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# Optimism leads to optimality: Ambiguity in network formation\*

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## Abstract

We analyze a model of endogenous two-sided network formation where players are affected by uncertainty in their opponents' decisions. We model this uncertainty using the notion of equilibrium under ambiguity (Eichberger and Kelsey, 2014). Unlike the set of Nash equilibria, the set of equilibria under ambiguity does not always include underconnected and thus inefficient networks such as the empty network. On the other hand, it may include networks with unreciprocated, one-way links, which comes with an efficiency loss as linking efforts are costly. We characterize equilibria under ambiguity and provide conditions under which increased player optimism comes with an increase in efficiency in equilibrium. Next, we analyze the dynamic situation with one-sided, myopic updating with regular optimistic shocks and derive a global stability condition of benefit-maximizing equilibrium networks.

## 1 Introduction

The question of how social networks form and which structures are both stable and efficient is of primary interest in economic literature. The use of classic equilibrium

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concepts such as the Nash equilibrium, however, invites a problem of multiplicity: the set of Nash equilibria encompasses both Pareto-optimal, as well as Pareto-dominated networks such as the empty network. Different refinements such as pairwise stability, (Jackson and Wolinsky, 1996), and strong stability (Jackson and van den Nouweland, 2005), have been considered in the literature, but the problem of equilibrium selection and conflict between efficiency and stability in case of pairwise stability, and existence in case of strong stability persists.

We highlight these issues in models of endogenous network formation where an investing player bears the cost even if her intended partner does not reciprocate (Bala and Goyal, 2000), but link formation is two-sided, hence consent (and investment) is required from both participants. The potential benefits of a link, as in Bloch and Dutta (2009); Ding (2019), are exogenous. Examples of such networks abide ranging from friendships, scientific collaborations and provider-client relationships. As usual, strategic complementarities give rise to multiple equilibria. Most crucially, the Nash equilibria of this game permit underconnected networks as, for any two individuals  $i$  and  $j$ ,  $i$  not linking to  $j$  is always a best response to the belief that  $j$  does not link to  $i$ , no matter how attractive the players are to each other. Resolving this multiplicity in favor of more efficient equilibria has been a long-standing issue in the economic literature on networks. As first suggested by Harsanyi and Selten (1988), what is at hand is the players' reaction to uncertainty in their opponents' strategies – their inability to uniquely predict the behavior of their potential partners.

Social psychology has offered some insight into the problem by noting that optimistic people might be more likely to form and sustain links which are beneficial to them (Brissette et al., 2002; Andersson, 2012). The “tendency to expect favorable outcomes” (Srivastava and Angelo, 2009) results in optimists attaching more weight to their partners reciprocating their efforts towards establishing a relationship. Such expectations may become self-fulfilling as opponents reciprocate these linking offers even when they themselves are not optimistic. In general, optimism is identified as a pervasive individual feature (Peterson, 2000), which is positive both for individuals and the society as a whole (Tiger, 1979; Freud, 1928).

Experimental studies have shown that ambiguity in opponents' strategies is indeed present and leads to deviations of observed behavior from Nash equilibrium (Pulford and Colman, 2007; Di Mauro and Castro, 2008; Eichberger et al., 2008;

Kelsey and Le Roux, 2015, 2017). Furthermore, attitudes towards such ambiguity are heterogeneous and show non-negligible rates of both optimism and pessimism (Camerer and Karjalainen, 1994; Ivanov, 2011; Eichberger et al., 2008; Li et al., 2019). Recent developments in decision theory under ambiguity (Chateauneuf et al., 2007) and games under ambiguity (Eichberger and Kelsey, 2000, 2002, 2014) have offered the possibility to model such optimistic and pessimistic expectations (see Beauchene (2014) for a review).

So far, economic applications of ambiguity have been mainly focused on pessimism. Ellsberg (1961)’s thought experiment indicates preferences for the non-ambiguous (versus the ambiguous) urn. Gilboa and Schmeidler (1989)’s seminal model of max-min expected utility captures an extreme form of aversion against ambiguity, whereas Schmeidler (1989)’s Choquet expected utility, although not restrictive in terms of attitude towards ambiguity, is in general associated with the axiom of ambiguity-aversion and thus, beliefs in the form of a convex capacity. Indeed, ambiguity aversion, or pessimism has been shown to play an important role in economic interactions, as, e.g., in financial markets (Collard et al., 2011) and to be robust to market selection, (Guerdjikova and Sciubba, 2015; Eichberger and Guerdjikova, 2018). However ambiguity aversion is “the exception, not the rule” and ambiguity seeking behavior is common, in particular in situations involving losses and low probabilities of success (Kocher et al., 2018). Notably, optimism can select Pareto-dominant equilibria in coordination games (Eichberger et al., 2008) and leads to an increase in payoffs in the maximal equilibrium in supermodular games (Schipper, 2019).

In this paper, we combine these approaches by introducing optimistic and pessimistic attitudes into a network formation game. Despite the complementarity of linking efforts, the game is not supermodular and thus, equilibrium existence cannot be established by Topkis (1979)’s theorem. Instead, we show that starting from any network, a sequence of unilateral best-responses (under ambiguity) leads to an equilibrium in a finite number of steps. This result provides both an algorithm for computing equilibria under ambiguity in a network formation game and proves that such equilibria always exist.

We next document the insight generated by social psychology that optimists tend to have more links in equilibrium. Optimism increases the weight players place on others reciprocating and thus increases the number of links a player

might optimally initiate. Sufficiently high degrees of optimism imply that links will form in every equilibrium, ruling out (some) underconnected equilibria. Despite this, as illustrated in Section 1.1, increasing optimism need not always increase connectivity: an optimistic player might sever an existing two-way link in favor of a more valuable one, which would not be reciprocated.

We call a network monotonic if for each player, any link she formed is at least as valuable to her as any of the links that she has not formed, i.e., each player's linking decision is a monotonic function of her opponents values. We show that this property is decisive for determining if optimism increases efficiency. Notably, starting from a monotonic equilibrium an increase in optimism leads to a game which has a more connected equilibrium than the initial one and thus, generates a higher value of established links.

In a second step, we address the issue of equilibrium selection. We identify a class of networks, called maximum-weight networks, which are Nash equilibria and at the same time maximize the total value of links in the society. Such networks are both socially desirable and can be decentralized provided that coordination on the specific Nash equilibrium is achieved. This allows us to resolve the conflict between efficiency and stability common for such games (for recent studies see Griffith (2019); Baumann (2019)). If these networks are monotonic, then they are also Pareto-optimal within the equilibrium set. In this case, social desirability aligns with individual incentives to link to one's most preferred opponents, hence we say that preferences are *aligned*.

As is well known from experimental studies, (Cooper et al., 1990; Van Huyk et al., 1990), it is not realistic to expect that coordination on a socially optimal equilibrium obtains in a one-shot game with multiple players. Following the approach of Kandori et al. (1993) (henceforth KMR) and Young (1993), the literature has studied best-response dynamics in network formation games (Watts, 2001) as well as in games on a network with simultaneous choice of links and actions (Ely, 2002; Jackson and Watts, 2002a,b; Goyal and Vega-Redondo, 2005; Hojman and Szeidl, 2006; Galeotti and Goyal, 2010; Dawid and Hellman, 2014), see Hellman and Staudigl (2012) for a review. In general, neither uniqueness of the stochastically stable state, nor its efficiency are guaranteed (see, however, Staudigl (2011); Luo et al. (2018)). Furthermore, the techniques developed by KMR present computational challenges when the set of equilibria is large as in most network formation

games.

In this paper, we study the best-response dynamic of the network formation game and consider the long-run outcome. As in KMR, we assume that players undergo occasional mutations during this process, but rather than making random mistakes in their strategies as is typically assumed, players experience surges of optimism, which leads to a change in their best-response correspondence. Baillon et al. (2013) shows that increases in optimism could be a consequence of emotions such as sadness, joy and fear. Thus, ambiguity attitude might indeed be subject to random shocks. A surge of optimism generates higher expectations that the player's partners will reciprocate a link and leads the player to more actively pursue high-value partnerships. Notably, an optimistic shock leads to a destruction of an existing link only if a link of a greater value is simultaneously initiated.

If all maximum-weight networks are monotonic, the Markov process defined by our best-response dynamic subject to optimistic shocks has an absorbing set – the set of directed networks whose underlying undirected graph is a maximum-weight network. This set is almost surely reached in finite time, regardless of the initial condition, thus maximizing the total value of links formed. When the probability of shocks is small, the fraction of time in the limit that the process spends in an undirected maximum-weight network (and thus, in a Pareto optimal equilibrium) is close to 1. In the long-run, players behave as if they are maximizing the total value of links in the society.

The rest of the paper is organized as follows: We present an illustrative example in Section 1.1. Section 2 introduces the network formation game, specifies the notion of a maximum-weight network and defines the equilibrium under ambiguity concept. Section 3 is devoted to the analysis of the static network formation game. We characterize the set of equilibrium networks without ambiguity and show that when preferences are aligned, maximum-weight networks are Pareto-optimal. In a second step, we extend the analysis to games with ambiguity, characterize the set of equilibrium networks under ambiguity, show existence and discuss comparative statics with respect to the players' attitude towards ambiguity. In Section 4, we turn towards a dynamic analysis, introduce a best-response dynamic with optimistic shocks and analyze its long-term behavior. We demonstrate that for aligned preferences, as well as for linear cost, the dynamic almost surely reaches the set of maximum-weight networks in finite time. We use examples to illustrate

the failure of this result when preferences are not aligned. Section 5 concludes. This can lead to the formation of reciprocal links, which are beneficial to both parties and which persist even after the player's expectations return to being realistic.

### 1.1 An illustrative example

We illustrate the best-response dynamics with surges of optimism using a matching example of four players:  $A$ ,  $B$ ,  $C$ ,  $D$ .  $A$  is the preferred choice of  $B$  and vice-versa, whereas  $C$  and  $D$  are each other's preferred choices. This is represented as follows: the match between  $A$  and  $B$  as well as the one between  $C$  and  $D$  create a high value  $H > 0$ , while all other matches create a low value  $L \in (0, H)$ . Each player is limited to have at most one link. The only efficient network is thus given by  $\{\{A, B\}, \{C, D\}\}$  and in this network each player is matched with their most preferred partner. Thus, the maximum-weight equilibrium network is monotonic. Yet, other complete matchings, such as  $\{\{A, D\}, \{B, C\}\}$ , are also pairwise stable. Starting from this network suppose that  $A$  receives an optimistic shock, which leads her to believe that  $B$  would be willing to replace her current match  $C$  out with her. She then severs her link to  $D$  and offers a link to  $B$ . If  $B$  gets a chance to respond next (whether as a realist, or as an optimist) she will accept the link with  $A$ , severing the link with  $C$  and thus forming the link  $\{A, B\}$ . Another optimistic shock to  $D$  will result in her offering a link to  $C$  which will be accepted, thus establishing the efficient network. In this special case, the efficient network is an absorbing state, which will persist regardless of any further optimistic shocks.

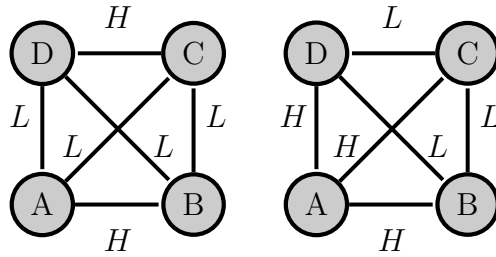


Figure 1: The weight network of the illustrative example. The left side shows a network with aligned preferences, the right side shows a network with non-aligned preferences.

We thus conclude that a dynamics based on optimistic surges can act as a refinement on the set of Nash equilibria by selecting those that maximize the total value of formed links. This, however, depends on the monotonicity of the

maximum-weight network. Otherwise, a surge of optimism might destroy existing links without the possibility of creating new ones. Consider, e.g., the situation in which every player prefers to be matched with  $A$  (all links with  $A$  offer value  $H$ , all other links offer value  $L$ ).  $\{\{A, B\}, \{C, D\}\}$  is again a maximum-weight network but no longer monotonic as both  $C$  and  $D$  are matched with someone other than their preferred match. Starting from this network, player  $C$ , under an optimistic shock, will sever her link with  $D$  and instead try to link to  $A$ , who, being indifferent may or may not reciprocate, cutting her link towards  $B$  if she does so. In turn,  $D$  will sever her now unreciprocated link towards  $C$ . Thus the link  $\{B, D\}$ , once dissolved, will never be reestablished. By symmetry we can conclude that any low-value link, once dissolved, will not be reestablished, thus no maximum-weight network is stable in this network. Optimism can thus prevent players from preserving a second-best link, even when their first-best match is not achievable.

