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Working Paper

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Working Paper, No. 338

Provided in Cooperation with:
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Suggested Citation: Bonanno, Giacomo (2020) : Filtered belief revision and generalized choice structures, Working Paper, No. 338, University of California, Department of Economics, Davis, CA

This Version is available at:
https://hdl.handle.net/10419/244606

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Filtered belief revision and generalized choice structures

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Abstract

In an earlier paper [Rational choice and AGM belief revision, *Artificial Intelligence*, 2009] a correspondence was established between the choice structures of revealed-preference theory (developed in economics) and the syntactic belief revision functions of the AGM theory (developed in philosophy and computer science). In this paper we extend the re-interpretation of (a generalized notion of) choice structure in terms of belief revision by adding: (1) the possibility that an item of "information" might be discarded as not credible (thus dropping the AGM success axiom) and (2) the possibility that an item of information, while not accepted as fully credible, may still be "taken seriously" (we call such items of information "allowable"). We establish a correspondence between generalized choice structures (GCS) and AGM belief revision; furthermore, we provide a syntactic characterization of the proposed notion of belief revision, which we call *filtered belief revision*.

*A first version of part of this paper was presented at the TARK 2019 conference and an extended abstract published in: Lawrence S. Moss (editor): *Proceedings Seventeenth Conference on Theoretical Aspects of Rationality and Knowledge* (TARK 2019), Electronic Proceedings in Theoretical Computer Science 297, pp. 82-90, July 2019.

1 Introduction

In Bonanno (2009) a correspondence was established between rational choice theory – also known as revealed-preference theory[1] – and the AGM theory of belief revision.[2]

Revealed-preference theory considers *choice structures* $\langle \Omega, \mathcal{E}, f \rangle$ consisting of a non-empty set Ω (whose elements are interpreted as possible alternatives to choose from), a collection $\mathcal{E}$ of subsets of Ω (interpreted as possible menus, or choice sets) and a function $f : \mathcal{E} \to 2^{\Omega}$ (2^{Ω} denotes the set of subsets of Ω), representing choices made by the agent, conditional on each menu. Given this interpretation, the following restriction on the function f is a natural requirement (the alternatives chosen from menu E should be elements of E): $\forall E \in \mathcal{E}$,

$$f(E) \subseteq E. \tag{1}$$

The objective of reveled-preference theory is to characterize choice structures that can be "rationalized" by a total pre-order $\succsim$ on Ω, interpreted as a preference relation,[3] in the sense that, for every $E \in \mathcal{E}$, $f(E)$ is the set of most preferred alternatives in E: $f(E) = \{\omega \in E : \omega \succsim \omega', \forall \omega' \in E\}$.

The AGM theory of belief revision is a syntactic theory that takes as starting point a consistent and deductively closed set K of formulas in a propositional language, interpreted as the agent's initial beliefs, and a function $B_K : \Phi \to 2^{\Phi}$ (where Φ denotes the set of formulas and 2^{Φ} the set of subsets of Φ), called a *belief revision function based on* K, that associates with every formula $\phi \in \Phi$ (interpreted as new information) a set $B_K(\phi) \subseteq \Phi$, representing the agent's revised beliefs in response to information ϕ. If the function B_K satisfies a set of six properties, known as the basic AGM postulates, then it is called a *basic AGM belief revision function*, while if it satisfies two additional properties (the so-called supplementary postulates) then it is called a *supplemented AGM belief revision function*. We denote a (basic or supplemented) AGM belief revision function by B^*_K.[4]

In Bonanno (2009) the two theories were linked by means of a re-interpretation of the set-theoretic structures of revealed-preference theory, as follows. The set

[1] See, for example, Rott (2001) and Suzumura (1983).

[2] Alchourrón et al. (1985), Gärdenfors (1988)

[3] Thus the intended meaning of $\omega \succsim \omega'$ is "alternative ω is considered to be at least as good as alternative ω'".

[4] In the literature it is common to denote an AGM belief revision function by $* : \Phi \to 2^{\Phi}$ and to denote by $K * \phi$ the belief set resulting from revising K by ϕ. However, we will continue to use the notation of Bonanno (2009).

Ω is interpreted as a set of *states*. A *model based on* (or an *interpretation of*) a choice structure $\langle \Omega, \mathcal{E}, f \rangle$ is obtained by adding a valuation V that assigns to every atomic formula p the set of states at which p is true. Truth of an arbitrary formula at a state is then defined as usual. Given a model $\langle \Omega, \mathcal{E}, f, V \rangle$, the *initial beliefs* of the agent are taken to be the set of formulas ϕ such that $f(\Omega) \subseteq \|\phi\|$, where $\|\phi\|$ denotes the truth set of ϕ; hence $f(\Omega)$ is interpreted as the set of states that are initially considered possible. The events (sets of states) in $\mathcal{E}$ are interpreted as possible items of information. If ϕ is a formula such that $\|\phi\| \in \mathcal{E}$, the *revised belief upon learning that* ϕ is defined as the set of formulas ψ such that $f(\|\phi\|) \subseteq \|\psi\|$. Thus the event $f(\|\phi\|)$ is interpreted as the set of states that are considered possible after learning that ϕ is the case. In light of this interpretation, condition (1) above corresponds to the *success postulate* of AGM theory (one of the six basic postulates): $\forall \phi \in \Phi$,

$$\phi \in B_K(\phi), \tag{2}$$

according to which any item of information is always accepted by the agent and incorporated into her revised beliefs.

The correspondence between choice structures and AGM belief revision is as follows. First of all, define a choice frame to be *supplemented AGM-consistent* if, for every interpretation of it, the associated partial belief revision function ('partial' because, typically, there are formulas ϕ such that $\|\phi\| \notin \mathcal{E}$) can be extended to a (full-domain) supplemented AGM belief revision function (that is, one that satisfies the six basic AGM postulates as well as the two supplementary ones). In Bonanno (2009) it is shown that a finite choice frame is strongly AGM-consistent if and only if it is "rationalizable", that is, if and only if there is a total pre-order $\succsim$ on Ω such that, for every $E \in \mathcal{E}$, $f(E) = \{\omega \in E : \omega \succsim \omega', \forall \omega' \in E\}$. In this context the interpretation of the relation $\succsim$ is no longer in terms of preference but in terms of plausibility: the intended meaning of $\omega \succsim \omega'$ is "state ω is considered to be at least as plausible as state ω'". Thus, for every item of information $E \in \mathcal{E}$, $f(E)$ is the set of most plausible states compatible with the information.

In this paper we continue the analysis of the relationship between choice structures and AGM belief revision by removing restrictions (1) and (2), thus considering a more general notion of belief revision.

The success axiom has been criticized in the AGM literature on the grounds that individuals may not be prepared to accept every item of "information" as credible. For example, during the U.S. Presidential campaign in 2016, a "news" item appeared on several internet sites under the title ""Pope Francis shocks

world, endorses Donald Trump for president".[5] While, perhaps, some people believed this claim, many discarded it as "fake news". In today's political climate, many items of "information" are routinely rejected as not credible.

There is a recent literature in the AGM tradition that relaxes the success axiom (2) and allows for some formulas to be treated as not credible, so that the corresponding "information" is not allowed to affect one's beliefs.[6] This paper's contribution follows this literature, while adding a further possibility.

First of all, we allow for some events – in the set of potential items of information $\mathcal{E}$ – to be treated as not credible, so that

$$f(E) = f(\Omega) \quad \text{if } E \in \mathcal{E} \text{ is rejected as not credible.} \tag{3}$$

Secondly, for information $E \in \mathcal{E}$ which *is* credible we postulate the "success" property (1):

$$f(E) \subseteq E, \quad \text{if } E \in \mathcal{E} \text{ is credible.}$$

Finally, we also add a third type of information, which is *taken seriously but not given the same status as credible information*. For example, a detective might have come to believe that of the three suspects suggested by preliminary evidence – Ann, Bob and Carla – Ann should be discarded in light of her impeccable past behavior, that is, the detective forms the belief that Ann is innocent. Suppose now that new evidence points to Ann as the person who committed the crime. In that case, while not forming the belief that Ann is indeed the culprit, the detective might now add Ann as a serious possibility, by no longer believing in her innocence; that is, the detective now *considers it possible* that Ann is the culprit. We call an item of information that is taken seriously, while not treated as fully credible, *allowable* and we capture possibility in terms of belief revision by the following condition that says that allowable information is not ruled out by the revised beliefs:

$$f(E) \cap E \neq \varnothing \quad \text{if } E \in \mathcal{E} \text{ allowable.} \tag{4}$$

We model credibility, allowability and rejection by partitioning the set $\mathcal{E}$ of possible items of information into three sets: the set $\mathcal{E}_C$ of *credible* items, the set $\mathcal{E}_A$ of *allowable* items and the set $\mathcal{E}_R$ of *rejected* items. Thus we consider *generalized choice structures* (GCS for short) $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ such that:

[5] See: https://www.cnbc.com/2016/12/30/read-all-about-it-the-biggest-fake-news-stories-of-2016.html (accessed June 3, 2019).

[6] See Booth et al. (2012; 2014), Boutilier et al. (1998), Fermé and Hansson (1999), Hansson (1999), Hansson et al. (2001), Makinson (1997), Schlechta (1997).

1. $\Omega \neq \emptyset$,
2. $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\emptyset \notin \mathcal{E}_C \cup \mathcal{E}_A$,[7]
3. $f : \mathcal{E} \to 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that
   (a) $f(\Omega) \neq \emptyset$,
   (b) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$,
   (c) if $E \in \mathcal{E}_C$ then $\emptyset \neq f(E) \subseteq E$,
   (d) if $E \in \mathcal{E}_A$ then $f(E) \cap E \neq \emptyset$.

Remark 1. *Note that if $\mathcal{E}_A = \mathcal{E}_R = \emptyset$ then the above definition of GCS coincides with the definition of choice frame in Bonanno (2009), which we will now call a* simple choice frame.

On the syntactic side we consider partitions of the set Φ of formulas into three sets: the set Φ_C of credible formulas (which contains, at least, all the tautologies), the set Φ_A of allowable formulas and the set Φ_R of rejected formulas (which contains, at least, all the contradictions). As in Bonanno (2009) we then use valuations to link syntax and semantics through interpretations and associate, with every interpretation of a GCS, a partial belief revision function. We then define a GCS to be *basic-AGM consistent* if, for every interpretation (or model) of it, the associated partial belief revision function can be extended to a full-domain belief revision function $B_K : \Phi \to 2^\Phi$ such that, for some basic AGM belief revision function $B^*_K : \Phi \to 2^\Phi$, $\forall \phi \in \Phi$:

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A. \end{cases}$$

Thus

1. if information ϕ is rejected then the original beliefs are maintained,
2. if ϕ is credible then revision is performed according to the basic AGM postulates, and

[7] These sets may be "small", that is, we do not assume that $\mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$ covers the entire set 2^Ω.

