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## Article

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# Numerical study of hybrid third order compact scheme for hyperbolic conservation laws

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## Abstract

In this paper, we present a numerical study to study the capability of the radius of curvature to detect the discontinuous point for hybrid high order schemes. This hybrid scheme primarily developed to reduce the computational time, but preserve the numerical accuracy. WENO-PZ3 is adopted to construct the physical flux in a small radius, and compact third-order TVD scheme otherwise. The numerical scheme is tested in quasi-one-dimensional Sod's shock tube problem. The results show that the hybrid schemes deliver 0.76 computational time ratios compared to the full WENO-PZ3 scheme and the numerical accuracy is maintained. We found that the radius of curvature is capable of detecting the discontinuous point, and the suitable radius of curvature to be categorized as the small radius is less than 0.025.

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**Keywords:** WENO; Hybrid numerical scheme; Hyperbolic conservation laws; Radius of curvature

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## 1. Introduction

The development of numerical schemes for hyperbolic conservation laws is one of the most challenging research since the solutions could contain very complex features that can induce non-physical phenomena such as oscillation near discontinuity, which can spread along with the solutions. However, numerical schemes should be able to encompass the complex feature of the solutions. In this situation, a high order scheme became a more efficient scheme to be implemented compared to the low order scheme, as mentioned by Jiang and Shu [1]. The most popular high order scheme i.e. weighted essentially non-oscillatory (WENO) scheme that originally introduced by Liu et al. [2], has a robust approach to handle the troublesome situation. There are many improved WENO-type schemes, e.g. WENO-M by Henrick et al. [3] and WENO-Z by Borges et al. [4] which improve the original WENO-JS scheme in [1]. The third order WENO-Z type scheme itself has an issue such as the accuracy near the critical point. This issue is fixed by Xu and Wu [5] which modify the WENO-Z3's reference smoothness indicator,

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and this scheme known as WENO-PZ3. The WENO-type schemes known as expensive numerical schemes since they need to compute the smoothness indicator in their reconstruction procedure. Some of the improvement to reduce the computational cost is using WENO-type scheme on appropriate region i.e. discontinuous region and low computational cost scheme for smooth region, e.g. in Guo and Jung [6].

In order to develop a hybrid scheme, the detection where discontinuity occurs is needed and the precision of discontinuity detection will lead to improving the scheme. In this paper, we study the capability of the radius of curvature to decide the proper numerical scheme. The hybrid scheme adopts the WENO-PZ3 scheme for the discontinuous points as the scheme has higher accuracy in the critical points and compact third-order limiter otherwise. We perform the numerical test in the compressible gas dynamics problem and we provide the detailed numerical experiments in the numerical result.

## 2. WENO-PZ3 scheme

As stated in the previous section, the WENO-PZ3 scheme modifies the classical WENO-Z reference smoothness indicator to improve the accuracy in the critical point, the improvement is done by Xu and Wu [5]. The improvement simply power the denominator and found the power parameter should set to 3/4 as the result in Xu and Wu [5] which gives good accuracy and improve the convergence order in the critical point.

$$\alpha_k = d_k \left( 1 + \frac{\tau_Z}{(\beta_k + \varepsilon)} \right), \quad \varepsilon = 10^{-40} \quad (1)$$

$$\tau_Z = |\beta_1 - \beta_2| \quad (2)$$

The smoothness indicator  $\beta_k$  is determined the same as the WENO-JS formulation. Reconstruction procedure also the same as the classical WENO-Z scheme and for conciseness, we will not explain the detailed reconstruction here.

## 3. Compact third-order scheme

Original compact third-order scheme ( $H_{CT}$ ) by Čada and Torrilhon [7] is the simple min–max function of the limiter-free double logarithmic reconstruction (LDLR) that introduced by Artebrant and Schroll [8]. This simplification has an effect on the computational cost, for comparison, LDL reconstruction has computational cost nearly equal as third-order ENO scheme as mentioned by Artebrant and Schroll [8]. In the next development, corrected  $H_{CT}$  reconstruction from the asymmetric diagonal issue is done by Schmidtman et al. [9] known by  $H_{3L}$  reconstruction. The WENO-PZ3 scheme would be adopted for critical point and for the other condition, an improved compact third-order scheme ( $H_{3L}$ ) is adopted.