Our matching example is rather special in that each player can only form a single link (i.e., the cost function is extremely convex) and thus, the condition of aligned preferences imposes a rather strict restriction on the individual rankings of matching partners. It is however well suited to provide an intuition for (i) how optimistic shocks can lead to the creation of new and more valuable links whenever preferences are aligned, and (ii) how optimism .

## 2 The network formation game

### 2.1 Notation and definitions

Let  $n > 1$  and let  $I = \{1, \dots, n\}$  be the finite set of players. A network on these players is represented by a binary  $n \times n$  matrix  $x$ . For a disjoint pair  $i, j \in I$  if  $x_{ij} = 1$ , then we say that  $i$  has a directed link towards  $j$ , or  $i$  links to  $j$ . If  $x_{ij} = x_{ji} = 1$ , we say that  $i$  and  $j$  are reciprocally linked to each other. As a shorthand we write  $\{i, j\} \in x$  to mean  $x_{ij} = x_{ji} = 1$  and  $\{i, j\} \notin x$  to mean  $x_{ij} = x_{ji} = 0$ . By convention, for every  $i \in I$  we set the values  $x_{ii}$  to 0. Let the set of networks of  $n$  players be denoted by  $X$ .

The network  $x \in X$  is undirected if for every disjoint  $i, j \in I$  it holds that  $x_{ij} = x_{ji}$ . Note that any such network can be identified by the set of all established links.



**Definition 2.1** (Neighborhoods and degrees in directed networks). For  $i \in I$  we let  $N_i^+(x) = \{j \in I : x_{ij} = 1\}$ ,  $N_i^-(x) = \{j \in I : x_{ji} = 1\}$  denote the set of players who player  $i$  links to and the set of players who link to player  $i$  in network  $x$ , respectively. The set  $N_i(x) = N_i^+(x) \cup N_i^-(x)$  is called the *neighborhood* of player  $i$ .

For  $i \in I$  let  $\deg_i^+(x) = |N_i^+(x)|$ ,  $\deg_i^-(x) = |N_i^-(x)|$ , and  $\deg_i(x) = |N_i(x)|$  denote player  $i$ 's *out-degree*, *in-degree*, and *degree*, respectively.

It is clear that for undirected networks, the three neighbor sets and degrees coincide.

**Definition 2.2** (Maximal undirected network contained in a network). Let the map  $y: X \rightarrow Y$  be defined as follows: for every disjoint  $i, j \in I$ ,  $y(x)_{ij} = y(x)_{ji} = x_{ij}x_{ji}$ . The network  $y(x)$  is the *maximal undirected network* contained in  $x$ .

For  $X' \subseteq X$  we let  $y(X')$  denote the image of  $X'$ . We introduce  $Y = y(X)$  to denote the set of undirected networks. For  $Y' \subseteq Y$  we let  $y^{-1}(Y')$  denote the pre-image of  $Y'$ , i.e. the set of directed networks, whose maximal undirected network is a member of  $Y'$ .

For an undirected network  $y \in Y$  let  $|y| = \sum_{i \in I} \deg_i(y)/2$  denote the number of links in  $y$ .

**Definition 2.3.** For two networks  $x, x'$  we say that  $x'$  is *more connected* than  $x$ , denoted by  $x \leq x'$  if for every  $i, j \in I$  we have  $x_{ij} \leq x'_{ij}$ .

As typical, we write  $x < x'$  to denote the asymmetric part of the relation  $x \leq x'$ .

For undirected networks  $x, x' \in Y$ , the union and set subtraction operators  $\cup$  and  $\setminus$  are interpreted the same way as for sets. Specifically, the network  $(x \setminus x') \cup (x' \setminus x)$ , containing all links that appear in exactly one of  $x$  and  $x'$  is called the symmetric difference between  $x$  and  $x'$ .

**Definition 2.4** (Distance between undirected networks). For two undirected networks  $x, x' \in Y$  the value  $\phi(x, x') = |(x \setminus x') \cup (x' \setminus x)|$  is called the *distance* between  $x$  and  $x'$ .

## 2.2 Game definition

We now define a game of network formation with additive benefits gained from direct interactions. Let  $W$  be an  $n \times n$  non-negative, symmetric, real matrix with

the convention  $w_{ii} = 0$  for  $i \in I$ , called the matrix of benefits. Let  $c_i: \mathbb{N} \rightarrow \mathbb{R}_+$  be a strictly monotonically increasing, weakly convex function, such that  $c_i(0) = 0$ , called player  $i$ 's cost function. A strategy of player  $i$  is a binary  $n$ -vector  $x_i = (x_{ij})_{j \in I}$  with  $x_{ij} = 1$  representing the choice by  $i$  to link to  $j$ , and  $x_{ij} = 0$  representing the choice not to link. By convention we take  $x_{ii} = 0$  for all  $i \in I$ . The aggregate of the players' decisions  $(x_i)_{i \in I}$  forms the directed network  $x$ . Let  $X_i$  denote player  $i$ 's set of strategies.

The payoff of player  $i \in I$  is given as follows.

$$u_i(x) = \sum_{j \neq i} w_{ij} x_{ij} x_{ji} - c_i(\deg_i^+(x)). \quad (1)$$

The interpretation is the following. The players may form costly links with other players to receive the benefit associated with that link as given by the weight matrix. Benefits of links are only realized by the participants if both decide to link to one another. If only one decides to link to the other, a one-sided link is formed without any benefits to either player, if both decide not to connect, no link is formed and no benefits are realized. Crucially, players incur costs of all outgoing links even ones that are not reciprocated.

Our game therefore bears close resemblance to Myerson (1991)'s independent link-formation process in which players independently make a list of their opponents with whom they wish to link and a mutual link ends up forming if and only if both participants name each other. In our game players realize benefits from direct neighbors in the resultant network but pay costs for the number of opponents added to their list.

It is useful to introduce the following definition for marginal links.

**Definition 2.5** (Marginal benefits, marginal costs). Let a network  $x \in X$  be given. Then, player  $i$ 's *benefit on her marginal link* is  $w'_i(x) = \min_{j \in N_i^+(x)} \{w_{ij}\}$ . For  $r \in \mathbb{N}$ , player  $i$ 's *marginal cost* function is given by  $c'_i(r) = c_i(r) - c_i(r-1)$  if  $r$  is positive and  $c'(0) = 0$ .

By convention we have  $\min \emptyset = \infty$ . Note that, while player  $i$ 's benefit on her marginal link equals the lowest value of her reciprocated link (i.e. her marginal benefit) if  $x$  is undirected, in directed networks we look at outgoing links. In the special case of linear cost functions, we denote constant marginal cost of player  $i$  by  $c_i$ .

We assume for simplicity that marginal costs are never equal to marginal benefits, that is, for every natural integer  $r$  and disjoint pair of players  $i, j \in I$  we have  $w_{ij} \neq c'_i(r)$ . We make this assumption to simplify characterizations by avoiding indifference, but it may also be interpreted as the players always preferring to connect (or not to connect) in case they are indifferent between linking to a player or not. Within the set of all possible matrices  $w$  and cost functions  $c$ , this property is generic.

### 2.3 Stable and efficient Nash equilibria

We now define the equilibrium concepts

For a player  $i \in I$ , let  $X_{-i} = \{0, 1\}^{(n-1) \times (n-1)}$  be the set of  $i$ -incomplete strategy combinations. As usual, the best response of a player  $i$  to the strategy combination chosen by her opponents is given by  $B_i(x_{-i})$  defined by:

$$B_i(x_{-i}) = \arg \max_{x_i \in \{0, 1\}} \left( \sum_{j \neq i} w_{ij} x_{ij} x_{ji} - c_i(\deg_i^+(x)) \right)$$

As is usual in the literature, we write  $B_i(x)$ , whenever appropriate.

**Definition 2.6.** A network  $x$  is a Nash equilibrium of the game of network formation iff  $x_i \in B_i(x)$  for all  $i \in I$ .

Let the set of Nash equilibrium networks be denoted by  $X^*$ . It is clear that, since linking to a player without reciprocation is costly without giving any immediate benefits, every Nash equilibrium is undirected. Hence, although in principle, we allow for the formation of undirected networks, as we will show below, such networks will not constitute a Nash equilibrium. However, they might still obtain as equilibria under ambiguity.

In general, network formation games have a multiplicity of Nash equilibria. Jackson and Wolinsky (1996) thus proposes a refinement called pairwise stability.

**Definition 2.7** (Pairwise stable equilibria). A Nash equilibrium network  $x$  is *pairwise stable* if for any two distinct players  $i, j \in I$   $x_{ij} = 0$  implies  $u_i(x \cup \{i, j\}) < u_i(x)$  or  $u_j(x \cup \{i, j\}) < u_j(x)$ .

Pairwise stable networks have two properties: no player should be better off by the deletion of an existing link, and no two unlinked players should be better off

by forming a reciprocated link between them. The first requirement is embedded in the definition of the Nash equilibrium, as severing a link is a unilateral decision, while the second distinguishes the notion of pairwise stability from that of a Nash equilibrium. Indeed, even for mutually profitable links,  $x_{ij} = x_{ji} = 0$  can be a part of a Nash equilibrium network.

One of the most common questions in the economics of network formation is whether efficient networks can form. In this paper we discuss two notions of efficiency, Pareto optimality, and utilitarian efficiency in realized benefits.

**Definition 2.8** (Pareto optimal network). A network  $x$  is *Pareto optimal* if there does not exist a network  $x'$  such that  $u_i(x') \geq u_i(x)$  for all  $i \in I$  with at least one strict inequality.

Clearly, a Pareto optimal Nash equilibrium network has to be pairwise stable, but not every pairwise stable equilibrium is Pareto optimal.

To allow for selection within Pareto optimal equilibria, we define efficiency notion. For a network  $x$ , let  $v(x) = \sum_{i \neq j} w_{ij} x_{ij} x_{ji}$  denote the total sum of realized benefits. It is clear that  $v(x) = v(y(x))$ .

**Definition 2.9** (Maximum-weight equilibrium networks). The network  $x^* \in X^*$  is called a maximum-weight equilibrium network if  $x \in \operatorname{argmax}_{x \in X^*} v(x)$ .

Let the set of maximum-weight equilibria be denoted by  $V^*$ . Such networks maximize total benefits from connections in the society, while ensuring that for each player  $i$ , the value she derives from her marginal link exceeds her cost, and  $i$  chooses her connections optimally given the behavior of the other players. Thus, they satisfy a participation constraint and an incentive constraint for each individual and can thus be, in principle, decentralized.

It is clear that maximum-weight equilibrium networks are also pairwise stable. Efficiency in realized benefits is a very similar notion to utilitarian efficiency but we do not explicitly consider costs – although they do play some role as we select from the set of Nash equilibrium networks. We raise two conceptual reasons for this: (1) we think of costs as player-specific constraints of maintaining social links, as such they are not transferable or easily measurable, (2) planners may be interested in the total weight that can be realized in the network without taking costs into account.

We conclude this section with the observation that if all players' cost functions are linear, the notions of pairwise stability, Pareto-efficiency, and maximum-weight equilibrium coincide in a unique network.

## 2.4 Ambiguity attitudes and equilibrium under ambiguity

The concept of Nash equilibrium presupposes that players have correct beliefs about their opponents' behavior and best-respond to these beliefs. In games with multiple Nash equilibria, such an assumption seems rather strong. While players might indeed entertain beliefs about the behavior of the other players in the game, they might have limited confidence in such beliefs. The theory of ambiguity allows us to model such uncertainty about opponents' strategies formally by the use of non-additive probabilities called capacities.

Let  $\mathcal{X}_{-i}$  denote the set of all subsets of  $X_{-i}$ .

**Definition 2.10** (Schmeidler (1989)). A *capacity* on  $(X_{-i}, \mathcal{X}_{-i})$  is a function  $\nu_i: \mathcal{X}_{-i} \rightarrow [0, 1]$  with  $\nu_i(\emptyset) = 0$ ,  $\nu_i(X_{-i}) = 1$  and  $\nu_i(A) \leq \nu_i(A')$ , whenever  $A \subseteq A'$ .

In words, a capacity is a non-additive measure, which is normalized between 0 and 1 and satisfies monotonicity with respect to set inclusion. In particular, a capacity will in general violate additivity: the measures assigned to an event  $A \subseteq X_{-i}$ ,  $\nu_i(A)$ , and its complement  $X_{-i} \setminus A$ ,  $\nu_i(X_{-i} \setminus A)$  need not sum up to 1.

The notion of a capacity is very general. In this paper, we will use a special class of capacities, called NEO-additive capacities which allow for a distinction between the strength of a players' perception of ambiguity and their attitude to such ambiguity, optimism or pessimism. These characteristics of the players are captured by two parameters,  $\alpha_i \in [0, 1]$ , and  $\delta_i \in [0, 1]$ .