3. if ϕ is allowable then revision is performed by contracting the original beliefs by the negation of ϕ (by the Harper identity the contraction by $\neg\phi$ coincides with taking the intersection of the original beliefs with the revision by ϕ).

Proposition 2 in Section 3 provides necessary and sufficient conditions for a GCS to be basic-AGM consistent. As a preliminary step, in Section 2 we define the syntactic notion of *filtered belief revision* and provide a characterization in terms of AGM consistency. In Section 4 we extend to the current framework the result of Bonanno (2009), concerning the correspondence between rationalizability by a plausibility order and supplemented AGM consistency.

2 The syntactic approach

Let Φ be the set of formulas of a propositional language based on a countable set A of atomic formulas.[8] Given a subset $K \subseteq \Phi$, its PL-deductive closure $[K]^{PL}$ (where 'PL' stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \psi$ is a tautology (that is, a theorem of Propositional Logic).[9] A set $K \subseteq \Phi$ is *consistent* if $[K]^{PL} \neq \Phi$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set $K \subseteq \Phi$ is *deductively closed* if $K = [K]^{PL}$.

Let K be a consistent and deductively closed set of formulas, representing the agent's initial beliefs, and let $\Psi \subseteq \Phi$ be a set of formulas representing possible items of information. A *belief revision function based on K and Ψ* is a function $B_{K,\Psi} : \Psi \rightarrow 2^{\Phi}$ that associates with every formula $\phi \in \Psi$ (thought of as new information) a set $B_{K,\Psi}(\phi) \subseteq \Phi$ (thought of as the revised beliefs upon learning that ϕ). If $\Psi \neq \Phi$ then $B_{K,\Psi}$ is called a *partial* belief revision function, while if $\Psi = \Phi$ then $B_{K,\Psi}$ is called a *full-domain* belief revision function and it is more simply denoted by B_K. If $B_{K,\Psi}$ is a partial belief revision function and B'_K is a full-domain belief revision function, we say that B'_K is an *extension of* $B_{K,\Psi}$ if, for all $\phi \in \Psi$, $B'_K(\phi) = B_{K,\Psi}(\phi)$.

A *full-domain* belief revision function $B^*_K : \Phi \rightarrow 2^{\Phi}$ is called a *basic AGM function* if it satisfies the first six of the following properties and it is called a *supplemented AGM function* if it satisfies all of them. The following properties are known as the *AGM postulates*: $\forall \phi, \psi \in \Phi$,

[8] Thus Φ is defined recursively as follows: if $p \in A$ then $p \in \Phi$ and if $\phi, \psi \in \Phi$ then $\neg\phi \in \Phi$ and $(\phi \vee \psi) \in \Phi$. The connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual.

[9] Note that, if F is a set of formulas, $\psi \in [F \cup \{\phi\}]^{PL}$ if and only if $(\phi \rightarrow \psi) \in [F]^{PL}$.

(AGM1) $B^*_K(\phi) = [B^*_K(\phi)]^{PL}$.

(AGM2) $\phi \in B^*_K(\phi)$.

(AGM3) $B^*_K(\phi) \subseteq [K \cup \{\phi\}]^{PL}$.

(AGM4) if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq B^*_K(\phi)$.

(AGM5) $B^*_K(\phi) = \Phi$ if and only if ϕ is a contradiction.

(AGM6) if $\phi \leftrightarrow \psi$ is a tautology then $B^*_K(\phi) = B^*_K(\psi)$.

(AGM7) $B^*_K(\phi \wedge \psi) \subseteq \left[B^*_K(\phi) \cup \{\psi\}\right]^{PL}$.

(AGM8) if $\neg\psi \notin B^*_K(\phi)$, then $\left[B^*_K(\phi) \cup \{\psi\}\right]^{PL} \subseteq B^*_K(\phi \wedge \psi)$.

AGM1 requires the revised belief set to be deductively closed.

AGM2 postulates that the information be believed.

AGM3 says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs.[10]

AGM4 says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs.

AGM5 requires the revised beliefs to be consistent, unless the information ϕ is a contradiction (that is, $\neg\phi$ is a tautology).

AGM6 requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ.

AGM1-AGM6 are called the *basic* AGM postulates, while AGM7 and AGM8 are called the *supplementary* AGM postulates.

AGM7 and AGM8 are a generalization of AGM3 and AGM4 that

> "applies to *iterated* changes of belief. The idea is that if $B_K(\phi)$ is a revision of K [prompted by ϕ] and $B_K(\phi)$ is to be changed by adding further sentences, such a change should be made by using expansions of $B_K(\phi)$ whenever possible.[11] More generally, the minimal change of K to include both ϕ and ψ (that is, $B_K(\phi \wedge \psi)$) ought to be the same as the expansion of $B_K(\phi)$ by ψ, so long as ψ does not

[10] Note that (see Footnote 9) $\psi \in [K \cup \{\phi\}]^{PL}$ if and only if $(\phi \rightarrow \psi) \in K$ (since, by hypothesis, $K = [K]^{PL}$).

[11] The expansion of $B^*_K(\phi)$ by ψ is $[B^*_K(\phi) \cup \{\psi\}]^{PL}$. Note, again, that, for every formula χ, $\chi \in [B^*_K(\phi) \cup \{\psi\}]^{PL}$ if and only if $(\psi \rightarrow \chi) \in B^*_K(\phi)$ (since, by AGM1, $B^*_K(\phi) = [B^*_K(\phi)]^{PL}$).

contradict the beliefs in $B_K(\phi)$" (Gärdenfors (1988), p. 55; notation changed to match ours).

For an extended discussion of the rationale behind the AGM postulates see Gärdenfors (1988).

We now extend the notion of belief revision by allowing the agent to discriminate among different items of information.

Definition 2.1. Let Φ be the set of formulas of a propositional language. A *credibility partition* is a partition of Φ into three sets Φ_C, Φ_A and Φ_R such that

1. Φ_C is the set of *credible* formulas and is such that
   (a) it contains all the tautologies,
   (b) if $\phi \in \Phi_C$ then ϕ is consistent,
   (c) if $\phi \in \Phi_C$ and $\vdash (\phi \leftrightarrow \psi)$ then $\psi \in \Phi_C$, that is, Φ_C is closed under logical equivalence.
2. Φ_A is the (possibly empty) set of *allowable* formulas. We assume that if $\phi \in \Phi_A$ then ϕ is consistent and that Φ_A is closed under logical equivalence.
3. Φ_R is the set of *rejected* formulas, which contains (at least) all the contradictions.

Definition 2.2. Let K be a consistent and deductively closed set of formulas (representing the initial beliefs). A (full-domain) belief revision function based on K, $B^\circ_K : \Phi \to 2^\Phi$, is called a *filtered belief revision function* if it satisfies the following properties: $\forall \phi, \psi \in \Phi$,

$(F1)$	if $\phi \in \Phi_R$	then $B^\circ_K(\phi) = K$,
$(F2)$	if $\neg\phi \notin K$	then (a) if $\phi \in \Phi_C$ then $B^\circ_K(\phi) = [K \cup \{\phi\}]^{PL}$ (b) if $\phi \in \Phi_A$ then $B^\circ_K(\phi) = K$,
$(F3)$	if $\neg\phi \in K$	then $B^\circ_K(\phi)$ is consistent and deductively closed and (a) if $\phi \in \Phi_C$ then $\phi \in B^\circ_K(\phi)$ (b) if $\phi \in \Phi_A$ then $B^\circ_K(\phi) \subseteq (K \setminus \{\neg\phi\})$ and $[B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL} = K$,
$(F4)$	if $\vdash \phi \leftrightarrow \psi$	then $B^\circ_K(\phi) = B^\circ_K(\psi)$.

By ($F1$), if information ϕ is rejected, then the original beliefs K are preserved.
($F2$) says that if, initially, the agent did not believe $\neg\phi$, then (a) if ϕ is credible then the new beliefs are given by the expansion of K by ϕ, while (b) if ϕ is allowable then the agent does not change her beliefs (since she already considered ϕ possible).
($F3$) says that if, initially, the agent believed $\neg\phi$, then (a) if ϕ is credible, then the agent switches from believing $\neg\phi$ to believing ϕ, while (b) if ϕ is allowable, then the agent revises her beliefs by removing $\neg\phi$ from her original beliefs in a minimal way (in the sense that she does not add any new beliefs and if she were to re-introduce $\neg\phi$ into her revised beliefs and close under logical consequence then she would go back to her initial beliefs).
By ($F4$) belief revision satisfies extensionality: if ϕ is logically equivalent to ψ then revision by ϕ coincides with revision by ψ.

The following proposition provides a characterization of filtered belief revision in terms of basic AGM belief revision.[12] The proof is given in Appendix A.

Proposition 1. *Let K be a consistent and deductively closed set of formulas and $B^\circ_K : \Phi \to 2^\Phi$ a belief revision function based on K. Then the following are equivalent:*

(A) B°_K is a filtered belief revision function,

(B) there exists a basic AGM belief revision function $B^*_K : \Phi \to 2^\Phi$ such that, $\forall \phi \in \Phi$,

$$B^\circ_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A \end{cases} \tag{5}$$

(5) says the following:

1. if information ϕ is rejected then the original beliefs are maintained,
2. if ϕ is credible then revision is performed according to the basic AGM postulates, and
3. if ϕ is allowable then revision is performed by contracting the original beliefs by the negation of ϕ (by the Harper identity the contraction by $\neg\phi$ coincides with taking the intersection of the original beliefs with the revision by ϕ).

[12] Note that if $\Phi_A = \varnothing$ then we are in the binary case of "credibility limited revision" of Makinson (1997), Fermé and Hansson (1999), Hansson et al. (2001).

Remark 2. *Note that if $\neg\phi \notin K$ then, by AGM3 and AGM4, $B^*_K(\phi) = [K \cup \{\phi\}]^{PL} \supseteq K$ and thus $K \cap B^*_K(\phi) = K$ so that information $\phi \in \Phi_A$ has no effect on the initial beliefs. Thus, if $\phi \in \Phi_A$, belief change occurs only when $\neg\phi \in K$, that is, when initially the agent believes $\neg\phi$; in this case, since $\neg\phi \in K$ (implying, by consistency of K, that $\phi \notin K$) and $\phi \in B^*_K(\phi)$ (implying, by consistency of $B^*_K(\phi)$, that $\neg\phi \notin B^*_K(\phi)$), it follows that $\phi \notin B^\circ_K(\phi)$ and $\neg\phi \notin B^\circ_K(\phi)$, so that the agent's reaction to being informed that ϕ (with $\phi \in \Phi_A$) is to* suspend judgment *concerning ϕ, in other words, to consider both ϕ and $\neg\phi$ as possible.*

3 Semantics: generalized choice structures

Definition 3.1. A *generalized choice structure* (GCS) is a tuple $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$ such that:

1. $\Omega \neq \varnothing$,[13]
2. $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\varnothing \notin \mathcal{E}_C \cup \mathcal{E}_A$,
3. $f : \mathcal{E} \to 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that
   (a) $f(\Omega) \neq \varnothing$,
   (b) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$,
   (c) if $E \in \mathcal{E}_C$ then $\varnothing \neq f(E) \subseteq E$,
   (d) if $E \in \mathcal{E}_A$ then $f(E) \cap E \neq \varnothing$.

