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Optimal Promotions of Competing Firms in a Frictional Labour Market with Organizational Hierarchies*

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Abstract

We study optimal promotion decisions of hierarchical firms, with one junior and one senior managerial position, which interact in a search and matching labour market. Workers acquire experience over time while being employed in a junior position and the firm has to determine the experience level at which the worker receives a promotion which allows her to fill a senior position. Promoted workers move to the senior position in their current firm, if it is vacant, otherwise they search for senior positions on the market. The promotion cut-offs of the competing firms exhibit strategic complementarity, but we show that generically a unique stable symmetric general equilibrium exists. If workers have homogeneous skills, then an increase in the skill level induces faster promotion. In the presence of two skill levels in the work force an increase of the fraction of high skilled leads to slower promotion of both types of workers, where the promotion threshold for high skilled workers is substantially below that for low skilled workers. This implies earlier promotions of high skill workers compared to the low skilled consistent with available empirical evidence. Finally, we show that a larger number of competitors in the market leads to earlier promotions. This finding extends to low skill workers in the market with skill heterogeneity. But the impact of competition on the promotions of high skill workers is non-monotone.

Keywords: promotions, theory of the firm, search and matching labour market, organizational hierarchies, job-to-job mobility

JEL Classification: C63, D21, J63, J64

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1 Introduction

Empirical evidence suggests that workers progress in their careers by means of internal promotions within firms, job-to-job transitions between firms and experience accumulation. However, existing research analyzes promotions and job-to-job mobility within different strands of literatures. Whereas search and matching studies developed strong techniques for the analytical treatment of on-the-job search and between-firm mobility of workers, research on internal promotions within firms is conducted in the literature on internal labour markets and principle agent models¹. In this study we develop a unified search and matching framework with hierarchical firms, experience accumulation, job-to-job mobility and internal promotions. A combination of these areas leads to new insights on how the composition of the applicant pool, strategic competition between firms and search frictions influence the optimal timing of promotions. In particular, our model shows that promotion times of competing firms are strategic complements meaning that later promotions of the market competitors lead to delayed promotions of a given firm. At the same time, in a market with heterogeneous workers, increasing the number of competing firms has qualitatively different effects on the optimal promotion threshold for low and high skilled workers and may have a non-monotone impact on the speed of promotions.

Modeling the impact of labour market competition on the decisions of firms is especially important in the view of the increasing shortage of (qualified) labour power observed in a number of developed economies. For example, in the USA and in Germany aggregate statistical data shows that the number of applicants per vacancy was falling and the average vacancy durations were increasing in the past decade due to the growing disbalance in the supply and demand of labour². At the same time, a rigorous analytical investigation of the impact of stronger labour market competition among firms on their internal promotion decisions is missing in the literature. Our paper aims to fill this gap by setting up a search and matching model with hierarchical firms, internal and external worker mobility and endogenous promotion times chosen by firms competing on the labour market.

More specifically, we develop a search and matching model with three hierarchical levels in the career ladder. The first level consists of non-managerial jobs available to all workers without frictions. In addition, there are firms in the market consisting of two professional positions: one junior position and one senior position. This structure implies that there are three hierarchical job levels and two submarkets in our model: the primary market for young inexperienced individuals applying for their first junior manager position and a secondary market for experienced workers applying for senior manager positions. Firms with open positions post vacancies in each of the two submarkets respectively. As in Gibbons and Waldman (1999), the productivity of junior managers is growing over time due to experience accumulation and there is complementarity between experience and the hierarchical layer the worker is assigned to.

The main choice variable of the firm is the promotion time. Specifically, firms choose the minimum experience cut-off which is necessary for the junior worker to be internally promoted to the senior level. Workers in junior positions achieve this level of experience by means of

¹Excellent surveys on both research directions are Rogerson et al. (2005) and Waldman (2009) respectively

²In Germany there was a dramatic change in the average duration of vacancies from 57 days in November 2010 to 126 days in November 2020. In the same period the number of unemployed persons per vacancy fell from 7.9 to 4.7. (Bundesagentur für Arbeit. Saisonbereinigte Zeitreihen). Also in the US the number of unemployed persons per job opening fell substantially from 4.8 in November 2010 to 2 in September 2020 (BLS JOLTS).

on-the-job training provided by the firm. The experience cut-off is set in the beginning of the employment relationship and is a part of the labour contract. Note that the actual promotion can only take place if the junior worker accumulated the minimum experience level set by the firm and there is an open senior position in this firm. This is different from the model of Gibbons and Waldman (1999), where every worker can always be promoted in every firm and promotions do not depend on the availability of open positions at higher hierarchical levels. The tradeoff for firms can be characterized in the following way: if the inexperienced worker is promoted too early in his/her career, this worker will have a relatively low productivity after the promotion because this worker's experience is too low for the senior level. In this situation it is a better strategy for the firm to wait and search for a more experienced worker in the secondary submarket for senior managers. This submarket exists because some workers have already reached sufficient experience to be promoted, but there are no open positions in their firms. Thus, these workers start searching for senior managerial jobs with alternative employers (on-the-job search). This is different from the classical model of on-the-job search by Burdett and Mortensen (1998), where all employees are searching for better paid jobs, and shows that promotions and on-the-job search are closely linked to each other, moreover, this link is missing in the previous studies.

Based on this model we find that the optimal promotion time of a given firm is increasing in the average promotion time of the market, so there is strategic complementarity between the promotion times of the different firms. This is because the optimal individual promotion time of the firm depends on the distribution of experience of managerial applicants in the secondary submarket, which again is determined by the promotion decisions of the other firms in the market. We account for this competition effect by characterizing Nash equilibrium assuming steady states of the labour flows. We find that there are two symmetric Nash equilibria but only one of them is stable. In addition, we analyze the steady state adjustment of worker stocks and transition probabilities in response to the optimal promotion time set by the firms. We find that this general equilibrium effect is mitigating the individual intentions of firms. In particular, if one firm has incentives to delay promotions of its' junior workers and hire more senior managers in the market, it will choose a higher experience requirement. Positive optimal response implies that other firms also delay promotions of their junior workers and require higher experience. Because of this, workers stay longer in junior positions and there are fewer applicants in the senior submarket, so job-to-job transitions between firms are substantially reduced and internal promotions become a more important source of upward mobility for workers. This shows how the general equilibrium effect counteracts the initial decision of firms.

We consider two extensions of our benchmark model. First, we consider a setup when additional output is generated if two workers (junior and senior) are working together as a team. We find that such team synergy is associated with earlier promotions. The reason is that search frictions in the senior submarket are more severe, so hiring junior workers is easier for firms than hiring experienced managers. So, in order to fill both positions, firms promote their own junior employees earlier compared to the benchmark case and try to hire another junior worker afterwards. This strategy leads to the highest gain from the team synergy for firms.

In the second extension we consider skill heterogeneity of workers, assuming that high skill workers are more productive than low skill workers only in senior managerial jobs. This model extension can explain the empirical evidence that high skill workers are promoted earlier than

low skill workers (Baker et al. (1994), McCue (1996) and Lluís (2005)). In addition, there is substitution between the two skill groups. If there are exogenous reasons forcing firms to promote one skill group earlier, they will delay promotions of the other skill group and let them accumulate more experience. We show that better quality of the labour force associated with a larger fraction of high skill workers induces slower promotions, whereas in a setting with homogeneous labour an increase of the skill level leads to faster promotions. Therefore, an improvement in the average skills of the labour force may have different implications for promotions depending on the exact reason for such an improvement. A larger fraction of high skill workers in the market raises the expected skill of an external hire *relative* to the skill of the incumbent junior worker under consideration for a promotion. In this situation it is optimal for firms to delay promotions gaining more time for an external hire. In the case of homogeneous workers, the skills of all workers are improving – external applicants *and* incumbent candidates – which leads to earlier promotions.

Finally, we investigate the impact of competition on the optimal timing of promotions, which is captured by increasing the number of competing firms in the market. Here our model delivers a number of testable implications. First, a larger number of firms in the market tightens the competition on both submarkets since there are more positions on all hierarchical levels. Our model predicts that this change should lead to earlier promotions because hiring workers to senior positions takes longer and is associated with a higher expected search cost. Thus firms promote their workers earlier in order to reduce the relatively stronger competition pressure in the senior submarket. Second, this result extends to the case of heterogeneous workers when it comes to promotions of low skill workers, which is the majority group. However, the impact of competition on the speed of promotions in the group of high skill workers is non-monotone. When the number of firms in the market is low, stronger competition may lead to delayed promotions of high skill workers. The reason is that the substitution effect between the optimal promotion times of high and low skill workers dominates the competition effect when the number of firms in the market is low.

Overall, our results indicate that the pressure of competition may lead to a situation when senior positions in firms are increasingly filled with low skill workers possessing low managerial experience which may harm the future prospects of the whole industry. However, this negative tendency may be mitigated by macroeconomic policies softening the shortage of labour and increasing the supply of workers in specific skill groups. Such policies can include, for example, targeted migration, higher participation of women in the labour market and extended educational infrastructures.

Our study is closely related to the literature on organizational hierarchies and internal labour markets. Organizational hierarchies are intensively studied since the seminal contribution by Garicano (2000). This paper considers an endogenous formation of firm hierarchies based on the time constraint for acquiring knowledge by workers. Some (ex-ante homogeneous) agents acquire special knowledge and are specializing in problem-solving; these agents are the managers and are situated on the top level of the firm hierarchy, while other agents are specialized on the actual production. This benchmark model is extended in different directions by Garicano and Rossi-Hansberg (2015). The literature on knowledge-based hierarchies is successful in explaining empirical facts and it is an appealing feature of this theory that hierarchies arise endogenously

when matching problems to those who know how to solve them. On the other hand, this research direction is lacking dynamics in individual careers, as workers assigned to different levels are never promoted within or across firms, thus there is no link between organizational hierarchies and career paths of individuals.

The second research stream is dealing with internal labour markets, so the main focus here is on individual career paths and promotions but the firm hierarchy is taken exogenously and fixed in this literature. One large research direction here includes tournament models in the spirit of Lazear and Rosen (1981). In their setting promotion decisions are modeled as a tournament in which workers exert costly effort to perform better than their coworkers and to be considered for promotion. Later tournament models include the fact that promotions can be used as a signal of higher ability, see for example, Zabojnik and Bernhardt (2001). Waldman (2003) finds that due to time inconsistency problem firms are likely to prefer internal candidates for promotions as to provide incentive to entry level workers to choose high effort. Recent studies, such as DeVaro (2006) confirm empirically that firms are choosing wage spreads strategically to elicit more effort from their employees. Chan (2006) further shows that firms prefer internal promotions because promotions are an important incentive mechanism for the employees. Therefore external hires usually have higher ability compared to internal candidates promoted to the same hierarchical level. In addition, DeVaro and Waldman (2012) find that promotions are sometimes used as a signal of worker's ability. While the role of competition in providing working incentives to employees must be acknowledged, we focus on human capital accumulation as a reason for promotion and analyse between-firm competition for experienced employees.

The literature on human capital accumulation and job assignments is more closely related to our research. The seminal contribution here is by Gibbons and Waldman (1999). In their study worker's productivity depends on the individual's skill level, accumulated experience and the hierarchical layer the worker is assigned to. As workers accumulate experience and knowledge they are optimally promoted by firms to higher positions due to the assumed complementarity between worker's productivity (skills and accumulated experience) and hierarchical layers within the firm. We use the same setup as a starting point in our model, however, we make a step further by embedding firms in a rich labour market with search frictions and worker mobility between firms in order to analyze the impact of labour market competition on the optimal promotion strategies of firms. Overall, the literature on career paths and promotions is successful in explaining wage dynamics of individuals within firms, whether due to experience accumulation or exerted effort. However, most of this literature is based on the principal agent modeling approach in isolation from the labour market and doesn't allow for the study of interaction between organizational structures and the economy. Most of these studies make restrictive assumptions on the model structure ensuring that there is no worker mobility between firms in the equilibrium.

Next, our study is conducted in the search and matching framework (Diamond (1982), Mortensen (1982), and Pissarides (1985)). We model job-to-job transitions following the approach of Burdett and Mortensen (1998). To the best of our knowledge the first study analyzing tenure in a search and matching framework with job-to-job transitions is Pissarides (1994). There are good and bad jobs in his setting, thus unemployed workers accept bad jobs but continue searching for good jobs. An important feature of the model is that workers accumulate job-

specific experience and their wage grows over time. In the equilibrium very experienced workers with high wages stop searching at all since the gain from moving to a good job becomes smaller than the cost of searching. The main difference of this study from current work is that we treat experience as transferable across firms while it is completely lost upon the quit in Pissarides (1994). Recent work in this field includes prominent extensions by Burdett and Coles (2003), Burdett et al. (2011) and Bagger et al. (2014). These studies analyse tenure accumulation with on-the-job search, but they do not consider internal promotions.

The rest of the paper is structured as follows. In section 2 we introduce the economic framework and analyse the dynamics of workers and firms across states. Section 3 presents the value functions of firms and their choice of the optimal promotion time as well as the emerging partial and general equilibrium in the benchmark setting. In section 4 we extend the model to two skill groups. Section 5 concludes the paper.

2 The Model

2.1 The economic framework

Time is continuous with an infinite horizon. There is a continuum of both firms and workers with a total measure of workers normalised to 1. The inflow of new workers into the labour market is denoted by d . In the benchmark model all entering workers are homogeneous with identical skills, however, in the extension we also analyze consequences of skill heterogeneity. Job ladders have three hierarchical levels. All young workers entering the market immediately take simple jobs on the low level. These are subsistence jobs that don't yield any professional experience. All entering firms are identical and every firm is a dyad consisting of two positions: one junior position and one senior (managerial) position. The inflow of new firms is denoted by n . Both positions are empty when the firm enters the market and can be posted simultaneously. Posting an open position (junior or senior) is associated with a flow cost s for the firm. For the purpose of tractability we assume that there are no dismissals, thus the pool of applicants for junior positions consists of young workers employed in low level jobs. Only workers with substantial professional experience are eligible to apply for senior positions. Let e_0 denote the stock of workers in low level jobs, e_1 – are workers employed in junior positions and e_2 denotes managers in senior positions, so that $e_0 + e_1 + e_2 = 1$ due to the normalisation.

Once accepted in the junior position young workers start accumulating professional experience $x \geq 0$ with $\dot{x} = 1$. We model experience accumulation as on-the-job training provided by firms. Thus every firm i decides about the length of training \bar{x}_i . Beyond this level of experience workers are expected to focus on their job tasks and firms do not permit further training activities at work. Intuitively, we capture situations when firms encourage junior workers to attend training courses taking a part of the working time (e.g. language and computer courses, MBA or CFA, dual studies). Experience x is transferable but not observable by other firms in the market until it becomes \bar{x}_i when the worker receives a certificate of completed training proving a high level of managerial experience. Even though \bar{x}_i is an endogenous choice variable of the firm, we assume that it is written down in the labour contract in the beginning of the employment relationship and verifiable by court. Moreover, an experience certificate makes the worker eligible for promotion to the senior position in all firms including the current employer.

Thus workers with experience \bar{x}_i are promoted to the senior position in their firm if the senior position in this firm is open. In the opposite case when the senior position is filled, the worker starts applying to senior positions in other firms. This is the process of on-the-job search. We assume that the training certificate is a sufficient proof of experience for other employers.

This model structure leads to the existence of two separate submarkets, one where firms are posting junior positions and anticipate a worker with $x = 0$ and another one where firms are posting their senior positions and anticipate workers searching on-the-job and possessing a proof of sufficient experience. Workers employed in junior positions produce output $d_1 + c_1 e^{\gamma x}$, whereas workers employed in senior positions produce output $d_2 + c_2 e^{\gamma x}$, where $d_1 > d_2$ and $c_1 < c_2$ as in Gibbons and Waldman (1999). Intuitively, this means that the fixed component of output d_j , $j = 1, 2$ is falling with a higher hierarchical level, while experience becomes more important, that is c_j , $j = 1, 2$ is increasing with j . The multiplier parameter γ is a proxy for the return to worker's experience. In a symmetric equilibrium all firms choose an identical promotion cut-off \bar{x} , thus firms correctly anticipate that applicants to senior positions achieved an experience level \bar{x} and their output in senior positions is $d_2 + c_2 e^{\gamma \bar{x}}$. There is no experience accumulation in senior positions in the sense that there is no training, so output is constant. Workers employed in senior managerial positions retire at an exogenous rate ρ . If the manager retires and the junior position is not filled, the firm is empty and exits the labour market. In our analysis we only consider the steady state, moreover the entry and exit parameters d and ρ are chosen to keep the population size constant.

Since the focus of the paper is on the optimal promotion decisions of firms and feedback effects of these decisions on the resulting structure of the labour market, we assume that workers don't act strategically in the model and take their behavior as given. Specifically, young workers without experience are always searching for their first professional job, accumulate experience till the level specified in their labour contract and start applying to managerial jobs if there is no open position in their firm. It is a simplifying assumption of the model that there is no labour market exit among younger workers employed in low level jobs and junior positions.

