

Shaik, Muneer; Maheswaran, S.

Article

Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices

Cogent Economics & Finance

Provided in Cooperation with:

Taylor & Francis Group

Suggested Citation: Shaik, Muneer; Maheswaran, S. (2019) : Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices, Cogent Economics & Finance, ISSN 2332-2039, Taylor & Francis, Abingdon, Vol. 7, Iss. 1, pp. 1-27, <https://doi.org/10.1080/23322039.2019.1597430>

This Version is available at:

<https://hdl.handle.net/10419/245218>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by/4.0/>



Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices

Muneer Shaik & S. Maheswaran |

To cite this article: Muneer Shaik & S. Maheswaran | (2019) Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices, Cogent Economics & Finance, 7:1, 1597430, DOI: [10.1080/23322039.2019.1597430](https://doi.org/10.1080/23322039.2019.1597430)

To link to this article: <https://doi.org/10.1080/23322039.2019.1597430>



© 2019 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.



Published online: 22 Apr 2019.



Submit your article to this journal [↗](#)



Article views: 377



View related articles [↗](#)



View Crossmark data [↗](#)



Citing articles: 1 View citing articles [↗](#)



Received: 02 November 2018
Accepted: 23 February 2019
First Published: 22 March 2019

*Corresponding author: Muneer Shaik,
IFMR Graduate School of Business,
KREA University, India
E-mail: muneer.shaik@ifmr.ac.in

Reviewing editor:
David McMillan, University of Stirling,
Stirling, UK

Additional information is available at
the end of the article

FINANCIAL ECONOMICS | RESEARCH ARTICLE

Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices

Muneer Shaik^{1*} and S. Maheswaran¹

Abstract: In this paper, we come up with an alternate theoretical proof for the independence and unbiased property of extreme value robust volatility estimator with respect to the standard robust volatility estimator. We show that the robust volatility ratio is unbiased both in the population and in the finite samples. We empirically test the robust volatility ratio on nine global stock indices from America, Asia Pacific and EMEA markets for the period from January 1996 to June 2017 based on daily open, high, low and close prices to understand the volatility behavior of stock returns over a period of time. Our results show that robust volatility ratio for different k-month periods is significantly less than 1 for all the global stock indices, thus finding the clear evidence of random walk behavior. This is possibly the first study based on robust volatility ratio to understand the volatility behavior of global stock indices.

Subjects: Mathematics & Statistics for Engineers; Economics; Finance

Keywords: volatility modeling; robust estimation; extreme value estimators; Brownian motion; volatility ratio

JEL Classification: C51; C58; C12; G15

Article Classification: Research paper

ABOUT THE AUTHOR

Muneer Shaik is currently working as a Senior Faculty Associate at IFMR Graduate School of Business KREA University, India. He holds a PhD in finance. He worked with J.P. Morgan in a managerial position in the equity and fixed income department. He has participated and presented his research papers at multiple conferences both in India and abroad. Some of his research papers have won the best paper awards. His research interests mainly include in the areas of volatility modeling, financial econometrics and market efficiency.

PUBLIC INTEREST STATEMENT

The volatility behavior of asset returns helps an investor to make better decisions with regard to his investment in the financial market. That is, if the movement of asset returns is efficient, in other words, if it follows the random walk behavior, we can say that there will not be any possibility for an informed investor to gain abnormal profits when compared to an uninformed investor. In order to understand the volatility behavior of asset returns, we have come up with a specification statistic named as robust volatility ratio. We have theoretically proved that the test statistic follows the properties like independence, un-biasedness and efficiency. Further, we have empirically tested on global stock indices and clearly found the evidence of random walk behavior.

1. Introduction

Estimation of volatility in asset returns has been an important area of research in the finance literature. Volatility is considered to be a valuable measure to estimate and used in diverse fields such as risk management, portfolio management, asset allocation, option pricing, foreign exchange and the term structure of interest rates. Since volatility is a measure of dispersion and not observable, an extensive research in this area has resulted in developing various volatility estimator models such as ARCH/GARCH-type models, stochastic volatility models, range-based volatility models and more recently high-frequency realized volatility models.

In general, most of the volatility estimation models in the financial markets are based on the two famous proxy variables: (1) the squared returns and (2) the absolute returns, i.e. absolute deviation (Ghysels, Harvey, & Renault, 1996). Statistically, when the distribution of the financial data follows Gaussian normal distribution and is free of outliers, the standard deviation is considered to be a more efficient measure of dispersion than the absolute mean deviation (Fisher, 1920; Stigler, 1973; Aldrich, 1997; Hinton, 1995). However, in “realistic situations” where some measurements are in error (i.e. in the presence of outliers) or for the distributions other than perfect normal, the superiority of standard deviation over the absolute mean deviation diminishes (Eddington, 1914; Fama, 1963; Barnett & Lewis, 1978; Huber, 1981). Even though the standard deviation based on squared returns as a proxy of volatility has a statistical drawback over absolute deviation, the latter has been used extensively in the advancement of volatility estimation models in the financial literature (Black & Scholes, 1973; Engle, 1982; Bollerslev, 1986; Hull & White, 1987; Andersen & Bollerslev, 1997, Andersen, Bollerslev, & Ebens, 2001; Adriana & Chris, 2014).

Modeling the volatility based on the absolute returns can be traced back to the work of Taylor (1986). The volatility model using absolute returns is found to be more robust against non-normality (Davidian & Carroll, 1987). The specification of volatility based on absolute returns is empirically found to produce better volatility forecasts relative to squared returns (Ding, Granger, & Engle, 1993; Ederington & Guan, 2004). Theoretically, it was proved that absolute returns are more persistent and better in predicting future volatility than squared returns (Forsberg & Ghysels, 2007). Absolute return volatility is easier to calculate and as a risk indicator has approximately the same sensitivity as realized volatility. However, realized volatility estimators require much effort and resources to implement as argued by Rogers and Zhou (2008). The usual absolute return volatility models are based on the closing prices alone.

It is well established in the literature that volatility estimation models based on extreme values of asset prices {High, Low, Open, Close} are more efficient and convenient when compared to the usual volatility estimator based on closing prices alone. The volatility estimators that use the extreme values of asset prices in the literature are Parkinson (1980) estimator that uses the {High, Low} prices, Garman and Klass (1980) estimator that uses {High, Low, Open, Close} prices, Rogers and Satchell (1991) estimator, Kunitomo (1992) estimator, Yang and Zhang (2000) estimator, Alizadeh, Brandt, and Diebold (2002), Chou (2005) and Maximum Likelihood estimator as in Ball and Torous (1984), Magdon Ismail and Atiya (2003) and Horst, Rodriguez, Gzyl, and Molina (2012). In this paper, we use Classical Robust Volatility Estimator (CRVE) and a new volatility estimator (i.e. Extreme Value Robust Volatility Estimator (EVRVE)) that uses a robust volatility proxy, i.e. absolute returns, along with extreme values of asset prices as mentioned in papers by Muneer and Maheswaran (2018b). These robust volatility estimators are also used in the research by Muneer and Maheswaran (2018a) to find the evidence of excess volatility based on the cross-sectional average of the constituent stocks of India's BSE Sensex and the USA's Dow Jones stock indices.

We provide an alternate proof to show that the proposed EVRVE is independent and unbiased relative to the CRVE. We find the closed form solution to the joint probability density of the running maximum and the drawdown of the standard Brownian motion with no drift parameter at the random stopping time. Based on the theoretical result, we show the independence of the proposed EVRVE relative to the CRVE. Since the proposed estimators are independent of each other, we

further propose a robust volatility ratio (RVR) to show that the EVRVE is unbiased relative to the CRVE both in the population and in finite samples.

The rest of the paper is organized as follows: We first discuss the methodology of this paper. Second, we discuss the robust volatility estimation and show the independence and bias properties of extreme value robust volatility estimators relative to the CRVEs. Finally, we provide the empirical findings of our model based on the global stock indices like CAC 40, DAX 30, DOW 30, FTSE 100, IBOVESPA, NIKKEI 225, NIFTY 50, S & P 500 and SET 50.

2. Methodology

In this section, we find the closed form solution for the joint probability density of the running maximum and the drawdown of the standard Brownian motion with no drift parameter at the random stopping time $\tau \sim \text{Exp}(\lambda)$ ¹ and is independent of the Brownian motion. Based on the theoretical result, we show that the proposed robust volatility estimator based on extreme values of asset prices is independent of the usual robust volatility estimator based on the absolute deviation. Further, we propose the RVR to show that the Extreme Value Robust Volatility Estimator (EVRVE) is unbiased relative to the CRVE in the population. Also in the finite sample, we propose the correction procedure to adjust the bias in the robust volatility estimator based on extreme values of asset prices.

In general, volatility estimation models are based on the assumption that asset returns follow Gaussian normal distribution. However, the distribution of the real-world financial asset returns data is found to exhibit substantial fat tails and also asymmetry around the mean relative to those of normal distribution. In this paper, we make use of a mixture of distribution hypothesis, which models the asset returns as a function of the random process of information arrival (Clark, 1973; Tauchen & Pitts, 1983; Harris, 2001; Richardson & Smith, 1994). The mixture of distribution models is more flexible to capture the leptokurtic and multimodal characteristics of real-world financial time series data (Kon, 1984). We choose the exponential mixture of the normal distribution as we are able to theoretically show that it is equal to the double exponential distribution [See, Proof of Claim 1 in the Appendix section]. In particular, we take insights from the work of Linden (2001) which mentions that double exponential distribution suits well for daily and weekly observations of stock return data.

2.1. Theoretical framework

Here, we demonstrate the theoretical setup that has been followed in this paper. We show the proofs of the reflection principle for the Brownian motion along with the joint probability of the running maximum and the drawdown of the Brownian motion in the below subsections which form the base for coming up with the robust volatility estimation technique as proposed in the following sections.