Next we introduce the notion of a model, or interpretation, of a GCS.

Fix a propositional language based on a countable set A of atomic formulas and let Φ be the set of formulas. A *valuation* is a function $V : A \to 2^\Omega$ that associates with every atomic formula $p \in A$ the set of states at which p is true. Truth of an arbitrary formula at a state is defined recursively as follows ($\omega \models \phi$ means that formula ϕ is true at state ω):
(1) for $p \in A$, $\omega \models p$ if and only if $\omega \in V(p)$,
(2) $\omega \models \neg\phi$ if and only if $\omega \not\models \phi$,
(3) $\omega \models (\phi \vee \psi)$ if and only if either $\omega \models \phi$ or $\omega \models \psi$ (or both).
The truth set of formula ϕ is denoted by $\|\phi\|$. Thus $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$.

[13] We do not assume that Ω is finite.

Given a valuation V, define:[14]

$$K = \left\{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\right\}, \tag{6}$$

$$\Psi = \left\{\phi \in \Phi : \|\phi\| \in \mathcal{E}\right\} \text{ and} \tag{7}$$

$$B_{K,\Psi} : \Psi \to 2^{\Phi} \text{ given by: } B_{K,\Psi}(\phi) = \left\{\chi \in \Phi : f\left(\|\phi\|\right) \subseteq \|\chi\|\right\}. \tag{8}$$

Since $f(\Omega)$ is interpreted as the set of states that the individual initially considers possible, (6) is the initial belief set. It is straightforward to show that K is consistent (since, by 3(a) of Definition 3.1, $f(\Omega) \neq \varnothing$) and deductively closed.

(7) is the set of formulas that are potential items of information.

(8) is the partial belief revision function encoding the agent's disposition to revise her beliefs in response to items of information in Ψ (for $E \in \mathcal{E}$, $f(E)$ is interpreted as the set of states that the individual considers possible after receiving information E).

Definition 3.2. Given a GCS $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$, a *model* or *interpretation* of it is obtained by adding to it a pair $(\{\Phi_C, \Phi_A, \Phi_R\}, V)$ (where $\{\Phi_C, \Phi_A, \Phi_R\}$ is a credibility partition of Φ, Definition 2.1, and V is a valuation) such that, $\forall \phi \in \Phi$,

1. if $\|\phi\| \in \mathcal{E}_C$ then $\phi \in \Phi_C$,
2. if $\|\phi\| \in \mathcal{E}_A$ then $\phi \in \Phi_A$,
3. if $\|\phi\| \in \mathcal{E}_R$ then $\phi \in \Phi_R$.

Definition 3.3. A generalized choice structure $C = \langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$ is *basic-AGM consistent* if, for every model $\langle C, \{\Phi_C, \Phi_A, \Phi_R\}, V\rangle$ of it, letting $B_{K,\Psi}$ be the corresponding partial belief revision function (defined by (8)), there exist

1. a full-domain belief revision function $B_K : \Phi \to 2^{\Phi}$ that extends $B_{K,\Psi}$ (that is, for every $\phi \in \Psi$, $B_K(\phi) = B_{K,\Psi}(\phi)$) and
2. a basic AGM belief revision function $B^*_K : \Phi \to 2^{\Phi}$

such that, for every $\phi \in \Phi$,

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{9}$$

[14] All these objects, including the truth sets of formulas, are dependent on the valuation V and thus ought to be indexed by it; however, in order to keep the notation simple, we will omit the subscript 'V.

That is, by Proposition 1, a GCS is basic-AGM consistent if, for every model of it, there exists a filtered belief revision function (Definition 2.2) that extends the partial belief revision function generated by the model.

The following proposition gives necessary and sufficient conditions for a GCS to be basic-AGM consistent. The proof is given in Appendix B.

Proposition 2. *Let $C = \langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ be a generalized choice structure. Then the following are equivalent:*

(A) C is basic-AGM consistent.

(B) C satisfies the following properties: for every $E \in \mathcal{E}_C \cup \mathcal{E}_A$,

1. *if $E \cap f(\Omega) \neq \varnothing$ then*
   - *(a) if $E \in \mathcal{E}_C$ then $f(E) = E \cap f(\Omega)$,*
   - *(b) $E \in \mathcal{E}_A$ then $f(E) = f(\Omega)$,*
2. *if $E \cap f(\Omega) = \varnothing$ and $E \in \mathcal{E}_A$ then $f(E) = f(\Omega) \cup E'$ for some $\varnothing \neq E' \subseteq E$.*

4 Rationalizability and supplemented AGM consistency

In this section we investigate what additional properties a GCS needs to satisfy in order to obtain a correspondence result analogous to Proposition 2 but involving *supplemented*, rather than basic, AGM belief revision (that is, belief revision functions that satisfy the six basic AGM postulates as well as the two supplementary ones).

From now on we will focus on basic-AGM-consistent GCS, which – in virtue of Definitions 3.1 and 3.3 and Proposition 2 – can be redefined as follows.

Definition 4.1. A *basic-AGM-consistent generalized choice structure* (BGCS) is a tuple $\langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ such that:

1. $\Omega \neq \varnothing$,
2. $\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R$ are mutually disjoint subsets of 2^Ω with $\Omega \in \mathcal{E}_C$ and $\varnothing \notin \mathcal{E}_C \cup \mathcal{E}_A$,
3. $f : \mathcal{E} \to 2^\Omega$ (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$) is such that
   - (a) $f(\Omega) \neq \varnothing$,

(b) if $E \in \mathcal{E}_R$ then $f(E) = f(\Omega)$,

(c) if $E \in \mathcal{E}_C$ then $\varnothing \neq f(E) \subseteq E$ and if $E \cap f(\Omega) \neq \varnothing$ then $f(E) = E \cap f(\Omega)$,

(d) if $E \in \mathcal{E}_A$ then

i. if $E \cap f(\Omega) \neq \varnothing$ then $f(E) = f(\Omega)$,

ii. if $E \cap f(\Omega) = \varnothing$ then $f(E) = f(\Omega) \cup E'$ for some $\varnothing \neq E' \subseteq E$.

In order to obtain a characterization in terms of supplemented AGM belief revision we need to add more structure.

Definition 4.2. A BGCS is called *partitioned* if there is a partition $\{\Omega_C, \Omega_A, \Omega_R\}$ of the set of states Ω (the elements of Ω_C are called *credible* states, the elements of Ω_A are called *allowable* states and the elements of Ω_R are called *rejected* states) such that

1. $\Omega_C \neq \varnothing$
2. If $E \in \mathcal{E}_C$ then
   (a) $E \cap \Omega_C \neq \varnothing$,
   (b) $E \cap \Omega_C \in \mathcal{E}_C$,[15]
   (c) $f(E) = f(E \cap \Omega_C) \subseteq \Omega_C$.[16]
3. If $\Omega_A \neq \varnothing$ then $\Omega_A \in \mathcal{E}_A$. Furthermore, if $E \in \mathcal{E}_A$ then
   (a) $E \cap \Omega_C = \varnothing$,
   (b) $E \cap \Omega_A \neq \varnothing$,
   (c) $E \cap \Omega_A \in \mathcal{E}_A$,
   (d) $f(E) = f(E \cap \Omega_A)$.
4. If $E \in \mathcal{E}_R$ then $E \subseteq \Omega_R$.

Note that

- by Point 2, if information E has a credible content ($E \cap \Omega_C \neq \varnothing$), then the agent revises her beliefs based exclusively on the credible content of the information ($f(E) = f(E \cap \Omega_C)$) and incorporates it into her revised beliefs ($f(E) \subseteq E \cap \Omega_C$),

[15] Note that, since $\Omega \in \mathcal{E}_C$, $\Omega_C \cap \Omega = \Omega_C$ and $\Omega_C \neq \varnothing$, it follows that $\Omega_C \in \mathcal{E}_C$.

[16] Since, by 3(c) of Definition 4.1, $f(E) \subseteq E$, it follows that $f(E) \subseteq E \cap \Omega_C$. In particular, $f(\Omega) = f(\Omega_C) \subseteq \Omega_C$.

- by Point 3, if information E does not have a credible content ($E \cap \Omega_C = \varnothing$) but does not consist entirely of rejected states either ($E \cap \Omega_A \neq \varnothing$), then the agent revises her beliefs based exclusively on the "allowable" content of the information ($f(E) = f(E \cap \Omega_A)$),
- by Point 4, if information E is rejected then it consists entirely of rejected states ($E \subseteq \Omega_R$).

We are interested in determining when a BGCS can be rationalized by a plausibility order.

Definition 4.3. Let Ω be a set and let $\{\Omega_C, \Omega_A, \Omega_R\}$ be a partition of Ω with $\Omega_C \neq \varnothing$. A *plausibility order* on Ω is a total pre-order[17] $\succsim \subseteq \Omega \times \Omega$ such that, letting $\succ$ denote the strict component of $\succsim$ and $\sim$ the equivalence component of $\succsim$,[18] $\forall \omega, \omega' \in \Omega$,

1. if $\omega \in \Omega_C$ and $\omega' \in \Omega_A \cup \Omega_R$ then $\omega \succ \omega'$,
2. if $\omega \in \Omega_A$ and $\omega' \in \Omega_R$ then $\omega \succ \omega'$.

Condition 1 says that credible states (those in Ω_C) are more plausible than allowable or rejected states (those in $\Omega_A \cup \Omega_R$) and Condition 2 says that allowable states (those in Ω_A) are more plausible than rejected states (those in Ω_R).

For every $F \subseteq \Omega$, we denote by $best_{\succsim} F$ the set of most plausible elements of F (according to $\succsim$), that is,

$$best_{\succsim} F := \{\omega \in F : \omega \succsim \omega', \forall \omega' \in F\}. \tag{10}$$

Definition 4.4. A partitioned BGCS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ is *rationalizable* if there exists a plausibility order $\succsim$ on Ω such that, $\forall E \in \mathcal{E}$, (where $\mathcal{E} = \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$)

$$f(E) = \begin{cases} best_{\succsim} E & \text{if } E \cap \Omega_C \neq \varnothing \\ best_{\succsim} \Omega \cup best_{\succsim} E & \text{if } E \cap \Omega_C = \varnothing \text{ and } E \cap \Omega_A \neq \varnothing. \\ best_{\succsim} \Omega & \text{if } E \subseteq \Omega_R. \end{cases} \tag{11}$$

If (11) is satisfied, we say that the plausibility order $\succsim$ *rationalizes* the partitioned BGCS.