**Definition 2.11** (Chateauneuf et al. (2007)). For a player  $i$ , given her ambiguity parameters  $\alpha_i, \delta_i$ , and a probability measure  $\pi_i$  on  $(X_{-i}, \mathcal{X}_{-i})$ , the *NEO-additive capacity*  $\nu_i(\cdot | \alpha_i, \delta_i, \pi_i)$  on  $(X_{-i}, \mathcal{X}_{-i})$  is defined as follows:

$$\nu_i(A | \alpha_i, \delta_i, \pi_i) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A = X_{-i}, \\ \delta_i \alpha_i + (1 - \delta_i) \pi_i(A) & \text{otherwise.} \end{cases}$$

A NEO-additive capacity can be interpreted as follows: the probability distribution  $\pi_i$  player  $i$ 's belief about the behavior of her opponents. The degree of

confidence of  $i$  in this belief is given by  $(1 - \delta_i)$ . In particular, if  $\delta_i = 0$ , the degree of confidence is 1,  $i$  is not affected by ambiguity, and her subjective beliefs are represented by  $\pi_i$  itself. When the degree of confidence  $(1 - \delta_i) < 1$ , the player perceives ambiguity  $\delta_i$  with respect to her best estimate  $\pi_i$ . Intuitively,  $\delta_i$  identifies a set of probability distributions given by the convex combination of  $\pi_i$  with the set of all possible probability measures on  $X_{-i}$ ,  $\Delta(X_{-i})$ ,

$$\Pi_i = (1 - \delta_i) \{\pi_i\} + \delta_i \Delta(X_{-i})$$

that the player takes into account when deciding on her strategy.

The effect of such ambiguity on  $i$ 's beliefs about an event  $A$  depends on the player's degree of optimism,  $\alpha_i$ . When  $\alpha_i = 0$ , the player is a complete pessimist and her beliefs scale down the probability of any event  $A$  (except for the whole event  $X_{-i}$ ) by her degree of confidence,  $(1 - \delta_i)$ . For a fully optimistic player with  $\alpha_i = 1$ , the probabilities of all events (except the impossible one,  $\emptyset$ ) are overweighted. More generally, for intermediate degrees of optimism,  $i$  will overweight events with low probability and underweight those with high probabilities.

In this paper, we will restrict attention to pure strategies and thus, to capacities for which  $\pi_i$  is a Dirac measure on some  $i$ -incomplete strategy combination  $x_{-i} \in X_{-i}$ ,  $\pi_i(x_{-i}) = 1$ . Slightly abusing notation, we will write  $\nu_i(\cdot | \alpha_i, \delta_i, x_{-i})$  to denote such a capacity.

Given  $i$ 's subjective beliefs expressed by NEO-additive capacity  $\nu_i(\cdot | \alpha_i, \delta_i, x_{-i})$ , her Choquet expected payoff from choosing a strategy  $x_i$  is given by the Choquet integral of the payoff function  $u_i$  with respect to  $\nu_i$ .

**Definition 2.12** (Chateauneuf et al. (2007)). The Choquet expected payoff with respect to a NEO-additive capacity  $\nu_i$  with parameters  $\alpha_i$ ,  $\delta_i$  and  $\pi_i$  such that  $\pi_i(x_{-i}) = 1$  for some  $x_{-i} \in X_{-i}$  is given by:

$$\begin{aligned} U_i(x_i, \nu_i(\cdot | \alpha_i, \delta_i, x_{-i})) &= \int u_i(x_i, \tilde{x}_{-i}) d\nu_i \\ &= \delta_i \left[ \alpha_i \max_{\tilde{x}_{-i} \in X_{-i}} u_i(x_i, \tilde{x}_{-i}) + (1 - \alpha_i) \min_{\tilde{x}_{-i} \in X_{-i}} u_i(x_i, \tilde{x}_{-i}) \right] + (1 - \delta_i) u_i(x_i, x_{-i}) \\ &= \alpha_i \max_{\pi \in (1 - \delta_i)\{\pi_i\} + \delta_i \Delta(X_{-i})} E_\pi[u_i(x_i, \tilde{x}_{-i})] + (1 - \alpha_i) \min_{\pi \in (1 - \delta_i)\{\pi_i\} + \delta_i \Delta(X_{-i})} E_\pi[u_i(x_i, \tilde{x}_{-i})] \end{aligned}$$

In words, player  $i$ 's Choquet expected payoff of strategy  $x_i$  given her subjective belief  $\nu_i$  is the sum of the expected payoff she would obtain under her objective

belief  $\pi_i$  with weight  $1 - \delta_i$ , the highest possible payoff attainable by  $x_i$  with weight  $\delta_i \alpha_i$ , and the lowest payoff attainable by  $x_i$ , with weight  $\delta_i(1 - \alpha_i)$ . There are three extreme cases: for a realistic player  $i$  ( $\delta_i = 0$ ), the Choquet expected payoff of  $x_i$  given  $\nu_i$  coincides with the payoff of  $x_i$  given  $x_{-i}$ , for an optimistic/pessimistic player  $i$  ( $\delta_i = \alpha_i = 1$  and  $\delta_i = 1, \alpha_i = 0$  respectively), the Choquet expected payoff of  $x_i$  ignores the objective belief  $\pi_i$  and equals the largest/smallest payoff available through  $x_i$ .

Alternatively, the Choquet expected payoff can be interpreted as an  $\alpha_i$ -max-min expected utility over the set of probability distributions centered around  $\pi_i$  with a “radius” of  $\delta_i$ .

Notice that in our network formation game  $\max_{x_{-i} \in X_{-i}} u_i(x_i, x_{-i})$  is always obtained at  $x_{ji} = 1$  for every  $j \neq i$ , whereas  $\min_{x_{-i} \in X_{-i}} u_i(x_i, x_{-i})$  obtains at  $x_{ji} = 0$  for every  $j \neq i$  and thus,

$$U_i(x_i, \nu_i(\cdot | \alpha_i, \delta_i, x_{-i})) = \sum_{j \neq i} x_{ij} w_{ij} (\delta_i \alpha_i + (1 - \delta_i) x_{ji}) - c_i(\deg_i^+(x)),$$

The best response correspondence of player  $i$  given that her beliefs are represented by a NEO-additive capacity  $\nu_i$  is defined as usual by

$$B_i(\nu_i(\cdot | \alpha_i, \delta_i, x_{-i})) = \arg \max_{x_i \in X_i} U_i(x_i, \nu_i(\cdot | \alpha_i, \delta_i, x_{-i})).$$

As most definitions of equilibrium, the equilibrium under ambiguity imposes consistency between equilibrium beliefs and equilibrium behavior. Differently from the standard Nash equilibrium concept, in our framework, players perceive ambiguity about the behavior of their opponents. Hence, their beliefs will in general assign positive weight to strategies, which are not used in equilibrium. The consistency notion is thus imposed on a subset of strategies assigned positive weights, on those that are in the support of the the objective beliefs,  $\pi_i$ . In the case, in which  $\pi_i$  is a Dirac measure on some  $x_{-i}$ , the support of  $\pi_i$  is given by  $x_{-i}$ , the  $i$ -incomplete strategy combination played by the other players.

We now adapt the equilibrium under ambiguity concept of Eichberger and Kelsey (2014) to our framework as follows:

**Definition 2.13** (Eichberger and Kelsey (2014)). A vector of NEO-additive capacities  $(\nu_i^*)_{i \in I}$  form an *equilibrium under ambiguity* (EUA) if for every  $i \in I$ ,  $x_{-i}^* \in B_{-i}(\nu_{-i}^*)$ .

EUA imposes constraints both on players' strategies and on players' beliefs. In particular, whenever player  $i$ 's best estimate is that strategy combination  $x_{-i}$  will be played by her opponents, this strategy combination is a vector of each of her opponent's best responses according to their respective equilibrium beliefs given by  $\nu_{-i}^*$ . It is easily seen that this condition can be equivalently rewritten as:

$$x_i^* \in B_i(\nu_i(\cdot \mid \alpha_i, \delta_i, x_{-i}^*))$$

for all  $i \in I$ , which illustrates the similarity of the concept to the Nash equilibrium.

### 3 Static results

In this section we characterize static solutions of the network formation game and explore comparative statics between network size and value with respect to player optimism.

#### 3.1 Equilibrium, stability and efficiency without ambiguity

We first introduce the following useful notation for the maximum degree of a player in an equilibrium network. Let

$$d_i = \max_{x \in X} \{\deg_i(x) : x_i \in B_i(x)\} = \{\deg_i(x) : x_i \in B_i(x \mid x_{ji} = 1 \text{ for all } j \neq i)\}$$

denote player  $i$ 's largest degree across all networks where she is at her best response. In particular, this is the degree of  $i$ 's best-response to all other players extending links to  $i$ . In any equilibrium network  $x^* \in X^*$  it is true that for each  $i \in I$  we have  $\deg_i(x^*) \leq d_i$ .

We next characterize a player's best response without ambiguity.

**Lemma 3.1** (Best response characterization). *For  $x \in X$  and  $i \in I$ , and  $x'_i \in X_i$  let  $x' = (x'_i, x_{-i})$ . Then,  $x'_i \in B_i(x)$  if and only if for every  $j \neq i$  we have*

$$x'_{ij} = 1 \Leftrightarrow x_{ji} = 1 \text{ and } w_{ij} > c'_i(\deg_i(x'))$$

*and there do not exist  $j, k$  such that  $x_{ki} = 1$ ,  $x'_{ik} = 0$ ,  $x'_{ij} = 1$  and  $w_{ik} > w_{ij}$ .*

The proof of Lemma 3.1 is postponed as it is implied by a later, more general statement, Lemma 3.12. Note that the best response of a player is weakly increasing in  $x_{-i}$ , thus, the game exhibits strategic complementarities.



We can therefore characterize the set of Nash equilibria.

**Lemma 3.2** (Nash equilibrium characterization). *A network  $x^*$  is a Nash equilibrium if and only if it is undirected and  $w'_i(x^*) > c'_i(\deg_i(x^*))$  for all  $i \in I$ .*

*Proof:* The fact that  $x^*$  is undirected is a clear necessary condition for a Nash equilibrium, while the second property ensures that no player has an incentive to delete any existing link. ■

Next, we characterize pairwise stable equilibria.

**Lemma 3.3** (Pairwise stable equilibrium characterization). *The network  $x^*$  is a pairwise stable equilibrium if and only if  $x^*$  is a Nash equilibrium and for every  $i, j \in I$  for which  $x_{ij}^* = x_{ji}^* = 0$  we have  $w_{ij} < \max\{c'_i(\deg_i(x^*) + 1), c'_j(\deg_j(x^*) + 1)\}$ .*

*Proof:* Only if direction: if  $x^*$  is not a Nash equilibrium then it cannot be pairwise stable by definition, while if there exists a pair  $i, j \in I$  with  $x_{ij}^* = x_{ji}^* = 0$  with  $w_{ij} > \max\{c'_i(\deg_i(x^*) + 1), c'_j(\deg_j(x^*) + 1)\}$ , then both participants would prefer to add the link  $\{i, j\}$ .

If direction: As the only if direction shows, no player has an incentive to add links, while  $x^*$  being a Nash equilibrium ensures that no player has an incentive to sever links either. ■

Pairwise stable networks, as demonstrated by Lemma 3.3, have desirable properties relating to efficiency as all links that do not form are not profitable for at least one participant, thus, grossly underconnected equilibria such as the empty network are ruled out. The hindrance in reaching efficiency by spontaneous pairwise coordination is that, the number of links formed may be far too low compared to efficient networks. For example, if benefits offered by the links are close to each other, players with low  $d_i$  values linking to each other decreases overall efficiency as players with high  $d_i$  values run out of players to link to and thus become unsatiated. This is illustrated by the following example.

**Example 3.4.** Let  $m > 1$  and consider an example of  $2m$  players. Suppose that benefits of all links are equal and assume that there are two groups of players,  $I_1$  and  $I_2$  with  $|I_1| = |I_2| = m$  such that  $d_i = m - 1$  if  $i \in I_1$  and  $d_i = 2m - 1$  if  $i \in I_2$ .

Then, the most efficient pairwise stable equilibrium is the one where every member of  $I_1$  is linked to  $m - 1$  members of  $I_2$  and every member of  $I_2$  is linked to each other. The total number of links formed is then  $1.5m(m - 1)$ . The least efficient pairwise stable equilibrium is the one where every member of  $I_1$  is linked to every other member of  $I_1$  and all members of  $I_2$  are linked to every member of  $I_2$ . The total number of links formed is then  $m(m - 1)$ , so a pairwise stable equilibrium may accommodate an efficiency loss as large as 33%.

