}g(z_1)-g(z_2)=&\left\{\frac{pDG(z_1)}{h+\lambda C+\lambda\kappa/\sigma f(z_1)}-\frac{pDG(z_2)}{h+\lambda C+\lambda\kappa/\sigma f(z_2)}\right\}\\&+\sqrt{\frac{pD\sigma}{h/2+\lambda C}}\sqrt{L(z_2)-L(z_1)}.\end{aligned}\tag{B-3}$$

Since $L(z_1)<L(z_2)$ for $z_1<z_2$, the second term on the right hand side of Eq. (B-3) is positive. The first term can be rewritten as

$$\begin{aligned}&\left\{\frac{pDG(z_1)}{h+\lambda C+\lambda\kappa/\sigma f(z_1)}-\frac{pDG(z_2)}{h+\lambda C+\lambda\kappa/\sigma f(z_2)}\right\}\\&=pD\left\{\frac{(h+\lambda C)\{G(z_1)-G(z_2)\}+\lambda\kappa/\sigma\{G(z_1)f(z_2)-G(z_2)f(z_1)\}}{\{h+\lambda C+\lambda\kappa/\sigma f(z_1)\}\{h+\lambda C+\lambda\kappa/\sigma f(z_2)\}}\right\}.\end{aligned}\tag{B-4}$$

From the definition of $G(z)$, we have $G(z_1)>G(z_2)$ when $z_1<z_2$. Thus, to show that $g(z_1)>g(z_2)$ when $z_1<z_2$, we are left with showing that $G(z_1)f(z_2)-G(z_2)f(z_1)$ is positive.

To this end, consider the following expression:

$$h(z_1)-h(z_2)\equiv\frac{G(z_1)f(z_2)-G(z_2)f(z_1)}{f(z_1)f(z_2)}=\frac{G(z_1)}{f(z_1)}-\frac{G(z_2)}{f(z_2)}.\tag{B-5}$$

Given that $f(z)>0$, to show $g(z_1)>g(z_2)$ when $z_1<z_2$ is equivalent to proving that $h(z_1)>h(z_2)$ when $z_1<z_2$. That is, $h(z)$ is a monotonically decreasing in z. Now, we express $G(z)/f(z)$ in terms of an infinite series in z by using [1]

$$h(z)\equiv\frac{G(z)}{f(z)}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{z^{2n+1}}>0,\tag{B-6}$$

where

$$(2n-1)!! \equiv 1\cdot 3\cdot 5\ldots(2n-3)\cdot(2n-1). \tag{B-7}$$

By letting $m=n-1$, $h(z)$ can be rewritten as

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{z^{2n+1}} = \frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{z^{2n+1}} \\ &= \frac{1}{z}+\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(2m+1)!!}{z^{2m+3}} \\ &= \frac{1}{z}-\sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!!}{z^{2m+3}} > 0. \end{aligned} \tag{B-8}$$

Since $h(z)>0$ for all z, we have, from Eq. (B-8),

$$\sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)!!}{z^{2m+3}} < \frac{1}{z}. \tag{B-9}$$

To show that $h(z)$ is monotonically decreasing in z, we will show that $dh(z)/dz<0$. From Eq. (B-6), the derivative becomes

$$\begin{aligned} \frac{dh(z)}{dz} &= \sum_{n=0}^{\infty}(-1)^n(2n-1)!!(-1)\frac{(2n+1)}{z^{2n+2}} \\ &= (-z)\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!!}{z^{2n+3}}. \end{aligned} \tag{B-10}$$

By using Eq. (B-9), $dh(z)/dz$ can be expressed as

$$\frac{dh(z)}{dz} = (-z)\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!!}{z^{2n+3}} < (-z)\frac{1}{z} < -1. \tag{B-11}$$

Since $dh(z)/dz<0$ for all z, we have $h(z_2)<h(z_1)$ when $z_2>z_1$. That is,

$$h(z_1)-h(z_2) \equiv \frac{G(z_1)}{f(z_1)}-\frac{G(z_2)}{f(z_2)} = \frac{G(z_1)f(z_2)-G(z_2)f(z_1)}{f(z_1)f_{z_2}} > 0. \tag{B-12}$$