Let $1 - \beta$ denote the fraction of output accruing to firms, thus the flow profit is equal to $(1 - \beta)(d_j + c_j e^{\gamma x})$ depending on the hierarchical level of the position $j = 1, 2$ and worker's experience x . This is a flow profit after subtracting all the costs (e.g. wages, capital costs and the costs of on-the-job training). Further, we introduce a profit synergy Δ if the firm is employing both workers simultaneously, that is, one junior worker accumulating experience and one senior manager. So the total profit of this firm is given by $(1 - \beta)(d_1 + c_1 e^{\gamma x} + d_2 + c_2 e^{\gamma \bar{x}}) + \Delta$. Intuitively, this is a synergy from team work because younger inexperienced workers gain from the advice of senior managers, whereas senior managers may gain from the innovative new ideas of younger workers³.

Variable d_{00} denotes the stock of empty new firms in the market, whereas d_{01} is the stock of firms with a senior manager but no junior worker. Since all these firms have an open junior position the total stock of open junior positions available for matching is equal to $d_{00} + d_{01}$. These positions are randomly matched with $z e_0$ searching inexperienced workers, where z denotes the search effort of workers. More precisely, z is the fraction of searching workers who prepare

³An alternative specification with a total profit given by $(1 - \beta)(d_1 + c_1 e^{\gamma x} + d_2 + c_2 e^{\gamma \bar{x}} + \Delta)$ does not alter our results.

and send an application at every instant of time. To determine the number of matches in the submarket for junior positions we use an urn-ball matching mechanism. Suppose some worker sends an application to one randomly chosen firm, then the probability that a given firm doesn't receive this application is $1 - \frac{1}{d_{00}+d_{01}}$. Since workers send their applications independently without coordination, the probability that this firm doesn't get any of the ze_0 applications is given by $(1 - \frac{1}{d_{00}+d_{01}})^{ze_0}$. Let q_1 be the job-filling rate resulting from this application process and λ_1 be the job-finding rate for inexperienced workers. They are given by:

$$q_1 = 1 - \left(1 - \frac{1}{d_{00} + d_{01}}\right)^{ze_0} \quad \lambda_1 = z \frac{q_1(d_{00} + d_{01})}{ze_0} = q_1 \frac{(d_{00} + d_{01})}{e_0} \quad (1)$$

The term $q_1(d_{00} + d_{01})$ is a total number of matches in the junior market, thus $\frac{q_1(d_{00}+d_{01})}{ze_0}$ is a probability of matching for workers conditional on sending an application in a given matching round. Multiplying this conditional matching probability with z we obtain the job-finding rate for junior workers. Further, let d_{10} denote firms with a junior worker but no senior manager. This means that the total number of open managerial positions is given by $d_{00} + d_{10}$. Finally, let d_{11}^N denote the stock of full firms with both employees, where the worker in the junior position is not yet eligible for promotion ($x < \bar{x}$). In a similar way, d_{11}^S is the stock of full firms, where the junior worker is already eligible for senior positions and searching on-the-job. This means that the stock of applicants in the managerial market is given by zd_{11}^S . So the job-filling rate in the managerial market q_2 and the workers' job-finding rate in this market λ_2 are given by:

$$q_2 = 1 - \left(1 - \frac{1}{d_{00} + d_{10}}\right)^{zd_{11}^S} \quad \lambda_2 = z \frac{q_2(d_{00} + d_{10})}{zd_{11}^S} = q_2 \frac{(d_{00} + d_{10})}{d_{11}^S} \quad (2)$$

Note that we assume the same search intensity parameter z in both markets. This setting can be generalized to different search intensities for experienced and inexperienced workers, however, it is not important for our main results. So we keep the model simple and consider only one search intensity parameter z .

The total number of firms in the market F is given by $d_{00} + d_{01} + d_{10} + d_{11}^N + d_{11}^S$. This notation also allows us to calculate the number of workers, so normalizing the population size to 1 yields:

$$e_0 + d_{10} + d_{01} + 2d_{11}^N + 2d_{11}^S = 1$$

Here $e_1 = d_{10} + d_{11}^N + d_{11}^S$ is the total number of employees in junior positions, and $e_2 = d_{01} + d_{11}^N + d_{11}^S$ is the total number of employees in senior positions.

2.2 Firm Dynamics

Transitions of firms are illustrated in figure 1. Consider changes in the stock of new empty firms d_{00} . The inflow of new firms into the market is given by n . Since every new firm posts both the junior and the senior position in the respective submarkets it exits the state d_{00} whenever it finds the first employee. So the outflow of firms from d_{00} takes place at rate $q_1 + q_2$. In this paper we restrict our analysis to the steady states and consider a stationary distribution of workers

and firms across states. This means that $\dot{d}_{00} = 0$ in the steady state:

$$0 = \dot{d}_{00} = n - (q_1 + q_2)d_{00} \quad \Rightarrow \quad d_{00} = \frac{n}{q_1 + q_2} \quad (3)$$

The entry of firms into the market is given by n , whereas the exit is ρd_{01} . These are the firms that lose their only employee due to retirement, which happens at rate ρ . Thus we get $d_{01} = n/\rho$ to guarantee a constant number of firms in the market. This is equivalent to the standard assumption of a constant population of workers.

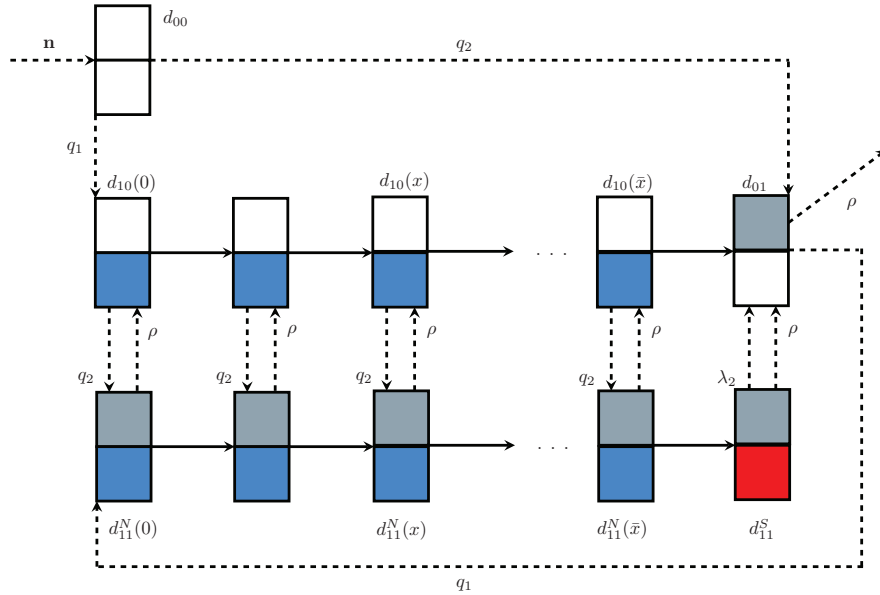


Figure 1: Types of firms and their transitions

Further, consider changes in the stocks of firms $d_{10}(x)$ and $d_{11}^N(x)$. Note that workers with experience $0 \leq x \leq \bar{x}$ are not yet searching on-the-job since their experience is not sufficient for managerial positions and there are no gains from changing to another junior job. Variable \bar{x} here denotes the equilibrium promotion cut-off and will be determined in section 3. This means that the inflow of firms into state $d_{10}(x)$ is equal to $\rho d_{11}^N(x)$. These are the firms where the manager retires at rate ρ and they are left with only one junior worker. At the same time $\rho d_{11}^N(x)$ is the outflow of firms from the state $d_{11}^N(x)$. If the manager retires firms post the open position in the second submarket for experienced workers and find a manager at rate q_2 . This means that the outflow of workers from the state $d_{10}(x)$ is equal to $q_2 d_{10}(x)$. This is also the inflow of firms into the state $d_{11}^N(x)$. So we get the following system of two first order linear differential equations⁴:

$$\begin{cases} \partial d_{10}(x)/\partial x &= -q_2 d_{10}(x) + \rho d_{11}^N(x) \\ \partial d_{11}^N(x)/\partial x &= q_2 d_{10}(x) - \rho d_{11}^N(x) \end{cases}$$

⁴In general the stock variable $d_{10}(x, t)$ may depend on time t , so the total derivative is given by:

$$\frac{\partial d_{10}(x, t)}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial d_{10}(x, t)}{\partial t} = -q_2 d_{10}(x) + \rho d_{11}^N(x)$$

Since the distribution of firms $d_{10}(x, t)$ is stationary in the steady state we set the time derivative $\dot{d}_{10} = \frac{\partial d_{10}(x, t)}{\partial t}$ equal to zero. Moreover, experience x is accumulating one to one with the time because $\dot{x} = \partial x / \partial t = 1$.

The coefficient matrix of this homogeneous system has eigenvalues 0 and $-(\rho+q_2)$, so the general solution is given by:

$$\begin{cases} d_{10}(x) &= k_1\rho + k_2e^{-(\rho+q_2)x} \\ d_{11}^N(x) &= k_1q_2 - k_2e^{-(\rho+q_2)x} \end{cases}$$

In order to find the constant terms k_1 and k_2 we use the following initial conditions: $q_1d_{00} = d_{10}(0)$ and $q_1d_{01} = d_{11}^N(0)$. The first condition implies that the stock of firms $d_{10}(0)$ always consists of new firms finding their first junior worker q_1d_{00} . The second condition implies that the stock of firms $d_{11}^N(0)$ consists of firms d_{01} who find a junior worker, that is q_1d_{01} . Using these initial conditions we find that:

$$k_1 = \frac{q_1n(\rho + q_1 + q_2)}{\rho(\rho + q_2)(q_1 + q_2)} > 0 \quad k_2 = -\frac{(q_1)^2n}{(\rho + q_2)(q_1 + q_2)} < 0$$

One can see that $k_2 < 0$, this means that $d_{10}(x)$ is increasing while $d_{11}^N(x)$ is decreasing in x . Intuitively this means that the flow $\rho d_{11}^N(x)$ due to retirement of senior managers always dominates the flow $q_2d_{10}(x)$ implying that finding senior managers is a difficult task for firms in the considered setting. Note that the sum of two variables is a constant, that is $d_{10}(x) + d_{11}^N(x) = k_1(\rho + q_2) \forall x \in [0..\bar{x}]$.

By integrating variables $d_{10}(x)$ and $d_{11}^N(x)$ over the interval $[0..\bar{x}]$ we find the total stocks of firms d_{10} and d_{11}^N :

$$d_{10} = \int_0^{\bar{x}} d_{10}(x)dx = k_1\rho\bar{x} + \frac{k_2}{\rho + q_2}(1 - e^{-(\rho+q_2)\bar{x}}) \quad (4)$$

$$d_{11}^N = \int_0^{\bar{x}} d_{11}^N(x)dx = k_1q_2\bar{x} - \frac{k_2}{\rho + q_2}(1 - e^{-(\rho+q_2)\bar{x}}) \quad (5)$$

The remaining unknown stock of firms is d_{11}^S . These are the firms with two employees, where the junior one is already searching for jobs with alternative employers. All firms of type $d_{11}^N(\bar{x})$ automatically enter the state d_{11}^S since the junior worker starts searching on-the-job upon attaining experience \bar{x} . This is the inflow of workers into the state d_{11}^S . At rate ρ the senior manager retires and the firm promotes the junior worker to the managerial job. In addition, it can also happen that the junior worker finds a new employer at rate λ_2 . As one can see from figure 1, in both cases the firm leaves the state d_{11}^S and enters the stock of firms d_{01} . Hence we get:

$$0 = \dot{d}_{11}^S = d_{11}^N(\bar{x}) - (\rho + \lambda_2)d_{11}^S \quad \Rightarrow \quad d_{11}^S = \frac{d_{11}^N(\bar{x})}{\rho + \lambda_2} = \frac{k_1q_2 - k_2e^{-(\rho+q_2)\bar{x}}}{\rho + \lambda_2} \quad (6)$$

Finally, recall that e_0 are the young individuals searching for their first professional job, so that $\dot{e}_0 = d - \lambda_1e_0$. In the steady state it should be that the inflow into this state d should be equal to the outflow λ_1e_0 , where the outflow are young inexperienced workers finding their first employer. So we get $e_0 = d/\lambda_1$. Variable d is the endogenous entry of young individuals, which we can find from normalising the total population of workers to 1:

$$\frac{d}{\lambda_1} = 1 - (d_{10} + d_{01} + 2d_{11}^N + 2d_{11}^S) \quad (7)$$

Solving jointly the system of equations (2)-(7), $d_{01} = n/\rho$, $e_0 = d/\lambda_1$ we can find the equilibrium distribution of firms $\{d_{00}, d_{10}, d_{11}^N, d_{11}^S, d_{01}\}$, as well as variables d and e_0 and the equilibrium transition rates λ_j , and q_j , $j = 1, 2$. Note that variable \bar{x} (promotion cut-off) is taken as given at this stage and will be endogenously derived in section 3.

2.3 Transition rates

We proceed by illustrating the mechanism of our model with a help of a numerical example which resembles realistic career paths of workers in developed economies. In this section we focus on the transitions of workers and firms for a given promotion cut-off \bar{x} . One period of time is set to be one quarter. Consider young workers entering the market at the age of 18 years. Variable z is the search intensity parameter which is the driving force behind the job-finding rate λ_1 . We set $z = 0.0146$, this corresponds to $\lambda_1 = 0.0145$ and implies that workers stay in level 0 jobs for approximately $1/\lambda_1 = 69$ quarters or 17.25 years. Intuitively, this means that workers find their first managerial job on level e_1 at the age of 35.25 years on average. In state e_1 workers start accumulating professional managerial experience x . We consider $\bar{x} = 45$, so that it takes 45 quarters or 11.25 years for workers to be eligible for the position of a senior manager. Thus workers reach the pre-specified necessary level of experience at the age of 46.5 years on average.

Recall that $d_{10}(\bar{x})$ is a stock of workers who are directly promoted to senior positions within their firm at every point in time. At the same time $d_{11}^N(\bar{x})$ is a stock of workers eligible for promotions, however, they can not be promoted directly within their firm since the senior position is occupied. These workers start searching on-the-job and enter the accumulated pool of workers searching and applying to senior positions d_{11}^S . So the total stock of workers eligible for promotion in a given period of time is $d_{10}(\bar{x}) + d_{11}^N(\bar{x}) + d_{11}^S = k_1(\rho + q_2) + d_{11}^S$. Out of these workers $d_{10}(\bar{x}) + (\rho + \lambda_2)d_{11}^S$ are actually promoted, where $d_{10}(\bar{x}) + \rho d_{11}^S$ are promoted directly within their firms and $\lambda_2 d_{11}^S$ make a transition to a senior position in another firm. So the average duration of time from the moment of becoming eligible \bar{x} till the actual promotion within or between firms is given by:

$$\frac{k_1(\rho + q_2) + d_{11}^S}{k_1\rho + k_2e^{-(\rho+q_2)\bar{x}} + (\rho + q_2)d_{11}^S}$$

In our model this duration is equal to 14 quarters or 3.5 years, so that workers become senior managers at the age of 50 years on average. This duration is achieved by setting the number of entering firms n equal to 0.0026. This also implies that the average stock of firms active in the market is equal to 0.6. So there are on average 600 active firms or 1200 positions per 1000 workers. However, not all of these positions are filled due to the search frictions and experience requirements. Further, we set $\rho = 0.015$, so the average time workers spend in senior positions till retirement is $1/\rho = 66.6$ quarters or 16.6 years. So workers retire on average at the age of 66.6 years. Finally, the total population is normalized to 1. Given that the exit rate of workers is $\rho = 0.015$, constant size of the population can be achieved by setting $d = 0.0052$. This means that 5.2 workers on average enter the market with a population of 1000 workers. Our choice of parameters at this stage is summarized in table 1. Note that variable \bar{x} is endogenous in the overall model, even though we keep it fixed at the current stage of analysis. Endogenous values of the quarterly transition rates in the steady-state are summarized on the right side of table 1.

| Parameter | Value | Interpretation | Variable | Value | Interpretation |
|-----------|--------|-----------------------------|-------------|--------|---------------------------|
| z | 0.0146 | Search intensity of workers | q_1 | 0.0171 | Job-filling rate, level 1 |
| ρ | 0.0150 | Exit/retirement rate | q_2 | 0.0036 | Job-filling rate, level 2 |
| n | 0.0026 | Entry of empty firms | λ_1 | 0.0145 | Job-finding rate, level 1 |
| d | 0.0052 | Entry of young workers | λ_2 | 0.0146 | Job-finding rate, level 2 |

Table 1: Values of exogenous parameters and quarterly transition rates

Table 2 shows the distributions of workers and firms in the steady-state. We can see that 35.7% of all workers remain on average in simple jobs e_0 . Further, 29.7% are employed in junior positions e_1 , where 6.3% of workers are searching on-the-job and applying to senior positions (d_{11}^S). 34.5% of workers occupy senior management positions e_2 . These numbers imply that $p_1 = 0.297/(0.297 + 0.345) = 0.462$, that is 46.2% of workers in professional jobs are employed in junior positions, with the remaining 53.7% being employed in senior positions. Considering transitions of workers, we can see that 1.2% of e_1 workers reach senior positions by changing employers. Another 5.7% of junior workers are internally promoted within their firms per year. Even though internal mobility of workers is not intensive, these numbers are close to the empirical findings. For example, Lluís (2005) finds that in Germany the annual probability of internal promotions is 5.7% for relatively young workers with less than 10 years of market experience and it falls afterwards with an average for all workers groups equal to 2.7%. The same study reports that internal mobility is more intensive in the US, with 6.7% for men and 6.2% for women with less than 10 years of experience and 5.0% on average for all men (4.6% for all women). A more recent study by Cassidy et al. (2016) reports an average probability of internal promotions equal to 4.6% in Finland.

| Variable | Value | Variable | Equation | Value |
|------------|--------|----------------------------------|---|--------|
| d_{00} | 0.1273 | Workers in simple jobs e_0 | $= 1 - e_1 - e_2$ | 0.3577 |
| d_{01} | 0.1760 | Workers in junior jobs e_1 | $= d_{10} + d_{11}^N + d_{11}^S$ | 0.2966 |
| d_{10} | 0.1270 | Workers in managerial jobs e_2 | $= d_{01} + d_{11}^S + d_{11}^N$ | 0.3456 |
| d_{11}^S | 0.0633 | Internally promoted (per year) | $= (d_{10}(\bar{x}) + \rho d_{11}^S)/e_1$ | 0.0576 |
| d_{11}^N | 0.1063 | Job-to-job movers (per year) | $= \lambda_2 d_{11}^S/e_1$ | 0.0124 |

Table 2: Stationary distributions of workers and firms for parameters from Table 1 and $\bar{x} = 45$

The left panel of figure 2 shows the stocks of firms $d_{10}(x)$ and $d_{11}^N(x)$ for different experience levels x of the junior worker. As expected $d_{10}(x)$ is increasing, while $d_{11}^N(x)$ is decreasing with x . Note that the starting ratio of these two stocks is $d_{10}(0)/d_{11}^N(0) = \rho/(q_1 + q_2)$ but the long-run ratio for larger values of x is: $\lim_{x \rightarrow \infty} d_{10}(x)/\lim_{x \rightarrow \infty} d_{11}^N(x) = \rho/q_2$. So the ratio is clearly increasing with higher experience levels. At the same time we know that the sum of these two stocks is fixed and equal to $k_1(\rho + q_2)$ and each of them is a monotonous function of x . This confirms again that $d_{10}(x)$ should be increasing. So as workers accumulate more and more experience they are more likely to find themselves in a situation with an open senior position. The reason is that senior managers retire over time, but the probability of substituting them with an external candidate is relatively low.