We address the question of achieving efficiency through optimistic shocks in section 4. We now move on to the notion of monotonic networks, a central notion in our paper.

**Definition 3.5** (Monotonic network). Network  $x$  is *monotonic in linking decisions* (monotonic, for short) for player  $i$  if for every  $j, k \in I$  such that  $w_{ij} \geq w_{ik}$  we have  $x_{ij} \geq x_{ik}$ . Network  $x$  is monotonic if for every  $i \in I$  it is monotonic for  $i$ .

If  $x$  is monotonic, then all players  $i$  link to some selection of their top  $\deg_i^+(x)$  choices of opponents. If an undirected network is monotonic, a player can only increase the value of her links by forming more links, not by replacing her existing links with better ones.

**Definition 3.6** (Aligned preferences). The players exhibit *aligned preferences* if every maximum-weight equilibrium network  $x^* \in V^*$  is monotonic.

Under aligned preferences, maximum-weight equilibrium networks are not only optimal for the society but also for the individuals; no player could be better off without increasing their number of neighbors, but increasing the number of neighbors in a way that is profitable for all participants is impossible. This is an appealing property for a game to have, as without it we observe a conflict between overall efficiency and the players' incentives, but it is not a general one. Indeed, it may be that none of the pairwise stable equilibria is monotonic. If preferences are not aligned, then conflicts may exist between different efficiency notions; maximum-weight equilibrium networks can be Pareto-dominated by other equilibrium networks, as shown in the following example.

**Example 3.7.** Consider a network of six players shown in Figure 2.

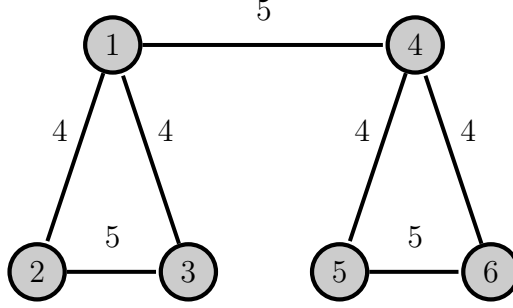


Figure 2: The weight network of example 3.7. Links not shown in the figure are assumed to offer no benefits.

Suppose that the cost structure is such that  $d_1 = d_4 = 2$ , while  $d_2 = d_3 = d_5 = d_6 = 1$ . Then, the only maximum-weight equilibrium network is the one given by  $x^* = \{\{1, 2\}, \{1, 3\}, \{4, 5\}, \{4, 6\}\}$  for a total weight of 16, but it is not monotonic for any player, hence preferences are not aligned.

Suppose that the cost structure of players 1 and 4 is such that they prefer a single link with value 5 to two links with values 4, e.g.  $c_1(1) = c_4(1) = 3.8$  and  $c_1(2) = c_4(2) = 7.7$ . Then, the network given by  $x = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\}$ , with its weight of 15 Pareto-dominates  $x^*$ .

In case preferences are aligned, maximum-weight equilibrium networks are Pareto-optimal, thus these efficiency notions mutually support each other.

**Lemma 3.8.** *Suppose that the players exhibit aligned preferences. Then every  $x^* \in V^*$  is Pareto-optimal in the set of Nash equilibria.*

*Proof:* Take an  $x^* \in V^*$ . For contradiction, assume an  $x \in X^*$  exists that Pareto-dominates  $x^*$ . Suppose that for every  $i \in I$  we have  $\deg_i(x) \geq \deg_i(x^*)$ . Then, for every  $i \in I$  we have  $c_i(x) \geq c_i(x^*)$ , thus, for  $u_i(x) \geq u_i(x^*)$  to hold we must have  $\sum_{j \in N_i(x)} w_{ij} \geq \sum_{j \in N_i(x^*)} w_{ij}$ , for each  $i \in I$  and strict inequality for at least one player, contradicting the weight-maximality of  $x^*$ . Thus, there exists a player  $i$  with  $\deg_i(x) < \deg_i(x^*)$ . However, since  $x^*$  is monotonic,  $u_i(x) \geq u_i(x^*)$  requires  $\deg_i(x) \geq \deg_i(x^*)$ , a contradiction. ■

The reverse of Lemma 3.8 is not true, that is, the existence of an equilibrium network which is monotonic and Pareto-undominated does not imply that preferences are aligned. In Example 3.7 the Pareto-dominant equilibrium network is

monotonic but preferences are not aligned as it is not a maximum-weight equilibrium.

In the special case where the benefits of all links are equal, preferences are always aligned and maximum-weight equilibrium networks are the largest equilibrium networks in the number of links. There we can derive an even stronger statement, every non-maximum-weight equilibrium network is Pareto-dominated by the maximum-weight equilibrium network closest to it.

**Lemma 3.9.** *If for every  $i, j$  we have  $w_{ij} = w$  for some  $w \in \mathbb{R}_+$ , then for every  $x' \in X^* \setminus V^*$  and every  $x \in V^*$  that is closest to  $x'$  in  $V^*$ ,  $x$  Pareto-dominates  $x'$ .*

*Proof:* Let  $x' \in X^* \setminus V^*$ , let  $x$  be an element of  $V^*$  that is closest to  $x'$ , and assume that  $x$  does not Pareto-dominate  $x'$ . Then, there exists  $i \in I$  such that  $\deg_i(x') > \deg_i(x)$ . Hence, there exists a player  $j \in N_i(x') \setminus N_i(x)$ . If  $\deg_j(x) < \deg_j(x')$ , then either  $\deg_j(x) < d_j$  and hence,  $\{i, j\}$  could be added to  $x$ , contradicting  $x \in V^*$ , or we have  $\deg_j(x') > d_j = \deg_j(x)$ , contradicting the fact that  $x'$  is an equilibrium network.

If  $\deg_j(x) > \deg_j(x')$ , then there must exist  $k \in N_j(x) \setminus N_j(x')$ . Consider the network  $x''$  which we get from  $x$  by removing the link  $\{j, k\}$  and adding the link  $\{i, j\}$ . It is clear that  $x'' \in X^*$ , since  $\deg_i(x'') \leq \deg_i(x') \leq d_i$ ,  $\deg_j(x'') = \deg_j(x') = d_j$ , and  $\deg_k(x'') = \deg_k(x) - 1 < d_k$ . Furthermore,  $x'' \in V^*$ , since  $x$  has the same number of links as  $x''$ . However, we have  $\phi(x', x'') = \phi(x', x) - 2$ , thus we contradict the choice of  $x$  as a closest element of  $V^*$  to  $x'$ . Thus,  $x$  must Pareto-dominate  $x'$ .

■

There is no equivalent statement for games with aligned preferences in general as a player  $i$ 's first-best network need not be a maximum-weight equilibrium network and thus it is possible for this player to be worse off in every maximum-weight equilibrium network.

As mentioned, uniform benefits of linking implies that preferences are aligned as the condition that all opponents who offer higher benefits than any neighbor must be neighbors as well is empty. To conclude this subsection, we derive a sufficient condition under which the unique maximum-weight equilibrium is monotonic, which in turn implies that preferences are aligned.

**Lemma 3.10.** *Suppose that*

1. *for every  $i \in I$  there exists a value  $\bar{w}_i$  such that  $|\{k: w_{ik} \geq \bar{w}_i\}| = d_i$ , and*
2. *for every  $i, j \in I$  we have  $w_{ij} \geq \bar{w}_i \Leftrightarrow w_{ij} \geq \bar{w}_j$ .*

*Let  $x^*$  be given as follows:  $x_{ij}^* = 1$  if  $w_{ij} \geq \bar{w}_i$ ,  $x_{ij}^* = 0$  otherwise. Then,  $x^*$  is the unique maximum-weight equilibrium network and it is monotonic.*

*Proof:* It is clear that  $x^*$  is monotonic and, by property (2), it is undirected. Furthermore, by property (1) it holds that for every  $i \in I$  we have  $\deg_i(x^*) = d_i$ . Thus, every player has the maximum number of neighbors and is linked to her best choice of  $d_i$  opponents. Thus,  $x^*$  is the only maximum-weight equilibrium network. ■

Lemma 3.10 lays out two sufficient conditions for preferences to be aligned: (1) each player must have a well-defined set of favorite opponents to link to that will satiate her, and (2) reciprocity in favorite opponents. The first condition precludes, among other settings, the uniform benefits case; even though uniform benefits represent aligned preferences they do not generally imply uniqueness of the maximum-weight equilibrium. If indifferences are not allowed, as is often the case in matching, this condition is automatically satisfied. The second condition ensures reciprocity of links if all players link to their set of favorite opponents.

**Lemma 3.11** (Unique efficient network in the linear cost case). *Let all player's cost functions be linear. Define the network  $x^*$  as follows:  $x_{ij}^* = 1$  if and only if  $w_{ij} > \max\{c_i, c_j\}$ . Then,  $x^*$  is the unique pairwise stable, Pareto efficient, and maximum-weight equilibrium network.*

The proof of Lemma 3.11 is straightforward. The uniqueness of the efficient equilibrium ensures that optimism has very clear, positive effects on efficiency in the linear case.

### 3.2 Equilibrium under ambiguity characterization and comparative statics

Analogously to the case without ambiguity, we begin by a characterization of a player's best response under ambiguity parameters  $\alpha_i$  and  $\delta_i$ . To simplify the analysis and avoid the cases of indifference we will discuss ambiguity parameters  $\delta_i, \alpha_i, i \in I$  such that for every  $i, j \in I$  and every  $r \in \mathbb{N}$  we have  $w_{ij}\delta_i\alpha_i \neq c'_i(r)$  and  $w_{ij}(\delta_i\alpha_i + 1 - \delta_i) \neq c'_i(r)$ , which is a generic setting of parameters.

**Lemma 3.12** (Best response under ambiguity). *For  $x \in X$  and  $i \in I$ , and  $x'_i \in X_i$  let  $x' = (x'_i, x_{-i})$ . Then,  $x'_i \in B_i(\nu_i(\cdot|\alpha_i, \delta_i, x_{-i}))$  if and only if for every  $j \neq i$  we have  $x_{ij} = 1$  if and only if the following two properties hold:*

1.  $w_{ij}(\delta_i\alpha_i + (1 - \delta_i)x_{ji}) > c'_i(\deg_i(x'))$  (the benefit of linking to  $j$  exceeds marginal cost),
2. for every  $k$  such that  $x'_{ik} = 0$  we have  $w_{ij}(\delta_i\alpha_i + (1 - \delta_i)x_{ji}) \geq w_{ik}(\delta_i\alpha_i + (1 - \delta_i)x_{ki})$  (none of the non-neighbors is a better pick than  $j$ ).

*Proof:* The  $\Leftarrow$  direction is clear; if the link towards a player  $j$  appears profitable and there is no better alternative to connect to, then player  $i$ 's best response is to link to  $j$ . The  $\Rightarrow$  direction consists of two simple parts: (1) linking to  $j$  cannot be a best response if it does not appear profitable, (2) linking to  $j$  but not linking to an apparently more profitable  $k$  cannot be a best response. ■

It is clear that Lemma 3.12 implies Lemma 3.1.

**Lemma 3.13** (EUA characterization). *The network  $x^*$  is an EUA of a game given by  $W$  and  $c$  if and only if for every trio  $i, j, k \in I$  such that  $x^*_{ij} = 1, x^*_{ik} = 0$  we have*

$$w_{ij}(\delta_i\alpha_i + x^*_{ji}(1 - \alpha_i)) \geq w_{ik}(\delta_i\alpha_i + x^*_{ki}(1 - \alpha_i))$$

*and every pair  $i, j \in I$   $x^*$  follows rules presented by Table 1.*

*Proof:* Both conditions follow from Lemma 3.12. ■

$w_{ij}$	$> \frac{c'_j(x^*)}{\delta_j \alpha_j}$	$\in \left( \frac{c'_j(x^*)}{\delta_j \alpha_j + 1 - \delta_j}, \frac{c'_j(x^*)}{\delta_j \alpha_j} \right)$	$< \frac{c'_j(x^*)}{\delta_j \alpha_j + 1 - \delta_j}$
$> \frac{c'_i(x^*)}{\delta_i \alpha_i}$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = 1, x_{ji}^* = 0$
$\in \left( \frac{c'_i(x^*)}{\delta_i \alpha_i + 1 - \delta_i}, \frac{c'_i(x^*)}{\delta_i \alpha_i} \right)$	$x_{ij}^* = x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 0 \text{ or } 1$	$x_{ij}^* = x_{ji}^* = 0$
$< \frac{c'_i(x^*)}{\delta_i \alpha_i + 1 - \delta_i}$	$x_{ij}^* = 0, x_{ji}^* = 1$	$x_{ij}^* = x_{ji}^* = 0$	$x_{ij}^* = x_{ji}^* = 0$

Table 1: EUA characterization. High-value directions always form, low-value directions never form in equilibrium. In-between-value directions form only if reciprocated, either because the opposite direction is high-value, or because the opposite direction is also an in-between-value. If, for two players, both directions are in-between-value, there exist multiple equilibria. Note: we use the convention  $1/0 = \infty$ .