[17] Thus $\succsim$ is complete or total ($\forall \omega, \omega' \in \Omega$ either $\omega \succsim \omega'$ or $\omega' \succsim \omega$ or both) and transitive ($\forall \omega, \omega', \omega'' \in \Omega$ if $\omega \succsim \omega'$ and $\omega' \succsim \omega''$ then $\omega \succsim \omega''$).

[18] That is, (1) $\omega \succ \omega'$ if $\omega \succsim \omega'$ and not $\omega' \succsim \omega$, and (2) $\omega \sim \omega'$ if both $\omega \succsim \omega'$ and $\omega' \succsim \omega$.

Note that, by 2(c) of Definition 4.3, $best_{\succsim}\,\Omega \subseteq \Omega_C$ and thus $best_{\succsim}\,\Omega = best_{\succsim}\,\Omega_C$; furthermore, Properties 2 and 3 of Definition 4.2 are consistent with (11): for example, if $E \cap \Omega_C \neq \varnothing$ then $best_{\succsim}\,E = best_{\succsim}\,(E \cap \Omega_C)$, so that $f(E) = f(E \cap \Omega_C)$.

The following proposition provides necessary and sufficient conditions for a BGCS to be rationalizable. The proof is given in Appendix C.

Proposition 3. *A partitioned BGCS* $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$ *is rationalizable if and only if, for every sequence* $\langle E_1, ..., E_n, E_{n+1}\rangle$ *in* $\mathcal{E}$ *with* $E_{n+1} = E_1$*, conditions (A) and (B) below are satisfied:*

(A) *if* $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) \neq \varnothing, \ , \forall k = 1, ..., n$*, then*
$(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) = f(E_k \cap \Omega_C) \cap (E_{k+1} \cap \Omega_C), \forall k = 1, ..., n.$

(B) *if* $E_k \cap \Omega_C = \varnothing, \forall k = 1, ..., n$*, and*
$(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) \neq \varnothing, \ , \forall k = 1, ..., n$*, then*
$(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) = f(E_k \cap \Omega_A) \cap (E_{k+1} \cap \Omega_A), \forall k = 1, ..., n.$

Conditions (A) and (B) in Proposition 3 are a generalization of what is known in the revealed preference literature as the Strong Axiom of Revealed Preference (SARP), which is a necessary, but not sufficient, condition for rationalizability by a total pre-order (see Hansson (1968)).[19]

The following definition mirrors Definition 3.3.

Definition 4.5. A partitioned BGCS $C = \langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$ is *supplemented-AGM consistent* if, for every model $\langle C, \{\Phi_C, \Phi_A, \Phi_R\}, V\rangle$ of it, letting $B_{K,\Psi}$ be the corresponding partial belief revision function, there exist

1. a full-domain belief revision function $B_K : \Phi \to 2^\Phi$ that extends $B_{K,\Psi}$ (that is, for every $\phi \in \Psi$, $B_K(\phi) = B_{K,\Psi}(\phi)$) and
2. two supplemented AGM belief revision functions $B_K^{*C} : \Phi \to 2^\Phi$ and $B_K^{*A} : \Phi \to 2^\Phi$

such that, for every $\phi \in \Phi$,

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B_K^{*C}(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B_K^{*A}(\phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{12}$$

[19] Let $\langle \Omega, \mathcal{E}, f\rangle$ be a *simple* choice structure (that is, in our context, $\Omega = \Omega_C$ and $\mathcal{E} = \mathcal{E}_C$) and let $\langle E_1, ..., E_n, E_{n+1}\rangle$ be a sequence in $\mathcal{E}$ with $E_{n+1} = E_1$. Then SARP is the following condition: if $E_k \cap f(E_{k+1}) \neq \varnothing, \ , \forall k \in \{1, ..., n\}$, then there exists a $j \in \{1, ..., n\}$ such that $E_j \cap f(E_{j+1}) = f(E_j) \cap E_{j+1}$.

The following proposition extends Propositions 7 and 8 in Bonanno (2009) to the current framework. The proof is given in Appendix D.

Proposition 4. *Let* $C = \langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ *be a partitioned BGCS where* $\Omega = \Omega_C \cup \Omega_A \cup \Omega_R$ *is finite. Then the following are equivalent:*

(A) C *is supplemented AGM consistent,*

(B) C *is rationalizable.*

As explained in Bonanno (2009), the assumption that Ω is finite is needed to ensure that $best_{\succsim} E \neq \emptyset$, for every $\emptyset \neq E \subseteq \Omega$. If one strengthens the definition of plausibility order by requiring that, $\forall E \subseteq \Omega$, if $E \neq \emptyset$ then $best_{\succsim} E \neq \emptyset$ then the assumption of finiteness of Ω can be dropped.

5 Summary and conclusion

We put forward a notion of belief revision that allows for two possibilities: (1) that an item of information be discarded as not credible and thus not allowed to affect one's beliefs and (2) that an item of information be treated as a serious possibility without assigning full credibility to it. We first defined the syntactic version of this notion, which we called "filtered belief revision" and characterized it in terms the notion of basic AGM belief revision. We then introduced the notion of generalized choice structure, which provides a simple set-theoretic semantics for belief revision and provided a characterization of filtered belief revision in terms of properties of generalized choice structures. Finally, we revisited, in this more general context, the notion of rationalizability of a choice structure in terms of a plausibility order and established a correspondence between rationalizability and AGM consistency in terms of full set of AGM postulates (that is, the six basic postulates together with the supplementary ones).

As noted in the introduction, this paper can be seen as an extension of the AGM-based literature on "credibility-limited" belief revision. The notion of filtered belief revision proposed here provides a hybrid approach to belief revision, based on the use of both revision and contraction. For future research it might be interesting to investigate alternative hybrid approaches to belief change.

A Proof of Proposition 1

(A) implies (B). Given a filtered belief revision function $B^\circ_K : \Phi \to 2^\Phi$, define the function $B^*_K : \Phi \to 2^\Phi$ as follows:

$$B^*_K(\phi) = \begin{cases} \Phi & \text{if } \phi \text{ is a contradiction} \\ [B^\circ_K(\phi) \cup \{\phi\}]^{PL} & \text{if } \phi \text{ is consistent.} \end{cases} \tag{13}$$

First we show that $B^*_K : \Phi \to 2^\Phi$ is a basic AGM belief revision function. Fix an arbitrary $\phi \in \Phi$.

1. Suppose first that ϕ is a contradiction, so that, by (13), $B^*_K(\phi) = \Phi$. Then
   - (AGM1) is satisfied since $\Phi = [\Phi]^{PL}$.
   - (AGM2) is satisfied since $\phi \in \Phi$.
   - (AGM3) is satisfied since $[K \cup \{\phi\}]^{PL} = \Phi$ (because, by hypothesis, ϕ is a contradiction).
   - (AGM4) is satisfied trivially, since, by hypothesis, $\neg\phi$ is a tautology and thus $\neg\phi \in K$ because K is deductively closed.
   - The 'if' part of (AGM5) is satisfied by construction.
   - (AGM6) is satisfied because if $\phi \leftrightarrow \psi$ is a tautology then ψ is also a contradiction and thus $B^*_K(\psi) = B^*_K(\phi) = \Phi$.
2. Suppose now that ϕ is consistent, so that, by (13), $B^*_K(\phi) = [B^\circ_K(\phi) \cup \{\phi\}]^{PL}$. Then
   - (AGM1) is satisfied because $[B^\circ_K(\phi) \cup \{\phi\}]^{PL} = \left[[B^\circ_K(\phi) \cup \{\phi\}]^{PL}\right]^{PL}$.
   - (AGM2) is satisfied because $\phi \in [B^\circ_K(\phi) \cup \{\phi\}]^{PL}$.
   - (AGM3) is satisfied because,
     (1) if $\neg\phi \in K$ then $[K \cup \{\phi\}]^{PL} = \Phi$ and,
     (2) by Definition 2.2,
     - if $\neg\phi \notin K$ and $\phi \in \Phi_C$ then $B^\circ_K(\phi) = [K \cup \{\phi\}]^{PL}$ and thus $B^*_K(\phi) = \left[[K \cup \{\phi\}]^{PL} \cup \{\phi\}\right]^{PL} = [K \cup \{\phi\}]^{PL}$,
     - if $\neg\phi \notin K$ and $\phi \in \Phi_A \cup \Phi_R$ then $B^\circ_K(\phi) = K$ and thus $[B^\circ_K(\phi) \cup \{\phi\}]^{PL} = [K \cup \{\phi\}]^{PL}$.
   - (AGM4) is satisfied, because - as shown above - if $\neg\phi \notin K$ then $[B^\circ_K(\phi) \cup \{\phi\}]^{PL} = [K \cup \{\phi\}]^{PL}$.
   - The 'only if' part of (AGM5) is satisfied because
     - if $\neg\phi \in K$ then, by Definition 2.2, $B^\circ_K(\phi)$ is consistent and thus not equal to Φ,

- if $\neg\phi \notin K$ then $B^\circ_K(\phi) = \begin{cases} [K \cup \{\phi\}]^{PL} & \text{if } \phi \in \Phi_C \\ K & \text{if } \phi \in \Phi_A \cup \Phi_R \end{cases}$ and thus, since K is consistent and ϕ is consistent, $B^\circ_K(\phi)$ is consistent so that $B^\circ_K(\phi) \neq \Phi$.

- (AGM6) is satisfied because if $\phi \leftrightarrow \psi$ is a tautology then, by ($F4$) of Definition 2.2, $B^\circ_K(\phi) = B^\circ_K(\psi)$ and thus $[B^\circ_K(\phi) \cup \{\phi\}]^{PL} = [B^\circ_K(\psi) \cup \{\psi\}]^{PL}$.

Next we need to show that

(a) if $\phi \in \Phi_C$ then $B^\circ_K(\phi) = B^*_K(\phi)$, and

(b) if $\phi \in \Phi_A$ then $B^\circ_K(\phi) = K \cap B^*_K(\phi)$.

(a) Fix an arbitrary $\phi \in \Phi_C$.

- If $\neg\phi \notin K$ then $B^\circ_K(\phi) = [K \cup \{\phi\}]^{PL} = \Big[[K \cup \{\phi\}]^{PL} \cup \phi\Big]^{PL} = B^*_K(\phi)$.

- If $\neg\phi \in K$ then, by ($F3a$) of Definition 2.2, $\phi \in B^\circ_K(\phi)$ and thus $[B^\circ_K(\phi) \cup \{\phi\}]^{PL} = [B^\circ_K(\phi)]^{PL} = B^\circ_K(\phi)$ (the last equality holds because, by ($F3$) of Definition 2.2, $B^\circ_K(\phi)$ is deductively closed); thus $B^*_K(\phi) = B^\circ_K(\phi)$.