The right panel of figure 2 shows comparative statics results with respect to the promotion cut-off \bar{x} . We vary this variable in the range [30..60] quarters or [7.5..15] years, with the bench-

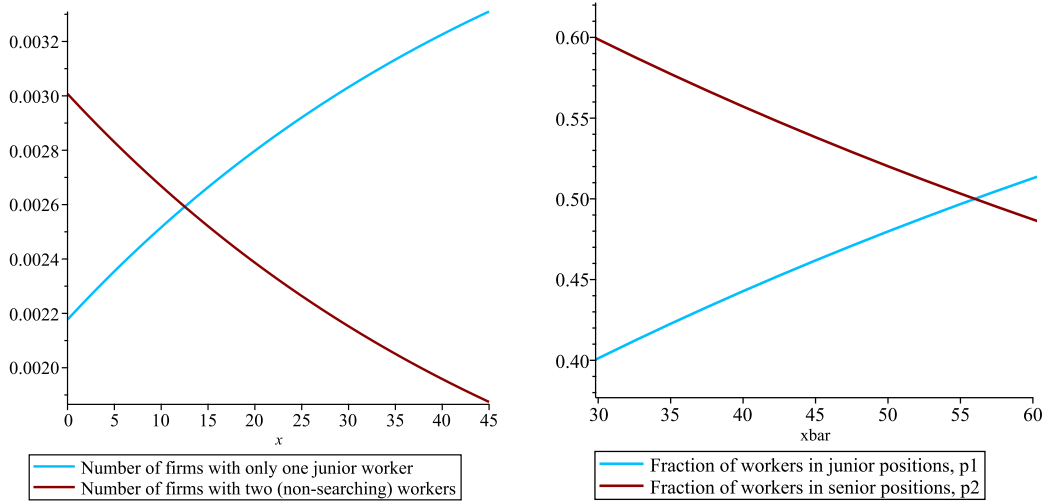


Figure 2: Left panel: Numbers of firms with only one worker in the junior position $d_{10}(x)$ and with two (non-searching) workers $d_{11}^N(x)$ as a function of worker's experience x ($\bar{x} = 45$). Right panel: Fractions of workers employed in the junior level $p_1 = e_1/(e_1 + e_2)$ and in the senior level $p_2 = 1 - p_1$ depending on the promotion cut-off \bar{x}

mark value $\bar{x} = 45$, that is 11.25 years. We can see that earlier promotions reduce the fraction of workers in junior positions p_1 and increase the fraction of workers in senior positions $p_2 = 1 - p_1$. If we consider the implications of earlier promotions for the pool of applicants to senior positions then there are two counteracting effects. If there are many open senior vacancies in the economy then a smaller \bar{x} will lead to many internal promotions, so the pool of external applicants to senior positions will diminish. But on the other hand, if the number of senior positions is limited and internal promotions are rare, a smaller \bar{x} will increase the pool of external applicants to senior positions. We find that the second effect is dominating in our setting. This is a general equilibrium effect, which is not anticipated by individual firms when they choose their optimal promotion cut-off.

The left panel of figure 3 shows changes in the mobility of workers between levels 1 and 2 with respect to the promotion cut-off \bar{x} . Later promotions reduce the intensity of transitions from junior to senior positions. Both internal promotions and job-to-job transitions are less frequent with a higher promotion cut-off. This is because workers have to wait longer for the experience certificate proving their skills to other employers. The same figure (right axis) also illustrates the relative fraction of internally promoted workers, we obtain it by dividing the number of promoted workers $d_{10}(\bar{x}) + \rho d_{11}^S$ with a total number of workers making it to the senior position $\lambda_2 d_{11}^S + d_{10}(\bar{x}) + \rho d_{11}^S$. We can see that this relative fraction is increasing from 77% when $\bar{x} = 25$ to 86% when $\bar{x} = 65$. This reveals an unusual general equilibrium effect in our model. If some firm i decides to delay internal promotions and wants to hire more senior managers on the external market it sets a higher cut-off value \bar{x}_i . However, if all firms follow the same strategy and set a higher cut-off \bar{x} then the relative fraction of senior managers reaching senior positions via internal promotions is increasing. Thus internal promotions become a more important source of upward mobility for workers even though the individual intention of every firm is different. The reason is that with a higher experience requirement \bar{x} , there are less applicants in the external market, so the job-to-job mobility rate declines stronger than the internal promotion rate.

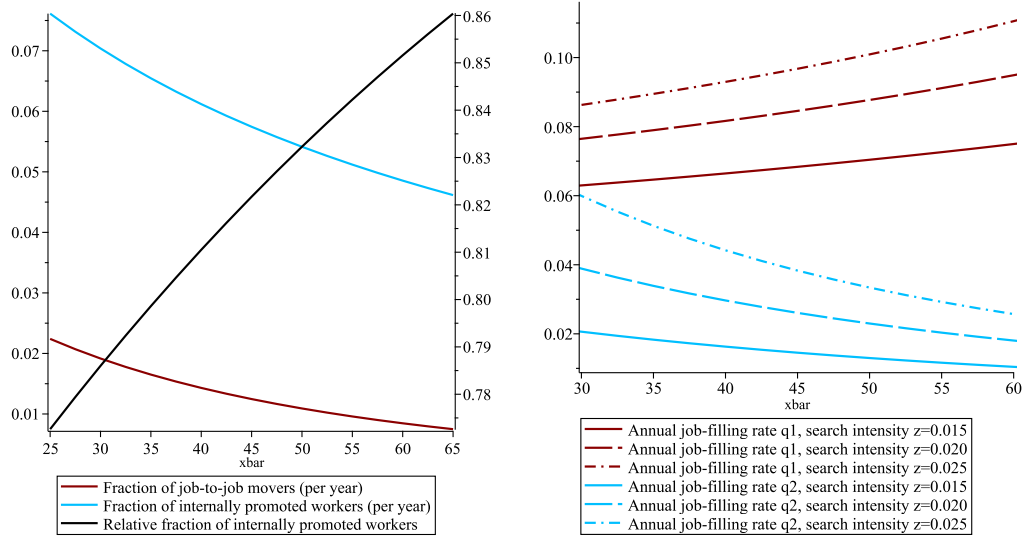


Figure 3: Selected variables for different values of the promotion cut-off \bar{x} and search intensity z . Left panel: Fractions of internally promoted workers $(d_{10}(\bar{x}) + \rho d_{11}^S)/e_1$ and job-to-job movers $\lambda_2 d_{11}^S/e_1$ per year. Right panel: Annual job-filling rates q_1 and q_2 .

The right panel of figure 3 shows changes in the job-filling rates q_1 and q_2 . More intensive job search by workers makes it easier for firms to fill their open positions, so q_1 and q_2 are both increasing in z . But there are adverse effects of the promotion cut-off \bar{x} . Later promotions reduce the pool of competing vacancies on level 1. Reduced competition of firms in this submarket improves their hiring chances, so the job-filling rate q_1 is increasing with \bar{x} . There is an opposite effect in the second submarket for experienced workers. Delayed promotions reduce the pool of applicants for senior positions which leads to the lower job-filling rate q_2 .

3 Optimal promotion by firms

In this section we analyze the optimal promotion strategy of firms in several steps. First, in Subsection 3.1 we determine the best response of an individual firm to a given promotion threshold used by all other firms. Second, in Subsection 3.2 we do a partial equilibrium analysis and show that for our calibration of the model there is a unique stable fixed point of the best response map for fixed values of the job-filling and job-finding rate. We then show that the transition rates generated under the (partial) equilibrium value of the promotion threshold actually coincide with the values underlying our partial equilibrium analysis. Hence, the obtained promotion threshold also constitutes a symmetric general equilibrium of our model. In Subsection 3.3 we then explore the implications of changes in key parameters on the optimal firm promotion threshold disentangling partial and general equilibrium effects.

3.1 Firm's best response

As a first step we characterize in this subsection the optimal promotion time chosen by an individual firm for a given promotion threshold of all competitors and for given job filling rates for junior and senior positions. Denoting by $J_{00}(\bar{x}_i, \bar{x})$ the present value of a firm starting to search for a worker, i.e. a firm with neither a junior nor a senior level worker, which uses a

promotion threshold \bar{x}_i , whereas all other firms on the market promote at \bar{x} . When a new firm opens it has to choose its promotion strategy and the optimal choice is given by

$$\bar{x}_i^*(\bar{x}) = \arg \max_{\bar{x}_i \geq 0} J_{00}(\bar{x}_i, \bar{x}). \quad (8)$$

In order to analyze this optimization problem the value function J_{00} has to be determined. When entering the market the firm has two open positions – one junior and one senior – so the firm is searching for workers in both markets simultaneously and has a double cost $2s$. Therefore,

$$rJ_{00}(\bar{x}_i, \bar{x}) = -2s + q_1(J_{10}(0|\bar{x}_i, \bar{x}) - J_{00}) + q_2(J_{01}(\bar{x}|\bar{x}_i, \bar{x}) - J_{00}),$$

where $J_{10}(x|\bar{x}_i, \bar{x})$ is the present value for a firm with only one junior worker, whose experience is x , and no senior level worker and $J_{01}(y|\bar{x}_i, \bar{x})$ is the present value for a firm with only one senior level worker, whose experience is y , and no junior worker. If the firm first finds an inexperienced worker, which happens at rate q_1 it moves to the state $J_{10}(0)$, since we know that $x = 0$. In contrast, if the firm first finds a senior manager which happens at rate q_2 it moves to the state $J_{01}(\bar{x})$ since we know that all managers in the senior market have experience \bar{x} .

To determine $J_{10}(x|\bar{x}_i, \bar{x})$ let $J_{11}^N(x, y|\bar{x}_i, \bar{x})$ be the present value of profits for a firm with a worker, whose experience is x , and a manager with experience y . Note that both value functions indirectly depend on the promotion cut-off \bar{x}_i chosen by firm i and on the market experience level \bar{x} chosen by competing firms. Let $\pi_1(x) = (d_1 + c_1 e^{\gamma x})(1 - \beta)$ and $\pi_2(y) = (d_2 + c_2 e^{\gamma y})(1 - \beta)$ denote the flow profits obtained by the firm from a filled junior and senior position respectively. The present value $J_{10}(x|\bar{x}_i, \bar{x})$ is given by the following equation:

$$rJ_{10}(x|\bar{x}_i, \bar{x}) = \pi_1(x) - s + q_2(J_{11}^N(x, \bar{x}|\bar{x}_i, \bar{x}) - J_{10}(x|\bar{x}_i, \bar{x})) + \frac{\partial J_{10}(x|\bar{x}_i, \bar{x})}{\partial x} \quad (9)$$

The firm receives a flow profit $\pi_1(x)$ by employing its worker in the junior position and the worker is accumulating experience x . In addition, the firm pays a flow cost s for posting a vacancy in the market for experienced workers. At rate q_2 the firm is successful in this market and moves to the state $J_{11}^N(x, \bar{x}|\bar{x}_i, \bar{x})$, where \bar{x} is the market level of experience set by other firms and guaranteeing workers' eligibility for senior positions. For the ease of exposition in the following we use $J_{10}(x)$ for $J_{10}(x|\bar{x}_i, \bar{x})$ and $J_{11}^N(x, y)$ for $J_{11}^N(x, y|\bar{x}_i, \bar{x})$ and omit the indirect dependence on $\{\bar{x}_i, \bar{x}\}$ in other value functions. We come back to the explicit notation when we determine the optimal promotion time \bar{x}_i^* of firm i and the equilibrium value of \bar{x} in the end of this section.

Next consider the present value $J_{11}^N(x, y)$, where x is the current experience of the worker in the junior position and y is the constant experience level of the manager. Note that $y = \bar{x}$ if the manager was hired in the market but it can be different from \bar{x} if the manager was promoted within the firm:

$$rJ_{11}^N(x, y) = \pi_1(x) + \Delta + \pi_2(y) - \rho(J_{11}^N(x, y) - J_{10}(x)) + \frac{\partial J_{11}^N(x, y)}{\partial x}$$

Here the firm receives additional profit Δ from teamwork, but may lose the manager due to retirement which happens at rate ρ . Let $\Delta J(x, \bar{x}) = J_{11}^N(x, \bar{x}) - J_{10}(x)$ be the capital gain of the

firm from filling a senior position in the market which guarantees experience $y = \bar{x}$, so that

$$(r + \rho + q_2)\Delta J(x, \bar{x}) = \pi_2(\bar{x}) + \Delta + s + \frac{\partial \Delta J(x, \bar{x})}{\partial x}$$

The general solution of this first order linear differential equation is given by:

$$\Delta J(x, \bar{x}) = \frac{\pi_2(\bar{x}) + \Delta + s}{r + \rho + q_2} + K e^{(r+\rho+q_2)x}$$

where K is the integration constant. This equation shows that the capital gain from hiring a manager in the market has three components: (1) the firm receives the flow profit $\pi_2(\bar{x})$ and (2) the additional profit Δ from team work and (3) the firm saves the cost of posting a vacancy s . Next insert $\Delta J(x, \bar{x})$ into equation (9), this yields:

$$rJ_{10}(x) = \pi_1(x) - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} + q_2 K e^{(r+\rho+q_2)x} + \frac{\partial J_{10}(x)}{\partial x} \quad (10)$$

This allows us to find the general solution for the present value of profits $J_{10}(x)$ (with A denoting the integration constant, see Appendix for the derivation) and $J_{11}^N(x, \bar{x})$. Recall that $J_{11}^N(x, \bar{x}) = \Delta J(x, \bar{x}) + J_{10}(x)$, so we get:

$$\begin{aligned} J_{10}(x) &= \frac{d_1(1-\beta) - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} + A e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r+\rho+q_2)x}}{\rho + q_2} \\ J_{11}^N(x, \bar{x}) &= \frac{(\pi_2(\bar{x}) + \Delta + s)(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{d_1(1-\beta) - s}{r} + A e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{r - \gamma} \end{aligned}$$

Next consider $J_{01}(y)$, which is the present value of profits for a firm with only one manager, whose experience level is y :

$$rJ_{01}(y) = \pi_2(y) - \rho J_{01}(y) - s + q_1(J_{11}^N(0, y) - J_{01}(y))$$

The firm receives the flow profit $\pi_2(y) = (d_2 + c_2 e^{\gamma y})(1 - \beta)$ generated by the manager and is continuously posting a vacancy in the market for junior workers, which is associated with a flow cost s . At rate q_1 the firm is successful in this market and moves to the state $J_{11}^N(0, y)$. This is because applicants to junior positions are young and inexperienced with $x = 0$. Finally, at rate ρ the firm may lose the senior manager and remains empty. All empty firms exit the market. Rewrite $J_{01}(y)$ in the following way:

$$J_{01}(y) = \frac{\pi_2(y) - s + q_1 J_{11}^N(0, y)}{r + \rho + q_1}$$

The last state for the firm is when the junior worker has already accumulated experience necessary for promotion. Recall that \bar{x}_i denotes promotion cut-off of some arbitrary firm i . This means that the junior worker obtains experience evaluation and becomes eligible for senior positions having accumulated experience \bar{x}_i . This promotion cut-off is chosen by the firm upon signing the employment contract. If the senior position is open in firm i , the worker with $x = \bar{x}_i$ is promoted immediately. However, it is also possible that the senior position is occupied, so the worker starts searching for alternative employment. Let $J_{11}^s(\bar{x}_i, y)$ be the present value of

profits for a firm with a searching worker whose experience is (\bar{x}_i) and a manager (y):

$$rJ_{11}^s(\bar{x}_i, y) = \pi_1(\bar{x}_i) + \Delta + \pi_2(y) - \rho(J_{11}^s(\bar{x}_i, y) - J_{01}(\bar{x}_i)) - \lambda_2(J_{11}^s(\bar{x}_i, y) - J_{01}(y))$$