There are two important differences between Nash equilibrium networks and EUA: First, the empty network is not always an EUA, since, as Table 1 shows, some players may find it optimal to link to an opponent under any circumstance. As a result, existence becomes a non-trivial issue. Note that whenever the cost function is strictly concave, the game is not supermodular in the players' own strategies and thus, the standard existence result of Topkis (1979) cannot be used. Second, as Table 1 shows once again, in general, EUA may contain directed links. As mentioned, a sufficiently optimistic player will link to a highly profitable opponent even without reciprocation, while if the opponent is sufficiently pessimistic or does not find the link profitable, she will not reciprocate.

**Proposition 3.14** (Existence of EUA). *An equilibrium under ambiguity exists for any network formation game and any set of ambiguity parameters that satisfy the assumptions of no indifference.*

*Proof:* We show this statement by the use of strict best-response paths. A strict best-response path is a sequence of networks  $(x^t)_{t \in \{0, \dots, T\}}$  such that for every  $t \in \{1, \dots, T\}$  there exists a player  $i^t$  such that  $x_{-i^t}^t = x_{-i^t}^{t-1}$ ,  $x_{i^t}^{t-1} \notin B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^t))$ , and  $x_{i^t}^t \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^t))$ . In a strict best-response path every period has exactly one player move from a strategy that isn't a best response to the current strategies of her opponents to a best response. We will show that no matter the starting network  $x^0$ , the order of updates, or the length of the sequence, there are no cycles in this process, i.e. there does not exist a non-trivial strict best-response path for which  $x^0 = x^T$ . Given this fact and due to the finiteness of the set of possible networks with  $n$  players, starting from any network, any sequence of such strict improvements by players who are not at their best response will lead to an EUA.

So, for contradiction suppose that such a cycle  $(x^t)_{t \in \{0, \dots, T\}}$  exists with  $x^0 = x^T$ .

We say that the link  $\{i, j\}$  is *changing* in this sequence if there exist two time periods  $t$  and  $t'$  such that  $x_{ij}^t \neq x_{ij}^{t'}$ . It is clear that such links must exist in any strict best-response path.

**Claim 3.15.** *In a strict best-response cycle there must exist a changing link  $\{i, j\}$  and a time period  $t$  such that  $x_{ij}^t = x_{ji}^t = 1$ .*

To show this, suppose for contradiction that such a changing link does not exist. Take a player  $i$  and two distinct time periods  $t < t'$  such that  $i^t = i^{t'} = i$  and for every  $t'' \in \{t+1, \dots, t'-1\}$  we have  $i^{t''} \neq i$ . For any cycle such periods may be found by an appropriate time shift. It is clear that  $t$  and  $t'$  cannot be consecutive time periods in a strict best-response path and that there must exist a link  $\{i, j\}$  such that  $x_{ij}^{t+1} \neq x_{ij}^{t'+1}$  otherwise  $x_i^{t'} = x_i^{t'+1}$ , contradicting that  $x_i^{t'} \notin B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t'}))$ . Since there are no changing reciprocal links we must have  $N_i(x^t) = N_i(x^{t+1}) = N_i(x^{t'}) = N_i(x^{t'+1})$ . Furthermore, for every  $k \in I$  with  $x_{ik}^t = 0$  and  $x_{ki}^t = 1$  we have  $x_{ik}^{t+1} = 0$ , otherwise a changing reciprocal link would appear. Similarly, for every  $k \in I$  with  $x_{ik}^{t'} = 0$  and  $x_{ki}^{t'} = 1$  we have  $x_{ik}^{t'+1} = 0$ .

It follows that  $x_i^{t+1} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^t))$  implies  $x_i^{t'+1} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t'}))$  as no player can form a link to  $i$ , nor delete a link to  $i$  that she would reciprocate forming or deleting in  $t'$  as there can be no changing reciprocated links. Since  $x_i^{t+1} = x_i^{t'}$  due to the fact that  $i$  hasn't changed her action between  $t$  and  $t'$  we have  $x_i^{t'} \in B_i(\nu_i(\cdot | \alpha_i, \delta_i, x^{t'}))$ , a contradiction by the fact that we assumed a strict best-response cycle.

■

Secondly, we show that a strict best-response cycle also cannot contain a reciprocated changing link. The two statements together imply that there are no strict best-response cycles.

**Claim 3.16.** *In a strict best-response cycle there cannot be a changing link  $\{i, j\}$  and a time period  $t$  such that  $x_{ij}^t = x_{ji}^t = 1$ .*

Once again, suppose for contradiction that such a link exists in a strict best-response cycle. Let  $i, j$  be a pair and  $t$  a time period such that  $\{i, j\}$  is a reciprocated changing link with the largest value  $w_{ij}$ ,  $i^t = i$ ,  $x_{ij}^t = x_{ji}^t = 1$ ,  $x_{ij}^{t+1} = 0$ , and  $|\{k \in N_i(x^t) : w_{ik} = w_{ij}\}| \geq |\{k \in N_i(x^{t+1}) : w_{ik} = w_{ij}\}|$ . In words,  $i, j$  is a



best-value reciprocated changing link that was deleted by  $i$  in period  $t$  and she does not have more links of value  $w_{ij}$  in period  $t + 1$  than in period  $t$ . Such  $i, j$  and  $t$  are guaranteed to exist in a strict best-response cycle.

Let time period  $t' < t$  be such that  $i^{t'} = i$ ,  $x_{ij}^{t'} = 0$ ,  $x_{ij}^{t'+1} = 1$ . As before, for any cycle, such periods may be found by an appropriate time shift.

Let  $K_i = \{k: w_{ik}\alpha_i\delta_i > w_{ij}\alpha_i\delta_i + 1 - \delta_i\}$  denote the set of players whom  $i$  strictly prefers to  $j$  even if  $j$  reciprocates but they do not. If this set is empty, player  $i$  will never delete a link with  $j$ . It is clear that we must have  $x_{ik}^{t'+1} = 1$  for every  $k \in K_i$ , otherwise we cannot have  $x_{ij}^{t'+1} = 1$  as a best response to  $x^{t'}$  by Lemma 3.12. This in turn implies that  $x_{ik}^t = 1$  for every  $k \in K_i$ , otherwise we would have a time period  $t'' \in \{t' + 1, \dots, t - 1\}$  where  $i$  deletes her link to a  $k \in K_i$  but keeps the link to  $j$ , which cannot happen in a best response.

So, at time  $t$ , when  $i$  deletes her link to  $j$ , every link that offers a better payoff was already formed, and since she cannot have more neighbors of equal value in  $t + 1$  by the choice of  $j$ , it is impossible that  $x_i^{t+1}$  is a best response to  $x^t$  provided that  $x_i^t$  was not a best response to  $x^t$ . This concludes the proof of Claim 3.16 and thus of Proposition 3.14. ■

We now turn our attention towards the comparative statics with respect to changes in the optimism parameters  $\alpha_i$ . We begin by the introduction of minimal and maximal out-degrees under ambiguity.

**Definition 3.17** (Minimal and maximal out-degree). For player  $i \in I$  her *minimal* and *maximal out-degree*, denoted by  $\underline{d}_i$  and  $\bar{d}_i$  are defined as

$$\begin{aligned}\underline{d}_i(\alpha_i, \delta_i) &= \deg_i^+(x) \text{ where } x_i \in B_i \left( \nu_i \left( \cdot | \alpha_i, \delta_i, (x_{ji} = 0)_{j \neq i} \right) \right), \\ \bar{d}_i(\alpha_i, \delta_i) &= \deg_i^+(x) \text{ where } x_i \in B_i \left( \nu_i \left( \cdot | \alpha_i, \delta_i, (x_{ji} = 1)_{j \neq i} \right) \right),\end{aligned}$$

respectively.

In words, the values  $\underline{d}_i$  and  $\bar{d}_i$  are the number of links player  $i$  offers in a best response to the belief that no players link to  $i$  and all players link to  $i$ , respectively. Since indifferences of linking and not linking are ruled out, these values are well-defined, with  $\underline{d}_i$  being the largest degree  $r$  such that  $r \geq |\{j \in I: \alpha_i\delta_i w_{ij} > c'_i(r)\}|$

holds and  $\bar{d}_i$  being the largest degree  $r$  such that  $r \geq |\{j \in I: (\alpha_i \delta_i + (1 - \delta_i))w_{ij} > c'_i(r)\}|$  holds.

It is clear that without ambiguity,  $\underline{d}_i = 0$  and  $\bar{d}_i = d_i$ , while for general ambiguity parameters we have  $\underline{d}_i \geq 0$  and  $\bar{d}_i \leq d_i$ . These values are crude lower and upper estimates of the number of out-links player  $i$  can have in any equilibrium. We now show that an increase of a player's optimism without changing her perception of ambiguity raises both these bounds.

**Lemma 3.18** (Minimal and maximal out-degree comparative statics). *If, for some  $i \in I$  we have  $\alpha'_i > \alpha_i$ , then  $\underline{d}_i(\alpha'_i, \delta_i) \geq \underline{d}_i(\alpha_i, \delta_i)$ , and  $\bar{d}_i(\alpha'_i, \delta_i) \geq \bar{d}_i(\alpha_i, \delta_i)$ .*

*Proof:* The statement follows from the fact that for every  $i, j \in I$  and every  $x_{-i} \in X_{-i}$  it holds that  $w_{ij}(\alpha_i \delta_i + x_{ji}(1 - \delta_i)) \geq w_{ij}(\alpha'_i \delta_i + x_{ji}(1 - \delta_i))$ , thus in best response to both situations, more optimism will lead to a weakly higher number of out-links. ■

By Lemma 3.18, a player's bounds on the number of out-links increase in any best response and in any equilibrium with the player's optimism as the perceived value of all potential links also increases. However, the number of links in the equilibrium network, or even the equilibrium degree of the player becoming more optimistic, may decrease. For this reason the comparative statics between player optimism and the size of the equilibrium network is not straightforward.

There are cases, however, where the equilibrium number of links, and thus the total benefits in the network, will increase as a result of an increase in optimism. We show that if an equilibrium is monotonic, then if the optimism parameter of all players increases, there will always exist an equilibrium under the new parameters with weakly more links.

**Proposition 3.19** (Monotonic equilibrium comparative statics). *Let  $x \in X^*(\delta, \alpha)$  be monotonic and consider an  $\alpha' \geq \alpha$ . Then, there exists an equilibrium under ambiguity,  $x' \in X^*(\alpha', \delta)$  such that  $y(x) \leq y(x')$ .*

*Proof:* We decompose the network formation game with parameters  $(\delta, \alpha')$  into two parts: the mutual links that are formed in the equilibrium  $x$  of the game  $(\delta, \alpha)$ ,  $y(x)$  and the remaining links. In particular, we will look at the network formation game with parameters  $(\delta, \alpha')$  such that

$$\begin{aligned}
\tilde{w}_{ij} &= 0 \text{ for all } \{i, j\} \in y(x) \\
\tilde{w}_{ij} &= w_{ij} \text{ for all } \{i, j\} \notin y(x) \\
\tilde{c}_i(r) &= c_i(r + \deg_i(x)) \text{ for } r \in \mathbb{N}, i \in I
\end{aligned} \tag{2}$$

Clearly, for every  $i \in I$  the maximal number of links in this game is given by  $\tilde{d}_i = \bar{d}_i - \deg_i(x)$ . Furthermore, by Proposition 3.14, this game has an equilibrium – denote such an equilibrium by  $\tilde{x}$ . Note that since  $\tilde{c}_i(r) > 0$  for all  $r$ ,  $\tilde{x}_{ij} = 0$  holds for all  $\{i, j\} \in y(x)$ .

For the original game with parameters  $(\delta, \alpha')$ , define the strategy combination  $x'$  by  $x'_{ij} = 1$  for all  $\{i, j\} \in y(x)$  and  $x'_{ij} = \tilde{x}_{ij}$  for all  $\{i, j\} \notin y(x)$ . Note that for each  $i$ ,

$$\deg_i^+(x') = \deg_i^+(\tilde{x}) + \deg_i(x) \tag{3}$$

We now show that  $x' \in X^*(\delta, \alpha')$  by showing that  $x'_i$  satisfies the two conditions in Lemma 3.12, i.e. it is a best response to  $x'_{-i}$ , starting with condition 1. Take an  $\{i, j\}$  such that  $x'_{ij} = 1$ .