(b) Fix an arbitrary $\phi \in \Phi_A$. We need to show that $B^\circ_K(\phi) = K \cap B^*_K(\phi)$. First of all, note that, by ($F2$) and ($F3$) of Definition 2.2, $B^\circ_K(\phi)$ is deductively closed, that is, $B^\circ_K(\phi) = [B^\circ_K(\phi)]^{PL}$.

- If $\neg\phi \notin K$ then, by ($F2$) of Definition 2.2, $B^\circ_K(\phi) = K$; furthermore, $B^\circ_K(\phi) \subseteq [B^\circ_K(\phi) \cup \{\phi\}]^{PL} = B^*_K(\phi)$; hence $B^\circ_K(\phi) = K \cap B^*_K(\phi)$.

- If $\neg\phi \in K$ then, by ($F3$) of Definition 2.2,

$$B^\circ_K(\phi) \subseteq (K \setminus \{\neg\phi\}) \text{ and} \tag{14}$$

$$[B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL} = K \tag{15}$$

Since $B^\circ_K(\phi) \subseteq [B^\circ_K(\phi) \cup \{\phi\}]^{PL} = B^*_K(\phi)$ it follows from (14) that $B^\circ_K(\phi) \subseteq K \cap B^*_K(\phi)$.
It remains to prove that $K \cap B^*_K(\phi) \subseteq B^\circ_K(\phi)$. By (15), $\forall \psi \in \Phi$,

$$\psi \in K \text{ if and only if } (\neg\phi \rightarrow \psi) \in B^\circ_K(\phi). \tag{16}$$

Fix an arbitrary $\psi \in K \cap B^*_K(\phi)$. Since $\psi \in K$, by (16), $(\neg\phi \rightarrow \psi) \in B^\circ_K(\phi)$. Since $\psi \in B^*_K(\phi) = [B^\circ_K(\phi) \cup \{\phi\}]^{PL}$, $(\phi \rightarrow \psi) \in B^\circ_K(\phi)$. Thus, since $B^\circ_K(\phi)$ is deductively closed $(\neg\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi) \in B^\circ_K(\phi)$; hence, since $(\neg\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi) \rightarrow \psi$ is a tautology and $B^\circ_K(\phi)$ is deductively closed, $\psi \in B^\circ_K(\phi)$.

(B) implies (A). Let $B^*_K : \Phi \to 2^\Phi$ be a belief revision function that satisfies the six basic AGM postulates and let $B^\circ_K : \Phi \to 2^\Phi$ be such that, $\forall \phi \in \Phi$,

$$B^\circ_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{17}$$

We need to show that B°_K is a filtered belief revision function, that is, that, $\forall \phi, \psi \in \Phi$,

$$\begin{array}{lll} (F1) & \text{if } \phi \in \Phi_R & \text{then } B^\circ_K(\phi) = K, \\ (F2) & \text{if } \neg\phi \notin K & \text{then} \\ & & (a) \text{ if } \phi \in \Phi_C \text{ then } B^\circ_K(\phi) = [K \cup \{\phi\}]^{PL} \\ & & (b) \text{ if } \phi \in \Phi_A \text{ then } B^\circ_K(\phi) = K, \\ (F3) & \text{if } \neg\phi \in K & \text{then } B^\circ_K(\phi) \text{ is consistent and deductively closed and} \\ & & (a) \text{ if } \phi \in \Phi_C \text{ then } \phi \in B^\circ_K(\phi) \\ & & (b) \text{ if } \phi \in \Phi_A \text{ then } B^\circ_K(\phi) \subseteq (K \setminus \{\neg\phi\}) \\ & & \qquad \text{and } [B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL} = K, \\ (F4) & \text{if } \vdash \phi \leftrightarrow \psi & \text{then } B^\circ_K(\phi) = B^\circ_K(\psi). \end{array} \tag{18}$$

$(F1)$ is the first line in (17). Fix an arbitrary $\phi \in \Phi_C \cup \Phi_A$; then, by Definition 2.1, ϕ is consistent.

Suppose first that $\neg\phi \notin K$. Then, by AGM3 and AGM4, $B^*_K(\phi) = [K \cup \{\phi\}]^{PL}$ so that, if $\phi \in \Phi_C$, Part (a) of $(F2)$ follows from the second line of (17) and, if $\phi \in \Phi_A$, then also Part (b) of $(F2)$ is satisfied because $K \subseteq [K \cup \{\phi\}]^{PL} = B^*_K(\phi)$ so that $K \cap B^*_K(\phi) = K$.

Suppose now that $\neg\phi \in K$. Since ϕ is consistent, by AGM1 and AGM5, $B^*_K(\phi)$ is deductively closed and consistent; since, by hypothesis, K is deductively closed and consistent it follows that $K \cap B^*_K(\phi)$ is also deductively closed and consistent, so that $B^\circ_K(\phi)$ is deductively closed and consistent. If $\phi \in \Phi_C$ then Part (a) of $(F3)$ is satisfied because, by AGM2, $\phi \in B^*_K(\phi)$. Suppose that $\phi \in \Phi_A$. Since $B^*_K(\phi)$ is consistent and $\phi \in B^*_K(\phi)$ it follows that $\neg\phi \notin B^*_K(\phi)$ and thus $\neg\phi \notin K \cap B^*_K(\phi) = B^\circ_K(\phi)$, so that $B^\circ_K(\phi) \subseteq K \setminus \{\neg\phi\}$. Next we show that $[B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL} = K$. Since, by hypothesis, $\neg\phi \in K$, and, by construction, $B^\circ_K(\phi) \subseteq K$, it follows that $(B^\circ_K(\phi) \cup \{\neg\phi\}) \subseteq K$ and thus $[B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL} \subseteq [K]^{PL} = K$. It remains to prove that $K \subseteq [B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL}$. Fix an arbitrary $\psi \in K$. Since, by hypothesis, $K = [K]^{PL}$ and $(\psi \to (\neg\phi \to \psi)$ is a tautology, $(\neg\phi \to \psi) \in K$. Since $\phi \in B^*_K(\phi)$ and $B^*_K(\phi)$ is deductively closed, $(\phi \vee \psi) \in B^*_K(\phi)$ and since $(\phi \vee \psi)$ is logically equivalent to $(\neg\phi \to \psi)$, it follows that $(\neg\phi \to \psi) \in B^*_K(\phi)$. Thus $(\neg\phi \to \psi) \in K \cap B^*_K(\phi)$ and thus $\psi \in [(K \cap B^*_K(\phi)) \cup \{\neg\phi\}]^{PL} = [B^\circ_K(\phi) \cup \{\neg\phi\}]^{PL}$.

Finally, if ψ is logically equivalent to ϕ then $\psi \in \Phi_C \cup \Phi_A$ because both sets are closed under logical equivalence and, by hypothesis, $\phi \in \Phi_C \cup \Phi_A$. Since, by AGM4, $B^*_K(\phi) = B^*_K(\psi)$ it follows that $(F4)$ is satisfied. □

B Proof of Proposition 2

(A) implies (B). Fix a basic-AGM-consistent GCS $\mathcal{C} = \langle \Omega, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$ and an arbitrary $E \in \mathcal{E}_C \cup \mathcal{E}_A$. Let p, q and r be atomic propositions and consider a model $(\{\Phi_C, \Phi_A, \Phi_R\}, V)$ where $\|p\| = E$, $\|q\| = f(E)$ and $\|r\| = f(\Omega)$. Let $K = \left\{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\right\}$, $\Psi = \left\{\phi \in \Phi : \|\phi\| \in \mathcal{E}\right\}$ and $B_{K,\Psi}(\phi) = \left\{\chi \in \Phi : f\left(\|\phi\|\right) \subseteq \|\chi\|\right\}$. Thus $r \in K$, $p \in \Psi$ and $q \in B_{K,\Psi}(p)$. Let $B_K : \Phi \to 2^\Phi$ be a full-domain extension of $B_{K,\Psi} : \Psi \to 2^\Phi$ and $B^*_K : \Phi \to 2^\Phi$ a basic AGM revision function such that, for every $\phi \in \Phi$,

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{19}$$

- Suppose first that

$$E \cap f(\Omega) \neq \varnothing. \tag{20}$$

We need to show that

$$\text{if } E \in \mathcal{E}_C \text{ then } f(E) = E \cap f(\Omega). \tag{21}$$

and

$$\text{if } E \in \mathcal{E}_A \text{ then } f(E) = f(\Omega). \tag{22}$$

By (20), $f(\Omega) \not\subseteq \Omega \setminus E = \Omega \setminus \|p\| = \|\neg p\|$, that is,

$$\neg p \notin K \tag{23}$$

so that, by AGM3 and AGM4,

$$B^*_K(p) = [K \cup \{p\}]^{PL}. \tag{24}$$

  - Consider first the case where $E \in \mathcal{E}_C$, so that $p \in \Phi_C$. Since $B_{K,\Psi}(p) = B^*_K(p)$ and $q \in B_{K,\Psi}(p)$, $q \in B^*_K(p)$, so that, by (24), $q \in [K \cup \{p\}]^{PL}$; hence $(p \to q) \in K$ (recall that K is deductively closed), that is, $f(\Omega) \subseteq \|\neg p \vee q\| = (\Omega \setminus E) \cup f(E)$; thus, intersecting both sides with E, $E \cap f(\Omega) \subseteq f(E) \cap E = f(E)$ (recall that, by Definition 3.1, since $E \in \mathcal{E}_C$, $f(E) \subseteq E$).

    Next we show that $f(E) \subseteq E \cap f(\Omega)$. Since $f(\Omega) = \|r\|$, $r \in K$ and thus, since K is deductively closed, $(p \to r) \in K$, from which it follows that $r \in [K \cup \{p\}]^{PL} = B^*_K(p)$ (by (24)); thus, since $B^*_K(p) = B_{K,\Psi}(p)$, $r \in B_{K,\Psi}(p)$, that is, $f(E) \subseteq \|r\| = f(\Omega)$. Hence, since $f(E) \subseteq E$, $f(E) \subseteq E \cap f(\Omega)$. This completes the proof of (21).

  - Consider next the case where $E \in \mathcal{E}_A$, so that $p \in \Phi_A$. By (19), since $q \in B_{K,\Psi}(p)$, $q \in B_K(p) = K \cap B^*_K(p)$. From $q \in K$ it follows that $f(\Omega) \subseteq \|q\| = f(E)$. It remains to prove that the converse is also true, namely that $f(E) \subseteq f(\Omega)$. Since

$f(\Omega) = \|r\|$, $r \in K$. Thus, since K is deductively closed, $(p \to r) \in K$, from which it follows that $r \in [K \cup \{p\}]^{PL} = B_K^*(p)$ (by (24)). Thus $r \in K \cap B_K^*(p)$, so that, since $B_{K,\Psi}(p) = B_K(p) = K \cap B_K^*(p)$ (by (19)), $r \in B_{K,\Psi}(p)$, that is, $f(E) \subseteq \|r\| = f(\Omega)$. This completes the proof of (22).