$H_{3L}$  has a smoothness indicator to choose whether to use the full third-order scheme or third-order limited scheme. In this section, we will briefly explain the  $H_{3L}$  reconstruction, but dropping its  $H_{3L}$ 's smoothness indicator.

$$H_{3L}(\delta_{i-\frac{1}{2}}, \delta_{i+\frac{1}{2}}) = \text{sgn}(\delta_{i+\frac{1}{2}}) \max(0, \min(\text{sgn}(\delta_{i+\frac{1}{2}}) H_3, \max(-\text{sgn}(\delta_{i+\frac{1}{2}}) \delta_{i+\frac{1}{2}}, \min(2\text{sgn}(\delta_{i+\frac{1}{2}}) \delta_{i+\frac{1}{2}}, \text{sgn}(\delta_{i+\frac{1}{2}}) H_3, 1.5 |\delta_{i+\frac{1}{2}}|))) \quad (3)$$

where

$$\delta_{i-\frac{1}{2}} = \bar{q}_i - \bar{q}_{i-1} \quad (4)$$

$$\delta_{i+\frac{1}{2}} = \bar{q}_{i+1} - \bar{q}_i \quad (5)$$

$$H_3(\delta_{i-\frac{1}{2}}, \delta_{i+\frac{1}{2}}) = \frac{2\delta_{i+\frac{1}{2}} + \delta_{i-\frac{1}{2}}}{3} \quad (6)$$

In Schmidtman et al. [9], the improved scheme does not have denominator from the slope  $\delta_{i\pm 1/2}$  that can be an advantage in computation to avoid division by zero.

#### 4. Edge detection via radius of curvature

The radius of curvature is adopted to detect the edge where the WENO-PZ3 scheme would be used in this condition. The formulation of the radius of curvature is presented in (7), the derivation can be found in [10].

$$R = \frac{[1 + (\bar{q}')^2]^{3/2}}{|\bar{q}''|} \quad (7)$$

The expansion of the first and second derivative of  $\bar{q}$  is formulated as follow

$$\bar{q}' = \frac{\bar{q}_{i+1} - \bar{q}_{i-1}}{2\Delta x} \quad (8)$$

$$\bar{q}'' = \frac{\bar{q}_{i-1} - 2\bar{q}_i + \bar{q}_{i+1}}{\Delta x^2} \quad (9)$$

#### 5. Numerical results

In this section, we perform numerical experiments on two-dimensional Euler equations for gas dynamics (10).

$$\frac{\partial \mathbf{q}(x, y, t)}{\partial t} + \nabla \cdot \Phi(\mathbf{q}) = 0 \quad (10)$$

where

$$\Phi(\mathbf{q}) = \mathbf{f}(\mathbf{q})i + \mathbf{g}(\mathbf{q})j \quad (11)$$

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix} \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (\rho e + p)u \end{bmatrix} \quad \mathbf{g}(\mathbf{q}) = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ (\rho e + p)v \end{bmatrix} \quad (12)$$

We use ideal gas relations as the equation of state (fifth equation):

$$\rho e = \rho(u^2 + v^2) + p/(\gamma - 1) \quad (13)$$

$\gamma$  denote a specific heat ratio with value 1.4.

The problem is Sod's shock tube, which is a quasi-one-dimensional problem with initial conditions given by

$$(\rho, u, v, p) = \begin{cases} (1.0, 0, 0, 1.0) & x < 0.5 \\ (0.125, 0, 0, 0.1) & x > 0.5 \end{cases} \quad (14)$$

The test conducted with  $CFL = 0.8$  and  $t_{final} = 0.1$  using different numbers of elements  $N$  to compare numerical error also computational time.