Case 1: If  $\{i, j\} \in y(x)$ , and there exists some  $k$  such that  $x'_{ik} = 1$  and  $\{i, k\} \notin y(x)$ , then we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] \geq w_{ik} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ki}] > \tilde{c}'_i(\deg_i^+(\tilde{x})) = c'_i(\deg_i^+(x')),$$

where the first inequality follows from the fact that  $x$  is monotonic and thus  $w_{ij} \geq w_{ik}$ , the second inequality follows from the fact that  $x'_{ki} = \tilde{x}_{ki}$  and  $\tilde{x}$  is an equilibrium of (2) and the equality is a consequence of the definition of  $\tilde{c}$  and (3).

Case 2: If  $\{i, j\} \in y(x)$  and  $N_i^+(x') = N_i(x)$  then we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i)] \geq w_{ij} [\alpha_i \delta_i + (1 - \delta_i)] > c'_i(\deg_i(x)) = c'_i(\deg_i^+(x')),$$

where the first inequality is due to  $\alpha'_i \geq \alpha_i$ , the second is due to the fact that  $\{i, j\} \in y(x)$  and  $x$  was an EUA of the game with  $\alpha$ , and the equality is by the definition of  $\tilde{c}$  and (3).

Case 3: If  $\{i, j\} \notin y(x)$ , then  $x'_{ij} = \tilde{x}_{ij}$

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] > \tilde{c}'_i(\deg_i(\tilde{x})) = c'_i(\deg_i(x)),$$

where the inequality holds due to the fact that  $\tilde{x}$  is an equilibrium of (2) and the equality is a consequence of the definition of  $\tilde{c}$  and (3).

Thus, condition 1 of Lemma 3.12 is satisfied whenever  $x'_{ij} = 1$ .

Take next an  $\{i, j\}$  such that  $x'_{ij} = 0$ . Clearly,  $\{i, j\} \notin y(x)$  and thus  $x'_{ij} = \tilde{x}_{ij}$ . Since  $\tilde{x}_{ij}$  is an equilibrium of the game defined by (2), we have

$$w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] < \tilde{c}'_i (\deg_i^+ (\tilde{x})) = c'_i (\deg_i^+ (x')) .$$

Thus, condition 1 of Lemma 3.12) is satisfied whenever  $x'_{ij} = 0$  as well.

Finally, we check condition 2. Suppose that there exist distinct players  $i, j$  and  $k$  such that  $w_{ij}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}) > w_{ik}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji})$ , but  $x'_{ik} = 1$  and  $x'_{ij} = 0$ . Clearly,  $\{i, j\} \notin y(x)$ . If  $\{i, k\} \in y(x)$  then  $x'_{ki} = 1$ , and since  $w_{ik} \geq w_{ij}$  by the fact that  $x$  is monotonic, this contradicts  $w_{ij}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}) > w_{ik}(\alpha'_i \delta_i + (1 - \delta_i) x'_{ji})$ . Hence,  $\{i, k\} \notin y(x)$ . Since  $\tilde{x}$  is an EUA of the game (2), by condition 2 of Lemma 3.12 we have

$$\begin{aligned} w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) \tilde{x}_{ji}] &= w_{ij} [\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}] \leq \\ &w_{ik} (\alpha'_i \delta_i + (1 - \delta_i) x'_{ji}) = w_{ik} (\alpha'_i \delta_i + (1 - \delta_i) \tilde{x}_{ji}), \end{aligned}$$

a contradiction.

Therefore, all conditions of Lemma 3.12 holds. ■

By Proposition 3.19, we thus obtain that if an equilibrium is monotonic, an increase in the players' optimism will lead to more reciprocated links, increasing the value of realized benefits in the network as well.

In the special case when all cost functions are linear, the game is supermodular and the result of Proposition 3.19 holds regardless of whether the initial equilibrium  $x$  is monotonic (Schipper, 2019).

## 4 Dynamic results

Static equilibrium concepts such as the Nash equilibrium and pairwise stability often suffer from multiplicity and thus result in indeterminacy of the equilibrium network. In contrast, strongly stable equilibria may fail to exist. Since the seminal

works of KMR and Young (1993), models of myopic adaptation with random mutations have been used as a selection device in games with multiple equilibria.

In general, such models postulate a best-response type of adaptive dynamics. In each period, a player is chosen at random and revises her strategy by playing a best response to the current strategy combination of her opponents. With strictly positive probability, mutations (such as mistakes, experimentation) occur; instead of playing her best response, the player chooses a strategy at random.

In this paper player mutations, instead of leading to a random choice of a strategy, will affect the player's ambiguity parameters, her perception of ambiguity and optimism. Specifically, we are interested in mutations that amount to optimistic shocks; the affected player best-responds to the belief that every one of her opponents is willing to form a link to her. We use this process to select for equilibria that are stable for such optimistic deviations. In addition, we allow for players to best-respond to the current network in the classic sense as well.

Our dynamic process takes place in discrete time,  $t \in \mathbb{N}$ . The network at time  $t$  will be denoted by  $x^t$ . Take an initial network structure  $x^{t-1}$ . In period  $t$ , a player  $i^t$  is chosen at random with uniform probability  $1/n$  and receives an opportunity to revise her strategy to  $x_{i^t}$ . All other players continue playing the strategies corresponding to  $x_{-i^t}^{t-1}$ .

For player  $i \in I$  let  $\bar{\delta}_i \in (0, 1)$  and  $\bar{\alpha}_i \in [0, 1]$  such that for every  $j, k \in I \setminus \{i\}$  such that  $w_{ij} > w_{ik}$  we have  $\bar{\delta}_i \bar{\alpha}_i w_{ij} > w_{ik}$ . Such values are guaranteed to exist with both  $\bar{\delta}_i$  and  $\bar{\alpha}_i$  being close to 1, as the player set is finite. These parameters represent highly optimistic behavior but not the complete ignorance of objective beliefs. This has three important effects on behavior: (1) if pressed to make a choice, the player will strictly prefer an unreciprocated link of a higher value to a reciprocated link of a lower value, (2) due to  $\bar{\delta}_i < 1$  the player will strictly prefer a reciprocated link to an unreciprocated one of equal value, (3) the player always finds it optimal to have exactly  $d_i$  out-links to her top  $d_i$  choices, i.e.  $\underline{d}_i = \bar{d}_i = d_i$ .

The main parameter governing our dynamic process is the frequency of optimistic shocks, denoted by  $\beta \in (0, 1)$ . If close to zero, our process resembles a best-response dynamic with regular, but infrequent optimistic shocks, and if close to 1, the players almost always react optimistically. The ambiguity parameters of player  $i^t$  at the time of revision are thus given by the following time-independent probability distribution:

$$\begin{aligned}\delta_{it}^t &= 0 \text{ and } \alpha_{it}^t \text{ arbitrary with probability } 1 - \beta, \\ \delta_{it}^t &= \bar{\delta}_{it}^t \text{ and } \alpha_{it}^t = \bar{\alpha}_{it}^t, \text{ with probability } \beta,\end{aligned}$$

Given her current ambiguity parameters, the revising player behaves myopically and chooses a best response to  $x_{-it}^{t-1}$ :

$$x_{it}^t \in B_{it}(\nu_{it}(\cdot | \alpha_{it}^t, \delta_{it}^t, x_{-it}^{t-1}))$$

If the best response  $B_{it}(\cdot)$  is not a singleton, player  $i^t$  chooses a best response in  $B_{it}(\cdot)$  uniformly at random. The resulting network at time  $t$ ,  $x^t$ , is then given by

$$x^t(x^{t-1}, i^t, \alpha_{it}^t, \delta_{it}^t) = (x_{it}^t, x_{-it}^{t-1}).$$

In words, any updating player is an optimist with uniform probability  $\beta$  and a best-responder with probability  $1 - \beta$ . In a naive learning and experimenting process, the case of a small, but positive  $\beta$  is of interest, meaning that players behave as myopic best-responders most of the time but sometimes experience optimistic shocks. Our results below, however, apply for general values of  $\beta$ .

In what follows, we will shorten the notation of an optimistic player's best response to  $x$  to  $B_i^o(x) = B_i(\nu_i(\cdot | \bar{\alpha}_i, \bar{\delta}_i, x_{-i}))$ .

This dynamic process describes a Markov chain on the set  $X$ . For  $x, x' \in X$  let  $p(x, x')$  denote the one-step transition probability from  $x$  to  $x'$ . It is clear that for every  $x, x' \in X$  with  $p(x, x') > 0$  there exists  $i \in I$  such that either  $x'_i \in B_i(x)$  or  $x'_i \in B_i^o(x)$  and for every  $j \neq i$  we have  $x'_{-i} = x_{-i}$ . For  $m \in \mathbb{N}$  let  $p^m(x, x')$  denote the transition probability from  $x$  to  $x'$  in exactly  $m$  steps. For a set of networks  $X' \subseteq X$  let  $p^m(x, X') = \sum_{x' \in X'} p^m(x, x')$  denote the probability of reaching this set from  $x$  in exactly  $m$  steps.

**Definition 4.1** (Successor). For  $x, x' \in X$  and  $m \in \mathbb{N}$  we say that  $x'$  is an  $m$ -successor of  $x$  if  $p^m(x, x') > 0$ .

We denote the set of  $m$ -successors of network  $x$  by  $S^m(x)$  and let  $\bigcup_{m \in \mathbb{N}} S^m(x) = S(x)$  denote the set of all successors of  $x$ . By convention we include  $x \in S(x)$ .

We now present our main result of this section: If preferences are aligned, then the dynamic described above always pushes the game towards networks that contain a maximum-weight equilibrium network, and, possibly, some additional one-way links.

**Proposition 4.2** (Optimism leads to benefit-maximizing equilibrium value under aligned preferences). *Suppose that the players exhibit aligned preferences. Then, for every  $x \in X$  we have  $\lim_{m \rightarrow \infty} p^m(x, y^{-1}(V^*)) = 1$*

To obtain the desired result, we first show that for sufficiently high degrees of ambiguity and optimism, an optimistic shock to a given player  $i$  incentivizes her to initiate her  $d_i$  most preferred links, even if they are currently not reciprocated. By definition, all such links have a value of at least  $c'_i(d_i)$ . Combined with an appropriate sequence of “realistic” best-responses in which other players are given the opportunity to reciprocate the links offered by  $i$  and subsequently  $i$  can withdraw all non-reciprocated links, this gives rise to a network, in which all of  $i$ ’s links are reciprocated and have a value of at least  $c'_i(d_i)$ . Repeating this procedure for all players results in a Nash equilibrium network, in which the links of any player have a value of at least  $c'_i(d_i)$ .

Once such an equilibrium is reached, and provided that it is not already a maximum-weight one, we identify a closest maximum-weight network (one that can be reached with a minimal number of changes in links) which is monotonic by the assumption that preferences are aligned. We then show that this maximum-weight equilibrium is reached by some series of updates.

Specifically, we show that we can choose a link  $\{i, j\}$  which is part of the maximum-weight network, but not formed at the current step, and a player  $i$  who would initiate this link under an optimistic shock. If this is followed by  $j$  being given a best-responding revision opportunity she may reciprocate this link, possibly severing some others. We show that there exist a series of best-responses by  $j$  and possibly by some other players through which we obtain a network which is strictly closer to the desired maximum-weight equilibrium network (notably by the inclusion of the link  $\{i, j\}$  and the exclusion of some unnecessary links).

Since the set of all networks is finite, repeating this process allows the dynamic to reach  $V^*$  in finite time. Finally, we show that once a network in  $V^*$  is reached, the value of the network cannot decrease on the path of the best-response dynamics: This is because realistic best-responders have no interest in severing any links in a maximum-weight equilibrium as they are all profitable and reciprocated, while optimists don’t sever links because maximum-weight equilibria are monotonic and hence they already link to a top selection of their opponents.

After a maximum-weight equilibrium is reached, optimistic shocks could lead

to the creation of additional non-reciprocated links and realistic best-responders may reciprocate these links while severing others that are equally valuable, thus the game may shift from one maximum-weight equilibrium to another. However, the total value of reciprocal links stays at its maximum thereafter.

We note the difference between our approach and that of KMR. Similarly to their approach, our best-response dynamic with optimistic shocks defines a Markov process on the set of strategy combinations and thus, on the set of directed networks. In KMR, the random mutations imply that any strategy can be chosen with strictly positive probability and thus, the Markov process visits any of the Nash equilibria infinitely often. When the probability of mutations goes to 0, only the equilibrium with the largest basin of attraction remains as a long-run equilibrium of the game. In 2x2 games, this dynamic selects for the risk-dominant (as opposed to the Pareto-dominant) equilibrium.