2.1.1. Reflection principle for Brownian motion

Lemma 1: The reflection principle for the Brownian motion states that when $\mu = 0$,

$$P(X_t \leq x, M_t \geq b) = P(X_t \geq u) \Big|_{u=2b-x} = 1 - \Phi\left(\frac{u}{\sqrt{t}}\right) \Big|_{u=2b-x}$$

for $b > 0$, $x \leq b$ where “ x ” and “ b ” are the specific levels on the Brownian motion path. That is to say, the joint probability of the terminal value (X_t) and the running maximum (M_t) of the standard Brownian motion at a fixed time “ t ” with no drift parameter (i.e. $\mu = 0$) will be equal to the univariate probability. Here, $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

Proof: In order to prove this, let us define the random stopping time of the Brownian motion for specified level b as

$$T_b = \inf\{t \geq 0, X_t \geq b\} \text{ for } b > 0$$

By using the symmetric property of the Brownian motion over the set $(t \geq T_b)$, we have, (1)

$$P(X_t \leq x | \mathcal{F}_{T_b}) = P(X_t \geq u | \mathcal{F}_{T_b})$$

Here, \mathcal{F}_{T_b} represents the filtration at the random stopping time T_b .

Now let us consider the L.H.S of Equation (1). We get that

Therefore, we have shown that

$$P(X_t \leq x | \mathcal{F}_{T_b})|_{t \geq T_b} = P(X_t \leq x, M_t \geq b)|_{t \geq T_b}$$

Now consider the R.H.S of Equation (1). We get that,

$$\begin{aligned} R.H.S &= P(X_t \geq u | \mathcal{F}_{T_b})|_{t \geq T_b} = \mathbb{E}\{\mathbf{1}_{(X_t \geq u)} | \mathcal{F}_{T_b}\}|_{t \geq T_b} = \mathbb{E}\{\mathbb{E}\{\mathbf{1}_{(X_t \geq u)} \mathbf{1}_{(t \geq T_b)} | \mathcal{F}_{T_b}\}\} \\ &= \mathbb{E}\{\mathbf{1}_{(X_t \geq u, t \geq T_b)}\} = P(X_t \geq u, t \geq T_b) = P(X_t \geq u, M_t \geq b) \\ &= P(X_t \geq u) = 1 - \Phi\left(\frac{u}{\sqrt{t}}\right) \end{aligned}$$

Thus, by making use of the symmetric property of the Brownian motion, we have shown that the joint probability of the terminal value and running maximum of the standard Brownian motion converges to a univariate probability. Thus, the reflection principle of the Brownian motion when $\mu = 0$ and for $x \leq b, b \geq 0$ is given as,

$$P(X_t \leq x, M_t \geq b) = P(X_t \geq u) \Big|_{u=2b-x} = 1 - \Phi\left(\frac{u}{\sqrt{t}}\right) \Big|_{u=2b-x} \tag{2}$$

Hence, Lemma 1 is proved.

2.1.2. Joint probability of the running maximum and the drawdown of the Brownian motion

Lemma 2: Let (M_τ, Y_τ) denote the value of the running maximum and the “drawdown” of the standard Brownian motion at a stochastic time τ .

The “drawdown” (Y) of the Brownian motion is defined as the difference between the running maximum (M) and the terminal value (X). That is to say, we have $Y = M - X$.

Let us assume that the stochastic time τ is independent of the Brownian motion and is distributed exponentially with the parameter λ i.e. $\tau \sim \text{Exp}(\lambda)$.

Then, the joint probability of M_τ and Y_τ where $b \geq 0, y \geq 0$ and $\tau \sim \text{Exp}(\lambda)$ is,

Proof: Let us recall the result of Lemma 1 of the ABC procedure paper (Maheswaran & Kumar, 2013). It says,

Let (X_t, M_t) denote the value of a stochastic process and its running maximum at a fixed point in time “ t ”. Let us say that $H(x, b) = P(X_t \leq x, M_t \geq b)$ for $b > 0, x \leq b$. If u is sufficient for $H(x, b)$ and H is differentiable with respect to both arguments, then for $b > 0, y < 0$, we have,

$$P(Y_t \geq y, M_t \geq b) = 2P(X_t \leq x, M_t \geq b)|_{x=b-y}$$

We make use of the above result to prove our Lemma 2.

$$P(M_\tau \geq b, Y_\tau \geq y) = e^{-\beta b} e^{-\beta y} \tag{3}$$

Now let us consider the L.H.S of Equation (3), we get,

$$\begin{aligned} L.H.S &= P(M_\tau \geq b, Y_\tau \geq y) = \int_0^\infty \lambda e^{-\lambda t} P(M_t \geq b, Y_t \geq y) dt \\ &= \int_0^\infty \lambda e^{-\lambda t} 2P(X_t \leq x, M_t \geq b) |_{x=b-y} dt = \int_0^\infty \lambda e^{-\lambda t} 2P(X_t \geq u) |_{u=b+y} dt \\ &= \int_0^\infty \lambda e^{-\lambda t} 2 \left\{ \int_0^\infty \frac{1}{\sqrt{t}} \Phi\left(\frac{z}{\sqrt{t}}\right) dt \right\} dz = \int_{Z=u}^\infty \left\{ \int_{t=0}^\infty 2\lambda e^{-\lambda t} \frac{1}{\sqrt{t}} \Phi\left(\frac{z}{\sqrt{t}}\right) dt \right\} dz \\ &= \int_{Z=u}^\infty 2 \left\{ \int_{t=0}^\infty \lambda e^{-\lambda t} \frac{1}{\sqrt{t}} \Phi\left(\frac{z}{\sqrt{t}}\right) dt \right\} dz = \int_{Z=u}^\infty 2 \left\{ \frac{1}{2} \beta e^{-\beta(|z|)} \right\} dz = \int_{Z=u}^\infty \beta e^{-\beta z} dz \end{aligned}$$

Now let us say $w = \beta z \Rightarrow z = \frac{w}{\beta} \Rightarrow dz = \frac{dw}{\beta}$. Also, if $z \in (u, \infty)$, then $w \in (\beta u, \infty)$.

Therefore, we have,

$$\begin{aligned} L.H.S &= P(M_\tau \geq b, Y_\tau \geq y) = \int_{\beta u}^\infty e^{-w} dw = e^{-\beta u} \\ &= e^{-\beta(b+y)} = e^{-\beta b} e^{-\beta y} \text{ over } b \geq 0, y \geq 0 = R.H.S \end{aligned}$$

Since L.H.S = R.H.S for $b \geq 0, y \geq 0$, we have,

$$\int_0^\infty \lambda e^{-\lambda t} P(M_t \geq b, Y_t \geq y) dt = e^{-\beta b} e^{-\beta y}$$

That is to say, when $\tau \sim \text{Exp}(\lambda)$, we have

$$P(M_\tau \geq b, Y_\tau \geq y) = e^{-\beta b} e^{-\beta y}$$

Therefore, the joint probability of the running maximum and the drawdown of the standard Brownian motion at a stochastic time $\tau \sim \text{Exp}(\lambda)$ which is independent of the Brownian motion are i.i.d exponential random variables with the parameter β where $\beta = \sqrt{2\lambda}$.

Hence, Lemma 2 is proved.

2.2. Robust estimation of volatility for Brownian motion

Here, we assume that the process X_τ follows Brownian motion with no drift parameter at a random stopping time τ . We suppose that the random stopping time τ is exponentially distributed with parameter λ and is independent of the Brownian motion. Based on this, we derive robust extreme value estimators and discuss their properties.

Suppose, $\{O, H, L, C\}_i$ denote the opening, high, low and close price of an asset for an $x_i = \ln(C_i) - \ln(O_i)$ day. Based on the set of i th price series, we derive the triplet $\{x, b, c\}_i$ such that

$$x_i = \ln(C_i) - \ln(O_i)$$

$$b_i = \ln(H_i) - \ln(O_i)$$

$$c_i = \ln(L_i) - \ln(O_i)$$

Here $\{x, b, c\}$ be specific levels on the Brownian motion path representing the terminal value, running maximum and running minimum, respectively. We have, $x > 0, b > 0, c < 0, b \geq x$ and $c \leq x$.

Now let us introduce the CRVE, denoted as “Sigx” by letting $Y_2 = |x|$ defined as,

$$Sigx = \frac{1}{N} \left[\sum_{i=1}^N |x_i| \right] \tag{4}$$

In Equation (1), it is important to note that the proposed CRVE, denoted as *Sigx*, only uses the open and close price series of an asset.

We further introduce the EVRVE, denoted as “*Sigux*” by letting $Y_1 = u - |x|_{|u=2b-x}$ defined as,

$$Sigux = \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \tag{5}$$

By using the symmetric property of the Brownian motion, we can define “*Sigvx*” by letting $Y_3 = v - |x|_{|v=2c-x}$ as,

$$Sigvx = \frac{1}{N} \left[\sum_{i=1}^N (v - |x_i|) \right] \tag{6}$$

Here, we have defined {u,v} as $u = 2b - x$ and $v = 2c - x$.

Based on the proposed extreme value robust volatility estimators, *Sigux* and *Sigvx* as mentioned above in Equations (5) and (6), we can further define another estimator as,

$$Siguxvx = avg\{Sigux, Sigvx\} = \frac{Sigux + Sigvx}{2} \tag{7}$$

It is important to note that the proposed EVRVE, *Sigux* uses open, high and close price series of an asset. *Sigvx* uses open, low and close price series of an asset and *Siguxvx* uses all the extreme value information of asset price series, i.e. open, high, low and close prices.

We further show the independence of Y_1 and Y_2 by using the joint probability densities and by applying the theoretical framework of Lemma 1 and Lemma 2 as shown in Section 2.2 of this paper. Later, we show the bias property of EVRVE relative to the CRVE both in the population and in the finite sample by proposing a RVR.

2.2.1. Independence property of Y_1 and Y_2

Let us introduce the generic terms X_1 and X_2 which are i.i.d exponential with parameter $\beta = \sqrt{2\lambda}$ and is defined as

$$X_1 = b = M_\tau X_2 = y = Y_\tau$$

Then, the joint probability density of $\{X_1, X_2\}$ at specific points $\{x_1, x_2\}$ for $x_1 > 0, x_2 > 0$ can be written as

$$f_{X_1, X_2}(x_1, x_2) = [\beta e^{-\beta x_1}] [\beta e^{-\beta x_2}] \tag{8}$$

since X_1, X_2 are i.i.d exponential with parameter β where $\beta = \sqrt{2\lambda}$ based on Lemma 2.

Now let us introduce random variables, Y_1 and Y_2 which are defined in terms of b, y as

$$Y_1 = u - |x|_{|u=2b-x} = 2min(b, y) \ \& \ Y_2 = x = b - y$$

In order to show the independence property of Y_1 and Y_2 , let us consider two cases.

Case 1: Let us consider the special case when $0 < x_2 < x_1$.