This equation shows the following. The firm obtains the flow profit generated by both workers $\pi_1(\bar{x}_i) + \pi_2(y)$ and additional profit Δ from teamwork. At rate ρ the manager may retire, so the searching worker is promoted to the senior position and the firm moves to the state $J_{01}(\bar{x}_i)$. Alternatively, it may happen that the worker finds alternative employment and quits at rate λ_2 . In this case the firm is left with only one manager and the present value of profits is $J_{01}(y)$. Next we know that other firms promote their workers at \bar{x} , so all managers hired in the market have experience $y = \bar{x}$. Then $J_{11}^S(\bar{x}_i, \bar{x})$ is given by:

$$J_{11}^S(\bar{x}_i, \bar{x}) = \frac{\pi_1(\bar{x}_i) + \Delta + \pi_2(\bar{x}) + \rho J_{01}(\bar{x}_i) + \lambda_2 J_{01}(\bar{x})}{r + \rho + \lambda_2}$$

In order to find the two integration constants A and K we use the following two boundary conditions: $J_{10}(\bar{x}_i) = J_{01}(\bar{x}_i)$ and $J_{11}^N(\bar{x}_i, \bar{x}) = J_{11}^S(\bar{x}_i, \bar{x})$. The first condition says that firms are committed to promote the worker upon experience \bar{x}_i if the senior position is open, so the present value of the firm changes from $J_{10}(\bar{x}_i)$ to $J_{01}(\bar{x}_i)$. The second condition says that workers with experience \bar{x}_i stop accumulating experience and start searching for alternative jobs at \bar{x}_i if the senior position is filled, so the present value of the firm is changing from $J_{11}^N(\bar{x}_i, \bar{x})$ to $J_{11}^S(\bar{x}_i, \bar{x})$.

The first boundary condition $J_{10}(\bar{x}_i) = J_{01}(\bar{x}_i)$ can be written as:

$$\begin{aligned} J_{10}(\bar{x}_i) &= \frac{d_1(1-\beta) - s}{r} + q_2 \frac{\pi_2(\bar{x}) + \Delta + s}{r(r + \rho + q_2)} + Ae^{r\bar{x}_i} + \frac{c_1(1-\beta)e^{\gamma\bar{x}_i}}{r - \gamma} - \frac{q_2Ke^{(r+\rho+q_2)\bar{x}_i}}{\rho + q_2} \\ &= \frac{\pi_2(\bar{x}_i) - s + q_1J_{11}^N(0, \bar{x}_i)}{r + \rho + q_1} = J_{01}(\bar{x}_i) \end{aligned}$$

The second boundary condition $J_{11}^N(\bar{x}_i, \bar{x}) = J_{11}^S(\bar{x}_i, \bar{x})$ becomes:

$$\begin{aligned} J_{11}^N(\bar{x}_i, \bar{x}) &= \frac{(\pi_2(\bar{x}) + \Delta + s)(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho Ke^{(r+\rho+q_2)\bar{x}_i}}{\rho + q_2} + \frac{d_1(1-\beta) - s}{r} + Ae^{r\bar{x}_i} + \frac{c_1(1-\beta)e^{\gamma\bar{x}_i}}{r - \gamma} \\ &= \frac{\pi_1(\bar{x}_i) + \Delta + \pi_2(\bar{x})}{r + \rho + \lambda_2} + \frac{\rho(\pi_2(\bar{x}_i) - s + q_1J_{11}^N(0, \bar{x}_i))}{(r + \rho + \lambda_2)(r + \rho + q_1)} + \frac{\lambda_2(\pi_2(\bar{x}) - s + q_1J_{11}^N(0, \bar{x}))}{(r + \rho + \lambda_2)(r + \rho + q_1)} = J_{11}^S(\bar{x}_i, \bar{x}) \end{aligned}$$

Note that one term which is still unknown in both boundary conditions is $J_{11}^N(0, \bar{x}_i)$. We derive this term in the Appendix. Solving these two boundary conditions for A and K we can see that both variables depend on the individual decision of firm i and on the behavior of other firms \bar{x} , that is $A(\bar{x}_i, \bar{x})$ and $K(\bar{x}_i, \bar{x})$.

Based on this analysis we can now write the firm's optimization problem (8) as

$$\bar{x}_i^*(\bar{x}) = \arg \max_{\bar{x}_i \geq 0} [q_1 J_{10}(0 | \{\bar{x}, A(\bar{x}_i, \bar{x}), K(\bar{x}_i, \bar{x})\}) + q_2 J_{01}(\bar{x} | \{\bar{x}, A(\bar{x}_i, \bar{x}), K(\bar{x}_i, \bar{x})\})],$$

where we show explicitly the arguments of functions $J_{10}(0)$ and $J_{01}(\bar{x})$. The solution of this maximization problem gives the optimal response function $\bar{x}_i(\bar{x})$ of firm i . Since firms are homogeneous with respect to their profit functions, they all have identical optimal response functions. In light of this in what follows we restrict our attention to symmetric Nash equilibria

and impose the equilibrium condition $\bar{x}_i^*(\bar{x}) = \bar{x}$ to find the equilibrium promotion time \bar{x} .

3.2 Partial and general equilibrium

The complexity of the expressions derived for J_{10} and J_{01} makes an analytical characterization of the best response function and the resulting equilibrium infeasible, even if we consider a partial equilibrium with fixed transition rates. Therefore, we illustrate the main properties of the best response function and the equilibrium by extending the calibration of our model developed in Section 2.3 (Table 1) and carrying out a numerical analysis. First, we consider a partial equilibrium framework with fixed transition rates $\{q_1, q_2, \lambda_1, \lambda_2\}$, with the corresponding values from table 1. We choose the annual discount rate equal to 4%, so that $r = 0.01$. Even though wage bargaining between the worker and the firm is not contained in our analysis, variable β can be seen as a proxy for the bargaining power parameter, so we choose $\beta = 0.5$ following Pissarides and Petrongolo (2001) and Pissarides (2009). The flow cost of an open vacancy is set low ($s = 0.1$). Later we consider a variation in this parameter in the range $[0.08..0.12]$. Further, parameters $d_2 < d_1$ and $c_2 > c_1$ are calibrated so that $\bar{x} = 45$, corresponding to a promotion time of 11.25 years, is an equilibrium outcome of the overall model. Even though it is an endogenous variable in the complete model, we keep it fixed in this section and analyse the optimal response of a single firm i . We start with a benchmark value $\Delta = 0$ and postpone the analysis of production complementarities to the next section.

The multiplier parameter γ can be seen as a proxy for the return to tenure because wages are paid out of the remaining output $\beta(d_j + c_j e^{\gamma x})$. We set $\gamma = 0.003$ which corresponds to the rate of return to tenure at 1.2% per year. According to Farber (1999) the usual OLS estimate of the return to tenure in the United States is 2% per year with the same employer. Empirical methods generally separate this number into two parts: 1. human capital accumulation within the firm and 2. selection component due to the fact that high ability workers stay longer in their jobs and earn more. Farber (1999) finds that 1.5% of the return to tenure is due to the accumulation of human capital and only 0.5% due to selection. In a more recent study Bingley and Westergaard-Nielsen (2003) report the same 2% return to tenure in Denmark, but the human capital component is estimated only at 0.5% per year. These numbers reveal that our parameter choice – 1.2% per year due to human capital accumulation within the firm – is in the middle range of the existing empirical estimates. Moreover, it coincides with the return to tenure estimated by Iftikhar and Zaharieva (2019) for Germany. The second set of parameters is summarized in table 3 below:

| | Value | Interpretation | | Value | Interpretation |
|---------|-------|------------------------------|----------|-------|------------------------------|
| r | 0.010 | Quarterly discount rate | γ | 0.003 | Quarterly return to tenure |
| β | 0.500 | Bargaining power | s | 0.100 | Flow cost of an open vacancy |
| c_1 | 0.500 | Slope parameter, level 1 | c_2 | 2.000 | Slope parameter, level 2 |
| d_1 | 0.200 | Intercept parameter, level 1 | d_2 | 0.100 | Intercept parameter, level 2 |

Table 3: Values of exogenous parameters

Figure 4 shows the objective function of firm i – $J_{00}(\bar{x}_i)$ – for a fixed market promotion time $\bar{x} = 45$ and for fixed transition rates $\{q_1, q_2, \lambda_1, \lambda_2\}$ (left panel). We can see that promoting junior workers too early is not optimal for the firm. This is despite the fact that $d_1 + c_1 < d_2 + c_2$, which

means that the flow profit of the firm is higher in the senior position even if the worker doesn't possess any managerial experience and $x = 0$. The reason is that firms are forward-looking and anticipate a larger gain from promotion once the worker accumulated some managerial experience. At the same time waiting too long is also suboptimal for the firm because the foregone profit is increasing. This is the indirect cost of delayed promotions. In addition, there is the direct flow cost of an open vacancy in the senior position s . As can be clearly seen for our considered parameter values the optimal promotion time is $\bar{x}_i^*(45) = 45$.

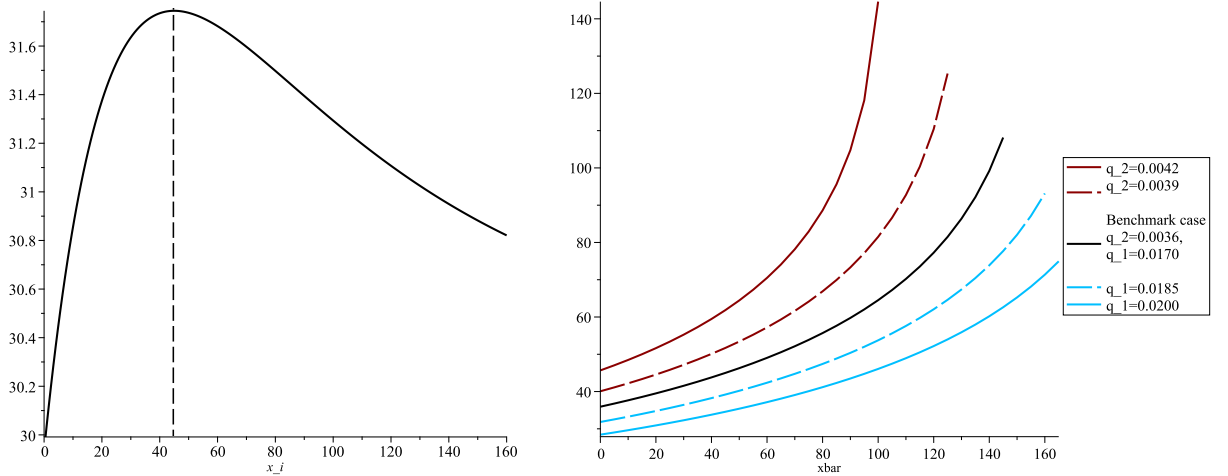


Figure 4: Left panel: Objective function of firm i and the optimal choice $\bar{x}_i^*(\bar{x})$ for a fixed market promotion cut-off $\bar{x} = 45$ and fixed transition rates. Right panel: Optimal response function $\bar{x}_i^*(\bar{x})$ for different values of \bar{x} , comparative statics with respect to the job-filling rates q_1 and q_2

The right panel of figure 4 shows the optimal response function $\bar{x}_i^*(\bar{x})$ for different values of the market promotion time \bar{x} and fixed transition rates (black solid curve). We can see that firm i has strong incentives to delay promotions if other firms in the market promote their junior workers later. Higher \bar{x} implies that managers applying externally are more experienced, so the quality of the candidate pool in the managerial market is better. In this situation it is optimal for firm i to wait longer because the marginal gain from waiting is increasing with \bar{x} due to the better quality of external candidates. Hence, we obtain that there is strategic complementarity between the promotion times of the different firms in the market.

Further, we consider the effect of increasing the job-filling rate q_1 keeping fixed all other transition rates. So it becomes easier for firms to fill their junior positions. The right panel of figure 4 shows that the optimal response curve $\bar{x}_i^*(\bar{x})$ is shifting downwards for all \bar{x} . Note that s/q_1 is the average cost of an open junior position because s is the cost per unit time and $1/q_1$ is the average duration of the vacancy. Higher q_1 lowers the cost of open junior positions, so it is optimal for the firm to promote its junior worker earlier. The opposite is true when we increase q_2 , so the optimal response curve $\bar{x}_i^*(\bar{x})$ is shifting upwards for all \bar{x} . In this case open senior positions become cheaper because s/q_2 is decreasing, so firm i finds it optimal to delay promotions. This shows that the two positions are substitutes from the perspective of the firm.

We already know that $\bar{x}_i^* = \bar{x}^{pe} = 45$ for all firms i is a symmetric partial equilibrium of the model for the given transition rates (values from table 1). But is it a unique partial equilibrium? Figure 5 shows that in addition to the low equilibrium $\bar{x}_i^{pe} = 45$ there also exists a second partial

equilibrium with $\bar{x}_h^{pe} = 157.6$ for these transition rates. Both equilibria are illustrated on the right panel of figure 5. In light of the strategic complementarity between the optimal promotion times of the firms it is not surprising that multiple equilibria exist in our model. However, as can be clearly seen in right panel of figure 5 only the low equilibrium is strategically stable. Any best response dynamics initialized with a market promotion level $\bar{x} \in [0, \bar{x}_h^{pe}]$ converges to the lower equilibrium $\bar{x}_l^{pe} = 45$.

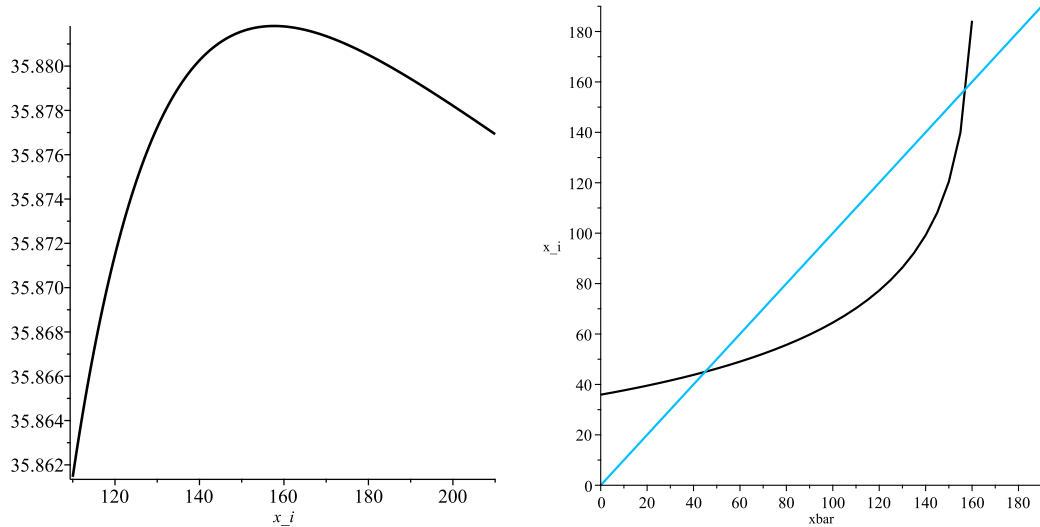


Figure 5: Left panel: Objective function of firm i for $\bar{x} = 157.6$. Right panel: Optimal response curve $\bar{x}_i^*(\bar{x})$ exhibiting the two partial equilibria $\bar{x}_l^{pe} = 45$ and $\bar{x}_h^{pe} = 157.6$ for fixed transition rates from table 1

In Section 2.3 we have shown that if all firms use a promotion threshold of $\bar{x} = 45$, then the transition rates under the stationary distribution are given by $\{q_1 = 0.0171, q_2 = 0.0036, \lambda_1 = 0.0145, \lambda_2 = 0.0146\}$ (see Table 1). Since these are exactly the transition rates under which we have carried out the partial equilibrium analysis above, it follows directly that $\bar{x}_i^* = \bar{x}_l^{pe} = 45, i \in [0, 1]$ is also a general equilibrium of the model. Similarly to the partial equilibrium setting, also with endogenous transition rates a second equilibrium with a very high promotion threshold exists, which however is unstable. Hence in what follows we focus on the lower equilibrium and in the following section examine how the equilibrium promotion threshold changes in response to a variation of key parameters in the model.