In contrast, in our model, the asymmetric nature of the optimistic shocks implies that, (as long as the probability of a shock is strictly positive), the Markov process reaches the set of maximal weight networks, with probability 1, regardless of initial conditions. Furthermore, the set  $y^{-1}(V^*)$  is absorbing. Thus, rather than selecting for a single equilibrium, our dynamic selects for a *set* of equilibrium networks which are Pareto-optimal.

We formally prove this result by a series of lemmata.

**Lemma 4.3.** *For every  $x \in X$  there exists  $x^* \in X^* \cap S(x)$  such that for all  $i \in I$  we have  $w'_i(x^*) > c'_i(d_i)$ .*

*Proof:* First we show that there exists  $x' \in X^* \cap S(x)$ . Take an arbitrary  $x \in X$ . Then, suppose that in the next  $n$  periods, all players  $\{1, \dots, n\}$  are given an ambiguity-neutral best-responding revision opportunity in increasing order. Call the resulting network  $x'$ . The probability of this happening is  $((1 - \beta)/n)^n > 0$ , so we must have  $x' \in S(x)$ . Then, for every  $i \in I$  we have  $w'_i(x') > c'_i(\deg_i^+(x'))$  and for every pair with  $i > j$  we can have  $x'_{ij} = 1$  only if  $x_{ji} = 1$ . Thus, repeating the same process in the reverse order will lead to an undirected network  $x''$  with  $w'_i(x'') > c'_i(\deg_i(x''))$  for all  $i \in I$ , satisfying the equilibrium conditions.

This shows that we can take  $x \in X^*$  at the start. Take a player  $i$  such that  $w'_i(x) < c'_i(d_i)$ . If none can be found, we are done. If such a player exists, suppose that she receives an optimistic revision opportunity. Then since for every



$x_i^o \in B_i^o(x)$  we have  $w'_i(x_i^o, x_{-i}) > c'_i(d_i)$ , she will sever all links with players giving her less benefits than  $c'_i(d_i)$  and possibly initiate some links to others. Then, allow all her neighbors with whom she severed her links to best-respond in an ambiguity-neutral way, their only best response being is to sever their unreciprocated links towards  $i$  and make no other changes. Finally, let player  $i$  best-respond to sever all unreciprocated links she had initiated as an optimist. Call the resulting network  $x'$ . Since  $p^m(x, x') > 0$  for some  $m \leq N_i(x) + 1$  we must have  $x' \in S(x)$ .

At  $x'$ , we clearly have  $w'_i(x') > c'_i(d_i)$  since all players  $j \neq i$  with  $w_{ij} < c'_i(d_i)$  now have no outgoing links towards  $i$  while  $N_i(x) \supseteq N_i(x')$ . Thus, we have  $x'_i \in B_i(x')$ , meaning that  $x' \in X^*$ . Furthermore,  $w'_i(x') > c'_i(d_i)$ , as well as  $|x'| < |x|$ , since player  $i$  lost links and no player gained any. If there exists  $j \neq i$  with  $w'_j(x') < c'_j(d_j)$ , we can repeat this process. If the process does not terminate sooner with a desired network, it will terminate by reaching the empty network which trivially satisfies the conditions of the statement. ■

The next lemma shows that the learning process enters the set of maximum-weight equilibria that are monotonic,  $V^*$ , for any starting state with a positive probability.

**Lemma 4.4.** *Suppose that the players exhibit aligned preferences. For every  $x \in X$  there exists an  $x^* \in V^* \cap S(x)$ .*

*Proof:* By Lemma 4.3 we may take  $x \in X^*$  such that  $w'_i(x) > c'_i(d_i)$  for all  $i \in I$ . Let  $x^* \in V^*$  be the closest element of  $V^*$  to  $x$ . If  $x \geq x^*$ , then we must have  $x = x^*$  otherwise we contradict the choice of  $x^*$  as an element of  $V^*$ , in which case we are done. If  $x \not\geq x^*$ , we have  $x^* \setminus x \neq \emptyset$ .

We will show that there exists an  $x'' \in X^* \cap S(x)$  which also satisfies  $w'_i(x'') > c'_i(d_i)$  for every  $i \in I$  and for which  $\phi(x'', x^*) < \phi(x, x^*)$ . Through a repeated application of this argument the distance eventually decreases to 0, meaning that we reach  $x^*$ . We distinguish two cases.

Case 1. There exists a player  $i$  and a link  $\{i, j\} \in x^* \setminus x$  such that  $\deg_i(x) < d_i$ . Let  $j \in \arg\max_{j' \in N_i(x^*) \setminus N_i(x)} w_{ij'}$  be the most valuable such link for player  $i$ . Then let  $x'_i$  be a strategy profile which is monotonic for  $i$ ,  $\deg_i^+(x'_i, x_{-i}) = d_i$ , and  $N_i^+(x'_i, x_{-i}) \supseteq (N_i(x) \cap N_i(x^*)) \cup \{j\}$ . By Lemma 3.12, and due to  $x$  satisfying

$w'_i(x) > c'_i(d_i)$ , we must have  $x'_i \in B_i^o(x)$ . Let player  $i$  receive an optimistic revision opportunity and suppose that she revises to  $x'_i$ .

Case 1a. If  $\deg_j(x) < d_j$ , then we must have  $w_{ij} \geq w'_j(x^*) > c'_j(d_j) \geq c'_j(\deg_j(x) + 1)$ . Therefore,  $B_j(x'_i, x_{-i})$  is a singleton with its only element, which we denote by  $x'_j$ , satisfying  $x'_{jk} = 1$  if and only if  $k \in N_j(x) \cup \{i\}$ . Suppose that player  $j$  receives a best-responding revision opportunity, and thus, selects the strategy  $x'_j$ . For each  $k \in N_i(x) \setminus N_i^+(x'_i, x_{-i})$  let  $k$  receive a best-responding revision opportunity, their only best response being is to keep all of their existing links except those with  $i$  who had severed her link to each  $k$ .

Finally, let player  $i$  also receive best-responding revision opportunity. Her only best response, which we denote by  $x''_i$ , is to delete all unreciprocated links she initiated when she moved to  $x'_i$ , that is,  $x''_{ik} = 1$  if and only if  $k \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$ . Let the resulting network be denoted by  $x''$ . It is clear that  $x''$  is an equilibrium network,  $w'(x'') \geq c'(d)$ , and  $\phi(x'', x^*) < \phi(x, x^*)$ . Since the above process leads to  $x''$  from  $x$ , we have  $x'' \in S(x)$  as well.

Case 1b. If  $\deg_j(x) = d_j \geq \deg_j(x^*)$ , then there must exist a link  $\{j, k\} \in x \setminus x^*$ . Since  $x^*$  is monotonic we must also have  $w_{ij} \geq w_{jk}$ . Thus, there must exist an  $x'_j \in B_j(x'_i, x_{-i})$  such that  $x'_{j\ell} = 1$  if and only if  $\ell \in (N_j(x) \setminus \{k\}) \cup \{i\}$ . As before let  $j$  receive a best-responding revision opportunity and suppose that she selects  $x'_j$ . Let this be followed by player  $k$  best-responding; her only best response being to cut her now unreciprocated link towards  $j$  and keeping all her other neighbors.

Then, as before, for each  $k' \in N_i(x) \setminus N_i^+(x'_i, x_{-i})$  let  $k'$  receive a best-responding revision opportunity, their only best response being to keep all of their existing links except those with  $i$ , who had severed her link to each  $k'$ . Finally, let  $i$  best-respond, her only best response being to delete all unreciprocated links she initiated when she moved to  $x'_i$ , that is,  $x''_{ik} = 1$  if and only if  $k \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$ . Let the resulting network be denoted by  $x''$ . As before,  $x'' \in X^*$ , satisfies  $w'(x'') \geq c'(d)$  and  $\phi(x'', x^*) < \phi(x, x^*)$ , as well as  $x' \in S(x)$ , completing Case 1.

Case 2. For every link  $\{i, j\} \in x^* \setminus x$  we have  $\deg_i(x) = d_i$  and  $\deg_j(x) = d_j$ . We will show that there must exist such a link  $\{i, j\}$  for which there exists  $\{i, k\} \in x \setminus x^*$  such that  $w_{ij} > w_{ik}$ .

Suppose this isn't true, i.e. for every  $i, j, k$  with  $\{i, j\} \in x^* \setminus x$  and  $\{i, k\} \in x \setminus x^*$  we have  $w_{ij} \leq w_{ik}$ . Since  $\deg_i(x) = d_i$  for all  $i$  with  $N_i(x) \neq N_i(x^*)$  we must have

$\deg_i(x) \geq \deg_i(x^*)$  for all  $i$ . These two facts imply that  $v(x) \geq v(x^*)$  which can only hold with equality as  $x^*$  is a maximum-weight equilibrium network. If  $\deg_i(x) > \deg_i(x^*)$  for some  $i \in I$  and  $\deg_j(x) \geq \deg_j(x^*)$  for all other  $j \neq i$ , then  $v(x) = v(x^*)$  can hold only if we have such a trio  $i, j, k$  with  $\{i, j\} \in x^* \setminus x$ ,  $\{i, k\} \in x \setminus x^*$ , and we have  $w_{ij} > w_{ik}$ , a contradiction. So suppose that  $\deg_i(x) = \deg_i(x^*)$  for all  $i$ .

Now we will show that under these conditions  $x$  is monotonic, a contradiction since that would mean  $x \in V^*$ . Take a player  $i$  and suppose that there exist  $j, k \neq i$  such that  $\{i, j\} \in x$ ,  $\{i, k\} \notin x$  and  $w_{ij} < w_{ik}$ . If no such player can be found,  $x$  is monotonic. We must have  $\{i, k\} \notin x^*$  by assumption thus  $\{i, k\} \in x \setminus x^*$ . However, due to  $\deg_i(x) = \deg_i(x^*)$  there must exist an  $\ell \in N_i(x^*) \setminus \{j\}$  and by assumption we must have  $w_{ik} > w_{ij} \geq w_{i\ell}$ . Thus, since  $\{i, k\} \notin x^*$ , this is a contradiction as  $x^*$  is monotonic.

Therefore if for every link  $\{i, j\} \in x^* \setminus x$  we have  $\deg_i(x) = d_i$  and  $\deg_j(x) = d_j$ , then there must exist  $\{i, j\} \in x^* \setminus x$  and a  $\{i, k\} \in x \setminus x^*$  such that  $w_{ij} > w_{ik}$ . In particular, let  $j \in \operatorname{argmax}_{j' \in N_i(x^*) \setminus N_i(x)} w_{ij'}$  and let  $k \in \operatorname{argmin}_{j' \in N_i(x) \setminus N_i(x^*)} w_{ij'}$ .

Construct a strategy  $x'_i$  such that it is monotonic and for every  $\ell \in (N_i(x) \cap N_i(x^*)) \cup \{j\}$  we have  $x'_{i\ell} = 1$ . By the fact that  $w_{ij} > w_{ik}$ , we must have such a strategy with  $\deg_i^+(x'_i, x_{-i}) = d_i$  as player  $i$  can include  $j$  to her set of neighbors and exclude  $k$  and still maintain an out-degree of  $d_i$ . By Lemma 3.12, we have  $x'_i \in B_i^o(x)$ . As in Case 1, let  $i$  receive an optimistic revision opportunity and suppose she picks the optimistic best response  $x'_i$ . From here by the same process as in Case 1b we can reach  $x'' \in X^* \cap S(x)$  with  $w'(x'') > c'(d)$  and  $\phi(x'', x^*) < \phi(x, x^*)$ . ■

Finally, we show that under aligned preferences, once a maximum-weight equilibrium value is reached, then all subsequent networks reached will contain a maximum-weight equilibrium networks as well as some additional one-way links. In other words, the set  $y^{-1}(V^*)$  is absorbing under our dynamic process.

**Proposition 4.5.** *Suppose that the players exhibit aligned preferences. For every  $x^* \in V^*$  and every  $x' \in S(x^*)$  we have  $x' \in y^{-1}(V^*)$ .*

We first prove a supporting statement.

