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RESEARCH ARTICLE

Credit risk in derivative securities: A simplified approach

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Abstract

The pricing of options and other derivatives which are subject to the default risk of the writer usually requires the calibration of a sophisticated model and substantial effort in determining the input parameters. We propose a very simple method to incorporate correlated credit risk into the pricing of vulnerable derivatives. The approach is based upon some approximations of more complex models and requires a minimum of input parameters. It is therefore easily applicable and maintains the accuracy of sophisticated models to a large extent, as shown in numerical studies for call options, put options, and discount certificates.

KEYWORDS

certificates, credit risk, option pricing, vulnerable options

JEL CLASSIFICATION

G13, G21

1 | INTRODUCTION

Derivative securities which are not settled by a clearing house with a central counterparty or guaranteed by a third party are subject to the credit risk of the writer. This holds true in particular for retail derivatives, which banks offer to individual private investors. Retail derivatives have attracted increasing interest, both in practice and in the academic literature, over the past two decades. Numerous studies have focused on the pricing of such products, for example, Henderson and Pearson (2011) and Egan (2019) for the U.S. market, Schertler (2016) and Pelster and Schertler (2019) for the German market, Wallmeier and Diethelm (2009) for the Swiss market, and Célérier and Vallée (2017) for several European markets. The goal of this paper is to provide a simple model to account for credit risk in the pricing of derivative products. In contrast to other approaches, the model allows for correlation between market risk and credit risk, but the application remains easily tractable in terms of input data and numerical complexity.

Banks offer their derivative retail products at prices that exceed the theoretical value by a certain amount—the so-called mark-up or gross margin. While private investors are willing to pay these margins to gain access to derivatives markets, they reflect the issuing bank's profit before structuring, hedging, and other costs. As margins have fallen below one percent of the product price for standard products in recent years, credit risk plays an increasingly important role in their analysis. Hull and White (1995) have proposed a simple way to incorporate credit risk into the theoretical value of a credit-risky derivative: Assuming independence between credit risk and market risk (i.e., between the issuer and the underlying security), the value is simply obtained by discounting the default-free value with the credit spread of the issuer.

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Assuming such independence, however, can cause seriously biased results. A prime example of this is a warrant written by a bank on its own stock. In the case of a call warrant, credit risk subtracts nothing from the default-free value, because in any case where the buyer has a claim, it follows that: the call is in the money, the stock price of the bank is above the strike price, and the bank has not defaulted. In the case of a put warrant, on the other hand, the claim of the buyer is largest when the stock price has fallen to zero, but this likely means insolvency of the bank and the loss of the claim.

These simple examples show that the actual (negative) value of credit risk depends upon the connection between issuer and underlying, as well as on the structure of the derivative. Baule et al. (2008) show that an assumption of independence according to Hull and White (1995) leads to a considerable overestimation of the credit risk for a sample of real-world derivatives. They build a structural model which allows for the incorporation of correlations between the issuer and the underlying. Interestingly, while many subsequent papers acknowledge their work, they continue to use the simple model of Hull and White. Papers that cite the Baule et al. (2008) paper, but nonetheless use the Hull and White (1995) approach, include Baule (2011), Baule and Tallau (2011), Baule and Blonski (2015), Entrop et al. (2016), Schertler (2016), Schertler and Stoerch (2015), Schertler and Stoerch (2018), and Pelster and Schertler (2019).

A likely reason for this persistence is the appealing simplicity of the Hull and White (1995) approach, which only requires information about the credit spread of the issuer. In contrast, with the Baule et al. (2008) approach, a structural model of the issuer has to be calibrated, requiring information about its asset volatility, default point, and recovery rate, as well as correlation of its asset value to the underlying price. There is thus a gap between awareness of the shortcomings of the Hull and White approach and the usability of the Baule et al. model. The aim of this paper is to close this gap by proposing an approach that is easily applicable but which takes into account the relation between market risk and credit risk. The basic idea is to approximate the payoff structure of the derivative by combining a position in the underlying and in a zero coupon bond. While no correlation needs to be taken into account for the latter, we derive a simple formula for the former which requires the correlation between underlying and issuer asset value as the only additional input variable. We show that this approximation vastly improves the accuracy of the credit risk value compared to the Hull and White approach for different derivatives, namely call warrants, put warrants, and discount certificates.