3.3 Comparative statics: partial and general equilibrium effects

Based on the benchmark numerical example developed in the previous section we now address two key questions of our study: (1) how promotion chances of junior workers are affected if there exist production complementarities and synergies from the team work and (2) what is the link between the optimal promotion time and the skill level of the worker?

In order to address the first question we gradually increase the synergy parameter Δ , which was fixed at 0 in the benchmark case. This is illustrated on the left panel of figure 6. If the synergy parameter is increasing from 0 to 0.6 the promotion cut-off \bar{x}^{ge} in the general equilibrium is decreasing from 45 down to 43.7. Stronger complementarities in the production process create stronger incentives for firms to employ a full team of two employees rather than having open

vacancies. In our setting the job-filling rate in the junior market $q_1 = 0.0171$ is substantially higher than the job-filling rate in the senior market $q_2 = 0.0036$ which means that hiring junior workers is easier than senior managers. In this situation firms prefer earlier promotions of junior employees in the hope that the junior position will be filled faster than the senior position and the firm can gain additional profits from the team production process. Note that this gain comes at the expense of accepting less experienced senior managers.

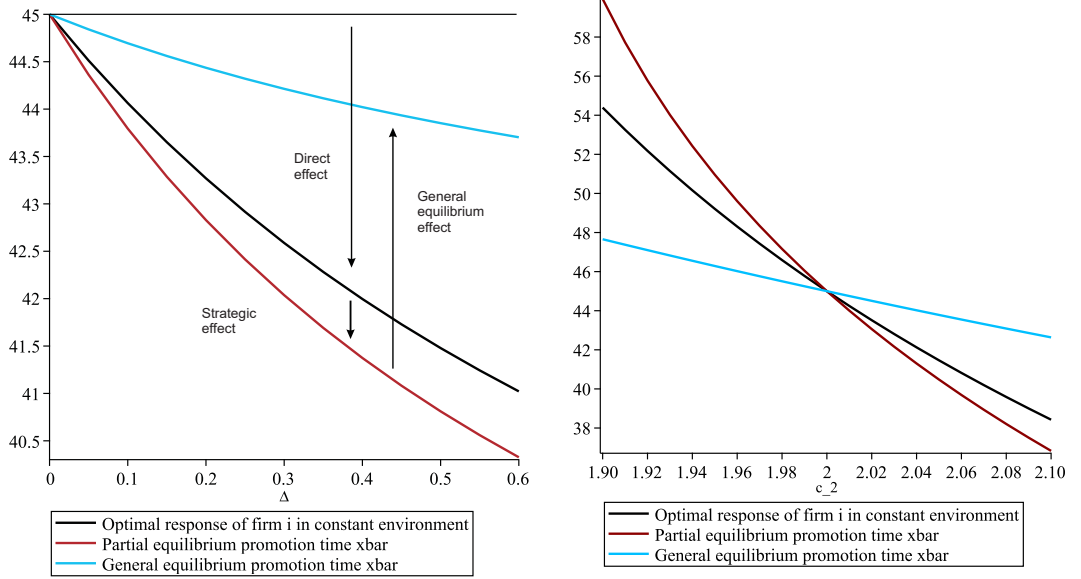


Figure 6: Left panel: Equilibrium promotion time $\bar{x}_i(\cdot)$ as a function of the synergy parameter Δ . Right panel: Equilibrium promotion time $\bar{x}_i(\cdot)$ as a function of the skill parameter c_2

Further, we decompose this effect into three parts. We write the individually optimal promotion threshold $\bar{x}_i^*(\bar{x}, \zeta, \Delta)$ as a function of the market promotion level \bar{x} as well as the vector of transition rates ζ and the synergy parameter Δ . Furthermore, $\bar{x}^{pe}(\zeta, \Delta)$ denotes the (partial) equilibrium market cutoff under transitions rates ζ and ζ^Δ the general equilibrium transition rates for the synergy parameter Δ . The general equilibrium cutoff under synergy Δ is then denoted as $\bar{x}^{ge}(\Delta) := \bar{x}^{pe}(\zeta^\Delta, \Delta)$. Hence $\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, 0) = \bar{x}^{pe}(\zeta^0, 0) = \bar{x}^{ge}(0) = 45$. Using this notation we obtain the following decomposition of the effect of a change in Δ :

$$\begin{aligned}
\bar{x}^{ge}(0) - \bar{x}^{ge}(\Delta) &= \bar{x}^{pe}(\zeta^0, 0) - \bar{x}^{pe}(\zeta^\Delta, \Delta) = \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta, 0) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^\Delta, \Delta), \zeta^\Delta, \Delta) = \\
&= \underbrace{[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, 0) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, \Delta)]}_{\text{Direct effect}} \\
&+ \underbrace{[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, 0), \zeta^0, \Delta) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^0, \Delta), \zeta^0, \Delta)]}_{\text{Strategic effect}} \\
&+ \underbrace{[\bar{x}_i^*(\bar{x}^{pe}(\zeta^0, \Delta), \zeta^0, \Delta) - \bar{x}_i^*(\bar{x}^{pe}(\zeta^\Delta, \Delta), \zeta^\Delta, \Delta)]}_{\text{General equilibrium effect}}
\end{aligned}$$

First, figure 6 (left panel) shows the *direct effect*, this is a change in the optimal promotion time of firm i as a function of Δ in a setting with constant environment. As we can see from the figure, the firm has very strong incentives to promote earlier. If the synergy parameter is increasing from 0 to 0.6 the optimal promotion cut-off of firm i is decreasing from 45 down to

41 (black curve). So the direct effect for $\Delta = 0.6$ is equal to $4 = 45 - 41$. Second, we allow for changes in the behavior of competing firms $\bar{x}^{pe}(\zeta^0, \Delta)$ but keep the set of transition rates ζ^0 fixed. This is the *strategic effect* reflecting strategic competition between firms. We already know from figure 4 that earlier promotions by the competitors lead to earlier promotions of firm i . This is illustrated by the red curve on figure 6. If the synergy parameter is increasing from 0 to 0.6 and the firm takes earlier promotions of competitors into account the optimal promotion cut-off is decreasing even stronger from 41 down to 40.3, so the strategic effect is equal to $0.7 = 41 - 40.3$. It makes promotions more sensitive to the production complementarity Δ . The sum of these two effects would be observed in a partial equilibrium setting, in which the transition rates are kept constant. Third, we analyze the *general equilibrium effect* and allow for the endogenous changes in the transition rates. From figure 3 we already know that if all firms set earlier promotion times then q_1 is decreasing and q_2 is increasing. Intuitively, this means that earlier promotions make it easier for firms to hire senior managers but hiring junior workers becomes more difficult. This general equilibrium effect mitigates the incentives of firm i to promote earlier and makes promotions less sensitive to the production complementarity Δ . The general equilibrium effect is illustrated by the blue curve and is equal to $-3.4 = 40.3 - 43.7$. Based on this decomposition we can conclude that the direct effect and the general equilibrium effect are quantitatively larger than the strategic effect in our setting.

Next we turn to the effect of education. We proxy this effect by changes in the parameter c_2 . The intuition behind this proxy is that more educated workers with higher skills will be more productive in senior positions than low skill workers even if they have similar practical experience. This is due to the methodological competence, broader knowledge and problem-solving skills associated with higher education. Following this logic we assume that higher c_2 corresponds to the labour market with more educated workers but there are no productivity differences in junior jobs (c_1). The right panel of figure 6 shows changes in the promotion times where $c_2 = 2$ is the benchmark case in the middle of the figure. We can see that higher education generally leads to earlier promotions. The effects are reversed when the labour force is less qualified: if c_2 is decreasing from 2 to 1.95, firm i responds by setting the equilibrium promotion time equal to 49.2 in a constant environment. If all competitors follow the same strategy and set longer promotion times the partial equilibrium is achieved at $\bar{x}_i^*(\bar{x}^{pe}) = \bar{x}^{pe} = 51$. The decomposition reveals again that the general equilibrium effect dampens the direct effect of the parameter change on the optimal promotion time and makes it less sensitive to the education parameter. We obtain for $c_2 = 1.95$ a general equilibrium cut-off of $\bar{x}^{ge} = 46.3$. Even though this result provides first evidence of the positive link between education and the speed of promotions in our model, it is only a comparative statics result and it is not clear if it will be confirmed in a setting where two skill types are mixed in the same labour market. We provide this analysis in section 4.

3.4 Number of firms and competition

In this section we analyze the impact of stronger competition in the labour market on the equilibrium promotion time \bar{x} . We capture the effect of stronger competition by increasing the entry of firms n . Given that the exit of firms is driven by the exogenous retirement of their employees ρ , a more intensive firm entry leads to a larger number of firms in the steady state

F . In figure 7 we illustrate the relationship between the equilibrium promotion cutoff \bar{x} and the number of firms in the market (F). We can see that the promotion time is decreasing from 50.77 when $F = 0.5$ to 42.18 when $F = 0.65$. Recall that the benchmark case is $F = 0.6$, meaning that there are 600 two-position firms per 1000 workers. This means that stronger competition in the market reflected in a larger number of competitors leads to earlier promotions.

The right panel of figure 7 reveals the underlying mechanism. It shows a response of the job-filling rates to a larger number of firms F . We can see that both rates are declining. The reason is that a larger number of firms leads to a larger stock of positions and vacancies in both submarkets, worsening the hiring chances of every single firm. This theoretical experiment resembles the situation in Germany and in the USA in the last decade, which we mentioned in the introduction, namely, that filling vacancies becomes increasingly more difficult for firms. From figure 4 (right panel) we already know that lower q_1 leads to delayed promotions of workers, and at the same time a lower rate q_2 leads to earlier promotions. Our results indicate that the second effect is dominating because the job-filing rate q_2 is smaller than q_1 , making the expected search cost s/q_2 larger than s/q_1 . Thus, filling positions for senior managers is not only more difficult for firms, but it is also associated with a higher expected search cost s/q_2 . In this situation making promotions earlier reduces the pressure on firms associated with the senior submarket and reduces the expected cost of applicant search.

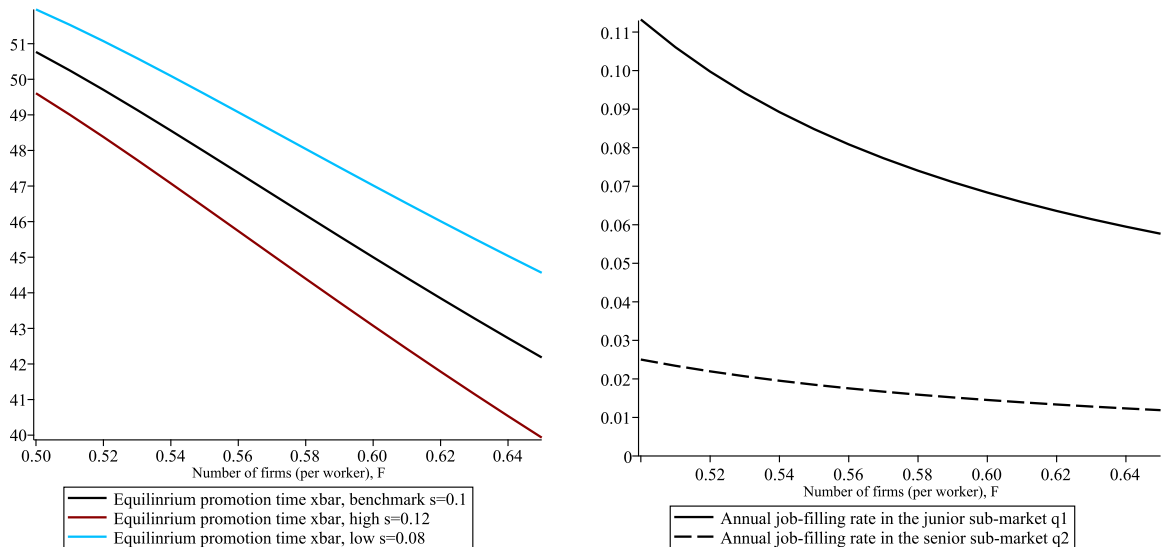


Figure 7: Left panel: The equilibrium promotion time \bar{x} as a function of the number of firms F and the search cost s . Right panel: Annual job-filling rates q_1 and q_2 .

In addition to the effect of competition, the left part of figure 7 shows comparative statics with respect to the flow cost of applicant search s . We vary s between 0.08 and 0.12 implying a 20% variation around the benchmark $s = 0.1$. We can see that a higher search cost s leads to earlier promotions. Even though a higher search cost s has a negative impact on the profits of firms on both submarkets, the expected cost s/q_j , $j = 1, 2$, is increasing stronger in the senior sub-market making unfilled senior positions more costly relatively to the open junior positions and leading to earlier promotions.

Our results from this section suggest that firms promote their employees earlier in more competitive labour markets reflected in a larger number of employing firms. This is also true

if the flow search cost is increasing. More generally, firms promote earlier if hiring becomes relatively more difficult for them on the upper levels of the job ladder. Earlier promotions also mean that workers obtain less training on-the-job, thus their experience in the senior level is lower. In the next section we extend the model to a setup with two different worker groups and investigate whether our results continue to hold in the presence of worker heterogeneity.

4 Two skill levels

4.1 Optimal promotion with two skill levels

In this section we extend the model to a setting with two skill groups and analyze the spillover effects that the presence of one skill group imposes on the other group. To keep the model tractable, we refrain from the synergy effect and set $\Delta = 0$ throughout this extension. Let c_2^L be the education parameter of low skill workers. Once employed in the senior job, they generate the flow profit $\pi_2^L(x) = (d_2 + c_2^L e^{\gamma x})(1 - \beta)$ for the firm. Further, $c_2^H > c_2^L$ denotes the education parameter of high skill workers, so they generate the flow profit $\pi_2^H(x) = (d_2 + c_2^H e^{\gamma x})(1 - \beta)$. We assume that the difference between c_2^L and c_2^H is sufficiently small so that firms do not reject low skill applicants. Moreover, c_1 remains the same for both worker groups, indicating that high and low skill workers are equally productive when performing junior level jobs. It is the difference in managerial abilities that we want to capture in this extension. Let a denote the fraction of low skill workers in the population. Variables \bar{x}_i^L and \bar{x}_i^H denote the promotion times set by firm i for each skill group respectively. As before, this decision is made upon the entry and there is full commitment on the side of the firm.

Further, let α_1 denote the fraction of low skill applicants in the junior market and α_2 be the fraction of low skill applicants in the senior market. Both variables are closely related to the share of low skill workers in the population a but can be larger or smaller than a . The entering firm solves the optimization problem

$$\begin{aligned} \{\bar{x}_i^{L*}, \bar{x}_i^{H*}\} &= \arg \max_{\bar{x}_i^L, \bar{x}_i^H} q_1[\alpha_1 J_{L0}(0|\{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\}) + (1 - \alpha_1)J_{H0}(0|\{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\})] \\ &+ q_2[\alpha_2 J_{0L}(\bar{x}_L|\{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\}) + (1 - \alpha_2)J_{0H}(\bar{x}_H|\{\bar{x}_i^L, \bar{x}_i^H, \bar{x}_L, \bar{x}_H\})], \quad (11) \end{aligned}$$

where $\{\bar{x}_i^{L*}, \bar{x}_i^{H*}\}$ denote the optimal choices, \bar{x}_j is the market experience level of applicants in the managerial market with a skill level $j = L, H$, J_{j0} is a firm with an inexperienced worker of skill $j = L, H$ and an open senior vacancy while J_{0f} is a firm with a senior worker of skill $f = L, H$ and a junior vacancy. The corresponding Bellman equations and the solution procedure for the two skill level case are shown in the Appendix. As before, we consider symmetric equilibria, so that $\bar{x}_i^{L*}(\bar{x}_L, \bar{x}_H) = \bar{x}_L$ and $\bar{x}_i^{H*}(\bar{x}_L, \bar{x}_H) = \bar{x}_H$, which guarantees that firms do not have incentives to deviate.

4.2 Partial equilibrium

To illustrate the implications of skill heterogeneity for our results, we first consider again a partial equilibrium framework with fixed transition rates from table 1. We set $c_2^L = 1.95$ and $c_2^H = 2.05$, so that high skill workers are more productive than low skill workers in senior jobs. From our

analysis in section 3.3 we know that for these parameters and the transition rates emerging from our default setting (see Table 1), in the absence of high-skill workers (i.e. $\alpha_1 = \alpha_2 = 1$), the partial equilibrium promotion threshold for low skill workers is $\bar{x}_L^{pe} = 51$. We start with a situation where $\alpha_1 = \alpha_2 = 0.8$, which implies that 80% (20%) of workers in both the junior and senior market are low (high) skilled. This roughly corresponds to the share of university/college master graduates in developed economies, for example, this share was 13.04% of the population aged 25 and above in the USA in 2018⁵. The left panel of figure 8 shows the objective function of the firm for the default transition rates. We find that the partial equilibrium is achieved for $\bar{x}_L^{pe} = 56.8$ and $\bar{x}_H^{pe} = 26.5$, which implies that high skill workers are promoted much earlier than low skill workers. Intuitively, a firm with a low skill worker in a junior position has a strong incentive to delay the promotion of this worker because this delay increases the chance for the firm to hire a high skill worker from the market for the senior position. Quite on the contrary, if the junior worker has high skills, then it is profitable for the firm to exploit these skills in the senior position rather than hiring from the market which comes at the risk of putting a low skill worker into the senior position. The fact that high skill workers are promoted faster is supported by the empirical evidence ((Baker et al. (1994), McCue (1996) and Lluís (2005))) and indicates a proper setup of our model.