The numerical error in Table 1 indicates that the hybrid scheme preserves the numerical accuracy and the convergence rate for more elements in the domain can be achieved because the WENO-PZ3 scheme is utilized

**Table 1.** Comparison of  $L_1$  (error) for Sod's shock tube problem with initial condition (14) at  $t = 0.1$ .

N	$L_1$ (error)		
	WENO-PZ3	$H_{3L}$	Hybrid
100	1.0468E-02	9.9586E-03	9.9751E-03
200	5.3284E-03	5.2763E-03	5.2969E-03
400	2.7435E-03	2.5578E-03	2.5646E-03
800	1.3443E-03	1.3671E-03	1.3479E-03

**Table 2.** Comparison of computational time for Sod's shock tube problem with initial condition (14) at  $t = 0.1$ .

N	Computational time (s)		
	WENO-PZ3	$H_{3L}$	Hybrid
100	4.62	3.07	3.52
200	18.08	12.05	14.25
400	75.56	48.76	57.15
800	325.43	211.73	247.65

in critical points. As shown in Table 2, the computational time for the hybrid scheme is closer to the  $H_{3L}$  scheme, about 15% higher than the  $H_{3L}$  scheme due to the edge detection and the WENO-PZ3 scheme.

Fig. 1 depicts the three different numerical results for Sod's shock tube problem using  $N = 800$  elements. In this numerical study, the hybrid scheme is resulting nearly the same as the WENO-PZ3 scheme in the critical points. In the near sharp discontinuities, the WENO-PZ3 scheme causing small overshoot and undershoot in the hybrid scheme but for overall  $L_1$  error, the WENO-PZ3 scheme give less error. In Fig. 1, we can relate that the overshoot and undershoot is the inheritance of the WENO-PZ3.

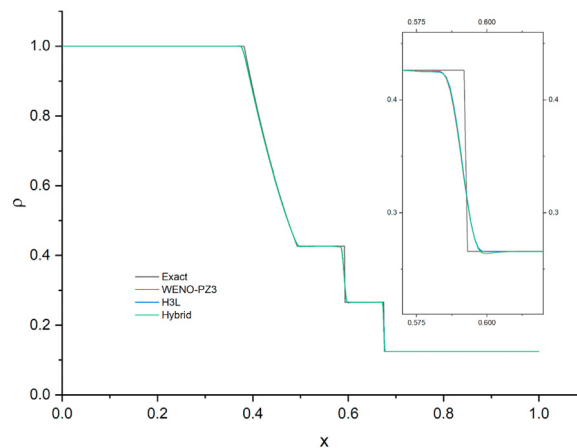
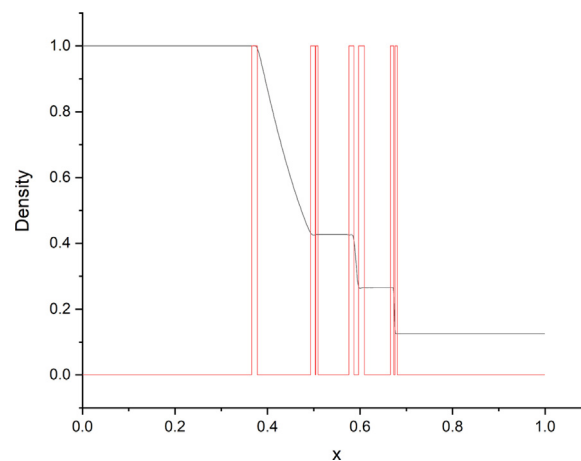
**Fig. 1.** The numerical result of Sod's shock tube problem with initial condition (14) at  $t = 0.1$ .**Fig. 2.** Edge detection using the radius of curvature in Sod's shock tube problem with initial condition (14) at  $t = 0.1$ .

Fig. 2 shows the detected edge of the density variable along  $x$ -direction at  $t = 0.1$  using the radius of curvature with edge radius is set to 0.025. The scheme detects the precise edge location and gives a low number of elements that use high computational cost schemes.

The low number of cells that use a higher computational cost scheme is one advantage of using this method. The numerical accuracy nearly the same as the  $H_{3L}$  scheme and the developed scheme can achieve better accuracy with  $N = 800$  elements. We believe the accuracy still can be improved by choosing a better scheme for the critical points.

## 6. Conclusions

In this paper, we have study the capability radius of curvature to detect edges in a function and implement the method to the hybrid numerical scheme for hyperbolic conservation laws. We found that the radius of curvature is capable to detect the edges (discontinuities) and radius that less than 0.025 is the suitable radius to capture the edge of a function. The accuracy of the hybrid scheme is preserved and the computational time ratio is 0.76 compared to the WENO-PZ3 scheme. The overshoot and undershoot near sharp discontinuities are an inheritance of the WENO-PZ3 scheme and the power parameter for the scheme should be studied further. Moreover, the numerical experiment in this study is very limited.

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