2 | MODELING CORRELATED CREDIT RISK

Johnson and Stulz (1987) pioneered the pricing of “vulnerable options” (i.e., options subject to default risk). In their structural model, the option is the sole liability of its writer (or issuer). Klein (1996) extended the structural model and allowed for exogenous recovery rates. The approach is further developed by Ammann (2001), for example, with stochastic interest rates. The model of Baule et al. (2008), sketched in the following, is basically set up along the lines given by the Klein (1996) approach, with the modification that absolute bankruptcy costs are fixed and do not depend on the actual issuer asset value at default. Regarding the underlying, the assumed process under the risk-neutral measure is a standard geometric Brownian motion as in the Black and Scholes (1973) model:

$$dS_t = r S_t dt + \sigma_S S_t dW^S, \quad (1)$$

where S_t is the underlying stock price¹ at time t , σ_S is its volatility, r is the risk-free rate, and W^S is a standard Wiener process. We should note that this assumption merely affects the valuation of credit risk; for determining the default-free derivative value, any sophisticated model can be used, for example, with time-varying or stochastic volatility. Regarding the writer or issuer of the derivative, its *asset value* also follows geometric Brownian motion, in line with the seminal approach of Merton (1974):

$$dV_t = r V_t dt + \sigma_V V_t dW^V, \quad (2)$$

where V_t is the issuer asset value at time t , σ_V is its volatility, and W^V is a second Wiener process. The two processes are correlated with parameter ρ :

$$\text{Corr}(dW^S, dW^V) = \rho. \quad (3)$$

A default event occurs if and only if the asset value at the expiry date T of the derivative is below the default point D . In the event of a default, the buyer of the derivative receives a constant fraction δ of the promised payoff.

¹In the following, we refer to the underlying as a stock. Of course, other underlying types such as stock indices are also possible.

Baule et al. (2008) show that in this framework, the value of a defaultable zero coupon bond with maturity T and notional 1 is given by

$$ZB_0^d = e^{-rT} [1 + (\delta - 1) N(-b_2)], \quad (4)$$

where $N(\cdot)$ represents the cumulative standard normal distribution function. $N(-b_2)$ equals the risk-neutral T -year default probability, with b_2 given by

$$b_2 = \frac{\ln(V_0/D) + (r - \sigma_V^2/2) T}{\sigma_V \sqrt{T}}. \quad (5)$$

The value of the defaultable zero coupon bond can also be expressed in terms of its credit spread s and the value ZB_0 of a risk-free zero coupon bond:

$$ZB_0^d = e^{-sT} \cdot ZB_0 = e^{-sT} \cdot e^{-rT}. \quad (6)$$

Comparing (4) with (6) yields

$$\begin{aligned} e^{-sT} &= 1 + (\delta - 1) N(-b_2) \\ \Rightarrow N(b_2) &= \frac{e^{-sT} - \delta}{1 - \delta}. \end{aligned} \quad (7)$$

Now we consider a defaultable *stock certificate*. At maturity T , this certificate pays the buyer an amount equal to the stock price S_T , as long as the issuer has not defaulted. In the case of an issuer default, it pays the fraction δS_T . The value of such a certificate is given by Baule et al. (2008):

$$S_0^d = S_0 [\delta + (1 - \delta) N(a_2)] \quad (8)$$

with

$$a_2 = \frac{\ln(V_0/D) + (r - \sigma_V^2/2 + \rho \sigma_V \sigma_S) T}{\sigma_V \sqrt{T}} = b_2 + \rho \sigma_S \sqrt{T}. \quad (9)$$

It follows

$$\begin{aligned} S_0^d &= S_0 \cdot [\delta + (1 - \delta) N(b_2 + \rho \sigma_S \sqrt{T})] \\ &= S_0 \cdot \left[\delta + (1 - \delta) N \left(N^{-1} \left(\frac{e^{-sT} - \delta}{1 - \delta} \right) + \rho \sigma_S \sqrt{T} \right) \right], \end{aligned} \quad (10)$$

using (7).

As a first approximation, we set the recovery rate $\delta = 0$. This might sound like a rough approximation; however, we will calibrate the model to the credit spread s of the issuer. As a benchmark, the model should perfectly explain this spread, or, equivalently, the price of a defaultable zero coupon bond of the issuer. This can be achieved by various combinations of the parameters σ_V (the asset volatility), V_0/D (the leverage ratio), and δ (the recovery rate). The tractability of the model relies upon the fact that *none* of these parameters has to be explicitly specified; only the credit spread s is required as an input. We therefore fix δ at an arbitrary value; for simplicity, we set it at $\delta = 0$. The numerical analysis will show that a different value of δ only has a minor impact on the results for defaultable derivatives.