In such a case, we have $x_1 - x_2 > 0$.

Therefore, $|x_1 - x_2| = x_1 - x_2$.

We have Y_1 based on robust extreme values defined as,

$$Y_1 = u - |x|_{u=2b-x} = (2b - x) - |x|_{x=b-y} = (b + y) - |b - y|_{b=x_1, y=x_2} \\ = (x_1 + x_2) - |x_1 - x_2| = (x_1 + x_2) - (x_1 - x_2) = 2\min(x_1, x_2)|_{0 < x_2 < x_1} = 2x_2$$

Similarly, let us define² Y_2 as

$$Y_2 = x = b - y = x_1 - x_2$$

That is to say, when $0 < x_2 < x_1$, we have,

$$Y_1 = 2 \cdot x_2 \text{ \& } Y_2 = x_1 - x_2$$

Now let us consider the inverse transformation of Y_1 and Y_2 , we get

$$x_2 = \frac{Y_1}{2} \text{ and } x_1 = Y_2 + x_2 = Y_2 + \frac{Y_1}{2} = \frac{1}{2} \cdot Y_1 + Y_2$$

Now let us represent the same in matrix notation, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \times \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The Jacobian of the transformation is given by

$$J = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

Thus, we have,

$$|\det(J)| = \frac{1}{2}$$

Now, the domain of the generic random variables X_1, X_2 is given by

$$\mathbb{D}_A = \{(x_1, x_2) : 0 < x_2 < x_1\}$$

The domain of the transformed random variables Y_1, Y_2 is given by

$$\mathbb{D}_B = \{(y_1, y_2) : y_1 > 0, y_2 > 0\}$$

Therefore, let us derive the joint probability density³ as

$$g_{Y_1, Y_2}(y_1, y_2) = |\det(J)| f_{X_1, X_2}(x_1, x_2) = \frac{1}{2} \beta e^{-\beta[\frac{1}{2}y_1 + y_2]} \beta e^{-\beta[\frac{1}{2}y_1]} \\ = \frac{1}{2} \beta^2 e^{-\beta[\frac{1}{2}y_1 + y_2 + \frac{1}{2}y_1]} = \frac{1}{2} \beta^2 e^{-\beta[y_1 + y_2]} = [\beta e^{-\beta y_1}] [\frac{1}{2} \beta e^{-\beta y_2}]$$

That is to say, the joint probability density of Y_1, Y_2 when $0 < x_2 < x_1$ is given by

$$g_{Y_1, Y_2}(y_1, y_2) = [\beta e^{-\beta y_1}] \left[\frac{1}{2} \beta e^{-\beta y_2} \right] \Big|_{\mathbb{D}_B} \tag{9}$$

Case 2: Now let us consider the special case when $0 < x_1 < x_2$.

In such a case we have $x_1 - x_2 < 0$.

Therefore

$$|x_1 - x_2| = -(x_1 - x_2) = x_2 - x_1.$$

We have Y_1 defined as

$$Y_1 = u - |x|_{u=2b-x} = (2b - x) - |x|_{x=b-y} = (b + y) - |b - y|_{b=x_1, y=x_2} = (x_1 + x_2) - |x_1 - x_2| = (x_1 + x_2) - (x_2 - x_1) = 2\min(x_1, x_2)|_{0 < x_1 < x_2} = 2x_1$$

Similarly, let us define Y_2 as

$$Y_2 = x = b - y = x_1 - x_2$$

That is to say, when $0 < x_1 < x_2$ we have,

$$Y_1 = 2 \cdot x_1 \text{ and } Y_2 = x_1 - x_2$$

Now let us consider the inverse transformation of Y_1 and Y_2 , we get

$$x_1 = \frac{Y_1}{2} \text{ and } x_2 = x_1 - Y_2 = \frac{1}{2} \cdot Y_1 - Y_2$$

Now let us represent the same in matrix notation, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \times \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The Jacobian of the transformation is given by

$$J = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

Thus, we have,

$$|\det(J)| = \frac{1}{2}$$

Now, the domain of the generic random variables X_1, X_2 is given by

$$\mathbb{D}_A = \{(x_1, x_2) : 0 < x_1 < x_2\}$$

The domain of the transformed random variables Y_1, Y_2 is given by

$$\mathbb{D}_B = \{(y_1, y_2) : y_1 > 0, y_2 < 0\}$$

Therefore, let us derive the joint probability density⁴ of Y_1, Y_2 as

$$\begin{aligned} g_{Y_1, Y_2}(y_1, y_2) &= |\det(J)| f_{X_1, X_2}(x_1, x_2) = \frac{1}{2} \beta e^{-\beta[\frac{1}{2}y_1]} \beta e^{-\beta[\frac{1}{2}y_1 - y_2]} \\ &= \frac{1}{2} \beta^2 e^{-\beta[\frac{1}{2}y_1 + \frac{1}{2}y_1 - y_2]} = \frac{1}{2} \beta^2 e^{-\beta[y_1 - y_2]} = [\beta e^{-\beta y_1}] [\frac{1}{2} \beta e^{-\beta y_2}] \end{aligned}$$

That is to say, the joint probability density of Y_1, Y_2 when $0 < x_1 < x_2$ is given by

$$g_{Y_1, Y_2}(y_1, y_2) = [\beta e^{-\beta y_1}] \left[\frac{1}{2} \beta e^{-\beta y_2} \right] \Big|_{\mathbb{D}_B, \mathbb{D}_B} \tag{10}$$

Finally, let us combine the results of both cases (1) & (2). Therefore, from Equations (9) and (10), we get the joint probability density of Y_1, Y_2 as

$$g_{Y_1, Y_2}(y_1, y_2) = [\beta e^{-\beta y_1}] \left[\frac{1}{2} \beta e^{-\beta |y_2|} \right]_{y_1 > 0, y_2 \in \mathbb{R}} \tag{11}$$

From Equation (11), it is clear that Y_1 & Y_2 are independent of each other. In particular, we have shown that Y_1 which is defined as $Y_1 = u - |x| = 2 \cdot \min(b, y)$ is independent of Y_2 which is defined as $Y_2 = x = b - y$ with specific exponential distributions as,

$$Y_1 \sim \text{Exp}(\beta)$$

and

$$Y_2 \sim \text{DExp}(\beta)$$

where “Exp” denotes exponential distribution and “DExp” denotes double exponential distribution.

2.3. Bias property

In this section, we theoretically check the bias property of the EVRVE relative to the CRVE both in the case of the population and in the case of the finite sample. We show that in the case of population, EVRVE is unbiased relative to the CRVE by finding the proposed RVR to be equal to 1. Also, we allow the Finite Sample Correction Procedure to adjust the insignificant bias of EVRVE relative to the CRVE in the case of the finite sample. We show that the proposed Modified Robust Volatility Ratio (MRVR) is unbiased and is equal to 1 in the finite sample.

2.3.1. Bias in the population

Theorem 1: In the population, the RVR is unbiased. That is to say, the EVRVE is unbiased relative to the CRVE in the case of population. That is,

$$\text{Robust Volatility Ratio (RVR)} = \left\{ \frac{\mathbb{E}(u - |x|)}{\mathbb{E}(|x|)} \right\} = 1.$$

Proof:

Let us consider Y_1 defined as

$$Y_1 = u - |x|$$

where.

$$Y_1 \sim \text{Exp}(\beta) \text{ and } \beta = \sqrt{2\lambda}$$

In order to get the EVRVE, we take the expected value of Y_1 . That is to say,

$$\mathbb{E}(Y_1) = \mathbb{E}(u - |x|) = \int_0^\infty w\beta e^{-\beta w} dw$$

Now put $y = \beta \cdot w \Rightarrow w = \frac{y}{\beta}$. Therefore,

$$\mathbb{E}(u - |x|) = \int_0^\infty \frac{y}{\beta} e^{-y} dy = \frac{1}{\beta} \int_0^\infty ye^{-y} dy = \frac{1}{\beta} (1) = \frac{1}{\sqrt{2\lambda}}$$

Hence, in the population, we define the EVRVE as,

$$\mathbb{E}(u - |x|) = \frac{1}{\sqrt{2\lambda}} \tag{12}$$

Now let us consider Y_2^5 defined as

$$Y_2 = |x|$$

where

$$Y_2 \sim \text{Exp}(\beta) \text{ and } \beta = \sqrt{2\lambda}$$

In order to get the CRVE, we take the expected value of Y_2 . That is to say,

$$\mathbb{E}(Y_2) = \mathbb{E}(|x|) = \frac{1}{\beta} = \frac{1}{\sqrt{2\lambda}}$$

Hence, in the population, we have the CRVE defined as,

$$\mathbb{E}(|x|) = \frac{1}{\sqrt{2\lambda}} \tag{13}$$

Therefore, from the results of Equations (12) and (13), we define the RVR in the population as

$$\text{Robust Volatility Ratio (RVR)} = \frac{\mathbb{E}(u - |x|)}{\mathbb{E}(|x|)} = \frac{\frac{1}{\beta}}{\frac{1}{\beta}} = 1 \tag{14}$$

We have found the RVR to be equal to 1. That is to say, EVRVE is unbiased relative to CRVE in the population at the random stopping time τ of the Brownian motion with no drift parameter.

Hence, we have proved the Theorem.

2.3.2. Bias in the finite sample

Here, we check the bias property in the case of the finite sample. Let us recall Y_1 defined as $Y_1 = u - |x| = 2 \cdot \min(b, y)$ and $Y_1 \sim \text{Exp}(\beta)$.