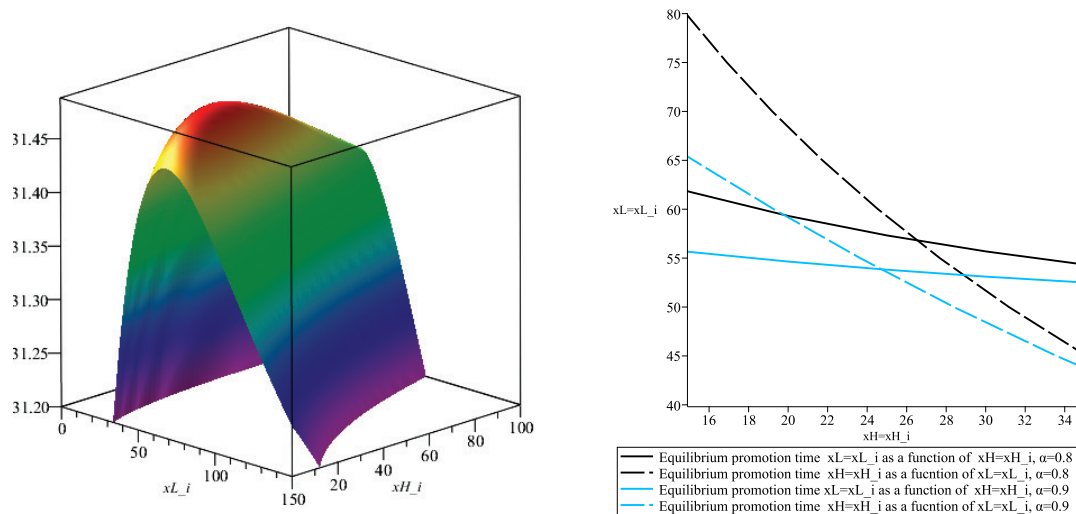


Figure 8: Left panel: Two-dimensional objective function of the firm in the space $\{\bar{x}_i^L, \bar{x}_i^H\}$ for $\alpha_1 = \alpha_2 = 0.8$ and market promotion cut-offs of $\bar{x}_L = 56.8, \bar{x}_H = 26.5$. Right panel: Sequence of partial equilibria for different values of $\alpha = \alpha_1 = \alpha_2$.

In the right panel of figure 8 we illustrate the nature of the partial equilibrium in the model with two worker groups. First, we find the optimal promotion cut-offs for high skill workers $\bar{x}_i^{H*}(\bar{x}_H, \bar{x}_i^L = \bar{x}_L) = \bar{x}_H$ for any given promotion cut-off of low skill workers $\bar{x}_i^L = \bar{x}_L$. If we exogenously decrease $\bar{x}_i^L = \bar{x}_L$ we can see that firms respond by later promotions of high skill workers (black dashed curve). Considering the left panel of figure 8 we can see that this negative dependence of the optimal threshold for x_i^L respectively x_i^H from the value of the other threshold also arises if we keep the thresholds of all other firms constant. Intuitively, faster promotion of own low-skill workers makes it more likely that the firm's senior position is filled at any point in

⁵”Educational Attainment in the United States: 2018”. U.S. Census Bureau.

time. A firm never wants to provide experience evaluation to high-skill workers and make them eligible for promotion at a point in time when its senior position is filled, due to the higher risk of losing these workers. Hence the increase of the probability of a filled senior position induced by a decrease of \bar{x}_L reduces the firm's incentive to set a low promotion cut-off for high-skill workers. Even though there are also other side effects, the numerical evidence shown in figure 8 suggests that the described mechanism dominates giving rise to strategic substitutability between the two promotion thresholds. A substitution effect also applies if we consider the impact of an exogenous decrease of $\bar{x}_i^H = \bar{x}_H$ on the optimal promotion threshold for low skilled, although the effect is much smaller in this case (black solid curve). The partial equilibrium is obtained at the intersection of the two curves, since no firm has incentives to deviate. Overall, we note that the optimal promotion timing of high skill workers is very sensitive to changes in $\bar{x}_i^L = \bar{x}_L$. On the contrary, promotions of low skill workers have low sensitivity in response in $\bar{x}_i^H = \bar{x}_H$.

If we increase $\alpha_1 = \alpha_2$ to 0.9 we find the equilibrium promotion cut-offs $\bar{x}_L^{pe} = 53.9$, $\bar{x}_H^{pe} = 24.7$, thus both types of workers are promoted earlier (blue curves). In the limiting case when $\alpha_1 = \alpha_2 = 1$ we arrive at the economy with only low skill workers with productivity level $c_2^L = 1.95$ and the corresponding equilibrium threshold is $\bar{x}_L^{pe} = 51$ (see section 3.3). Hence, we can conclude that a lower average skill level in the labour force (due to the larger share of low skill workers) is associated with earlier promotions. In the next section we check if this result will persist after the general equilibrium adjustment in the transition rates.

4.3 General equilibrium

In order to find the general equilibrium in the model with heterogeneous skills, we need to characterize the steady state distributions of firms and workers. Let d_{ij}^f be a firm with a junior worker with skill level $i = L, H$ and a senior worker with skill $j = L, H$. In case one of the positions is vacant, the corresponding subscript becomes 0 as in the benchmark model. Also, $f = N, S$ indicates whether the junior worker is already searching for a job on the senior market or not and this superscript is dropped if the junior position is vacant. Further, let e_0^L denote the stock of young and inexperienced workers searching for their first professional job who are low skilled. Similarly, let e_0^H denote the stock high skill workers who are not yet employed in a hierarchical firm. Re-writing the equations for the transition rates to take into account worker skill heterogeneity then yields the following expressions for the vacancy filling and job finding rates:

$$q_1 = 1 - \left(1 - \frac{1}{d_{00} + d_{0L} + d_{0H}}\right)^{z(e_0^L + e_0^H)} \quad \lambda_1 = q_1 \frac{(d_{00} + d_{0L} + d_{0H})}{e_0^L + e_0^H} \quad (12)$$

$$q_2 = 1 - \left(1 - \frac{1}{d_{00} + d_{L0} + d_{H0}}\right)^{z(d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S)} \quad \lambda_2 = q_2 \frac{(d_{00} + d_{L0} + d_{H0})}{d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S} \quad (13)$$

As mentioned above, $\alpha_2 = \frac{d_{LL}^S + d_{LH}^S}{d_{LL}^S + d_{LH}^S + d_{HH}^S + d_{HL}^S}$ is the fraction of low skill applicants on the senior market and $\alpha_1 = \frac{e_0^L}{e_0^L + e_0^H}$ is the fraction of low skill applicants on the junior market. Recall that a is the number of low skill workers in the population. This means that $a = e_0^L + d_{L0} + 2d_{LL}^N + d_{LH}^N + d_{HL}^N + 2d_{LL}^S + d_{LH}^S + d_{HL}^S + d_{0L}$. As in our benchmark model total population

is normalized to 1 such that $e_0^H + d_{H0} + 2d_{HH}^N + d_{HL}^N + d_{LH}^N + 2d_{HH}^S + d_{HL}^S + d_{LH}^S + d_{0H} = 1 - a$, i.e the number of high skill workers equals $1 - a$. The steady-state distribution of firms and workers is found by setting $\dot{d}_{ij}^f = 0$, $i = 0, L, H$, $j = 0, L, H$, $f = N, S$. The total number of firms F is given by:

$$d_{00} + d_{L0} + d_{H0} + d_{LL}^N + d_{LH}^N + d_{HL}^N + d_{HH}^N + d_{LL}^S + d_{LH}^S + d_{HL}^S + d_{HH}^S + d_{0L} + d_{0H} = F \quad (14)$$

The differential equations for all types of firms and the steady-state solution are described in detail in the Appendix.

| | | $a = 0.9$ | $a = 0.8$ |
|--------|----------------------------|---|---|
| F=0.5 | $\{\bar{x}_L, \bar{x}_H\}$ | {56, 20.3} | {60.99, 22.4} |
| | ζ^{ge} | $\alpha_1 = 0.8844; \lambda_1 = 0.0145; q_1 = 0.02877$ $\alpha_2 = 0.8510; \lambda_2 = 0.0146; q_2 = 0.00617$ | $\alpha_1 = 0.7706; \lambda_1 = 0.0145; q_1 = 0.02895;$ $\alpha_2 = 0.7132; \lambda_2 = 0.0146; q_2 = 0.00619$ |
| | Distribution | $e^{0L} = 0.336; e^{1L} = 0.340; e^{2L} = 0.324;$ $e^{0H} = 0.396; e^{1H} = 0.223; e^{2H} = 0.381$ | $e^{0L} = 0.329; e^{1L} = 0.354; e^{2L} = 0.316$ $e^{0H} = 0.392; e^{1H} = 0.231; e^{2H} = 0.377$ |
| F=0.55 | $\{\bar{x}_L, \bar{x}_H\}$ | {52.3, 24.6} | {55.8, 26.4} |
| | ζ^{ge} | $\alpha_1 = 0.8878; \lambda_1 = 0.0145; q_1 = 0.02143;$ $\alpha_2 = 0.8612; \lambda_2 = 0.0146; q_2 = 0.00453$ | $\alpha_1 = 0.7775; \lambda_1 = 0.0145; q_1 = 0.02147$ $\alpha_2 = 0.7312; \lambda_2 = 0.0146; q_2 = 0.00455$ |
| | Distribution | $e^{0L} = 0.344; e^{1L} = 0.324; e^{2L} = 0.332$ $e^{0H} = 0.391; e^{1H} = 0.231; e^{2H} = 0.378$ | $e^{0L} = 0.339; e^{1L} = 0.334; e^{2L} = 0.327$ $e^{0H} = 0.388; e^{1H} = 0.238; e^{2H} = 0.374$ |
| F=0.6 | $\{\bar{x}_L, \bar{x}_H\}$ | {49, 24.9} | {51.9, 26.6} |
| | ζ^{ge} | $\alpha_1 = 0.8893; \lambda_1 = 0.0145; q_1 = 0.01723$ $\alpha_2 = 0.8650; \lambda_2 = 0.0146; q_2 = 0.00355$ | $\alpha_1 = 0.7803; \lambda_1 = 0.0145; q_1 = 0.01724$ $\alpha_2 = 0.7379; \lambda_2 = 0.0146; q_2 = 0.00356$ |
| | Distribution | $e^{0L} = 0.351; e^{1L} = 0.309; e^{2L} = 0.339$ $e^{0H} = 0.394; e^{1H} = 0.226; e^{2H} = 0.380$ | $e^{0L} = 0.347; e^{1L} = 0.318; e^{2L} = 0.335$ $e^{0H} = 0.390; e^{1H} = 0.233; e^{2H} = 0.377$ |
| F=0.65 | $\{\bar{x}_L, \bar{x}_H\}$ | {46.1, 24.0} | {48.6, 25.6} |
| | ζ^{ge} | $\alpha_1 = 0.8900; \lambda_1 = 0.0145; q_1 = 0.01452$ $\alpha_2 = 0.8667; \lambda_2 = 0.0146; q_2 = 0.00289$ | $\alpha_1 = 0.7818; \lambda_1 = 0.0145; q_1 = 0.01452$ $\alpha_2 = 0.7411; \lambda_2 = 0.0146; q_2 = 0.00290$ |
| | Distribution | $e^{0L} = 0.358; e^{1L} = 0.296; e^{2L} = 0.346$ $e^{0H} = 0.398; e^{1H} = 0.218; e^{2H} = 0.385$ | $e^{0L} = 0.353; e^{1L} = 0.305; e^{2L} = 0.342;$ $e^{0H} = 0.395; e^{1H} = 0.224; e^{2H} = 0.382$ |

Table 4: Optimal promotion thresholds $\{\bar{x}_L, \bar{x}_H\}$ varying the number of low skill workers in the population a and the number of firms F . ζ^{ge} : general equilibrium transition rates. In the rows "Distribution": $e^{0L} = e_0^L/a$; $e^{1L} = (d_{L0} + d_{LL}^N + d_{LH}^N + d_{LL}^S + d_{LH}^S)/a$; $e^{2L} = (d_{LL}^N + d_{HL}^N + d_{LL}^S + d_{HL}^S + d_{0L})/a$; $e^{0H} = e_0^H/(1 - a)$; $e^{1H} = (d_{H0} + d_{HH}^N + d_{HL}^N + d_{HH}^S + d_{HL}^S)/(1 - a)$ and $e^{2H} = (d_{HH}^N + d_{LH}^N + d_{HH}^S + d_{LH}^S + d_{0H})/(1 - a)$.

In table 4 the general equilibrium promotion strategies, the corresponding transition rates and worker distribution are displayed for different fractions of low-skill workers (a) in the population and for different numbers of firms (F). In all scenarios the fraction of low-skill workers among the applicants for junior positions (α_1) are close to their average fraction in the workforce (a), whereas the fraction of low-skilled among the applicants for senior positions (α_2) are significantly smaller: ($\alpha_2 < a$). This effect is due to the slower promotion of low-skill workers compared to their high-skill peers, which makes them under-represented in the market for senior positions. For instance in the case $a = 0.8$, $F = 0.6$, even though 80% of the agents are low skill, only 73.8% or of the applicants to senior positions are also low skill.

Focusing on the case $F = 0.6$ we can compare the general equilibrium thresholds with the partial equilibrium values discussed in section 4.2. We observe that the promotion thresholds for high-skill workers are hardly affected by general equilibrium effects, whereas the promotion

thresholds for low-skill worker are significantly lower in general equilibrium compared to the partial equilibrium. For the case of $a = 0.8, F = 0.6$ we obtain $\bar{x}_L(\zeta^{ge}) = 51.9$ in general equilibrium compared to a threshold of $\bar{x}_L^{pe}(\zeta^0) = 56.8$ obtained for the partial equilibrium under the benchmark transition rates and the assumption that both for the junior and the senior positions the fraction of low-skill workers is given by $\alpha_1 = \alpha_2 = a = 0.8$. Intuitively, the reason for this difference is that under the partial equilibrium values $(\bar{x}_L^{pe}, \bar{x}_H^{pe}) = (56.8/26.5)$ the firm's actual job filling rate for senior positions on the market q_2 is lower and that for junior positions q_1 is higher compared to the value assumed in the partial equilibrium (see Table 1). As we know from figure 4, this induces the firm to promote the majority group of low-skill workers earlier and as a result \bar{x}_L is lower in general equilibrium than under partial equilibrium.

Analyzing the impact of a , we can see that qualitatively, the result that higher share of low skill workers is associated with earlier promotions remains unchanged after endogenizing the transition rates. Recall that in section 3.3 we have shown that lower quality of the homogeneous labour force is associated with later promotions. How can these two findings be reconciled? The key difference between these settings is that under worker heterogeneity an increase of the fraction of low skill workers reduces the expected skill of a worker hired from the market *relative* to the skill of the junior worker under consideration for internal promotion, regardless of the actual type of the junior worker. So the internal candidate becomes better in relative terms compared to the average external candidate. This induces earlier internal promotions. With homogeneous workers by definition the skill of an outside hire is always identical to that of an internal candidate. So when the skill level is falling, firms want to compensate for the lower qualification of their internal candidates and let them accumulate more experience by delaying internal promotions. Thus, changes in the quality of the labour force can have principally different implications for promotions in the two settings with homogeneous and heterogeneous workers. Taking into account that the firm's senior job filling rate decreases with the fraction of low skill workers we observe that the general equilibrium reinforces the partial equilibrium effect and leads to even earlier promotions of low skill workers. Overall, this discussion highlights that explicitly considering potential heterogeneities in the workforce is essential for understanding the relationship between the (average) skill level in the worker population and the firms' optimal promotion thresholds.

Table 4 also displays the distribution of high and low skill workers across hierarchical levels. In equilibrium larger fraction of high skill workers are in managerial positions. For instance, considering the case in which 80% of the agents are low skill ($a = 0.8, F = 0.6$), approximately 61.8% ($= e^{2H}/(e^{1H} + e^{2H})$) of high skill workers who are employed in professional jobs are on level 2 (62.7% when $a = 0.9$). This follows from the earlier promotion time firms set for high skill workers. As the fraction of low skill workers (a) increases from 0.8 to 0.9, the equilibrium promotion cut-offs \bar{x}^L and \bar{x}^H decrease which leads to more workers in senior positions (e^{2L} and e^{2H}) for both skill groups. This result corresponds to the findings from the benchmark model that earlier promotions decrease the fraction of workers employed in junior jobs and increase the fraction of senior workers.

Finally, we extend our results of the effects of firm competition on optimal promotion strategies to the model with skill heterogeneity. The results are also displayed in table 4 where we vary the number of firms F between 0.5 and 0.65. Firstly, stronger firm competition leads to

faster promotions for the majority group of low skill workers. This result corresponds to the one found under skill homogeneity. However, we observe that the equilibrium promotion time for high skill workers is increasing in the number of firms at first and then declines. In the case of high skill workers there are two counteracting effects that influence firm's optimal promotion strategy. Stronger competition induces earlier promotions since filling senior vacancies becomes more difficult, which is reflected in the declining senior vacancy filling rate q_2 . On the other hand, as discussed in section 4.2, a decrease of \bar{x}_L reduces firm's incentive to lower \bar{x}_H due to the strategic substitutability between the promotion times of the two skill groups. Our results show that the substitution effect is dominating when competition is weak leading to later promotion of high skill workers. However, the substitution effect is dominated when firm competition intensifies leading to earlier promotions also for high skill workers.