- Suppose now that $E \in \mathcal{E}_A$ (thus, by Point 2 of Definition 3.1, $E \neq \varnothing$) and

$$E \cap f(\Omega) = \varnothing. \tag{25}$$

We need to show that $f(E) = f(\Omega) \cup E'$ for some $\varnothing \neq E' \subseteq E$. Since $E \in \mathcal{E}_A$ and $\|p\| = E$, $p \in \Phi_A$. Thus, by (19),

$$B_{K,\Psi}(p) = B_K(p) = K \cap B_K^*(p). \tag{26}$$

Since $E = \|p\|$ and $f(E) = \|q\|$, $q \in B_{K,\Psi}(p)$ and thus, by (26), $q \in K$, that is, $f(\Omega) \subseteq \|q\| = f(E)$. It follows from this and the fact that $f(E) \cap E \subseteq f(E)$, that

$$f(\Omega) \cup (f(E) \cap E) \subseteq f(E). \tag{27}$$

Next we show that $f(E) \subseteq f(\Omega) \cup (f(E) \cap E)$. Since $f(\Omega) = \|r\|$, $r \in K$ and thus, since K is deductively closed, $(r \vee p) \in K$. Since $p \in B_K^*(p)$ and $B_K^*(p)$ is deductively closed, $(r \vee p) \in B_K^*(p)$. Thus, by (26), $(r \vee p) \in B_{K,\Psi}(p)$, that is, $f(E) \subseteq \|r \vee p\| = \|r\| \cup \|p\| = f(\Omega) \cup E$; hence (intersecting both sides with $\Omega \setminus E$),

$$\begin{aligned} f(E) \cap (\Omega \setminus E) &\subseteq (f(\Omega) \cup E) \cap (\Omega \setminus E) \\ &= (f(\Omega) \cap (\Omega \setminus E)) \cup (E \cap (\Omega \setminus E)) \\ &= f(\Omega) \cap (\Omega \setminus E) =_{\text{(by (25))}} f(\Omega). \end{aligned} \tag{28}$$

Thus,

$$f(E) = (f(E) \cap (\Omega \setminus E)) \cup (f(E) \cap E) \subseteq_{\text{(by (28))}} f(\Omega) \cup (f(E) \cap E). \tag{29}$$

If follows from (27) and (29) that $f(E) = f(\Omega) \cup E'$ with $E' = f(E) \cap E$. Finally, by (d) of definition of GCS (Definition 3.1), $f(E) \cap E \neq \varnothing$.

(B) implies (A). Fix a GCS that satisfies the properties of part (B) of Proposition 2 and an arbitrary model $(\{\Phi_C, \Phi_A, \Phi_R\}, V)$ of it. As usual, let

$$K = \left\{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\right\}, \tag{30}$$

$$\Psi = \left\{\phi \in \Phi : \|\phi\| \in \mathcal{E}\right\}, \tag{31}$$

$$B_{K,\Psi} : \Psi \to 2^{\Phi} \text{ given by: } B_{K,\Psi}(\phi) = \left\{\chi \in \Phi : f\left(\|\phi\|\right) \subseteq \|\chi\|\right\}. \tag{32}$$

Let $B^*_K : \Phi \to 2^\Phi$ be the following (full domain) belief revision function: $\forall \phi \in \Phi$,

$$B^*_K(\phi) = \begin{cases} 1.\ \Phi & \text{if and only if } \phi \text{ is a contradiction} \\ 2.\ [\phi]^{PL} & \text{if } \|\phi\| \notin \mathcal{E}_C \cup \mathcal{E}_A \\ 3.\ [K \cup \{\phi\}]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_C \cup \mathcal{E}_A \text{ and } \neg\phi \notin K \\ 4.\ \left[\{\psi \in \Phi : f(\|\phi\|) \subseteq \|\psi\|\}\right]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_C \text{ and } \neg\phi \in K \\ 5.\ \left[\{\psi \in \Phi : (f(\|\phi\|) \cap \|\phi\|) \subseteq \|\psi\|\}\right]^{PL} & \text{if } \|\phi\| \in \mathcal{E}_A \text{ and } \neg\phi \in K \end{cases} \tag{33}$$

First we show that B^*_K is a basic AGM belief revision function.

- AGM1 is satisfied by construction.
- AGM2 is clearly satisfied in cases 1-3 and 5 of (33). As for case 4, since $\|\phi\| \in \mathcal{E}_C$, by definition of GCS $f(\|\phi\|) \subseteq \|\phi\|$.
- AGM3 is clearly satisfied in cases 1-3 of (33). In cases 4 and 5, since $\neg\phi \in K$, $[K \cup \{\phi\}]^{PL} = \Phi$ and the property holds trivially.
- AGM4 is clearly satisfied in cases 1-3 of (33) since $[B^*_K(\phi) \cup \{\phi\}]^{PL} = B^*_K(\phi)$. In cases 4 and 5 the property holds trivially since $\neg\phi \in K$.
- AGM5 is satisfied by construction.
- AGM6 is satisfied because if $\phi \leftrightarrow \psi$ is a tautology then
  1. if ϕ is a contradiction then so is ψ and thus, by construction, $B^*_K(\phi) = B^*_K(\psi) = \Phi$.
  2. $[\phi]^{PL} = [\psi]^{PL}$.
  3. $[K \cup \{\phi\}]^{PL} = [K \cup \{\psi\}]^{PL}$.
  4. and 5. $\|\phi\| = \|\psi\|$.

Next define the following (full-domain) belief revision function: $\forall \phi \in \Phi$,

$$B^\circ_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B^*_K(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B^*_K(\phi) & \text{if } \phi \in \Phi_A \end{cases} \tag{34}$$

where $B^*_K(\phi)$ is given by (33). Then, by definition of basic-AGM consistent GCS (Definition 3.3), it only remains to prove that B°_K is an extension of $B_{K,\Psi}$ (given by (30)), that is, that, for every $\phi \in \Psi$, $\chi \in B_{K,\Psi}(\phi)$ if and only if $\chi \in B^\circ_K(\phi)$. Fix an arbitrary $\phi \in \Psi$, that is, a formula ϕ such that $\|\phi\| \in \mathcal{E}$.

- If $\|\phi\| \in \mathcal{E}_R$ (so that $\phi \in \Phi_R$) then (by definition of GCS: Definition 3.1) $f(\|\phi\|) = f(\Omega)$ and thus, $\forall \chi \in \Phi$, $\chi \in B_{K,\Psi}(\phi)$ if and only if $f(\Omega) \subseteq \|\chi\|$ if and only if $\chi \in K$ and, by (34), $B^\circ_K(\phi) = K$.

- If $\|\phi\| \in \mathcal{E}_C$ (so that $\phi \in \Phi_C$) then, $\forall \chi \in \Phi$, $\chi \in B_{K,\Psi}(\phi)$ if and only if $f(\|\phi\|) \subseteq \|\chi\|$; thus
  - if $\neg\phi \in K$ then, by 4 of (33), $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $\chi \in B^*_K(\phi) = B^\circ_K(\phi)$,
  - if $\neg\phi \notin K$ then $f(\Omega) \cap \|\phi\| \neq \varnothing$ and thus, by hypothesis (1(a) of Part (B) of Proposition 2), $f(\|\phi\|) = f(\Omega) \cap \|\phi\|$ so that $\chi \in B_{K,\Psi}(\phi)$ if and only if $f(\Omega) \cap \|\phi\| \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq \Big(\Omega \setminus \|\phi\|\Big) \cup \|\chi\| = \|\phi \to \chi\|$ if and only if $(\phi \to \chi) \in K$, if and only if $\chi \in [K \cup \{\phi\}]^{PL} = B^*_K(\phi) = B^\circ_K(\phi)$.
- If $\|\phi\| \in \mathcal{E}_A$ (so that $\phi \in \Phi_A$) then,
  - if $\neg\phi \in K$, that is, $f(\Omega) \cap \|\phi\| = \varnothing$ then, by hypothesis, $f(\|\phi\|) = f(\Omega) \cup E'$ for some $\varnothing \neq E' \subseteq \|\phi\|$ so that $E' = f(\|\phi\|) \cap \|\phi\|$; hence $\chi \in B_{K,\Psi}(\phi)$ if and only if $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq \|\chi\|$ and $f(\|\phi\|) \cap \|\phi\| \subseteq \|\chi\|$, if and only if $\chi \in K$ and, by 5 of (33), $\chi \in B^*_K(\phi)$, that is, $\chi \in K \cap B^*_K(\phi) = B^\circ_K(\phi)$,
  - if $\neg\phi \notin K$ then $f(\Omega) \cap \|\phi\| \neq \varnothing$ and thus, by hypothesis, $f(\|\phi\|) = f(\Omega)$ so that $\chi \in B_{K,\Psi}(\phi)$ if and only if $f(\Omega) \subseteq \|\chi\|$ if and only if $\chi \in K = K \cap [K \cup \{\phi\}]^{PL} = K \cap B^*_K(\phi) = B^\circ_K(\phi)$. □

C Proof of Proposition 3

The proof of Proposition 3 makes repeated use of the following proposition due to Hansson (Hansson (1968), Theorem 7, p. 455). We begin with a definition.

Definition C.1. A *simple choice structure* is a triple $\langle W, \mathcal{F}, h\rangle$ where W is a non-empty set, $\mathcal{F} \subseteq 2^W$, with $\varnothing \notin \mathcal{F}$ and $W \in \mathcal{F}$, and $h : \mathcal{F} \to 2^W$ satisfies $\varnothing \neq h(F) \subseteq F$, for all $F \in \mathcal{F}$.