Since we know that the exponential distribution with parameter β is same⁶ as the Gamma distribution with parameter $\{\alpha, N\}$ when $\alpha = \beta$ and $N = 1$. We can say that

$$Y_1 \sim \text{Exp}(\beta) \text{ is same as } Y_1 \sim \Gamma(\alpha, N) |_{(\alpha=\beta, N=1)}$$

That is to say, if we consider $\{y_i : 1 \leq i \leq N\} \sim \text{iid Exp}(\beta)$, then each individual

$$y_i \sim \Gamma(\alpha, N) |_{(\alpha=\beta, N=1)}$$

Therefore, their sum will also have the probability density function of gamma distribution. That is,

$$\sum_{i=1}^N y_i \sim \Gamma(\alpha, N) |_{(\alpha=\beta)}$$

Now let us find about the distribution of the average of the individual

$$y_i \sim \Gamma(\alpha, N) |_{(\alpha=\beta, N=1)}$$

In order to find that, let us consider the generic random variables, namely,

$$X = \sum_{i=1}^N y_i \sim \Gamma(\alpha, N) |_{(\alpha=\beta)}$$

$$W = \frac{1}{N} \left[\sum_{i=1}^N y_i \right] = \frac{1}{N} X$$

Now let us take the inverse transformation and we get

$$X = N \cdot W$$

The Jacobian of the transformation can be written as

$$\frac{\partial X}{\partial W} = N$$

We derive the joint probability density as,

$$g_w(w) = |N|f_X(x)|_{x=N.W} = N \frac{\alpha^N}{\Gamma(N)} e^{-\alpha x} x^{N-1} \Big|_{x=N.W}$$

$$= N \frac{\alpha^N}{\Gamma(N)} e^{-\alpha N W} (N W)^{N-1} = \frac{(\alpha N)^N}{\Gamma(N)} e^{-(\alpha N)W} (W)^{N-1} = \Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N}$$

That is to say, we have proved that if the individual y_i is gamma distributed, then the average is also gamma distributed,

$$\frac{1}{N} \left[\sum_{i=1}^N y_i \right] \sim \Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N}$$

Therefore, we define the EVRVE in the finite sample as

$$Sigux = \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \sim \Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N} \tag{15}$$

Now let us find the expected value of the estimator $Sigux$. We get,

$$\begin{aligned} \mathbb{E}[Sigux] &= \mathbb{E}[\Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N}] = \int_0^\infty w \frac{(\alpha^*)^N}{\Gamma(N)} e^{-\alpha^* w} w^{N-1} dw = \frac{(\alpha^*)^N}{\Gamma(N)} \int_0^\infty e^{-\alpha^* w} w^N dw \\ &= \frac{(\alpha^*)^N \Gamma(N+1)}{\Gamma(N) (\alpha^*)^{N+1}} = \frac{\Gamma(N+1)}{\Gamma(N)} \frac{(\alpha^*)^N}{(\alpha^*)^{N+1}} = \frac{N\Gamma(N)}{\Gamma(N)} \frac{(\alpha N)^N}{(\alpha N)^{N+1}} = \frac{1}{\alpha} \end{aligned}$$

Therefore, in the finite samples, we have,

$$\mathbb{E}[Sigux] = \mathbb{E}[\Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N}] = \mathbb{E} \left\{ \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \right\} = \frac{1}{\alpha}$$

Thus, the expected value of the EVRVE in the finite sample is given by,

$$\mathbb{E}[Sigux] = \frac{1}{\alpha} \text{ where } \alpha = \beta = \sqrt{2\lambda} \tag{16}$$

Now let us recall Y_2 defined as $Y_2 = |x|$ and $Y_2 \sim Exp(\beta) \Big|_{\beta=\sqrt{2\lambda}}$

We know that

$$Y_2 \sim Exp(\beta) \text{ is same as } Y_2 \sim \Gamma(\alpha, N) \Big|_{(\alpha=\beta, N=1)}$$

That is to say,

$$|x_i| \sim \Gamma(\alpha, N) \Big|_{(\alpha=\beta, N=1)}$$

Therefore, their sum will also have the probability density function of gamma distribution.

$$\sum_{i=1}^N |x_i| \sim \Gamma(\alpha, N) \Big|_{(\alpha=\beta)}$$

We get that their average is also gamma distributed. That is

$$Sigx = \frac{1}{N} \left[\sum_{i=1}^N |x_i| \right] \sim \Gamma(\alpha^*, N) \Big|_{\alpha^*=\alpha N} \tag{17}$$

Now let us find the expected value of the CRVE denoted by $Sigx$ as

$$\mathbb{E}[Sigx] = \mathbb{E}[\Gamma(\alpha^*, N) \Big|_{\alpha^*=N.\alpha}] = \int_0^\infty w \frac{(\alpha^*)^N}{\Gamma(N)} e^{-\alpha^* w} w^{N-1} dw = \frac{(\alpha^*)^N}{\Gamma(N)} \int_0^\infty e^{-\alpha^* w} w^N dw = \frac{1}{\alpha}$$

That is to say, the expected value of the CRVE is given by,

$$\mathbb{E}[Sigx] = \frac{1}{\alpha} \text{ where } \alpha = \beta = \sqrt{2\lambda} \tag{18}$$

Therefore, from Equations (16) and (18), we have found that the expected value of the EVRVE is same as the expected value of the CRVE.

$$i.e. \mathbb{E}[Sigux] = \mathbb{E}[Sigx] = \frac{1}{\alpha}$$

Now let us define the RVR in the finite sample as

$$RVR = \left\{ \frac{\frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right]}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} \tag{19}$$

Now let us find the expected value of RVR, we get

$$\mathbb{E}[RVR] = \mathbb{E} \left\{ \frac{\frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right]}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\}$$

We know that the numerator and denominator are independent of each other.⁷ We make use of the independence property and write the expected value of the RVR estimator as,

$$\mathbb{E}[RVR] = \mathbb{E} \left\{ \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \right\} \times \mathbb{E} \left\{ \frac{1}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} \tag{20}$$

From Equations (15) and (16), we have,

$$\mathbb{E}[Sigux] = \mathbb{E} \left\{ \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \right\} = \frac{1}{\alpha} \tag{21}$$

From Claim 4, mentioned in the Appendix section of this paper, we have

$$\mathbb{E} \left\{ \frac{1}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} = \left(\frac{N}{N-1} \right) . \alpha \tag{22}$$

The RVR in the finite sample will be unbiased if the expected value will be equal to 1. In order to check the unbiasedness property, let us find the expected value of RVR in the finite sample. That is,

$$\mathbb{E}[RVR] = \mathbb{E} \left\{ \frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right] \right\} \times \mathbb{E} \left\{ \frac{1}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} = \left(\frac{1}{\alpha} \right) \left(\frac{N}{N-1} \right) . \alpha = \left(\frac{N}{N-1} \right)$$

That is to say, we have found that the RVR in the finite sample is biased insignificantly and is equal to $\left(\frac{N}{N-1} \right)$.

2.3.3. Modified Robust Volatility Ratio (MRVR)

In order to make the estimator unbiased in the finite sample, we introduce the modified version of the RVR defined as

$$\text{Modified RVR} = \left\{ \frac{N-1}{N} \right\} \{ RVR \} = \left\{ \frac{N-1}{N} \right\} \times \left\{ \frac{\frac{1}{N} \left[\sum_{i=1}^N (u - |x_i|) \right]}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} \tag{23}$$

The MRVR Estimator will be unbiased if the expected value of the estimator will be equal to 1. That is to say,

$$\mathbb{E}[\text{Modified RVR}] = \left\{ \frac{N-1}{N} \right\} \mathbb{E}\{ RVR \} = \left\{ \frac{N-1}{N} \right\} \left\{ \frac{N}{N-1} \right\} = 1$$

Hence, we have shown that the MRVR is unbiased in the finite sample.

3. Empirical study

In this section, we check the empirical behavior of the proposed RVR by using the daily {Open, High, Low, Close} prices of the global stock indices. In this study, we undertake the empirical work on nine stock indices covering three different markets [Americas, Asia Pacific and EMEA (European, Middle Eastern & Africa)] as mentioned in Bloomberg. Table 1 provides the information on the global stock indices considered in our study.

In Table 2, we provide a summary of the descriptive statistics for the global stock indices considered in our study. The summary statistics are provided for return series calculated based on the closing prices of the respective stock indices. The sample period for all the stock indices is from the period January 1996 to June 2017. The daily price series data are collected from the Bloomberg source. We observe that the standard deviation is highest for the IBOVESPA index with the value of 2.048 and the lowest is for the FTSE 100 stock index with the value of 1.177. We find that all the stock indices are highly kurtotic. Of the nine global indices, we find that seven stock indices are negatively skewed and only two stock indices, namely, IBOVESPA and SET 50 are positively skewed. The count of the daily data observations for the stock indices ranges between 5259 and 5476 for the sample period from 1996 to 2017.

In Table 3, we provide the results of normality and stationary tests performed on the asset returns of global stock indices. It is clear from the Jarque–Bera normality test that all the global stock index returns are not normally distributed as all the values are statistically significant at 99% level of confidence. Hence, we reject the null hypothesis of normality for all the nine stock indices. We further perform the Augmented Dickey-Fuller test under three different scenarios, i.e. (with no drift & no trend, with drift & no trend, with drift & trend) to check whether the asset returns of stock indices are stationary or not. We find that under all the scenarios, the asset returns of stock indices are stationary at 99% confidence level.

3.1. Hypothesis testing

Theoretically, we have shown that the proposed two versions of RVRs as mentioned in Equations (14) and (23) are unbiased. The objective of our hypothesis is to find out empirically whether the unbiasedness in the proposed RVR exists even in case of global stock indices data or not. That is to say, we would like to check whether the EVRVE is unbiased relative to the CRVE with regard to the daily global stock indices data.

We test the below two hypothesis. In the first hypothesis test, we check whether Plain Robust Volatility Ratio (as mentioned in Equation (14)) is unbiased or not

$$H_0: \text{Plain Robust Volatility Ratio (PRVR)} = 1$$

$$H_1: \text{Plain Robust Volatility Ratio (PRVR)} \neq 1$$

In the second hypothesis test, we check whether MRVR (as mentioned in Equation (23)) is unbiased or not.

$$H_0: \text{Modified Robust Volatility Ratio (MRVR)} = 1$$

$$H_1: \text{Modified Robust Volatility Ratio (MRVR)} \neq 1$$

3.2. Interpretation

If the *t*-test statistic for both the hypothesis tests is found to be significant and negative, we conclude that the EVRVE is downward biased relative to the CRVE. That is to say, PRVR or MRVR will be less than 1 and it happens due to the random walk effect. If the test statistic is found to be insignificant, then we conclude that EVRVE is unbiased relative to the CRVE. That is to say, PRVR or MRVR will be equal to 1 and it happens due to the Brownian motion.