This completes the proof. □

Appendix C. Derivation of stochastic budget constraint

The stochastic budget constraint is given as

$$Prob\left\{C_v(Q_v+r_v-X_v)+\sum_{j=1}^{m}C_{oj}(Q_{oj}+r_{oj}-X_j)+\kappa_v F_{X_v}(r_v) + \sum_{j=1}^{m}\kappa_{oj}F_{X_j|X_v}(r_{oj}|r_v)\le\beta\right\}\ge\eta. \tag{C-1}$$

The above expression Eq. (C-1) can be rewritten as

$$Prob\left\{C_vX_v+\sum_{j=1}^{m}C_{oj}X_j\ge A-\beta\right\}\ge\eta, \tag{C-2}$$

where

$$A \equiv C_v(Q_v+r_v)+\sum_{j=1}^{m}C_{oj}(Q_{oj}+r_{oj})+\kappa_vF_{X_v}(r_v) + \sum_{j=1}^{m}\kappa_{oj}F_{X_j|X_v}(r_{oj}|r_v). \tag{C-3}$$

Since $X_v\sim N(\mu_v,\sigma_v^2)$ and $X_j\sim N(\mu_{oj},\sigma_{oj}^2)$, we have

$$C_vX_v\sim N(C_v\mu_v,C_v^2\sigma_v^2)$$

$$\text{and}\quad \sum_{j=1}^{m}C_{oj}X_j\sim N\left(\sum_{j=1}^{m}C_{oj}\mu_{oj},\sum_{j=1}^{m}C_{oj}^2\sigma_{oj}^2\right), \tag{C-4}$$

by assuming that demands for optional components are independent each other. From Eq. (C-4), we have

$$C_vX_v+\sum_{j=1}^{m}C_{oj}X_j\sim N\left(C_v\mu_v+\sum_{j=1}^{m}C_{oj}\mu_{oj},C_v^2\sigma_v^2+\sum_{j=1}^{m}C_{oj}^2\sigma_{oj}^2\right). \tag{C-5}$$

By defining $Y\equiv C_vX_v+\sum_{j=1}^{m}C_{oj}X_j$, we can normalize Y as

$$Z\equiv\frac{Y-\mu_Y}{\sigma_Y}\sim N(0,1), \tag{C-6}$$

where

$$\mu_Y\equiv C_v\mu_v+\sum_{j=1}^{m}C_{oj}\mu_{oj},\quad \sigma_Y^2\equiv C_v^2\sigma_v^2+\sum_{j=1}^{m}C_{oj}^2\sigma_{oj}^2. \tag{C-7}$$

Using Eq. (C-6), the stochastic budget constraint of Eq. (C-2) can be expressed as

$$Prob\left\{Z\ge\frac{A-\beta-\mu_Y}{\sigma_Y}\right\} = 1-F\left(\frac{A-\beta-\mu_Y}{\sigma_Y}\right)\ge\eta, \tag{C-8}$$

where $F(\cdot)$ is the standard normal cumulative density function. Furthermore, Eq. (C-8) can be rewritten as

$$F\left(\frac{A-\beta-\mu_Y}{\sigma_Y}\right)\le 1-\eta,\ \text{or}\ \ \frac{A-\beta-\mu_Y}{\sigma_Y}\le F^{-1}(1-\eta) \tag{C-9}$$

Hence, we have

$$A\le\beta+\mu_Y+F^{-1}(1-\eta)\sigma_Y \tag{C-10}$$

By the definition of Eq. (C-3) and letting $F^{-1}(1-\eta)\equiv z_{1-\eta}$, the stochastic budget constraint [Eq. (C-1)] becomes

$$\begin{aligned} &C_v(Q_v+r_v)+\sum_{j=1}^{m}C_{oj}(Q_{oj}+r_{oj})+\kappa_vF_{X_v}(r_v)+\sum_{j=1}^{m}\kappa_{oj}F_{X_j|X_v}(r_{oj}|r_v) \\ &\le\beta+\mu_Y+z_{1-\eta}\sigma_Y. \end{aligned} \tag{C-11}$$

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