With this approximation, the value of a defaultable stock certificate simplifies to

$$S_0^d = S_0 \cdot N(N^{-1}(e^{-sT}) + \rho \sigma_S \sqrt{T}). \quad (11)$$

This equation yields an “effective credit spread” for the stock certificate, s_S^{eff} , which we label the discount factor to obtain the defaultable value from the default-free value:

$$\begin{aligned} S_0^d &= e^{-s_S^{\text{eff}} T} S_0 \\ \Rightarrow s_S^{\text{eff}} &= -\frac{1}{T} \ln N(N^{-1}(e^{-sT}) + \rho \sigma_S \sqrt{T}). \end{aligned} \quad (12)$$

The effective credit spread is specific for the stock certificate.

Now we consider an arbitrary derivative on the stock, with price S_t , written by the defaultable issuer, with asset value V_t . As Baule et al. (2008) show, the valuation requires the calibration of the complete model, and involves a bivariate normal distribution. The extent to which the correlation between the underlying stock price and the issuer asset value influences the value of credit risk (i.e., the derivative-specific effective credit spread) depends upon the payoff structure of the derivative. We will approximate this payoff structure by combining a stock certificate and a zero coupon bond.

To this end, we assume that the default-free value $f_0 := f_0(S_0)$ of the derivative and its delta Δ_0 , the first derivative with respect to the underlying price, are given. These values can be obtained from the Black and Scholes (1973) model or from any other pricing model. The function $f_0(\cdot)$ denotes the default-free value dependent on the underlying level. As a first attempt, we approximate the payoff π_T by Δ_0 positions in the underlying and a remaining position in a zero coupon bond:

$$\pi_T \approx \Delta_0 \cdot S_T + (f_0 - \Delta_0 S_0) \cdot e^{rT} \cdot ZB_T. \quad (13)$$

(Note that $ZB_T = 1$.) Under this approximation, the value of the defaultable derivative is given by

$$\begin{aligned} f_0^d &\approx \Delta_0 \cdot S_0^d + (f_0 - \Delta_0 S_0) \cdot e^{rT} \cdot ZB_0^d \\ &= \Delta_0 \cdot S_0^d + (f_0 - \Delta_0 S_0) \cdot e^{-sT}. \end{aligned} \quad (14)$$

When there is no default risk, $S_0^d = S_0$ and $s = 0$. Therefore, (14) yields $f_0^d = f_0$.

However, the delta Δ_0 is a symmetrical measure, which gives an approximation of the payoff profile according to (13) both for upward and downward movements of the underlying. For the value of default risk, the promised payoff of the derivative dependent on the default of the issuer is of primary interest. The linear approximation should therefore be particularly good in cases when the issuer defaults. Default occurs when the asset value at time T falls below the default point, D . We thus consider the conditional expected underlying level, given that the issuer asset value V_T reaches this default point:²

$$S^- := E[S_T | V_T = D]. \quad (15)$$

In Appendix A, it is shown that this underlying level is given by

$$S^- = S_0 \cdot \exp\left(\left(r - \rho^2 \sigma_S^2 / 2\right) T - \rho \sigma_S \sqrt{T} N^{-1}(e^{-sT})\right). \quad (16)$$

With this value, we calculate a “downside delta” as

$$\tilde{\Delta}_0 = \frac{f_0(S_0) - f_0(S^-)}{S_0 - S^-}. \quad (17)$$

The motivation for the downside delta is as follows: We want to approximate the derivative payoff by a position in the underlying and a position in the zero coupon bond, similar to the usual replication approach in derivatives pricing. In a dynamic replication, the number of positions in the underlying is given by the standard delta. But this delta varies over time, and the approximation requires a static replication. Most relevant for the approximation are underlying paths that correspond to a default of the issuer. This event is related to an expected underlying level S^- . The downside delta can therefore be interpreted as an average delta over all underlying paths that start at the current level, S_0 , and end at the expected level at default, S^- .³

Thus, the downside delta can be used instead of the original delta to obtain a better approximation than by using (14):

$$f_0^d \approx \tilde{\Delta}_0 \cdot S_0^d + (f_0 - \tilde{\Delta}_0 S_0) \cdot e^{-sT}. \quad (18)$$

Figure 1 visualizes the approximation of the payoff profile for a discount certificate by the delta approach (14) and the downside delta approach (18).

