

Persson, Håkan

Working Paper

A Fixed Point Theorem for Monotone Functions equivalent to Browsers theorem

Working Paper, No. 1/2005

Provided in Cooperation with:

Örebro University School of Business

Suggested Citation: Persson, Håkan (2005) : A Fixed Point Theorem for Monotone Functions equivalent to Browsers theorem, Working Paper, No. 1/2005, Örebro University School of Business, Örebro

This Version is available at:

<https://hdl.handle.net/10419/244413>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

WORKING PAPER SERIES

WORKING PAPER NO 1, 2005



ESI

A Fixed Point Theorem for
Monotone Functions equivalent to
Browers theorem

by

Håkan Persson

<http://www.oru.se/esi/wps>

SE-701 82 Örebro
SWEDEN

ISSN 1403-0586

A Fixed Point Theorem for Monotone Functions equivalent to Brouwer's theorem

Håkan Persson*

June 2003

Abstract

The main theorem of this paper states that if $f : R_+^m \rightarrow R_+^m$ is increasing and continuous and the set $S = \{x \geq 0 : f(x) \leq x\}$ is bounded and contains some $x' > 0$ then there is a non-zero fixed point of f . If in addition to this there is $x'' \in S$, $x'' < x'$ there are multiple fixed points. The main theorem is equivalent to the Brouwer fixed point theorem.

Keywords: Increasing function, Fixed point

In this paper we study the existence of fixed points of continuous increasing functions defined on finite dimensional spaces. A function $f : A \rightarrow R^m$, where $A \subset R^m$, is increasing or monotone if for x' and x in A ,

$$x' \geq x \Rightarrow f(x') \geq f(x). \quad (1)$$

The main result of this paper states that if $f : R_+^m \rightarrow R_+^m$ is increasing and continuous and the set $S = \{x \geq 0 : f(x) \leq x\}$ is bounded and contains some $x' > 0$ then there is a non-zero fixed point x of f , $x = f(x) \neq 0$.

The theorem was proved, using degree theory, and applied to economic models in [1]. However, in this paper the proof uses the Knaster-Kuratowski-Mazurkiewicz lemma (KKM) (see e.g. [2]), which is equivalent to the Brouwer fixed point theorem. A corollary to the theorem is added in which we obtain the Brouwer fixed point theorem. Thus, our

*I have benefitted from comments by Hans Keiding. The address of the author is Department of Economics, University of Örebro, S-701 82 Örebro, Sweden. hakan.persson@esi.oru.se

theorem is equivalent to Brouwer and this shows the “non-simplicity” of our theorem, and its possible use in other applications.

Furthermore, if there are x'' and x' , both in S such that $x'' < x'$, then there are at least two fixed points of f .

Theorem 1 *Assume that $f : R_+^m \rightarrow R_+^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x \geq 0 : f(x) \leq x\}$. If S is bounded, and if there is $x' > 0, x' \in S$, then there is $x \geq 0, x \neq 0$, such that $x = f(x)$.*

Proof: For each $x' \in S, x' > 0$, f takes $\{x \geq 0 : x \leq x'\}$ to itself. Let $M = \text{cl}\{x \geq 0 : x' \in S, x' > 0, x \leq x'\}$, then f takes M to M . To see this, assume that there is $x \in M$ such that $f(x) \in \sim M$ which implies that there is $\epsilon > 0$ such that $|f(x) - y| > \epsilon$ for all $y \in M$. For each $\delta > 0$ there is $x'' \in M$ such that $|x - x''| < \delta, x'' \leq x', x' \in M$ and thus $f(x'') \leq f(x') \leq x'$ and $f(x'') \in M$ which thus contradicts the continuity of f .

Let ∂M to be the boundary of M relative to R_+^m and let $\Delta = \{x \in R_+^m : \sum_i x_i = 1\}$ is the unit $(m - 1)$ -simplex. For $x \in \Delta$ we have that there is $\alpha > 0$ such that $\alpha x \leq x'$ for some $x' \in S$. There is also $\bar{\alpha} = \max \alpha$ for $\alpha x \in M$ since M is compact. Furthermore, $\bar{\alpha}x \in \partial M$ and $\alpha x \notin \partial M$ for $\alpha \neq \bar{\alpha}$. Then the function $h : \Delta \rightarrow \partial M$

$$h(x) = \left\{ y \in \partial M : y \left(\sum_i y_i \right)^{-1} = x \right\}$$

is well defined.

h is also continuous. Assume that h is not continuous at $x \in \Delta$. At $x, y = kx$ for some constant $k > 0$. Then, there is $\epsilon > 0$ such that for each $\delta > 0$ there is $x' \in \Delta, x' > 0$ such that $|x - x'| < \delta$ and $|y - y'| > \epsilon$. Let $y' = k'x', k' > 0$. Then, $|x - x'| = |y/k - y'/k'| < \delta$ and since $|y - y'| > \epsilon$ we must have $y \geq y'$ or $y \leq y'$ with $y_i = y'_i$ for some i , only if $y_i = y'_i = 0$. Clearly, y and y' can not both be in ∂M which contradicts the definition of h .

The sets $C_i \subset \Delta$ defined by

$$C_i = \{x \in \Delta : f_i(h(x)) \geq h_i(x)\}, \quad i = 1, \dots, m$$

is a closed covering of Δ . If $x \notin C_i$ for all i , then $f(h(x)) < h(x)$ contradicting the definition of ∂M . For each C_i we have that $C_i \supset \Delta^i = \{x \in \Delta : x_i = 0\}$. By the KKM lemma there is $x \in \cap_i C_i$, and then $f(y) \geq y, y = h(x)$. Continuity of f and the definition of M then yield $y = f(y)$. Q.E.D.