Proposition 5 (Hansson (1968))**.** *Let $\langle W, \mathcal{F}, h\rangle$ be a simple choice structure. Then the following conditions are equivalent:*

1. *there exists a total pre-order $\succsim \subseteq W \times W$ such that, for all $F \in \mathcal{F}$,*
   $h(F) = best_{\succsim} F \overset{def}{=} \{\omega \in F : \omega \succsim \omega', \forall \omega' \in F\}$,
2. *for every sequence $\langle F_1, ..., F_n, F_{n+1}\rangle$ in $\mathcal{F}$ with $F_{n+1} = F_1$, if $F_k \cap h(F_{k+1}) \neq \varnothing$, $\forall k = 1, ..., n$, then $F_k \cap h(F_{k+1}) = h(F_k) \cap F_{k+1}$, $\forall k = 1, ..., n$.*

First we show that if $\succsim$ rationalizes the partitioned BGCS $\langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f\rangle$ then (A) of Proposition 3 is satisfied. Construct the following simple choice frame $\langle W, \mathcal{F}, h\rangle$:

$$\begin{cases} W = \Omega_C \\ \mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\} \\ h : \mathcal{F} \to 2^W \text{ is the restriction of } f \text{ to } \mathcal{F}. \end{cases} \tag{35}$$

By Definition 4.2, $\Omega_C \neq \varnothing$. By 2 of Definition 4.2, if $E \in \mathcal{E}_C$ then $E \cap \Omega_C \in \mathcal{E}_C$ and by 2 of Definition 4.1, $\varnothing \notin \mathcal{F}$ and $\Omega_C \in \mathcal{F}$. By 3(c) of Definition 4.1, $h(F) \neq \varnothing, \forall F \in \mathcal{F}$. By hypothesis, since the BGCS is rationalized by the total pre-order $\succsim \subseteq \Omega \times \Omega$, if $E \in \mathcal{E}_C$ then $f(E) = f(E \cap \Omega_C) = best_{\succsim}(E \cap \Omega_C) \subseteq E \cap \Omega_C$ and thus, letting $F = E \cap \Omega_C$, $h(F) \subseteq F$. Hence we have indeed defined a simple choice structure.
Let $\succsim_C$ be the restriction of $\succsim$ to Ω_C (that is, $\succsim_C = \succsim \cap (\Omega_C \times \Omega_C)$). By construction, since $\succsim$ rationalizes the given GCS, we have that

$$\forall F \in \mathcal{F}, \quad h(F) = best_{\succsim_C} F \overset{def}{=} \{\omega \in F : \omega \succsim_C \omega', \forall \omega' \in F\}. \tag{36}$$

Now fix an arbitrary sequence $\langle E_1, ..., E_n, E_{n+1}\rangle$ in $\mathcal{E}_C$ with $E_{n+1} = E_1$ such that, $\forall k = 1, ..., n$, $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) \neq \varnothing$. Let $\langle F_1, ..., F_n, F_{n+1}\rangle$ be the corresponding sequence in $\mathcal{F}$, that is, for every $k = 1, ..., n$, $F_k = E_k \cap \Omega_C$ (thus $F_{n+1} = F_1$). Then, for every $k = 1, ..., n$, $F_k \cap h(F_{k+1}) \neq \varnothing$. Thus, by (36) and Proposition 5, $F_k \cap h(F_{k+1}) = h(F_k) \cap F_{k+1}$, $\forall k = 1, ..., n$, that is, $(E_k \cap \Omega_C) \cap f(E_{k+1} \cap \Omega_C) = f(E_k \cap \Omega_C) \cap (E_{k+1} \cap \Omega_C)$, $\forall k = 1, ..., n$; that is, (A) of Proposition 3 holds.

Next we show that Part (B) of Proposition 3 is satisfied. If $\Omega_A = \varnothing$ there is nothing to prove. Assume, therefore, that $\Omega_A \neq \varnothing$ (so that, by 3 of Definition 4.2, $\Omega_A \in \mathcal{E}$). Construct the following choice frame $\langle W, \mathcal{G}, g\rangle$:

$$\begin{cases} W = \Omega_A \\ \mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \\ g : \mathcal{G} \to 2^W \text{ is the restriction of } f \text{ to } \mathcal{G}. \end{cases} \tag{37}$$

By 3(d) of Definition 4.1, for every $G \in \mathcal{G}$, $g(G) \neq \varnothing$; furthermore, by hypothesis, since the BGCS is rationalized by the total pre-order $\succsim \subseteq \Omega \times \Omega$, if $E \in \mathcal{E}_A$ then $f(E) = f(E \cap \Omega_A) = best_{\succsim}(E \cap \Omega_A) \subseteq E \cap \Omega_A$ and thus, letting $G = E \cap \Omega_A$, $g(G) \subseteq G$. Hence we have indeed defined a simple choice structure.
Let $\succsim_A$ be the restriction of $\succsim$ to Ω_A (that is, $\succsim_A = \succsim \cap (\Omega_A \times \Omega_A)$). By construction, since $\succsim$ rationalizes the given GCS, we have that

$$\forall G \in \mathcal{G}, \quad g(G) = best_{\succsim_A} G \overset{def}{=} \{\omega \in G : \omega \succsim_A \omega', \forall \omega' \in G\}. \tag{38}$$

Now fix an arbitrary sequence $\langle E_1, ..., E_n, E_{n+1}\rangle$ in $\mathcal{E}_A$ with $E_{n+1} = E_1$ (thus, by 3(a) of Definition 4.2 $E_k \cap \Omega_C = \varnothing$, $\forall k = 1, ..., n$) such that $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) \neq \varnothing$, $\forall k = 1, ..., n$. Let $\langle G_1, ..., G_n, G_{n+1}\rangle$ be the corresponding sequence in $\mathcal{G}$, that is, for every $k = 1, ..., n$, $G_k = E_k \cap \Omega_A$ (thus $G_{n+1} = G_1$). Then, for every $k = 1, ..., n$, $G_k \cap g(G_{k+1}) \neq \varnothing$. Thus, by (38) and Proposition 5, $G_k \cap g(G_{k+1}) = g(G_k) \cap G_{k+1}$, $\forall k = 1, ..., n$, that is, $(E_k \cap \Omega_A) \cap f(E_{k+1} \cap \Omega_A) = f(E_k \cap \Omega_A) \cap (E_{k+1} \cap \Omega_A)$, $\forall k = 1, ..., n$, that is, (B) of Proposition 3 holds.

Next we show that if the partitioned BGSC $\langle\{\Omega_C,\Omega_A,\Omega_R\},\{\mathcal{E}_C,\mathcal{E}_A,\mathcal{E}_R\},f\rangle$ satisfies Properties (A) and (B) of Proposition 3 then it can be rationalized by a plausibility order $\succsim \Omega\times\Omega$.

Let $\langle W,\mathcal{F},h\rangle$ be the simple choice frame defined in (35). Fix an arbitrary sequence $\langle E_1,...,E_n,E_{n+1}\rangle$ in $\mathcal{E}_C$ with $E_{n+1}=E_1$ such that $(E_k\cap\Omega_C)\cap f(E_{k+1}\cap\Omega_C)\neq\varnothing,,\forall k=1,...,n$, and let $\langle F_1,...,F_n,F_{n+1}\rangle$ be the corresponding sequence in $\mathcal{F}$ (that is, $F_k=E_k\cap\Omega_C$, for all $k=1,...,n$). By the Property (A) of Proposition 3, $(E_k\cap\Omega_C)\cap f(E_{k+1}\cap\Omega_C)=f(E_k\cap\Omega_C)\cap(E_{k+1}\cap\Omega_C),\forall k=1,...,n$, that is, $F_k\cap h(F_{k+1})=h(F_k)\cap F_{k+1},\forall k=1,...,n$. Hence, since the sequence was chosen arbitrarily, it follows from Proposition 5 that there exists a total pre-order $\succsim_C$ on $W\times W=\Omega_C\times\Omega_C$ such that

$$\forall F\in\mathcal{F},\ \ h(F)=best_{\succsim_C}F\overset{def}{=}\{\omega\in F:\omega\succsim_C\omega',\forall\omega'\in F\}. \tag{39}$$

Two cases are possible.

Case 1: $\Omega_A=\varnothing$. In this case, define $\succsim\subseteq\Omega\times\Omega$ as follows:

$$\succsim\ =\ \succsim_C\ \cup\ \{(\omega,\omega'):\omega\in\Omega_C\text{ and }\omega'\in\Omega_R\}\ \cup\ \{(\omega,\omega'):\omega,\omega'\in\Omega_R\}. \tag{40}$$

Then $\succsim$ satisfies the properties that define a plausibility order (Definition 4.3). Fix an arbitrary $E\in\mathcal{E}$. If $E\cap\Omega_C\neq\varnothing$ then, by 2(c) of Definition 4.2, $f(E)=f(E\cap\Omega_C)\overset{def}{=}h(E\cap\Omega_C)$ and by (39) $h(E\cap\Omega_C)=best_{\succsim_C}(E\cap\Omega_C)$. By (40), if $\omega\in E\cap\Omega_C$ and $\omega'\in E\setminus\Omega_C$ then $\omega\succ\omega'$ so that $best_{\succsim}E=best_{\succsim}(E\cap\Omega_C)=best_{\succsim_C}(E\cap\Omega_C)$; thus $f(E)=best_{\succsim}E$. If $E\cap\Omega_C=\varnothing$ then $E\subseteq\Omega_R$ and, by 3(b) Definition 3.1, $f(E)=f(\Omega)$. Since $\Omega\cap\Omega_C=\Omega_C\neq\varnothing$, $f(\Omega)=f(\Omega\cap\Omega_C)\overset{def}{=}h(\Omega_C)$ and by (39) $h(\Omega_C)=best_{\succsim_C}(\Omega_C)$; by (40), $best_{\succsim}\Omega=best_{\succsim_C}\Omega_C$, so that $f(\Omega)=best_{\succsim}\Omega$.

Case 2: $\Omega_A\neq\varnothing$. In this case let $\langle W,\mathcal{G},g\rangle$ be the choice frame defined in (37). Fix an arbitrary sequence $\langle E_1,...,E_n,E_{n+1}\rangle$ in $\mathcal{E}$ with $E_{n+1}=E_1$ such that $E_k\cap\Omega_C=\varnothing$, $\forall k=1,...,n$, and $(E_k\cap\Omega_A)\cap g(E_{k+1}\cap\Omega_A)\neq\varnothing,\forall k=1,...,n$ and let $\langle G_1,...,G_n,G_{n+1}\rangle$ be the corresponding sequence in $\mathcal{G}$ (that is, $G_k=E_k\cap\Omega_A$, for all $k=1,...,n$). By Property (B) of Proposition 3, $(E_k\cap\Omega_A)\cap g(E_{k+1}\cap\Omega_A)=g(E_k\cap\Omega_A)\cap(E_{k+1}\cap\Omega_A),\forall k=1,...,n$, that is, $G_k\cap g(G_{k+1})=g(G_k)\cap G_{k+1},\forall k=1,...,n$. Hence, since the sequence was chosen arbitrarily, it follows from Proposition 5 that there exists a total pre-order $\succsim_A$ on $W\times W=\Omega_A\times\Omega_A$ such that

$$\forall G\in\mathcal{G},\ \ g(G)=best_{\succsim_A}G\overset{def}{=}\{\omega\in G:\omega\succsim_A\omega',\forall\omega'\in G\}. \tag{41}$$

Define $\succsim\subseteq\Omega\times\Omega$ as follows (where $\succsim_C$ is the total pre-order on $\Omega_C\times\Omega_C$ that satisfies (39)):

$$\begin{aligned}\succsim\ =\ &\succsim_C\ \cup\succsim_A\\ &\cup\{(\omega,\omega'):\omega\in\Omega_C\text{ and }\omega'\in\Omega_A\cup\Omega_R\}\\ &\cup\{(\omega,\omega'):\omega\in\Omega_A\text{ and }\omega'\in\Omega_R\}\\ &\cup\{(\omega,\omega'):\omega,\omega'\in\Omega_R\}.\end{aligned} \tag{42}$$