²Alternatively, we have also considered the expected underlying level, given that the issuer asset value V_T falls below the default point, $E[S_T | V_T \leq D]$. However, this conditional expectation yields slightly worse results on average.

³In this interpretation, the dependence of delta upon the remaining time to maturity is neglected.

With the approximation based on the downside delta $\tilde{\Delta}_0$, the effective credit spread s^{eff} is given by

$$\tilde{\Delta}_0 \cdot e^{-s^{eff}T} S_0 + (f_0 - \tilde{\Delta}_0 S_0) \cdot e^{-sT} = e^{-s^{eff}T} f_0. \quad (19)$$

Using the first-order Taylor approximation of the exponential function, $e^x \approx 1 + x$, the effective credit spread of the derivative is a weighted average of the actual credit spread s and the effective credit spread of a stock certificate s_S^{eff} (12):

$$s^{eff} \approx \frac{\tilde{\Delta}_0 S_0 \cdot s_S^{eff} + (f_0 - \tilde{\Delta}_0 S_0) \cdot s}{f_0}. \quad (20)$$

Note that this value can fall below zero in certain cases when f_0 is small. Thus the effective credit spread used for valuation should be the maximum of 0 and the value given by (20).

Summing up, the effective credit spread of a vulnerable derivative can be calculated as follows:

- Input for credit risk valuation: unconditional credit spread, s ; correlation between underlying and issuer, ρ .
- Calculate effective credit spread for a vulnerable stock certificate, s_S^{eff} (12).
- Calculate conditional stock price, S^- (16).
- Calculate downside delta, $\tilde{\Delta}_0$ (17).
- Calculate effective credit spread, s^{eff} (20), subject to a minimum value of 0.

The risk-adjusted value of the derivative is then obtained by discounting the risk-free value with the effective credit spread.

It should be noted that this approach nests the Hull and White (1995) model as a special case: When the correlation parameter ρ is 0, Equation (12) yields $s_S^{eff} = s$ and the default-free value is discounted with the unconditional spread. Furthermore, for debt instruments whose payoff is uncorrelated with the issuer asset value (in particular for straight bonds), the downside delta is 0, and Equation (20) also yields the unconditional credit spread.

What are the benefits of this simple approach over the Baule et al. (2008) model? First, it gets by with a minimum of input requirements: Besides the unconditional credit spread s , only the correlation ρ between the underlying stock price and the issuer asset value is required as an additional input parameter. In contrast, the Baule et al. model requires the issuer asset volatility σ_V , its default point D/V_0 , and the recovery rate δ . Second, it is computationally easier to handle, since it does not need a bivariate normal distribution.

Of course, the simplification comes at the cost of a potential inaccuracy. In the next section we conduct several analyses about the additional model errors and their sources and relate them to the basic Hull and White (1995) model error when correlation between market risk and credit risk is ignored.