Table 1. Information on global stock indices

S. No	Name of the stock index	Traded on exchanges	Country	Constituents	Established	Index symbol
American markets						
1	Dow Jones Industrial Average Index	New York Stock Exchange, NASDAQ	USA	30	1896	DOW 30
2	Ibovespa Brasil Sao Paulo Stock Exchange Index	Sao Paulo Stock Exchange	Brazil	70	1968	IBOVESPA
3	Standard & Poor's 500 Index	New York Stock Exchange, NASDAQ	USA	505	1957	S&P 500
Asia Pacific (APAC) markets						
4	Nikkei 225 Stock Index	Tokyo Stock Exchange	Japan	225	1950	NIKKEI 225
5	CNX Nifty 50 Index	National Stock Exchange of India Limited	India	50	1992	NIFTY 50
6	Stock Exchange of Thailand 50 Index	Stock Exchange of Thailand	Thailand	50	1995	SET 50
Europe, Middle Eastern & African (EMEA) markets						
7	French Stock Market Index	Euronext Paris	France	40	1987	CAC 40
8	German Stock Market Index	Frankfurt Stock Exchange	Germany	30	1988	DAX 30
9	The Financial Times Stock Exchange 100 Index	London Stock Exchange	UK	100	1984	FTSE 100

Note: The information of the global stock indices is collected from fact sheets available in the respective stock index websites and Bloomberg source.

Table 2. Descriptive statistics of global stock indices

	CAC 40	DAX 30	DOW 30	FTSE 100	IBOVESPA	NIKKEI 225	NIFTY 50	S&P 500	SET 50
Mean	0.018	0.031	0.026	0.013	0.050	-0.001	0.044	0.025	0.003
Median	0.045	0.105	0.047	0.047	0.103	0.031	0.082	0.055	0.017
SD	1.445	1.505	1.137	1.177	2.048	1.526	1.548	1.205	1.564
Kurtosis	4.552	4.148	7.788	5.685	13.103	5.544	7.060	7.988	7.719
Skewness	-0.063	-0.145	-0.159	-0.154	0.279	-0.306	-0.223	-0.242	0.046
Range	20.066	19.672	18.709	18.650	46.051	25.346	29.388	20.427	27.413
Minimum	-9.472	-8.875	-8.201	-9.266	-17.226	-12.111	-13.054	-9.470	-16.063
Maximum	10.595	10.797	10.508	9.384	28.825	13.235	16.334	10.957	11.350
Sum	98.703	168.533	141.672	68.456	266.221	-2.876	234.999	136.203	17.386
Count	5476	5450	5412	5431	5315	5279	5337	5412	5259

Note: Here SD stands for standard deviation.

Summary statistics are provided for returns calculated based on closing prices of an asset.

The formula used for calculating asset returns at time t is $r_t = \log(P_t) - \log(P_{t-1})$, where P_t, P_{t-1} are the prices at time period $t, t - 1$, respectively.

Source: Developed by the authors.

Table 3. Normality and stationarity tests on asset returns of global stock indices

	CAC 40	DAX 30	DOW 30	FTSE 100	IBOVESPA	NIKKEI 225	NIFTY 50	S&P 500	SET 50
Normality test									
Jarque-Bera test	4718.00*	3916.763*	13671.56*	7317.78*	38014.79*	6826.59*	11104.00*	14410.80*	13028.81*
Stationarity test									
ADF test									
No drift no trend	-74.90*	-74.20*	-78.00*	-75.10*	-71.70*	-76.10*	-68.60*	-78.50*	-66.30*
Drift and no trend	-74.90*	-74.20*	-78.00*	-75.10*	-71.70*	-76.10*	-68.50*	-78.50*	-66.30*
Drift and trend	-74.90*	-74.20*	-78.00*	-75.10*	-71.70*	-76.10*	-68.50*	-78.50*	-66.30*

Note: * represents the significance level at 99% level of confidence.

Here ADF stands for Augmented Dickey-Fuller test.

Source: Developed by the authors.

3.3. Findings

Here, we provide the results of our empirical work on the global stock indices data. We perform the k -month analysis for different k -month time periods {1,2,3,6,12,24,36,48,60, Full Sample}. The idea of k -month analysis is to fix the parameters (namely, mean and variance) to be constant for a particular k -month rolling period. Also, we perform the Bootstrap analysis with 1000 replications to find the bootstrap standard error. We have calculated the t -statistic based on the robust bootstrap standard errors. We observe that the findings of Plain and Modified RVRs are similar to the findings of Variance Ratio's based on Rogers and Satchell (1991) estimator as mentioned in Maheswaran, Balasubramanian, and Yoonus (2011) and Maheswaran and Kumar (2014).

In Table 4, we present the results of Plain RVR for different k -month periods considered at a time. We find that the Plain RVR is significantly <1 for all the global indices. For example, in the case of the CAC 40 stock index, the Plain RVR for k -month = 1 is 0.944 with a bootstrap standard error of 0.016 and t -statistic value of -3.44 relative to 1. If we observe the column of stock index CAC 40, we can find that Plain RVR remains to be significant at 99% level of confidence for all different k -month time periods (i.e. for k -month = {1, 2, 3, 6, 12, 24, 36, 60, Full Sample}). Also, the test statistics mentioned in the parenthesis remain to be negative. This suggests that for index CAC 40, the Plain RVR remains to be significantly less than 1.

In case of other stock indices, namely, {DAX 30, DOW 30, FTSE 100, IBOVESPA, NIKKEI 225, NIFTY 50, S&P 500 and SET 50}, the values of the Plain RVR as mentioned in Table 4 based on k -month analysis of Full Sample are {0.868, 0.749, 0.747, 0.700, 0.761, 0.844, 0.718, 0.668}, respectively. We find that the t -statistic values for all the global stock indices for different k -month periods are significantly less than 1 and the extent of the bias depends on which data set we consider. In other words, we can say that the EVRVE is significantly downward biased relative to the CRVE. The downward bias in the stock index can intuitively happen due to the random walk effect.

Similarly, in Table 5, we present the results of MRVR for different k -month periods considered at a time. We find MRVR to be significantly <1 for all the global stock indices data. For example, in case of NIFTY 50 stock index, the value of MRVR for k -month = 1 is 0.839 with a bootstrap standard error of 0.020 and t -statistic of -8.16 relative to 1. If we observe the column of stock index NIFTY 50, we can find that the values of MRVR remain to be significant at 99% level of confidence for all different k -month time periods (i.e. for k -month = {1, 2, 3, 6, 12, 24, 36, 60, Full Sample}). Also, the test statistics mentioned in the parenthesis remain to be negative. This suggests that for index NIFTY 50, the MRVR remains to be significantly less than 1. In other words, we can say that the EVRVE is significantly downward biased relative to the CRVE. The downward bias in the stock index can intuitively happen due to the random walk effect. In case of other stock indices, namely, {CAC 40, DAX 30, DOW 30, FTSE 100, IBOVESPA, NIKKEI 225, S&P 500 and SET 50}, the values of the MRVR based on k -month analysis of Full Sample are 0.902, 0.867, 0.749, 0.747, 0.700, 0.761, 0.718 and 0.688, respectively. We find that the t -statistic values for all the global stock indices are significantly less than 1 and for all the different k -month periods.

In Figure 1, we display the plots for PRVR for different k -month periods considered at a time for all the nine global stock indices considered in our study. Similarly, in Figure 2, we show the plots for MRVR. In both the figures, we can clearly observe that all the lines of different stock indices considered in our study are significantly less than the benchmark value of 1. One interesting finding is that we observe the downward bias is found to be more in case of MRVR for all the global stock indices compared to the PRVR.

In order to check whether the proposed RVRs are robust in case of the presence of outliers in the data or not, we have performed the similar empirical analysis on Nifty 50 stock index excluding outliers in the data. We have removed the outliers in the NIFTY 50 stock index data for the period from January 1996 to June 2017 and computed the Plain and Modified RVR for different k -month periods. In Table 6, we find that the t -statistics are significant and negative and conclude that even after

Table 4. Plain robust volatility ratio when k-months are considered at a time

k-Months	CAC 40	DAX 30	DOW 30	FTSE 100	IBOVESPA	NIKKEI 225	NIFTY 50	S&P 500	SET 50
1	0.944*	0.902*	0.814*	0.793*	0.738*	0.824*	0.882*	0.777*	0.743*
	(0.016, -3.44)	(0.017, -5.87)	(0.016, -11.51)	(0.015, -13.58)	(0.015, -17.85)	(0.015, -11.34)	(0.016, -7.23)	(0.016, -, -14.10)	(0.014, -18.64)
2	0.922*	0.884*	0.791*	0.771*	0.722*	0.802*	0.867*	0.758*	0.728*
	(0.016, -4.73)	(0.016, -7.50)	(0.015, -13.67)	(0.015, -15.29)	(0.014, -20.19)	(0.014, -13.72)	(0.016, -8.56)	(0.015, -16.01)	(0.014, -19.82)
3	0.916*	0.877*	0.782*	0.764*	0.719*	0.793*	0.862*	0.751*	0.720*
	(0.016, -5.26)	(0.016, -7.60)	(0.015, -14.32)	(0.015, -15.62)	(0.014, -20.05)	(0.015, -13.98)	(0.017, -8.30)	(0.015, -16.59)	(0.013, -20.98)
6	0.908*	0.869*	0.772*	0.755*	0.714*	0.782*	0.858*	0.741*	0.707*
	(0.016, -5.56)	(0.016, -8.24)	(0.016, -14.33)	(0.015, -16.42)	(0.014, -21.13)	(0.014, -15.09)	(0.016, -8.75)	(0.015, -17.34)	(0.013, -22.01)
12	0.903*	0.867*	0.759*	0.749*	0.712*	0.774*	0.857*	0.729*	0.699*
	(0.017, -5.77)	(0.017, -8.04)	(0.016, -15.18)	(0.015, -16.94)	(0.014, -21.07)	(0.015, -15.24)	(0.016, -9.04)	(0.015, -17.60)	(0.014, -22.10)
24	0.898*	0.869*	0.747*	0.749*	0.710*	0.767*	0.857*	0.717*	0.657*
	(0.016, -6.19)	(0.017, -7.74)	(0.017, -15.14)	(0.016, -16.08)	(0.014, -20.23)	(0.015, -15.02)	(0.016, -8.86)	(0.016, -18.05)	(0.013, -20.78)
36	0.895*	0.875*	0.744*	0.750*	0.711*	0.766*	0.860*	0.714*	0.691*
	(0.016, -6.46)	(0.017, -7.45)	(0.016, -15.90)	(0.015, -16.45)	(0.014, -20.33)	(0.015, -15.70)	(0.016, -8.71)	(0.016, -18.01)	(0.014, -22.82)
48	0.892*	0.880*	0.742*	0.750*	0.712*	0.765*	0.859*	0.713*	0.687*
	(0.017, -6.44)	(0.018, -6.69)	(0.016, -15.90)	(0.016, -15.99)	(0.015, -19.27)	(0.016, -15.03)	(0.017, -8.30)	(0.016, -18.06)	(0.014, -22.30)
60	0.889*	0.882*	0.740*	0.751*	0.712*	0.764*	0.857*	0.711*	0.684*
	(0.018, -6.20)	(0.017, -6.78)	(0.016, -15.81)	(0.016, -15.60)	(0.015, -19.11)	(0.015, -15.40)	(0.017, -8.43)	(0.016, -17.56)	(0.015, -22.49)
Full sample	0.902*	0.868*	0.749*	0.747*	0.700*	0.761*	0.844*	0.718*	0.668*
	(0.016, -6.12)	(0.015, -8.64)	(0.015, -16.53)	(0.015, -17.23)	(0.014, -21.38)	(0.014, -16.52)	(0.016, -9.88)	(0.015, -18.45)	(0.013, -25.92)

Note: * represents the confidence level at 99% level of significance.
 The term in the parenthesis are denoted in the form (x,y), where x represents the bootstrap standard error and y represents the value of t-statistic.
 The formula for calculating the t-statistic is $\frac{\text{Actual Value} - 1}{\text{Bootstrap Standard Error}}$
 The formula for Plain RVR is based on Equation (14) mentioned in this paper.
Source: Developed by the authors.