Then $\succsim$ satisfies the properties that define a plausibility order (Definition 4.3). Fix an arbitrary $E \in \mathcal{E}$. If $E \cap \Omega_C \neq \varnothing$ or $E \subseteq \Omega_R$, then $f(E) = best_{\succsim} E$ by the argument developed in Case 1. If $E \cap \Omega_C = \varnothing$ and $E \cap \Omega_A \neq \varnothing$, then, by 3(d) of Definition 4.2, $f(E) = f(E \cap \Omega_A) \overset{def}{=} g(E \cap \Omega_A)$ and by (41) $g(E \cap \Omega_A) = best_{\succsim_A}(E \cap \Omega_A)$. By (42), if $\omega \in E \cap \Omega_A$ and $\omega' \in \Omega_R$ then $\omega \succ \omega'$ so that $best_{\succsim} E = best_{\succsim}(E \cap \Omega_A) = best_{\succsim_A}(E \cap \Omega_A)$; thus $f(E) = best_{\succsim} E$. □

D Proof of Proposition 4

(A) implies (B). Let $C = \langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$, with $\Omega \overset{def}{=} \Omega_C \cup \Omega_A \cup \Omega_R$ finite, be a partitioned BGCS which is supplemented AGM consistent, that is, there exist supplemented AGM belief revision functions $B_K^{*C} : \Phi \to 2^\Phi$ and $B_K^{*A} : \Phi \to 2^\Phi$ such that the function $B_K : \Phi \to 2^\Phi$ defined by

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B_K^{*C}(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B_K^{*A}(\phi) & \text{if } \phi \in \Phi_A \end{cases} \tag{43}$$

is an extension of $B_{K,\Psi}$, where, as usual, $K = \left\{\phi \in \Phi : f(\Omega) \subseteq \|\phi\|\right\}$, $\Psi = \left\{\phi \in \Phi : \|\phi\| \in \mathcal{E}\right\}$ and $B_{K,\Psi}(\phi) = \left\{\chi \in \Phi : f(\|\phi\|) \subseteq \|\chi\|\right\}$. We need to show that C is rationalizable by a plausibility order $\succsim$ on Ω (Definition 4.3), in the sense that, for every $E \in \mathcal{E} \overset{def}{=} \mathcal{E}_C \cup \mathcal{E}_A \cup \mathcal{E}_R$,

$$f(E) = \begin{cases} best_{\succsim} E & \text{if } E \cap \Omega_C \neq \varnothing \\ best_{\succsim} \Omega \cup best_{\succsim} E & \text{if } E \cap \Omega_C = \varnothing \text{ and } E \cap \Omega_A \neq \varnothing \\ best_{\succsim} \Omega & \text{if } E \subseteq \Omega_R \end{cases} \tag{44}$$

(where, for every $F \subseteq \Omega$, $best_{\succsim} F \overset{def}{=} \{\omega \in F : \omega \succsim \omega'\}$).
Extract from C the simple choice frame $\langle \Omega_C, \mathcal{F}, h \rangle$ where $\mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\}$ and h is the restriction of f to $\mathcal{F}$. By 2(c) of Definition 4.2, (1) $f(\Omega) = f(\Omega \cap \Omega_C) = f(\Omega_C) = h(\Omega_C)$, so that $\left\{\phi \in \Phi : h(\Omega_C) \subseteq \|\phi\|\right\} = K$ and (2) $\Psi_{\mathcal{F}} \overset{def}{=} \left\{\phi \in \Phi : \|\phi\| \in \mathcal{F}\right\} \subseteq \Psi$. For every $\phi \in \Phi$ let $B_{K,\Psi_{\mathcal{F}}}(\phi) = \left\{\chi \in \Phi : h(\|\phi\|) \subseteq \|\chi\|\right\}$. Then, by (43), the supplemented AGM function B_K^{*C} is an extension of $B_{K,\Psi_{\mathcal{F}}}$ and thus the simple frame $\langle \Omega_C, \mathcal{F}, h \rangle$ is AGM consistent in the sense of Definition 3 in Bonanno (2009) so that, by Proposition 8 in Bonanno (2009), there exists a total preorder $\succsim_C$ on Ω_C such that, for every $F \in \mathcal{F}$,

$$h(F) = best_{\succsim_C} F \overset{def}{=} \{\omega \in F : \omega \succsim_C \omega', \forall \omega' \in F\}. \tag{45}$$

Two cases are possible.

Case 1: $\Omega_A = \varnothing$. In this case, define $\succsim \subseteq \Omega \times \Omega$ as in (39) and the argument to show that, $\forall E \in \mathcal{E}$, $f(E) = best_{\succsim} E$ is a repetition of the argument used in Case 1 of the proof of Proposition 3.

Case 2: $\Omega_A \neq \varnothing$. In this case extract from C the simple choice frame $\langle \Omega, \mathcal{G}, g \rangle$ where $\mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \cup \{\Omega\}$ and g is the restriction of f to $\mathcal{G}$. By 3(d) of Definition 4.2, $\Psi_{\mathcal{G}} \overset{def}{=} \left\{\phi \in \Phi : \|\phi\| \in \mathcal{G}\right\} \subseteq \Psi$. By construction, since $g(\Omega) = f(\Omega)$, $\{\phi \in \Phi : g(\Omega) \subseteq \|\phi\|\} = K$. Then, by (43), the supplemented AGM function B_K^{*A} is an extension of $B_{K,\Psi_{\mathcal{G}}}$ and thus the simple frame $\langle \Omega, \mathcal{G}, g \rangle$ is AGM consistent in the sense of Definition 3 in Bonanno (2009) so that, by Proposition 8 in Bonanno (2009), there exists a total preorder $\succsim'_A$ on Ω_C such that, for every $G \in \mathcal{G}$,

$$g(G) = best_{\succsim'_A} G \overset{def}{=} \{\omega \in G : \omega \succsim'_A \bar{\omega}, \forall \bar{\omega} \in G\}. \tag{46}$$

Let $\succsim_A = \succsim'_A \cap (\Omega_A \times \Omega_A)$ and define $\succsim \subseteq \Omega \times \Omega$ as in (42). Then the argument to show that, $\forall E \in \mathcal{E}$, $f(E) = best_{\succsim} E$ is a repetition of the argument used in Case 2 of the proof of Proposition 3.

(B) implies (A). Let $C = \langle \{\Omega_C, \Omega_A, \Omega_R\}, \{\mathcal{E}_C, \mathcal{E}_A, \mathcal{E}_R\}, f \rangle$, with Ω finite, be a partitioned BGCS which is rationalized by a plausibility order $\succsim$ on Ω (Definition 4.3), in the sense that, for every $E \in \mathcal{E}$,

$$f(E) = \begin{cases} best_{\succsim} E & \text{if } E \cap \Omega_C \neq \varnothing \\ best_{\succsim} \Omega \cup best_{\succsim} E & \text{if } E \cap \Omega_C = \varnothing \text{ and } E \cap \Omega_A \neq \varnothing \\ best_{\succsim} \Omega & \text{if } E \subseteq \Omega_R. \end{cases} \tag{47}$$

We want to show that C is supplemented AGM consistent (Definition 4.5). Let $\langle \Omega_C, \mathcal{F}, h \rangle$ be the simple choice frame described above ($\mathcal{F} = \{E \cap \Omega_C : E \in \mathcal{E}_C\}$ and h is the restriction of f to $\mathcal{F}$). Then, by (47), $\langle \Omega_C, \mathcal{F}, h \rangle$ is rationalized by the total preorder $\succsim_C \overset{def}{=} \succsim \cap (\Omega_C \times \Omega_C)$, so that, by Proposition 7 in Bonanno (2009), there exists a supplemented AGM function B_K^{*C} that extends the function $B_{K,\Psi_{\mathcal{F}}}$ defined above ($B_{K,\Psi_{\mathcal{F}}}(\phi) = \left\{\chi \in \Phi : h(\|\phi\|) \subseteq \|\chi\|\right\}$). Similarly, let $\langle \Omega, \mathcal{G}, g \rangle$ be the other simple choice frame described above ($\mathcal{G} = \{E \cap \Omega_A : E \in \mathcal{E}_A\} \cup \{\Omega\}$ and g is the restriction of f to $\mathcal{G}$). Then, by (47), $\langle \Omega, \mathcal{G}, g \rangle$ is rationalized by the total preorder $\succsim_A \overset{def}{=} \succsim \cap (\Omega_A \times \Omega_A)$, so that, by Proposition 7 in Bonanno (2009), there exists a supplemented AGM function B_K^{*A} that extends the function $B_{K,\Psi_{\mathcal{G}}}$ defined above ($B_{K,\Psi_{\mathcal{G}}}(\phi) = \left\{\chi \in \Phi : g(\|\phi\|) \subseteq \|\chi\|\right\}$). Now define $B_K : \Phi \to 2^{\Phi}$ by

$$B_K(\phi) = \begin{cases} K & \text{if } \phi \in \Phi_R \\ B_K^{*C}(\phi) & \text{if } \phi \in \Phi_C \\ K \cap B_K^{*A}(\phi) & \text{if } \phi \in \Phi_A. \end{cases} \tag{48}$$

We need to show that B_K is an extension of $B_{K,\Psi}$. This is a consequence of the following facts:

1. by Definitions 3.1 and 3.2, if $\|\phi\| \in \mathcal{E}_R$, then $\phi \in \Phi_R$ and $B_{K,\Psi}(\phi) = \{\chi \in \Phi : f(\Omega) \subseteq \|\chi\|\} = K$,
2. B^{*C} is an extension of $B_{K,\Psi_{\mathcal{F}}}$ (recall that, by Definition 3.2, if $\|\phi\| \in \mathcal{E}_C$ then $\phi \in \Phi_C$),
3. B^{*A} is an extension of $B_{K,\Psi_{\mathcal{G}}}$ (recall that, by Definition 3.2, if $\|\phi\| \in \mathcal{E}_A$ then $\phi \in \Phi_A$),
4. by Definition 4.2, if $E \in \mathcal{E}_A$ then $\varnothing \neq E \cap \Omega_A \in \mathcal{E}_A$ and $E \cap \Omega_C = \varnothing$, so that by (47) $f(E) = best_{\gtrsim} \Omega \cup best_{\gtrsim} E = f(\Omega) \cup best_{\gtrsim} E$; thus if $\|\phi\| \in \mathcal{E}_A$ then $f(\|\phi\|) \subseteq \|\chi\|$ if and only if $f(\Omega) \subseteq \|\chi\|$ (that is, $\chi \in K$) and $best_{\gtrsim} E \subseteq \|\chi\|$ (so that $\chi \in B^{*A}_K(\phi)$) and thus $B_{K,\Psi}(\phi) \subseteq K \cap B^{*A}_K(\phi)$. □

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