Corollary 2 *The Brouwer Fixed Point Theorem. Let $f : \Delta \rightarrow \Delta$ be continuous, where $\Delta = \{x \in R_+^m : \sum_i x_i = 1\}$ is the unit $(m - 1)$ -simplex. Then, there is $x \in \Delta$ such that $x = f(x)$.*

Proof: Define $g : R_+^m \rightarrow R_+^m$ by

$$g_i(x) = \begin{cases} \inf\{f_i(x') : x' \in \Delta, x \leq x'\} & \text{if } \sum_j x_j \leq 1 \\ \sup\{f_i(x') : x' \in \Delta, x \geq x'\} & \text{if } \sum_j x_j \geq 1 \end{cases} \quad (2)$$

Then, g is continuous and monotone. Finally, define $h : R_+^m \rightarrow R_+^m$ by

$$h(x) = \left(\sum_j x_j \right)^\alpha g(x)$$

which implies that h satisfies the conditions of Theorem 1 for $\alpha > 1$ sufficiently large. Then there is $\bar{x} \neq 0$ such that $\bar{x} = h(\bar{x})$ or

$$\bar{x} = \left(\sum_j \bar{x}_j \right)^\alpha g(\bar{x})$$

Assume next that $\sum_j \bar{x}_j > 1$. From the definition of g we have that $g(\bar{x}) \geq f(x')$ for some $x' \in \Delta$ and thus $\sum_i g_i(\bar{x}) \geq 1$ and

$$\sum_i \bar{x}_i < \left(\sum_j \bar{x}_j \right)^\alpha \sum_i g_i(\bar{x})$$

which contradicts \bar{x} being a fixed point. Thus, $0 < \sum_j \bar{x}_j \leq 1$. Due to this we find that $\bar{x} \leq g(\bar{x})$. $\Delta' = \{x \geq \bar{x} : \sum_i x_i = 1\}$ is a proper subsimplex of Δ . For $x \in \Delta'$ and the monotonicity of g we have that $g(x) \geq g(\bar{x})$. Furthermore, from the definition of g we have $f(x) \geq g(x)$. Combining all inequalities we find that $f(x) \geq \bar{x}$. Thus,

$$f(\Delta') \subset \Delta'$$

Define X to be the set of vectors of all monotone sequences $\{x^n\}$, $x^{n+1} \geq x^n$ such that $x^0 \geq \bar{x}$ and $\sum_i x_i^n \leq 1$. Specifically, the elements of the sequence defined by $x^0 = \bar{x}$, $x^{n+1} = g(x^n)$ is in this set, and since g is continuous, this sequence converges to some $x^* = g(x^*)$. From Zorn's lemma (see e.g. [3]) it follows that there is $y \in X$ such that y is greater than all elements of X . Clearly, $y \in \Delta'$ and thus $f(y) \in \Delta'$ and from the definition of g it follows that $x^* \leq g(y) = f(y)$ and hence, $f(y) \in X$, and $f(y) \leq y$ which implies that $y = f(y)$. Q.E.D.

The following proposition deals with functions which map the whole of R^m into itself. The result is a consequence of Tarski's fixed point theorem (see e.g. [4]) and it is easy to prove directly as in [5].

Proposition 3 Assume that $f : R^m \rightarrow R^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x : f(x) \leq x\}$ be non-empty and bounded from below. Then, for each $x'' \in S$, there is $\bar{x} \leq x''$ such that $\bar{x} = f(\bar{x})$.

We can now combine the results of these propositions to obtain conditions for the existence of more than one fixed point.

Proposition 4 Assume that $f : R^m \rightarrow R^m$ is continuous and satisfies $x' \geq x \Rightarrow f(x') \geq f(x)$. Let $S = \{x \geq 0 : f(x) \leq x\}$. Assume that S is bounded and contains two points x' and x'' such that $x'' < x'$. Then, there are two fixed points, \bar{x} and \tilde{x} , of f for which $\bar{x} \leq \tilde{x}$ ($\bar{x} \neq \tilde{x}$).

Proof: The fixed point \bar{x} is given by a direct application of Proposition 3. Next, set $z = x' - \bar{x}$ which yields $z' = x' - \bar{x} > 0$. Define $g(z) = f(x) - \bar{x}$. Due to monotonicity of f , $g(z)$ maps R_+^m into itself and by assumption the set $S = \{z \geq 0 : g(z) \leq z\}$ contains $z' > 0$. Theorem 1 can now be applied and we thus have a fixed point $\tilde{z} = g(\tilde{z}) \geq 0$, $\tilde{z} \neq 0$, i.e. $\tilde{x} - \bar{x} = f(\tilde{x}) - \bar{x} \geq 0$, $\tilde{x} \neq \bar{x}$. Q.E.D.

References

- [1] Håkan Persson, *Theory and Applications of Multisectoral Growth Models*, Ph.D. Thesis, University of Gothenburg, 1983
- [2] Ross M. Starr, *General Equilibrium Theory : An Introduction*, Cambridge University Press, Cambridge, 1997
- [3] J.L. Kelly, *General Topology*, Van Nostrand, New York, 1955
- [4] G. Birkhoff, *Lattice Theory*, American Mathematical Society, Providence, R.I., 1967
- [5] S. Lahiri, "Input-Output Analysis with Scale-Dependent Coefficients", *Econometrica*, **44**, 1976, 947-961.