Table 5. Modified robust volatility ratio when k-months are considered at a time

k-Months	CAC 40	DAX 30	DOW 30	FTSE 100	IBOVESPA	NIKKEI 225	NIFTY 50	S&P 500	SET 50
1	0.900*	0.860*	0.775*	0.755*	0.702*	0.784*	0.839*	0.740*	0.706*
	(0.020, -4.98)	(0.020, -6.99)	(0.019, -11.73)	(0.019, -12.69)	(0.018, -16.63)	(0.019, -11.58)	(0.020, -8.16)	(0.019, -13.92)	(0.017, -17.54)
2	0.900*	0.863*	0.772*	0.753*	0.705*	0.782*	0.846*	0.740*	0.710*
	(0.019, -5.37)	(0.017, -7.89)	(0.017, -13.56)	(0.017, -14.88)	(0.016, -18.91)	(0.017, -12.73)	(0.018, -8.77)	(0.016, -15.77)	(0.015, -18.97)
3	0.902*	0.863*	0.770*	0.751*	0.708*	0.780*	0.848*	0.739*	0.708*
	(0.018, -5.61)	(0.018, -7.71)	(0.016, -14.13)	(0.016, -15.26)	(0.015, -18.99)	(0.017, -13.24)	(0.018, -8.30)	(0.017, -15.80)	(0.015, -19.83)
6	0.901*	0.862*	0.766*	0.749*	0.708*	0.776*	0.851*	0.735*	0.701*
	(0.018, -5.59)	(0.017, -8.11)	(0.017, -13.66)	(0.016, -15.48)	(0.015, -19.50)	(0.015, -14.59)	(0.017, -8.59)	(0.017, -15.82)	(0.014, -20.97)
12	0.899*	0.864*	0.756*	0.746*	0.709*	0.771*	0.853*	0.726*	0.696*
	(0.018, -5.69)	(0.018, -7.65)	(0.017, -14.33)	(0.016, -16.11)	(0.015, -19.53)	(0.016, -14.25)	(0.017, -8.70)	(0.016, -16.76)	(0.014, -21.11)
24	0.896*	0.868*	0.746*	0.747*	0.708*	0.766*	0.856*	0.716*	0.691*
	(0.017, -5.94)	(0.018, -7.40)	(0.017, -14.61)	(0.016, -15.34)	(0.015, -19.21)	(0.017, -14.12)	(0.017, -8.42)	(0.017, -17.18)	(0.014, -21.71)
36	0.893*	0.874*	0.743*	0.749*	0.710*	0.765*	0.858*	0.713*	0.690*
	(0.018, -6.01)	(0.018, -7.03)	(0.017, -14.73)	(0.016, -15.57)	(0.015, -19.36)	(0.016, -14.58)	(0.017, -8.15)	(0.017, -17.04)	(0.014, -21.77)
48	0.891*	0.879*	0.741*	0.749*	0.711*	0.764*	0.858*	0.712*	0.687*
	(0.018, -6.13)	(0.019, -6.44)	(0.017, -15.04)	(0.016, -15.29)	(0.016, -18.04)	(0.017, -14.24)	(0.018, -7.81)	(0.017, -17.09)	(0.015, -21.23)
60	0.889*	0.881*	0.739*	0.750*	0.712*	0.763*	0.856*	0.711*	0.683*
	(0.019, -5.97)	(0.019, -6.42)	(0.017, -14.90)	(0.017, -14.80)	(0.016, -18.08)	(0.016, -14.66)	(0.018, -7.99)	(0.017, -16.57)	(0.015, -20.63)
Full sample	0.902*	0.867*	0.749*	0.747*	0.700*	0.761*	0.844*	0.718*	0.668*
	(0.016, -6.13)	(0.015, -8.65)	(0.015, -16.54)	(0.015, -17.24)	(0.014, -21.39)	(0.014, -16.53)	(0.016, -9.89)	(0.015, -18.46)	(0.013, -25.94)

Note: * represents the confidence level at 99% level of significance.
 The terms in the parenthesis are denoted in the form (x, y), where x represents the bootstrap standard error and y represents the value of t-statistic.
 The formula for calculating the t-statistic is $\frac{\text{Actual Value} - \text{Bootstrap Standard Error}}{\text{Bootstrap Standard Error}}$.
 The formula for Modified Robust Volatility Ratio is based on Equation (23) mentioned in this paper.
Source: Developed by the authors

Figure 1. Plots of k-month plain robust volatility ratio for global stock indices.

Source: Developed by the authors.

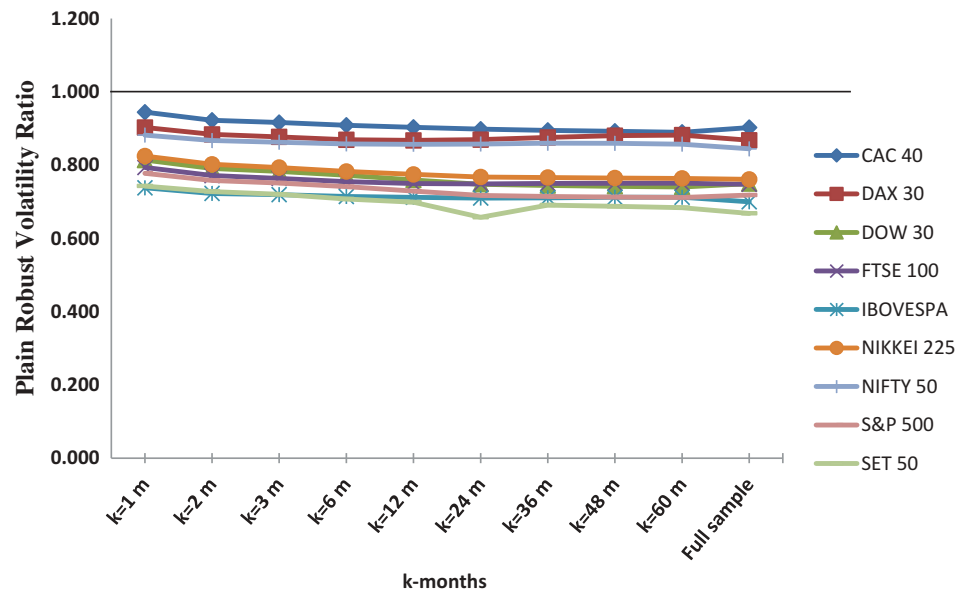
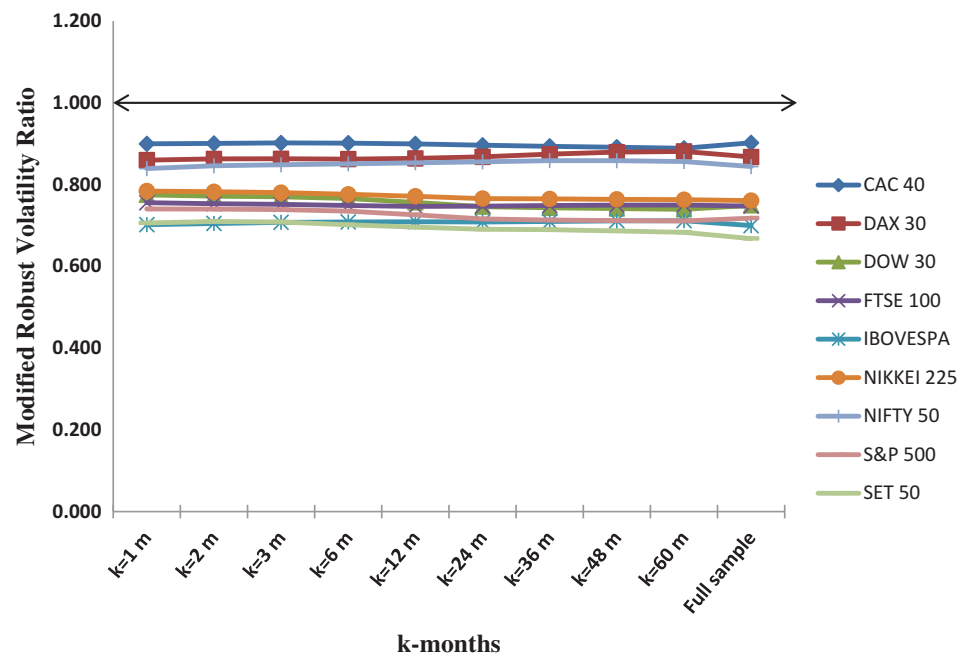


Figure 2. Plots of k-month modified robust volatility ratio for global stock indices.

Source: Developed by the authors.



removing the outliers, we get the similar result that Plain and Modified RVR is significantly downward biased in case of the Nifty Stock Index. For example, for k-month equal to 1, the value of Plain RVR is 0.876 with bootstrap standard error to be 0.016 and t-statistic value of -7.81 . Similarly, the MRVR is 0.833 with bootstrap standard error to be 0.019 and t-statistic value of -8.81 . Hence, we can say that the proposed RVRs remain to be robust even with and without the presence of outliers in the data.