Before concluding this section, we like to mention that our finding that high skill workers are promoted faster than low skilled ones, have also been derived in existing models of internal labour markets. For instance, in a context of asymmetric learning, it has been shown that workers with higher ability (Bernhardt (1995)) or more schooling (DeVaro and Waldman (2012)) are promoted earlier. In both models promotions reveal information about workers ability and upon promotion firms offer higher wages as to prevent competitors from hiring the workers. In a context of symmetric learning, Gibbons and Waldman (2006) similarly derive the result that schooling is positively related to promotion probabilities since workers with more education accumulate human capital faster. However, in these frameworks there is no turnover in equilibrium. Integrating promotions and job-to-job transitions we are able to endogenize the rates at which firms meet workers of a specific type, either for their junior or senior vacancies. We show how firm's promotion strategies are then altered by general equilibrium effects. More specifically, the promotion requirement for the majority group of low skill workers responds strongly to endogenizing the market transition rates. Moreover, this allows us to explore how changes in the distribution of worker types and the degree of firm competition affect promotion timing for all skill groups which are novel testable empirical implications.

5 Conclusions

In this paper we develop and analyze a model which incorporates the choice of optimal promotion times by hierarchical firms in a search and matching labour market with on-the-job search and captures the option of a firm to fill senior positions through outside recruiting rather than internal promotion. Embedding firms and their individual decisions into a rich labour market with search frictions and heterogeneous worker groups allows us to study the impact of competition and labour force composition on the timing of promotions. Our findings about the effect of the level and heterogeneity of worker skills and of the strategic competition between firms on optimal promotion times deliver innovative insights into the determinants of firm behavior on the labour market and into the resulting implications for labour flows. They provide theory-based explanations for empirical observations, e.g. that high skill workers are promoted earlier than low skill workers. Furthermore, our results also give rise to several innovative testable implications about the impact of different factors on promotion strategies, which can be used as the theoretical basis for future empirical work in this area. Our insight that the effects of parameter changes

on promotion cut-offs are typically much smaller in a general equilibrium framework than under the assumption of fixed job-filling/job finding rates at the different hierarchical levels, highlights the importance of endogenizing the supply side of the labour market when analyzing the design of promotion strategies.

The analysis presented in this paper can be extended in several promising directions. Apart from empirical work building on our results, endogenizing wages and considering a simultaneous setting of wages and promotion cut-offs by firms may provide further economic insights. Moreover, the impact of promotion strategies on wage inequality is a related promising area. Extending the framework developed in this paper allows to study the role of the promotion channel for transforming different types of skill heterogeneities into wage inequalities under different assumptions about the firms' hierarchical structure. In that respect also the role of professional networks for job transitions and emerging wage inequality might be considered. These networks might evolve endogenously through employment at the same company and influence the potential of workers for finding senior positions outside the own firm. Finally, the fact that individual firms do not internalize the general equilibrium effects in our model is likely to create a deadweight loss of welfare which opens space for the discussion of policy and regulation.

Appendix: Additional Calculations

Benchmark case

First, we solve equation (9), which is a first-order linear differential equation. This equation has the form $J'_{10}(x) = rJ_{10}(x) + g(x)$, where $g(x)$ is given by:

$$-g(x) = (d_1 + c_1 e^{\gamma x})(1 - \beta) - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} + q_2 K e^{(r+\rho+q_2)x}$$

With A denoting the integration constant the general solution of this equation is given by $J_{10}(x) = A e^{rx} + e^{rx} \int g(x) e^{-rx}$. The second part of this expression is given by:

$$\begin{aligned} e^{rx} \int g(x) e^{-rx} &= -e^{rx} \left[\int \left(d_1(1 - \beta) - s + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r + \rho + q_2} \right) e^{-rx} dx \right. \\ &\quad \left. + \int c_1(1 - \beta) e^{(\gamma-r)x} dx + \int q_2 K e^{(\rho+q_2)x} dx \right] \\ &= \frac{d_1(1 - \beta) - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} - \frac{e^{rx} c_1(1 - \beta) e^{(\gamma-r)x}}{\gamma - r} - \frac{e^{rx} q_2 K e^{(\rho+q_2)x}}{\rho + q_2} \\ &= \frac{d_1(1 - \beta) - s}{r} + q_2 \frac{(\pi_2(\bar{x}) + \Delta + s)}{r(r + \rho + q_2)} + \frac{c_1(1 - \beta) e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r+\rho+q_2)x}}{\rho + q_2} \end{aligned}$$

Further, we determine the function $J_{11}^N(0, x_i)$. To do so recall that $J_{11}^N(x, y)$ is given by:

$$rJ_{11}^N(x, y) = \pi_1(x) + \Delta + \pi_2(y) - \rho(J_{11}^N(x, y) - J_{10}(x)) + \frac{\partial J_{11}^N(x, y)}{\partial x}$$

Inserting $J_{10}(x)$ into this equation we get:

$$(r + \rho)J_{11}^N(x, y) = \pi_1(x) + \Delta + \pi_2(y) + \frac{\partial J_{11}^N(x, y)}{\partial x} + \rho \left[\frac{d_1(1 - \beta) - s}{r} + q_2 \frac{\pi_2(\bar{x}) + \Delta + s}{r(r + \rho + q_2)} + Ae^{rx} + \frac{c_1(1 - \beta)e^{\gamma x}}{r - \gamma} - \frac{q_2 K e^{(r + \rho + q_2)x}}{\rho + q_2} \right]$$

The general solution of this linear first order differential equation is given by:

$$J_{11}^N(x, y) = \frac{s(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K e^{(r + \rho + q_2)x}}{\rho + q_2} + \frac{d_1(1 - \beta) - s}{r} + Ae^{rx} + \frac{c_1(1 - \beta)e^{\gamma x}}{r - \gamma} + \frac{\pi_2(y) + \Delta}{r + \rho} + \frac{\rho q_2(\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)} + D e^{(r + \rho)x}$$

with D being the integration constant. Evaluating this equation at $y = \bar{x}$, we should get $J_{11}^N(x, \bar{x})$, which implies that $D = 0$, because:

$$\frac{(\pi_2(\bar{x}) + \Delta)}{r + \rho} + \frac{\rho q_2(\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)} = \frac{(\pi_2(\bar{x}) + \Delta)(r + q_2)}{r(r + \rho + q_2)}$$

Inserting $x = 0$ and $y = x_i$, we get the function $J_{11}^N(0, x_i)$:

$$J_{11}^N(0, x_i) = \frac{s(r + q_2)}{r(r + \rho + q_2)} + \frac{\rho K}{\rho + q_2} + \frac{d_1(1 - \beta) - s}{r} + A + \frac{c_1(1 - \beta)}{r - \gamma} + \frac{(\pi_2(x_i) + \Delta)}{r + \rho} + \frac{\rho q_2(\pi_2(\bar{x}) + \Delta)}{r(r + \rho + q_2)(r + \rho)}$$

Two skill levels: present value equations

Consider some firm with an inexperienced worker of skill $j = L, H$ employed in the junior position and an open vacancy on the senior level. The present value of discounted future profits of this firm is denoted by J_{j0} and given by:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2[\alpha_2 J_{jL}^N(x, \bar{x}_L) + (1 - \alpha_2)J_{jH}^N(x, \bar{x}_H) - J_{j0}(x)] + \frac{\partial J_{j0}(x)}{\partial x} \quad (15)$$

With probability α_2 the firm will hire another low skill worker for the senior position, which generates the present value of profits $J_{jL}^N(x, \bar{x}_L)$, while with the opposite probability $1 - \alpha_2$ the firm will hire a high skill worker which generates the present value of profits $J_{jH}^N(x, \bar{x}_H)$. Recall that \bar{x}_j denotes the market experience level of applicants in the managerial market with a skill level $j = L, H$. Variables $J_{jf}^N(x, y)$, $j, f = L, H$ can be found as:

$$rJ_{jf}^N(x, y) = \pi_1(x) + \pi_2^f(y) - \rho(J_{jf}^N(x, y) - J_{j0}(x)) + \frac{\partial J_{jf}^N(x, y)}{\partial x} \quad (16)$$

Here $\pi_2^f(y)$ is the flow profit generated by the senior manager who may retire and exit the market at rate ρ . In this case the firm is left with the inexperienced junior worker and the corresponding present value $J_{j0}(x)$. Further, we define an auxiliary variable $\bar{J}_j(x) \equiv \alpha_2 J_{jL}^N(x, \bar{x}_L) + (1 -$

$\alpha_2 J_{jH}^N(x, \bar{x}_H)$ which is a weighted average between the two present values and is given by:

$$r\bar{J}_j(x) = \pi_1(x) + \alpha_2\pi_2^L(\bar{x}_L) + (1 - \alpha_2)\pi_2^H(\bar{x}_H) - \rho(\bar{J}_j(x) - J_{j0}(x)) + \frac{\partial\bar{J}_j(x)}{\partial x} \quad (17)$$

Note that formally, $\bar{J}_j(x, \bar{x}_L, \bar{x}_H)$ depends on \bar{x}_L and \bar{x}_H but this dependence is suppressed for the ease of exposition. Equation (15) can then be written as:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2[\bar{J}_j(x) - J_{j0}(x)] + \frac{\partial J_{j0}(x)}{\partial x} \quad (18)$$

In addition, define another auxilliary variable $\Delta J_j(x) \equiv \bar{J}_j(x) - J_{j0}(x)$, this is the average present value gain of finding a manager. Taking difference between equations (17) and (18) it becomes:

$$(r + \rho + q_2)\Delta J_j(x) = \alpha_2\pi_2^L(\bar{x}_L) + (1 - \alpha_2)\pi_2^H(\bar{x}_H) + s + \frac{\partial\Delta J_j(x)}{\partial x}$$

The general solution of this first order linear differential equation is:

$$\Delta J_j(x) = \frac{\alpha_2\pi_2^L(\bar{x}_L) + (1 - \alpha_2)\pi_2^H(\bar{x}_H) + s}{r + \rho + q_2} + K_j e^{(r+\rho+q_2)x} \quad (19)$$

where K_j is the integration constant. Let $\bar{\pi}_2(\bar{x}_L, \bar{x}_H) = \alpha_2\pi_2^L(\bar{x}_L) + (1 - \alpha_2)\pi_2^H(\bar{x}_H)$ denote the average flow profit of the firm associated with hiring a manager in the market. With this notation we can rewrite equation (18) for $J_{j0}(x)$ by inserting $\Delta J_j(x)$ in the following way:

$$rJ_{j0}(x) = \pi_1(x) - s + q_2\left[\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r + \rho + q_2} + K_j e^{(r+\rho+q_2)x}\right] + \frac{\partial J_{j0}(x)}{\partial x} \quad (20)$$

With A_j denoting the integration constant, the general solution of this differential equation can be written as:

$$J_{j0}(x) = \frac{d_1(1 - \beta) - s}{r} + q_2\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} + A_j e^{rx} + \frac{c_1(1 - \beta)e^{\gamma x}}{r - \gamma} - \frac{q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2} \quad (21)$$

Finally, inserting $J_{j0}(x)$ into equation (16) we get the last differential equation for $J_{jf}^N(x, y)$ which allows us to solve the main part of the model. The differential equation for $J_{jf}^N(x, y)$ is given by:

$$(r + \rho)J_{jf}^N(x, y) = \pi_1(x) + \pi_2^f(y) + \frac{\rho(d_1(1 - \beta) - s)}{r} + \rho q_2\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} \quad (22)$$

$$+ \rho A_j e^{rx} + \frac{\rho c_1(1 - \beta)e^{\gamma x}}{r - \gamma} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{\partial J_{jf}^N(x, y)}{\partial x} \quad (23)$$

It can be rewritten as:

$$(r + \rho)J_{jf}^N(x, y) = d_1(1 - \beta) + \pi_2^f(y) + \frac{\rho(d_1(1 - \beta) - s)}{r} + \rho q_2\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r + \rho + q_2)} + \rho A_j e^{rx} + \frac{(\rho + r - \gamma)c_1(1 - \beta)e^{\gamma x}}{r - \gamma} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{\rho + q_2} + \frac{\partial J_{jf}^N(x, y)}{\partial x}$$

Let D_{jf} denote the integration constant, so the general solution of the above equation becomes:

$$\begin{aligned}
J_{jf}^N(x, y) &= \frac{d_1(1-\beta) + \pi_2^f(y)}{r+\rho} + \frac{\rho(d_1(1-\beta) - s)}{r(r+\rho)} + \rho q_2 \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r+\rho+q_2)(r+\rho)} \\
&+ \frac{\rho A_j e^{rx}}{r+\rho-r} + \frac{(\rho+r-\gamma)c_1(1-\beta)e^{\gamma x}}{(r-\gamma)(r+\rho-\gamma)} - \frac{\rho q_2 K_j e^{(r+\rho+q_2)x}}{(\rho+q_2)(r+\rho-(r+\rho+q_2))} + D_{jf} e^{(r+\rho)x} \\
&= \frac{d_1(1-\beta)}{r} + \frac{\pi_2^f(y)}{r+\rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} + \left[\frac{-s}{r} + \frac{rs}{r(r+\rho)} + \frac{\rho q_2 s}{r(r+\rho+q_2)(r+\rho)} \right] \\
&+ A_j e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{(r-\gamma)} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{(\rho+q_2)} + D_{jf} e^{(r+\rho)x} \\
&= \frac{d_1(1-\beta) - s}{r} + \frac{\pi_2^f(y)}{r+\rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} + \frac{s(r+q_2)}{r(r+\rho+q_2)} \\
&+ A_j e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{(r-\gamma)} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{(\rho+q_2)} + D_{jf} e^{(r+\rho)x}
\end{aligned}$$

where D_{jf} is the corresponding integration constant. Next we combine equations (19) and (21) to find solution for the auxilliary variable $\bar{J}_j(x) = J_{j0}(x) + \Delta J_j(x)$:

$$\bar{J}_j(x) = \frac{d_1(1-\beta) - s}{r} + (r+q_2) \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H) + s}{r(r+\rho+q_2)} + A_j e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{r-\gamma} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{\rho+q_2} \quad (24)$$

Evaluating $J_{jL}^N(x, y)$ at $y = \bar{x}_L$ with the corresponding term D_{jL} , $J_{jH}^N(x, y)$ at $y = \bar{x}_H$ with the corresponding term D_{jH} and taking a weighted average between the two we get $(1 - \alpha_2)J_{jH}^N(x, \bar{x}_H) = \bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L)$. The right-hand side of this equation is given by:

$$\begin{aligned}
\bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L) &= (r+q_2) \frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)} - \alpha_2 \frac{\pi_2^L(\bar{x}_L)}{r+\rho} - \alpha_2 \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} - \alpha_2 D_{jL} e^{(r+\rho)x} \\
&+ (1-\alpha_2) \left[\frac{d_1(1-\beta) - s}{r} + \frac{(r+q_2)s}{r(r+\rho+q_2)} + A_j e^{rx} + \frac{c_1(1-\beta)e^{\gamma x}}{r-\gamma} + \frac{\rho K_j e^{(r+\rho+q_2)x}}{\rho+q_2} \right]
\end{aligned}$$

Consider the first four terms of this equation:

$$\begin{aligned}
&\frac{\bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r+\rho} + \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} - \alpha_2 \frac{\pi_2^L(\bar{x}_L)}{r+\rho} - \alpha_2 \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} - \alpha_2 D_{jL} e^{(r+\rho)x} \\
&= (1-\alpha_2) \frac{\rho q_2 \bar{\pi}_2(\bar{x}_L, \bar{x}_H)}{r(r+\rho+q_2)(r+\rho)} + (1-\alpha_2) \frac{\pi_2^H(\bar{x}_H)}{r+\rho} - \alpha_2 D_{jL} e^{(r+\rho)x}
\end{aligned}$$

because $\bar{\pi}_2(\bar{x}_L, \bar{x}_H) - \alpha_2 \pi_2^L(\bar{x}_L) = (1-\alpha_2) \pi_2^H(\bar{x}_H)$. Comparing $\bar{J}_j(x) - \alpha_2 J_{jL}^N(x, \bar{x}_L)$ with $(1 - \alpha_2)J_{jH}^N(x, \bar{x}_H)$ we can see that $(1 - \alpha_2)D_{jH} e^{(r+\rho)x} = -\alpha_2 D_{jL} e^{(r+\rho)x}$, so that $(1 - \alpha_2)D_{jH} + \alpha_2 D_{jL} = 0$.