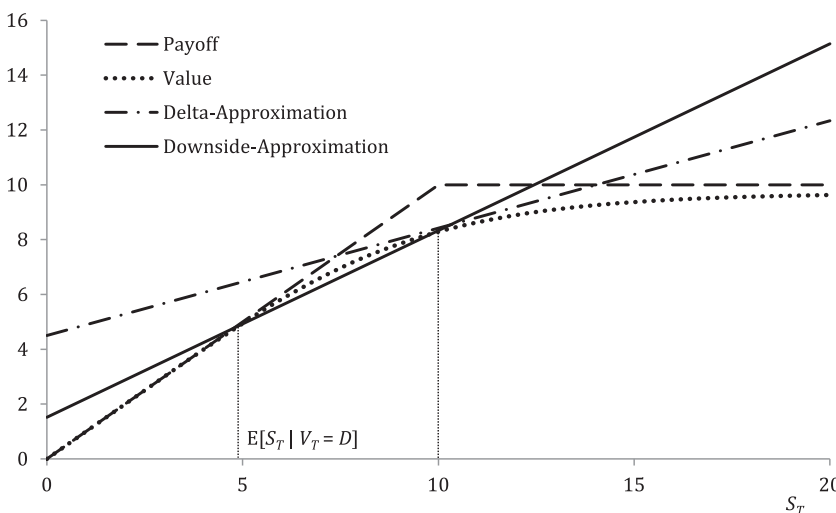


FIGURE 1 Approximation of the payoff profile for a discount certificate. At maturity a discount certificate pays the holder an amount equal to the level of the underlying stock (S_T) or a fixed cap (here, 10), whichever is smaller. The payoff is given by the dashed line, the value by the dotted line. A first linear approximation of the payoff is based on the delta of the certificate (dash-dotted line). When the stock price risk is positively correlated with the issuer's credit risk, the pattern for decreasing underlying levels is more crucial. Therefore, the approximation based upon the downside delta (solid line) is better suited to the valuation of credit risk. In this example, the underlying level is 10, and the time to maturity is 1 year

3 | NUMERICAL STUDY

3.1 | The impact of correlation

Before going into the details of a model analysis, we take a look at the general impact of correlation on the price of a vulnerable derivative. We consider three basic types of derivatives: call options, put options, and discount certificates. For initial insight, the derivatives are at the money (underlying level and strike price equal to 10) and time to maturity is 1 year. The underlying volatility is 0.2.

As a benchmark, we calibrate the Baule et al. (2008) model to an unconditional credit spread of $s = 0.01$. With a recovery rate of $\delta = 0.5$, this is achieved with a default point $D/V_0 = 0.8$ and an issuer asset volatility of $\sigma_V = 0.1197$.⁴

Figure 2 shows the derivative values dependent on the correlation between the underlying and the issuer asset value. For call options, the value lies between the bounds determined by the Black and Scholes (1973) value and the Hull and White (1995) value. For correlations above 0.5, the true value converges to the Black and Scholes value. Thus, as long as the correlation is not very small, adjusting this value with the Hull-White approach is worse than no adjustment at all.

For put options, the true value is always lower than the Hull and White value. Therefore, the Hull and White approach highly underestimates the necessary adjustment. Already for a correlation of 0.2, the correct adjustment must be twice as large as in the Hull and White model.

For discount certificates, the impact is not so large. The value is again between the bounds of the Black and Scholes and the Hull and White model. For a correlation of 0.5, the correct adjustment is about 80% of the Hull and White adjustment.

3.2 | Effective credit spreads

Deriving an effective credit spread for a derivative according to (20) involves a number of approximations: To obtain the simple formula, we have assumed a zero recovery rate and approximated the payoff profile by a straight line. This static replication of the derivative by a stock certificate and a zero coupon bond is the basic model simplification. For numerical convenience, we additionally used the first-order Taylor series. In the following, we provide some numerical tests to judge whether the simplifications still yield reasonable results. To this end, we compare the effective credit spread for a vulnerable derivative obtained by the approximation outlined in this paper with that of the Baule et al. (2008) model. We use the calibration of the previous section (unconditional credit spread of $s = 0.01$, recovery rate of $\delta = 0.5$, default point $D/V_0 = 0.8$, and issuer asset volatility $\sigma_V = 0.1197$).

Figure 3 shows the results for call options. The three graphs relate to correlations between underlying and issuer of 0.2, 0.5, and 0.8, respectively. Each graph shows the effective credit spread dependent upon the strike price of the call for underlying volatilities of 0.1, 0.2, and 0.4. Most important is that the approximation (solid line) is fairly close to the actual model value (dashed line) in all cases. Thus, the approach outlined here is a clear improvement over the Hull and White (1995) approach (constant spread of 0.01). As for the size of the spread, the Hull and White assumption of zero correlation considerably overestimates the actual spread, provided that the correlation or the strike price is not very low. For a medium correlation of 0.5, the effective credit spread for an at-the-money call option is below 0.002 in all cases.

Figure 4 shows the results for put options, replicating the correlation and volatility scenarios of Figure 3. Unlike call options, the Hull and White (1995) approach now underestimates the actual credit spread considerably, in particular when the put is out of the money. (Note that the scale of the y-axis is 10 times larger than in Figure 3). For a correlation of 0.5, the effective credit spread is larger than 0.03 in all cases. The approximation is still close to the Baule et al. (2008) model, although some larger deviations occur for deep-out-of-the-money puts. However, it is noteworthy that the absolute value of these puts is very small; for a medium volatility of 0.2, the put value falls below 0.01 when the underlying level falls below 7.