4. Conclusion

In this paper, we derive the reflection principle for the standard Brownian motion and find the joint probability of the terminal value and the running maximum at a fixed time with no drift parameter. We then find the closed-form solution for the joint probability of the running maximum and the

Table 6. Plain and modified robust volatility ratios when k-months are considered at a time in case of Nifty 50 index without outliers

k-Months	1	2	3	6	12	24	36	48	60	Full sample
Plain Robust Volatility Ratio										
NIFTY excluding outliers	0.876* (0.016, -7.81)	0.860* (0.015, -9.02)	0.855* (0.016, -9.13)	0.851* (0.016, -9.58)	0.851* (0.016, -9.49)	0.852* (0.017, -8.95)	0.854* (0.016, -9.08)	0.854* (0.016, -8.91)	0.852* (0.016, -9.18)	0.841* (0.016, -10.05)
Modified Plain Robust Volatility Ratio										
NIFTY excluding outliers	0.833* (0.019, -8.81)	0.840* (0.018, -9.16)	0.842* (0.018, -9.01)	0.844* (0.017, -9.33)	0.848* (0.017, -9.11)	0.850* (0.018, -8.47)	0.853* (0.017, -8.44)	0.853* (0.017, -8.44)	0.851* (0.017, -8.74)	0.841* (0.016, -10.06)

Note: * represents the confidence level at 99% level of significance. The terms in the parenthesis are denoted in the form (x, y) , where x represents the bootstrap standard error and y represents the value of t-statistic. The formula for calculating the t-statistic is $\frac{\text{Actual Value} - \text{Bootstrap Standard Error}}$. The formula for Plain and Modified Robust Volatility Ratio is based on Equations (14) and (23), respectively, as mentioned in this paper. **Source:** Developed by the authors.

drawdown of the standard Brownian motion at a stochastic time τ which is independent of the Brownian motion and is distributed exponentially with the parameter λ .

In this paper, we have provided an alternate proof to show that the EVRVE is found to be independent of the CRVE with specific exponential distributions. We further found that the robust volatility ratio is unbiased both in the population and in the finite sample.

We have empirically checked the Plain and Modified RVRs on real-world financial data of global stock indices, namely, CAC 40, DAX 30, DOW 30, FTSE 100, IBOVESPA, NIKKEI 225, NIFTY 50, S & P 500 and SET 50 for the period from the year 1996 to 2017. We find that the RVRs are downward biased in case of all the global stock indices due to the random walk effect. That is to say, the proposed EVRVE is found to be significantly downward biased relative to the CRVE in global stock indices. In particular, the significant downward bias is more in case of MRVR when compared to the Plain RVR.

The study has implications for the policymakers and practitioners who would like to understand the volatility behavior in asset returns based on the outcome of the RVR. If the RVR is less than 1, then it intuitively means that the stock price process follows random walk movement due to which the stock indices tend to be efficient. Further research can be extended to find the efficiency and the performance of this robust volatility estimator with regard to the volatility estimators in the existing literature.

Funding

The authors received no direct funding for this research.

Author details

Muneer Shaik¹

E-mail: muneer.shaik@ifmr.ac.in

ORCID ID: <http://orcid.org/0000-0002-3508-1096>

S. Maheswaran¹

E-mail: mahesh@ifmr.ac.in

¹ IFMR Graduate School of Business, KREA University, 5655, Central Express Way, Sector 24, Sri City, Andhra Pradesh 517646, India.

Citation information

Cite this article as: Volatility behavior of asset returns based on robust volatility ratio: Empirical analysis on global stock indices, Muneer Shaik & S. Maheswaran, *Cogent Economics & Finance* (2019), 7: 1597430.

Notes

1. Here λ is the exponential parameter.
2. We define “y” as the drawdown of the Brownian motion, i.e. $y = b - x$. Therefore, we can write $x = b - y$.
3. In case 1, we have $x_1 = \frac{1}{2}y_1 + y_2$ and $x_2 = \frac{1}{2}y_1$.
4. In case 2, we have $x_1 = \frac{1}{2}y_1$ and $x_2 = \frac{1}{2}y_1 - y_2$.
5. In Section 2.3.1, from Equation (11), we have shown, $Y_2 = x$ is double exponentially distributed. When we have its modulus, i.e. $Y_2 = |x|$, then Y_2 will be exponentially distributed.
6. We use the mathematical result that $Exp(\beta) = \Gamma(\alpha, N)_{(\alpha=\beta, N=1)}$.
7. If X and Y are independent of each other, then $\mathbb{E}(\frac{X}{Y}) = \mathbb{E}(X) \cdot \mathbb{E}(\frac{1}{Y})$.
8. We make use of the integral property $\int_0^{\infty} e^{-mx} dx = \frac{1}{m}$ in solving the integral.
9. See the Appendix section, Claim 2 & Claim 3 results to get the values of conditional and unconditional characteristic function of x_t where $x_t \sim N(0, \sigma^2 = Y_t)$ and $Y_t \sim Exp(\lambda)$.

References

- Adriana, S. C., & Chris, K. (2014). Discrete stochastic autoregressive volatility. *Journal of Banking & Finance*, 43, 160–178. doi:10.1016/j.jbankfin.2014.03.020
- Aldrich, J. (1997). R. A. Fisher and the making of maximum likelihood 1912–1922. *Statistical Science*, 12, 162–176. doi:10.1214/ss/1030037906
- Alizadeh, S., Brandt, M. W., & Diebold, F. X. (2002). Range-based estimation of stochastic volatility models. *Journal of Finance*, 57, 1047–1091. doi:10.1111/1540-6261.00454
- Andersen, T. G., & Bollerslev, T. (1997). Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns. *Journal of Finance*, 52, 975–1005. doi:10.1111/j.1540-6261.1997.tb02722.x
- Andersen, T. G., Bollerslev, T. X. D., & Ebens, H. (2001). The distribution of realized stock return volatility. *Journal of Financial Economics*, 61, 43–76. doi:10.1016/S0304-405X(01)00055-1
- Ball, C. A., & Torous, W. N. (1984). The maximum likelihood estimation of security price volatility: Theory, evidence, and an application to option pricing. *Journal of Business*, 57, 97–112. doi:10.1086/296226
- Barnett, V., & Lewis, T. (1978). *Outliers in statistical data*. Chichester: John Wiley and Sons.
- Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637–654. doi:10.1086/260062
- Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31, 307–327. doi:10.1016/0304-4076(86)90063-1
- Chou, R. Y. (2005). Forecasting financial volatilities with extreme values: The conditional autoregressive range (CARR) model. *Journal of Money, Credit, and Banking*, 37, 561–582. doi:10.1353/mcb.2005.0027
- Clark, P. K. (1973). A subordinate stochastic process model with finite variance for speculative prices. *Econometrica*, 41, 135–155. doi:10.2307/1913889
- Davidian, M., & Carroll, R. J. (1987). Variance function estimation. *Journal of the American Statistical*

- Association, 82, 1079–1091. doi:10.1080/01621459.1987.10478543
- Ding, Z., Granger, C. W. J., & Engle, R. F. (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, 1, 83–106. doi:10.1016/0927-5398(93)90006-D
- Eddington, A. (1914). *Stellar movements and the structure of the universe*. London: Macmillan.
- Ederington, L. H., & Guan, W. (2004). Forecasting volatility. *Journal of Futures Markets*, 25, 465–490. doi:10.1002/fut.20146
- Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50, 987–1008. doi:10.2307/1912773
- Fama, E. F. (1963). Mandelbrot and the stable paretian hypothesis. *The Journal of Business*, 36, 420–429. doi:10.1086/jb.1963.36.issue-4
- Fisher, R. (1920). A mathematical examination of the methods of determining the accuracy of observation by the mean error and the mean square error. *Monthly Notes of the Royal Astronomical Society*, 80, 758–770. doi:10.1093/mnras/80.8.758
- Forsberg, L., & Ghysels, E. (2007). Why do absolute returns predict volatility so well? *Journal of Financial Econometrics*, 5, 31–67. doi:10.1093/jfinec/nbl010
- Garman, M., & Klass, M. J. (1980). On the estimation of security price volatilities from historical data. *Journal of Business*, 53, 67–78. doi:10.1086/296072
- Ghysels, E., Harvey, A., & Renault, E. (1996). *Statistical methods in finance (Handbook of statistics, volume 14)*. Amsterdam: North Holland.
- Harris, L. (2001). Cross-security tests of the mixture of distribution hypothesis. *Journal of Financial and Quantitative Analysis*, 21, 39–46. doi:10.2307/2330989
- Hinton, P. (1995). *Statistics Explained*. London: Routledge.
- Horst, E. T., Rodriguez, A., Gzyl, H., & Molina, G. (2012). Stochastic volatility models including open, close, high and low prices. *Quantitative Finance*, 12, 199–212. doi:10.1080/14697688.2010.492233
- Huber, P. (1981). *Robust statistics*. New York: John Wiley and Sons.
- Hull, J. C., & White, A. (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42, 381–400. doi:10.1111/j.1540-6261.1987.tb02568.x
- Kon, S. J. (1984). Models of stock returns – A comparison. *Journal of Finance*, 39, 147–165.
- Kunitomo, N. (1992). Improving the parkinson method of estimating security price volatilities. *Journal of Business*, 65, 295–302. doi:10.1086/296570
- Linden, M. (2001). A model for stock return distribution. *International Journal of Finance & Economics*, 6, 159–169. doi:10.1002/ijfe.149
- Magdon Ismail, M., & Atiya, A. F. (2003). A maximum likelihood approach to volatility estimation for a brownian motion using high, low and close price data. *Quantitative Finance*, 3, 376–384. doi:10.1088/1469-7688/3/5/304
- Maheswaran, S., Balasubramanian, G., & Yoonus, C. A. (2011). Post-colonial finance. *Journal of Emerging Market Finance*, 10, 175–196. doi:10.1177/097265271101000202
- Maheswaran, S., & Kumar, D. (2013). An automatic bias correction procedure for volatility estimation using extreme values of asset prices. *Economic Modelling*, 33, 701–712. doi:10.1016/j.econmod.2013.05.019
- Maheswaran, S., & Kumar, D. (2014). A reflection principle for a random walk with implications for volatility estimation using extreme values of asset prices. *Economic Modelling*, 38, 33–44. doi:10.1016/j.econmod.2013.11.045
- Muneer, S., & Maheswaran, S. (2018a). Evidence of excess volatility based on a new robust volatility ratio. *Journal of Economic Studies*, 45, 855–875. doi:10.1108/JES-06-2017-0150
- Muneer, S., & Maheswaran, S. (2018b). Robust volatility estimation with and without the drift parameter. *Journal of Quantitative Economics*, 1–35. doi:10.1007/s40953-018-0129-4
- Parkinson, M. (1980). The extreme value method for estimating the variance of the rate of return. *Journal of Business*, 53, 61–65. doi:10.1086/296071
- Richardson, M., & Smith, T. (1994). A direct test of the mixture of distribution hypothesis: Measuring the daily flow of information. *Journal of Financial and Quantitative Analysis*, 29, 101–116. doi:10.2307/2331193
- Rogers, L. C. G., & Satchell, S. E. (1991). Estimating variance from high, low and closing prices. *The Annals of Applied Probability*, 1, 504–512. doi:10.1214/aop/1177005835
- Rogers, L. C. G., & Zhou, F. (2008). Estimating correlation from high, low, opening and closing prices. *The Annals of Applied Probability*, 18, 813–823. doi:10.1214/07-AAP460
- Stigler, S. (1973). Studies in the history of probability and statistics XXXII: Laplace, fisher and the discovery of the concept of sufficiency. *Biometrika*, 60, 439–445.
- Tauchen, G., & Pitts, M. (1983). The price variability-volume relationship on speculative markets. *Econometrica*, 51, 485–505. doi:10.2307/1912002
- Taylor, S. (1986). *Modeling financial time series*. New York: Wiley.
- Yang, D., & Zhang, Q. (2000). Drift-independent volatility estimation based on high, low, open, and closing prices. *Journal of Business*, 73, 477–491. doi:10.1086/209650