In the next step we consider the last Bellman equations for firms with experienced junior workers and senior managers. Let $J_{0f}(y)$ denote the present value of future profits for a firm with only one senior manager whose experience is y :

$$rJ_{0f}(y) = \pi_2^f(y) - s - \rho J_{0f}(y) + q_1[\alpha_1 J_{Lf}^N(0, y) + (1 - \alpha_1)J_{Hf}^N(0, y) - J_{0f}(y)]$$

With probability α_1 the firm fills its junior position with a low skill worker, while with probability

$(1 - \alpha_1)$ the open position is filled with a high skill worker. The last state that we have to take into account is $J_{jf}^S(x, y)$, where the junior worker accumulated sufficient experience and is already searching for senior positions in competing firms. It is given by:

$$rJ_{jf}^S(x, y) = \pi_1(x) + \pi_2^f(y) - \rho(J_{jf}^S(x, y) - J_{0j}(x)) - \lambda_2(J_{jf}^S(x, y) - J_{0f}(y))$$

If the senior manager retires, the remaining worker is promoted to the senior position, so the firm ends up with a present value of profits $J_{0j}(x)$. In contrast, if the junior worker quits the firm ends up with a present value of profits $J_{0f}(y)$.

As before we impose several boundary conditions:

$$J_{j0}(\bar{x}_i^j) = J_{0j}(\bar{x}_i^j) \quad J_{jf}^N(\bar{x}_i^j, \bar{x}_f) = J_{jf}^S(\bar{x}_i^j, \bar{x}_f) \quad j, f = L, H$$

These conditions imply that firms commit to promoting workers whenever they reach a pre-specified skill-specific experience level \bar{x}_i^j depending on their skills $j = L, H$. However, if the senior position is filled the worker starts searching on-the-job. Combining this set of 6 equations with 2 equations $\alpha_2 D_{jL} + (1 - \alpha_2) D_{jH} = 0$ we can find a vector of 8 integration constants $\{A_j, K_j, D_{jf}\}$ for the optimal skill-specific promotion times \bar{x}_i^j of firm i and market experience cut-offs \bar{x}_j .

In the final step we consider the objective function of firm i . Given that the firm has to determine its strategy upon the entry, it aims at maximizing the present value of expected future profits J_{00} given by:

$$rJ_{00} = -2s + q_1[\alpha_1 J_{L0}(0) + (1 - \alpha_1) J_{H0}(0) - J_{00}] + q_2[\alpha_2 J_{0L}(\bar{x}_L) + (1 - \alpha_2) J_{0H}(\bar{x}_H) - J_{00}]$$

This equation shows that there are four sources of uncertainty for the firm at this stage: which position will be filled first – junior or senior – and which type of worker will be hired – high or low skilled. The choice variables of the firm are \bar{x}_i^L and \bar{x}_i^H which are the promotion cut-offs for each of the two skill groups.

Two skill levels: differential equations

Similarly to the case with homogeneous workers, in the steady state we have $d_{00} = \frac{n}{q_1 + q_2}$. Let $d_{H0}(x)$ be the number of firms which have a high skill junior worker with experience x and a senior vacancy. Next, $d_{HL}^N(x)$ ($d_{HH}^N(x)$) are the firms with a high skill junior workers with experience x who is not searching and low (high) skill senior worker. Consider the changes in the stock of those firms. The inflow into state $d_{H0}(x)$ is equal to $\rho(d_{HH}^N(x) + d_{HL}^N(x))$ which shows that whenever the senior worker in firm $d_{HL}^N(x)$ or $d_{HH}^N(x)$ retires the firm transitions into $d_{H0}(x)$ state. At the same time $\rho d_{HL}^N(x)$ and $\rho d_{HH}^N(x)$ are the rates of outflow of $d_{HL}^N(x)$ and $d_{HH}^N(x)$ firms, respectively. On the other hand, $d_{H0}(x)$ firms find a senior worker from the external market at rate q_2 which implies that the outflow of this state is $q_2 d_{H0}(x)$. Then the inflow into the state $d_{HL}^N(x)$ is equal to $q_2 \alpha_2 d_{H0}(x)$, i.e. the firm that is in state $d_{H0}(x)$ finds a low skill senior worker from the external market and becomes of type $d_{HL}^N(x)$. Similarly, $(1 - \alpha_2)$ is the fraction of high skill applicants to senior positions which implies that the inflow into state $d_{HH}^N(x)$ is equal to $q_2(1 - \alpha_2) d_{H0}(x)$. We then get the following system of three first order linear

differential equations:

$$\begin{cases} \partial d_{H0}(x)/\partial x &= -q_2 d_{H0}(x) + \rho(d_{HH}^N(x) + d_{HL}^N(x)) \\ \partial d_{HH}^N(x)/\partial x &= q_2(1 - \alpha_2)d_{H0}(x) - \rho d_{HH}^N(x) \\ \partial d_{HL}^N(x)/\partial x &= q_2\alpha_2 d_{H0}(x) - \rho d_{HL}^N(x) \end{cases}$$

The coefficient matrix is given by:

$$\begin{pmatrix} -q_2 & \rho & \rho \\ q_2(1 - \alpha_2) & -\rho & 0 \\ q_2\alpha_2 & 0 & -\rho \end{pmatrix}$$

The eigenvalues are: $r_1 = 0$, $r_2 = -(q_2 + \rho)$, $r_3 = -\rho$ with corresponding eigenvectors:

$$v_1 = \begin{pmatrix} \frac{\rho}{q_2\alpha_2} \\ \frac{1-\alpha_2}{\alpha_2} \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} -\frac{1}{\alpha_2} \\ \frac{1-\alpha_2}{\alpha_2} \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Thus the general solution is:

$$\begin{cases} d_{H0}(x) &= k_1^H \frac{\rho}{\alpha_2 q_2} - k_2^H \frac{1}{\alpha_2} e^{-(\rho+q_2)x} \\ d_{HH}^N(x) &= k_1^H \frac{(1-\alpha_2)}{\alpha_2} + k_2^H \frac{(1-\alpha_2)}{\alpha_2} e^{-(\rho+q_2)x} - k_3^H e^{-\rho x} \\ d_{HL}^N(x) &= k_1^H + k_2^H e^{-(\rho+q_2)x} + k_3^H e^{-\rho x} \end{cases}$$

We find k_1^H , k_2^H and k_3^H using the initial conditions: $d_{H0}(0) = q_1(1 - \alpha_1)d_{00}$, $d_{HH}^N(0) = q_1(1 - \alpha_1)d_{0H}$ and $d_{HL}^N(0) = q_1(1 - \alpha_1)d_{0L}$. The first initial condition implies that the stock of $d_{H0}(0)$ consists of new firms which just hired a junior worker and this worker happens to be high skilled. The second and third initial conditions show that the stock of firms $d_{HH}^N(0)$ ($d_{HL}^N(0)$) equals the firms with a junior vacancy and a high (low) skill senior worker who filled their junior position with a high skill worker. This gives the following results:

$$\begin{aligned} k_1^H &= \frac{q_1 q_2 \alpha_2 (1 - \alpha_1)}{q_2 + \rho} (d_{00} + d_{0H} + d_{0L}) \\ k_2^H &= \frac{q_1 \alpha_2 (1 - \alpha_1)}{q_2 + \rho} (\rho(d_{0H} + d_{0L}) - q_2 d_{00}) \\ k_3^H &= q_1(1 - \alpha_1)((1 - \alpha_2)d_{0L} - \alpha_2 d_{0H}) \end{aligned}$$

Let $d_{L0}(x)$ be the number of firms with a low skill junior worker with experience x and a senior vacancy. Further, $d_{LL}^N(x)$ and $d_{LH}^N(x)$ denote the firms with a junior worker with experience x and senior worker with low and high skill, respectively. The dynamic equations for these firms are defined analogously to the ones for $d_{H0}(x)$, $d_{HH}^N(x)$ and $d_{HL}^N(x)$ and lead to the following system of first order differential equations:

$$\begin{cases} \partial d_{L0}(x)/\partial x &= -q_2 d_{L0}(x) + \rho(d_{LL}^N(x) + d_{LH}^N(x)) \\ \partial d_{LL}^N(x)/\partial x &= q_2\alpha_2 d_{L0}(x) - \rho d_{LL}^N(x) \\ \partial d_{LH}^N(x)/\partial x &= q_2(1 - \alpha_2)d_{L0}(x) - \rho d_{LH}^N(x) \end{cases}$$

The coefficient matrix is given by:

$$\begin{pmatrix} -q_2 & \rho & \rho \\ q_2\alpha_2 & -\rho & 0 \\ q_2(1-\alpha_2) & 0 & -\rho \end{pmatrix}$$

The eigenvalues are the same as for the previous system: $r_1 = 0$, $r_2 = -(q_2 + \rho)$, $r_3 = -\rho$ and the corresponding eigenvectors are given by:

$$v_1 = \begin{pmatrix} \frac{\rho}{q_2(1-\alpha_2)} \\ \frac{\alpha_2}{1-\alpha_2} \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} -\frac{1}{1-\alpha_2} \\ \frac{\alpha_2}{1-\alpha_2} \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The general solution is thus:

$$\begin{cases} d_{L0}(x) &= k_1^L \frac{\rho}{q_2(1-\alpha_2)} - k_2^L \frac{1}{1-\alpha_2} e^{-(\rho+q_2)x} \\ d_{LL}^N(x) &= k_1^L \frac{\alpha_2}{(1-\alpha_2)} + k_2^L \frac{\alpha_2}{(1-\alpha_2)} e^{-(\rho+q_2)x} - k_3^L e^{-\rho x} \\ d_{LH}^N(x) &= k_1^L + k_2^L e^{-(\rho+q_2)x} + k_3^L e^{-\rho x} \end{cases}$$

We then find k_1^L , k_2^L and k_3^L using the following initial conditions: $d_{L0}(0) = q_1\alpha_1 d_{00}$, $d_{LL}^N(0) = q_1\alpha_1 d_{0L}$ and $d_{LH}^N(0) = q_1\alpha_1 d_{0H}$. The initial conditions are interpreted analogously to the ones for k_1^H , k_2^H and k_3^H . The results are given by:

$$\begin{aligned} k_1^L &= \frac{q_1 q_2 \alpha_1 (1-\alpha_2)}{q_2 + \rho} (d_{00} + d_{0H} + d_{0L}) \\ k_2^L &= \frac{q_1 \alpha_1 (1-\alpha_2)}{q_2 + \rho} (\rho(d_{0H} + d_{0L}) - q_2 d_{00}) \\ k_3^L &= q_1 \alpha_1 (\alpha_2 d_{0H} - (1-\alpha_2) d_{0L}) \end{aligned}$$

Further, the steady state values for the stocks of d_{H0} , d_{HH}^N and d_{HL}^N firms are found by integrating $d_{H0}(x)$, $d_{HH}^N(x)$ and $d_{HL}^N(x)$ over the interval $[0..\bar{x}_H]$:

$$\begin{aligned} d_{H0} &= \int_0^{\bar{x}_H} d_{H0}(x) dx = \frac{k_1^H \rho}{q_2 \alpha_2} \bar{x}_H - \frac{k_2^H}{\alpha_2(\rho+q_2)} (1 - e^{-(\rho+q_2)\bar{x}_H}) \\ d_{HH}^N &= \int_0^{\bar{x}_H} d_{HH}^N(x) dx = \frac{k_1^H (1-\alpha_2)}{\alpha_2} \bar{x}_H + \frac{k_2^H (1-\alpha_2)}{\alpha_2(\rho+q_2)} (1 - e^{-(\rho+q_2)\bar{x}_H}) - \frac{k_3^H}{\rho} (1 - e^{-\rho\bar{x}_H}) \\ d_{HL}^N &= \int_0^{\bar{x}_H} d_{HL}^N(x) dx = k_1^H \bar{x}_H + \frac{k_2^H}{\rho+q_2} (1 - e^{-(\rho+q_2)\bar{x}_H}) + \frac{k_3^H}{\rho} (1 - e^{-\rho\bar{x}_H}) \end{aligned}$$

Similarly we derive the steady state equations for the stock of d_{L0} , d_{LL}^N and d_{LH}^N firms by integrating $d_{L0}(x)$, $d_{LL}^N(x)$ and $d_{LH}^N(x)$ over the interval $[0..\bar{x}_L]$:

$$\begin{aligned} d_{L0} &= \int_0^{\bar{x}_L} d_{L0}(x) dx = \frac{k_1^L \rho}{q_2(1-\alpha_2)} \bar{x}_L - \frac{k_2^L}{(1-\alpha_2)(\rho+q_2)} (1 - e^{-(\rho+q_2)\bar{x}_L}) \\ d_{LL}^N &= \int_0^{\bar{x}_L} d_{LL}^N(x) dx = \frac{k_1^L \alpha_2}{(1-\alpha_2)} \bar{x}_L + \frac{k_2^L \alpha_2}{(1-\alpha_2)(\rho+q_2)} (1 - e^{-(\rho+q_2)\bar{x}_L}) - \frac{k_3^L}{\rho} (1 - e^{-\rho\bar{x}_L}) \\ d_{LH}^N &= \int_0^{\bar{x}_L} d_{LH}^N(x) dx = k_1^L \bar{x}_L + \frac{k_2^L}{\rho+q_2} (1 - e^{-(\rho+q_2)\bar{x}_L}) + \frac{k_3^L}{\rho} (1 - e^{-\rho\bar{x}_L}) \end{aligned}$$

Next, let us consider the firms which have a junior worker who is already searching in the external market: d_{LL}^S , d_{HL}^S , d_{HH}^S and d_{LH}^S . The inflow into these states happens whenever the junior worker of a d_{ij}^N ($i = L, H$, $j = L, H$) firm reaches the promotion cut-off \bar{x}_L or \bar{x}_H , depending on the worker's type. The outflow of these states occurs either when the senior worker

retires which happens at a rate ρ or when the searching junior worker finds a senior job at a different firm which occurs at a rate λ_2 . Hence, we get:

$$\begin{aligned} d_{LL}^{\dot{S}} &= d_{LL}^N(\bar{x}_L) - (\rho + \lambda_2)d_{LL}^S \\ d_{HL}^{\dot{S}} &= d_{HL}^N(\bar{x}_H) - (\rho + \lambda_2)d_{HL}^S \\ d_{HH}^{\dot{S}} &= d_{HH}^N(\bar{x}_H) - (\rho + \lambda_2)d_{HH}^S \\ d_{LH}^{\dot{S}} &= d_{LH}^N(\bar{x}_L) - (\rho + \lambda_2)d_{LH}^S \end{aligned}$$

Inserting $d_{LL}^N(x)$ evaluated at \bar{x}_L and solving for $d_{LL}^{\dot{S}} = 0$ we find the following steady state equation for the stock of d_{LL}^S firms:

$$d_{LL}^S = \frac{k_1^L \alpha_2 + k_2^L \alpha_2 e^{-(\rho+q_2)\bar{x}_L} - k_3^L (1 - \alpha_2) e^{-\rho\bar{x}_L}}{(1 - \alpha_2)(\rho + \lambda_2)}$$

Analogously, the steady state equations for the stock of d_{HL}^S , d_{HH}^S and d_{LH}^S firms are:

$$\begin{aligned} d_{HL}^S &= \frac{k_1^H + k_2^H e^{-(\rho+q_2)\bar{x}_H} + k_3^H e^{-\rho\bar{x}_H}}{\rho + \lambda_2} \\ d_{HH}^S &= \frac{k_1^H (1 - \alpha_2) + k_2^H (1 - \alpha_2) e^{-(\rho+q_2)\bar{x}_H} - k_3^H \alpha_2 e^{-\rho\bar{x}_H}}{\alpha_2 (\rho + \lambda_2)} \\ d_{LH}^S &= \frac{k_1^L + k_2^L e^{-(\rho+q_2)\bar{x}_L} + k_3^L e^{-\rho\bar{x}_L}}{(\rho + \lambda_2)} \end{aligned}$$

Finally, let us consider the firms which have only a senior worker. Firms in state $d_{H0}(\bar{x}_H)$ enter state d_{0H} because the junior worker is promoted. Also, firms in state d_{00} enter state d_{0H} when they find a high skill senior worker from the external market which happens at a rate $q_2(1 - \alpha_2)$. Further, at a rate ρ the senior worker in firms d_{HL}^S or d_{HH}^S retires and the junior worker is automatically promoted. Then these firms also enter state d_{0H} . Next, at a rate λ_2 the searching worker in firms d_{LH}^S or d_{HH}^S finds a job in a new firm and leaves so that the firms transition into state d_{0H} . On the other hand, the outflow from state d_{0H} happens either when the senior worker retires at a rate ρ or when the firm finds a junior worker from the market which occurs at a rate q_1 . This leads to the following dynamic equation:

$$\dot{d}_{0H} = d_{H0}(\bar{x}_H) + q_2(1 - \alpha_2)d_{00} + \rho d_{HL}^S + \lambda_2 d_{LH}^S + (\rho + \lambda_2)d_{HH}^S - (\rho + q_1)d_{0H}$$

Similarly we can write the dynamic equation for d_{0L} firms:

$$\dot{d}_{0L} = d_{L0}(\bar{x}_L) + q_2 \alpha_2 d_{00} + \rho d_{LH}^S + \lambda_2 d_{HL}^S + (\rho + \lambda_2)d_{LL}^S - (\rho + q_1)d_{0L}$$

Substituting in all expressions we have already derived and solving for $\dot{d}_{0H} = \dot{d}_{0L} = 0$ then gives us the steady state values for the stock of d_{0H} and d_{0L} firms.

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