**Lemma 4.6.** *Suppose that the players exhibit aligned preferences. For a  $T \in \mathbb{N}$  let  $(x^t)_{t \in \{0, \dots, T\}}$  be a finite sequence of networks obeying the defined updating rule such that  $x^0 \in V^*$  and  $x^t \in y^{-1}(V^*)$  for  $t \in \{0, \dots, T\}$ . Then, for every  $t \in \{0, \dots, T\}$  and every  $i \in I$  we have  $w'_i(x^t) > c'_i(d_i)$ .*

*Proof:* Suppose that we have  $w'_i(x^t) < c'_i(d_i)$ . Let  $t$  denote the first such period, i.e.  $x^t_i \in B_i(x^t)$  or  $x^t_i \in B_i^o(x^t)$ . It is clear that the latter is impossible as optimistic players never initiate or accept such links, thus we must have  $x^t_i \in B_i(x^t)$ . Since  $y(x^t) \in V^*$  and since preferences are aligned  $x^t_i$  is monotonic for  $i$ . This means that for every  $k$  such that  $w'_i(x^t) < w_{ik}$  we have  $\{i, k\} \in x^t$ . However, by the definition of  $d_i$ , this has to imply  $\deg_i^+(x) > d_i$ , which is also impossible as a best-responding player would not initiate more than  $d_i$  links. ■

*Proof of Proposition 4.5:* We first show that  $y(x') \in X^*$ . Take a sequence  $(x^t)_{t \in \{0, \dots, T\}}$  obeying the defined updating rule with  $x^0 \in V^*$ . Since  $x^0$  is a Nash equilibrium, then by the updating rule, for every  $x^t$  we have  $w'_{it}(x^t) > c'_{it}(\deg_{it}(x^t))$ . Thus, for every  $t' \in \{0, \dots, T\}$  and every  $i, j \in I$  with  $x^{t'}_{ij} = x^{t'}_{ji} = 1$  we have  $w_{ij} > c'_i(\deg_i^+(x^{t'})) \geq c'_i(\deg_i(x^{t'}))$ , satisfying the conditions of a Nash equilibrium.

Thus, for contradiction, suppose that  $x' \in S(x^*)$  and  $y(x') \notin V^*$ . Then we must have  $v(x') < v(x^*)$ , as  $v(x') = v(y(x')) > v(x^*)$  would contradict the choice of  $x^*$  as a maximum-weight equilibrium, while  $v(x') = v(x^*)$  would imply  $y(x') \in V^*$ . Thus, the value of the network must decrease along the sequence between  $x^*$  and  $x'$ . Suppose that  $x'$  is the earliest element of the sequence with  $v(x') < v(x^*)$ , let  $x$  denote the element immediately preceding it and let  $i$  denote the player who updated between  $x$  and  $x'$ , i.e.  $x_{-i} = x'_{-i}$ . Since  $v(x) > v(x')$  there must exist  $\{i, j\} \in x \setminus x'$  and by Lemma 4.6 it must satisfy  $w_{ij} > c'_i(d_i)$ .

Suppose that  $x'_i \in B_i(x)$ . Then, since  $i$  drops her link with  $j$  as a best response even though  $j$  reciprocates, we must have  $w_{ij} < c'_i(\deg_i^+(x') + 1) = c'_i(\deg_i(x') + 1)$ . Thus,  $\deg_i(x') + 1 > d_i$ , meaning that  $\deg_i(x') = d_i$ . Since  $i$  made a best response,  $u_i(x) \leq u_i(x')$ , and since the number of neighbors in  $x'$  is at maximum, and thus the weak increase in utility did not come through a decrease in costs, the total value of  $i$ 's links must have weakly increased from  $x$  to  $x'$ . Hence, we have

$$v(x) = \sum_{j \neq i} w_{ij} x_{ij} x_{ji} + \sum_{j \neq k \neq i} w_{jk} x_{jk} x_{kj} \geq \sum_{j \neq i} w_{ij} x'_{ij} x_{ji} + \sum_{j \neq k \neq i} w_{jk} x_{jk} x_{kj} = v(x'),$$

a contradiction to  $v(x') < v(x)$ .

Suppose that  $x'_i \in B_i^o(x)$ . Then, since  $x'_i$  is monotonic for  $i$  and  $i$  drops her link with  $j$ ,  $\deg_i^+(x') = d_i$ . Since  $w_{ij} > c'_i(d_i)$ , and yet  $i$  chose not to reciprocate  $j$ 's link, there must exist  $k \in I$  with  $w_{ij} = w_{ik}$  such that  $\{i, k\} \in x'$ . Since  $y(x)_i$  is monotonic, and  $\{i, j\} \in x$  for every  $\ell \in I$  with  $w_{i\ell} > w_{ij}$  we have  $\{i, \ell\} \in x$ . Since  $x'_i$  is monotonic for  $i$ ,  $\{i, \ell\} \in x'$  as well. Furthermore, for every  $\ell \in I$  with  $w_{ij} = w_{i\ell}$  and  $x'_{i\ell} = 1$  we must have  $x'_{\ell i} = x_{\ell i} = 1$  as well, otherwise deleting a link to an unreciprocated opponent of equal value is better for  $i$  than deleting her link to  $j$ . Thus,  $\deg_i^+(x') = \deg_i(x') = d_i$ . Finally,  $x'_i$  being monotonic means that  $i$  has a reciprocal link to exactly  $d_i$  of her most favored opponents. So as before we have

$$v(x) = \sum_{j \neq i} w_{ij} x_{ij} x_{ji} + \sum_{j \neq k \neq i} w_{jk} x_{jk} x_{kj} \geq \sum_{j \neq i} w_{ij} x'_{ij} x_{ji} + \sum_{j \neq k \neq i} w_{jk} x_{jk} x_{kj} = v(x'),$$

a contradiction to  $v(x') < v(x)$ . ■

*Proof of Proposition 4.2.* By Lemma 4.4 for every  $x \in X$  we have a  $x^* \in V^*$  with  $x^* \in S(x)$ . Since there are finitely many networks, with probability one, the process enters  $V^*$  and thus  $y^{-1}(V^*)$ . By Proposition 4.5, the game never leaves  $y^{-1}(V^*)$ . ■

Proposition 4.2 shows that if weight-efficient networks are monotonic, then, by the individual updates of players, an efficient network will eventually form and its value is never lost in future periods. If there is more than one efficient network, the dynamic process may shift from one to another and back as unsatiated optimistic players keep forming directed links and satiated best-responders randomly reciprocate them while deleting links towards existing neighbors, thus the game may not converge to a single equilibrium network.

If preferences are not aligned, then maximal-weight equilibrium networks are not stable for optimistic shocks; a player who is not linked to her best choice of neighbors, upon experiencing an optimistic shock, will invariably delete her links to her existing neighbors and make offers to her best choices (see the illustrative matching example of the introduction).

We conclude this section by showing convergence in the linear case, which is not covered by the case of aligned preferences.

**Proposition 4.7.** *Let the cost functions be linear. Then, for every  $x \in X$  and the unique  $x^* \in V^*$  we have  $\lim_{m \rightarrow \infty} p^m(x, y^{-1}(x^*)) = 1$ .*

As before, we prove the statement by showing that the dynamic system will take the game into  $x^*$  from any starting network.

**Claim 4.8.** *Let the cost functions be linear. For every  $x \in X$  and the unique  $x^* \in V^*$  we have  $x^* \in S(x)$ .*

*Proof:* By Lemma 4.3 we can take  $x \in X^*$ . We show that if  $x = x^*$  there exists  $x' \in S(x) \cap X^*$  such that  $\phi(x', x^*) < \phi(x, x^*)$ . Take a link  $\{i, j\} \in x^* \setminus x$ . Then, we must have  $w_{ij} > \max\{c_i, c_j\}$ . Take the following series of updates:  $i$  receives a revision opportunity as an optimist,  $j$  receives a revision opportunity as a best-responder, then  $i$  receives a revision opportunity as a best-responder. It is clear that an optimistic player  $i$  will link to  $j$ , as well as keep all of her links to her neighbors in  $x$ . A best-responder  $j$  will make no changes except reciprocate player  $i$ 's link. Finally, player  $i$  deletes all the unreciprocated links she made as an optimist. For the resulting network,  $x' = x \cup \{i, j\}$ , we have  $x' \in X^*$  as well as  $\phi(x', x^*) = \phi(x, x^*) - 1$ . ■

Secondly, we show that once  $x^*$  is reached, the game never leaves the set  $y^{-1}(x^*)$ .

**Claim 4.9.** *Let the cost functions be linear. For the unique  $x^* \in V^*$  we have  $S(x^*) \subseteq y^{-1}(x^*)$ .*

*Proof:* Suppose for contradiction that there exist a pair  $x, x' \in S(x^*)$  such that  $x \in y^{-1}(x^*)$ ,  $x' \in S^1(x)$ , and  $x' \notin y^{-1}(x^*)$ . Then, there must exist  $\{i, j\} \in x' \setminus x$ , with  $w_{ij} > \max\{c_i, c_j\}$ . Clearly, one of  $i$  and  $j$  must have updated to get from  $x$

to  $x'$  so suppose it was  $i$ , i.e.  $x_{-i} = x'_{-i}$ . If  $x'_i \in B_i(x)$ , then, since  $x'_{ji} = x_{ji} = 1$ , deleting the link to  $j$  cannot be a best response. If  $x'_i \in B_i^o(x)$ , then, since  $w_{ij} > c_i$ , deleting the link to  $j$  cannot be an optimistic best response, so we have arrived at a contradiction. ■

Proposition 4.7 extends the convergence result laid out in Proposition 4.2 to all games with linear costs, without the precondition of aligned preferences. This is due to the fact that, under linear costs the unique maximum weight equilibrium, which coincides with the unique pairwise stable equilibrium, is stable for both optimistic and best-responding deviations.

## 5 Conclusion

In this paper we investigate a dynamic model of network formation with players facing ambiguity with respect to their beliefs on their opponents' strategies. We model this uncertainty using NEO-additive beliefs: players respond to mixtures of their rational beliefs as expressed by a probability measure of their opponents' actions, their optimistic beliefs, corresponding to beliefs in which opponents are trying to maximize their payoffs, and their pessimistic beliefs, corresponding to beliefs in which opponents are trying to minimize their payoffs.

To our knowledge, ours is the first project to consider ambiguity in strategies in a network formation game. We intuit that such a combination of frameworks is natural and fits well with behavioral considerations of network formation: optimistic players believe their linking offers are always reciprocated, pessimistic players believe that opponents never reciprocate. These considerations provide behavioral explanation to the existence of asymmetric interactions in social networks. Similar explanations are absent in models focusing on pairwise improvements or larger coalitional improvements. Directed interactions may arise due to optimism as a player believes the opponent will reciprocate and thus make an offer to connect without coordinating with the opponent.

By explicitly modeling player optimism we also allow the link-formation process to be governed proactively by the players. Under the classical tool, that of the Nash equilibrium, link formation was governed by the players passively best-responding

to each other's actions or their beliefs about their opponents. In what has proven to be the modern benchmark tool, pairwise stability, link-formation is governed by the logic of random pairwise interactions where, upon meeting a mutually beneficial opponent, the pair decides to form a link. Here, while the decision to form a new link is an active one once the interaction happens, the players remain passive in their opportunities to create these interactions. Under optimism, players proactively seek out their best possible links and the most possible links, and create linking opportunities through a costly investment.

A natural definition for socially optimal networks is the total value gained from connections by the players. We show that if costs of connection are linear such networks are also Pareto-optimal. The same holds under convex costs provided that individual benefits are aligned with the social optimum. It is well-known that pairwise coordination is generally not enough to reach the social optimum as an improving coalition may be arbitrarily large.

Our second main result concerns the dynamic process of network formation. If all players are best-responders but receive optimistic shocks to their beliefs in certain time periods, then all links in a social optimum will form. If the frequency of shocks is high, optimum links are reached faster but players will spend more time in networks where they are not at their objective best response. If this frequency is lower, optimum links are reached later but players spend more time at their objective best response – i.e. in the actual social optimum.

Our results suggest that optimism plays a crucial role in the formation of social networks. Generally, optimal coordination across very large networks is achieved by selfish, myopic improvements by the players. Without coordination, efficient outcomes of network formations cannot be expected. For instance, it is well-known that in two-sided network formation games there is a multiplicity of Nash equilibria with many being inefficient, e.g., the empty network is always an equilibrium. Player optimism improves social gains in two ways: (1), it motivates players to offer links to opponents without an immediate reciprocation, and (2), it motivates players to seek out the most profitable links of their opponents. Through optimism, the players, and thus society will improve social value from connections until only their best links remain. These links, however, will only be reciprocated if both players find it optimal to connect to each other. Thus, whether the outcome is a social optimum depends on whether individual preferences align with social



ones to reach the maximum-weight network. Unless this condition is met, however, the social optimum is generally not Pareto-optimal nor pairwise stable, hence we cannot expect it to be self-enforcing.

Several questions remain to be addressed in future research. First, it would be of interest to allow also for pessimistic shocks and study their impact on network formation. We conjecture that as long as pessimistic shocks are relatively rare with respect to optimistic ones, the optimality results established above will continue to hold. Second, it would be of interest to examine further the properties of the networks which form as a result of our adaptive dynamics and check in how far they satisfy other refinement criteria suggested in the literature, such as, e.g., strong stability as defined by Jackson and van den Nouweland (2005). Finally, generalizing the shape of the link-strength function, similarly to the works of Griffith (2019); Baumann (2019); Ding (2019) might be of interest as well.

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