Appendix

Claim 1: The Exponential Mixture of the Normal distribution is the Double Exponential.

Proof: Let us suppose that x_t is double exponentially distributed. Therefore, the unconditional probability density function of double exponential distribution with the exponential parameter β is given as,

$$f_{x_t}(x) = \frac{1}{2}\beta e^{-\beta(|x|)} \text{ for } \beta < 0, x \in \mathbb{R}.$$

We know that the unconditional characteristic function of x_t (i.e. φ_{x_t}) for a random variable ζ is

$$\varphi_{x_t}(\zeta) = \mathbb{E}[e^{i\zeta x_t}] = \int_{-\infty}^{\infty} e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{-\beta(|x|)} \right\} dx$$

Now let us decompose the above definite integral into parts as I(+) and I(-). That is to say,

$$\int_{-\infty}^{\infty} e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{-\beta(|x|)} \right\} dx = I(+) + I(-)$$

where we have,

$$I(+) = \int_0^{\infty} e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{-\beta x} \right\} dx$$

$$I(-) = \int_{-\infty}^0 e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{\beta x} \right\} dx$$

Now, let us solve I(+) and I(-) separately.⁸

$$I(+) = \int_0^{\infty} e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{-\beta x} \right\} dx = \frac{1}{2}\beta \int_0^{\infty} e^{-[\beta - i\zeta]x} dx = \frac{1}{2}\beta \left[\frac{1}{\beta - i\zeta} \right]$$

$$\text{Also, we have, } I(-) = \int_{-\infty}^0 e^{i\zeta x} \left\{ \frac{1}{2}\beta e^{\beta x} \right\} dx.$$

Now put $w = -x \Rightarrow x = -w$. Also, if $x \in (-\infty, 0) \Rightarrow w \in (0, \infty)$ and $dx = dw$.

Therefore, we have,

$$I(-) = \int_0^{\infty} e^{-i\zeta w} \left\{ \frac{1}{2}\beta e^{-\beta w} \right\} dw = \frac{1}{2}\beta \int_0^{\infty} e^{-[\beta + i\zeta]w} dw = \frac{1}{2}\beta \left[\frac{1}{\beta + i\zeta} \right]$$

Now add both the integral parts I(+) and I(-) to get the value of the unconditional characteristic function $\varphi_{x_t}(\zeta)$. We get,

$$\varphi_{x_t}(\zeta) = I(+) + I(-) = \frac{1}{2}\beta \left[\frac{1}{\beta - i\zeta} + \frac{1}{\beta + i\zeta} \right] = \frac{1}{1 + \left(\frac{1}{\beta^2}\right)\zeta^2}$$

That is to say, we have shown that if the probability density function of x_t is double exponentially distributed as $f_{x_t}(x) = \frac{1}{2}\beta e^{-\beta(|x|)}$ for $\beta > 0$ and $x \in \mathbb{R}$, then the unconditional characteristic function is given as

$$\varphi_{x_t}(\zeta) = \frac{1}{1 + \left(\frac{1}{\beta^2}\right)\zeta^2} \tag{24}$$

We know that the unconditional characteristic function of x_t where $x_t \sim N(0, \sigma^2 = Y_t)$ and $Y_t \sim \text{Exp}(\lambda)$ is given as⁹

$$\varphi_{x_t}(\zeta) = \frac{1}{1 + (\frac{1}{2\lambda})\zeta^2} \tag{25}$$

Therefore, by comparing Equations (24) and (25), we get

$$\beta^2 = 2\lambda \Rightarrow \beta = \sqrt{2\lambda}$$

Hence, we have shown that the exponential mixture of the normal distribution is the double exponential distribution with the parameter $\beta = \sqrt{2\lambda}$.

Claim 2: The conditional characteristic function of x_t is

$$\varphi_{x_t}(\zeta|Y_t = y) = e^{[-\frac{1}{2\zeta^2}y]}$$

Proof:

Let us suppose that the conditional daily returns x_t are normally distributed with mean 0 and variance Y_t , i.e..

$$(x_t|Y_t) \sim N(0, \sigma^2 = Y_t)$$

Also, the unobserved stochastic volatility $Y_t \sim \text{Exp}(\lambda)$.

We know that the characteristic function for a random variable (ζ) is given as

$$\varphi_{x_t}(\zeta) = \mathbb{E}[e^{i\zeta x_t}]$$

Therefore, we have,

$$\varphi_{x_t}(\zeta|Y_t = y) = \mathbb{E}[e^{i\zeta x_t|Y_t=y}]$$

We know that the moment generating function for a random variable (ζ) is given as,

$$\mathbb{E}[e^{\tau\zeta}] = e^{(\mu\tau + \frac{1}{2}\sigma^2\tau^2)}$$

Now let us replace $\tau = i\zeta$, $\mu = 0$ and $\sigma^2 = y$.

Thus, we get the conditional characteristic function of x_t as

$$\varphi_{x_t}(\zeta|Y_t = y) = \mathbb{E}[e^{i\zeta x_t|Y_t=y}] = e^{[-\frac{1}{2\zeta^2}y]}$$

Hence, claim is proved.

Claim 3: The unconditional characteristic function of x_t is

$$\varphi_{x_t}(\zeta) = \frac{1}{1 + (\frac{1}{2\lambda})\zeta^2}$$

Proof:

$$\begin{aligned} L.H.S &= \varphi_{x_t}(\zeta) = \mathbb{E}[e^{i\zeta x_t}] = \mathbb{E}\{\mathbb{E}[e^{i\zeta x_t}]\} = \int_0^\infty \lambda e^{-\lambda y} \{\varphi_{x_t}(\zeta | Y_t = y)\} dy \\ &= \int_0^\infty \lambda e^{-\lambda y} e^{[-\frac{1}{2s^2}y]} dy \quad (\text{from claim 2}) = \int_0^\infty \lambda e^{-(\lambda + \frac{1}{2s^2})y} dy \\ &= \lambda \int_0^\infty e^{-(\lambda + \frac{1}{2s^2})y} dy = \frac{1}{1 + (\frac{1}{2\lambda})\zeta^2} = R.H.S \end{aligned}$$

Since L.H.S = R.H.S, the claim is proved.

Claim 4:

$$\mathbb{E} \left\{ \frac{1}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} = \left(\frac{N}{N-1} \right) \cdot \alpha$$

Proof:

$$\begin{aligned} L.H.S &= \mathbb{E} \left\{ \frac{1}{\frac{1}{N} \left[\sum_{i=1}^N |x_i| \right]} \right\} = \int_0^\infty \frac{1}{w} \cdot \frac{(\alpha^*)^N}{\Gamma(N)} \cdot e^{-\alpha^* \cdot w} \cdot w^{N-1} dw = \frac{(\alpha^*)^N}{\Gamma(N)} \cdot \int_0^\infty e^{-\alpha^* \cdot w} \cdot w^{N-2} dw \\ &= \frac{(\alpha^*)^N}{\Gamma(N)} \cdot \frac{\Gamma(N-1)}{(\alpha^*)^{N-1}} = \frac{\Gamma(N-1)}{(N-1) \cdot \Gamma(N-1)} \cdot \frac{(\alpha^*)^N}{(\alpha^*)^{N-1}} = \frac{(\alpha \cdot N)^N}{(N-1) \cdot (\alpha \cdot N)^{N-1}} = \frac{\alpha \cdot N}{N-1} \\ &= \left(\frac{N}{N-1} \right) \cdot \alpha = R.H.S \end{aligned}$$

Since L.H.S = R.H.S, the claim is proved.



© 2019 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.

You are free to:

Share — copy and redistribute the material in any medium or format.

Adapt — remix, transform, and build upon the material for any purpose, even commercially.

The licensor cannot revoke these freedoms as long as you follow the license terms.

Under the following terms:

Attribution — You must give appropriate credit, provide a link to the license, and indicate if changes were made.

You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.

No additional restrictions

You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits.



***Cogent Economics & Finance* (ISSN: 2332-2039) is published by Cogent OA, part of Taylor & Francis Group.**

Publishing with Cogent OA ensures:

- Immediate, universal access to your article on publication
- High visibility and discoverability via the Cogent OA website as well as Taylor & Francis Online
- Download and citation statistics for your article
- Rapid online publication
- Input from, and dialog with, expert editors and editorial boards
- Retention of full copyright of your article
- Guaranteed legacy preservation of your article
- Discounts and waivers for authors in developing regions

Submit your manuscript to a Cogent OA journal at www.CogentOA.com