Finally, Figure 5 shows the results for discount certificates. A discount certificate is a popular retail derivative that has been the focus of several academic studies (including Baule, 2011; Baule et al., 2008; Entrop et al., 2016; Schertler, 2016,

⁴Note that other combinations of D/V_0 and σ_V lead to exactly the same results, as long as the credit spread and the recovery rate are matched.

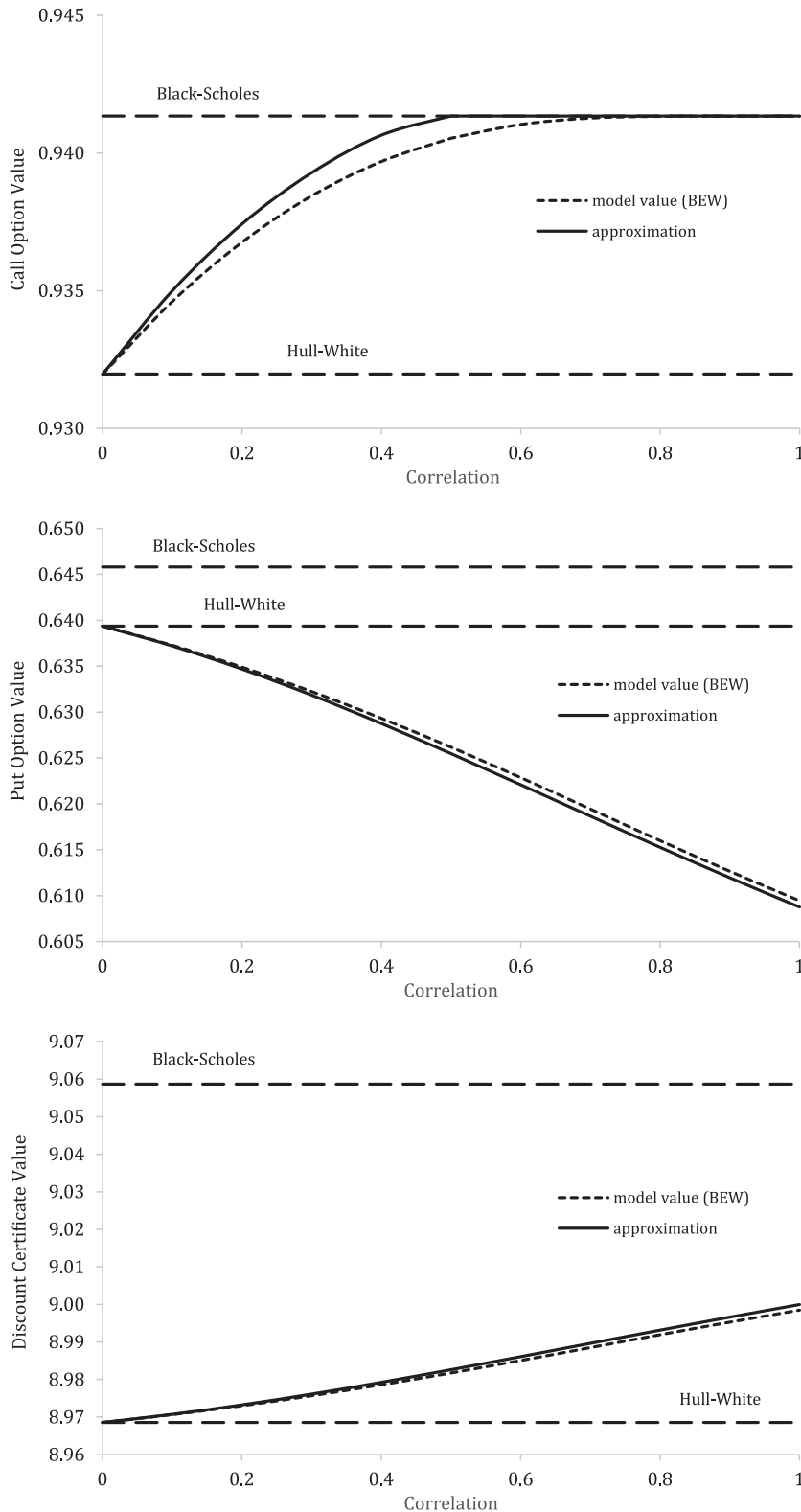
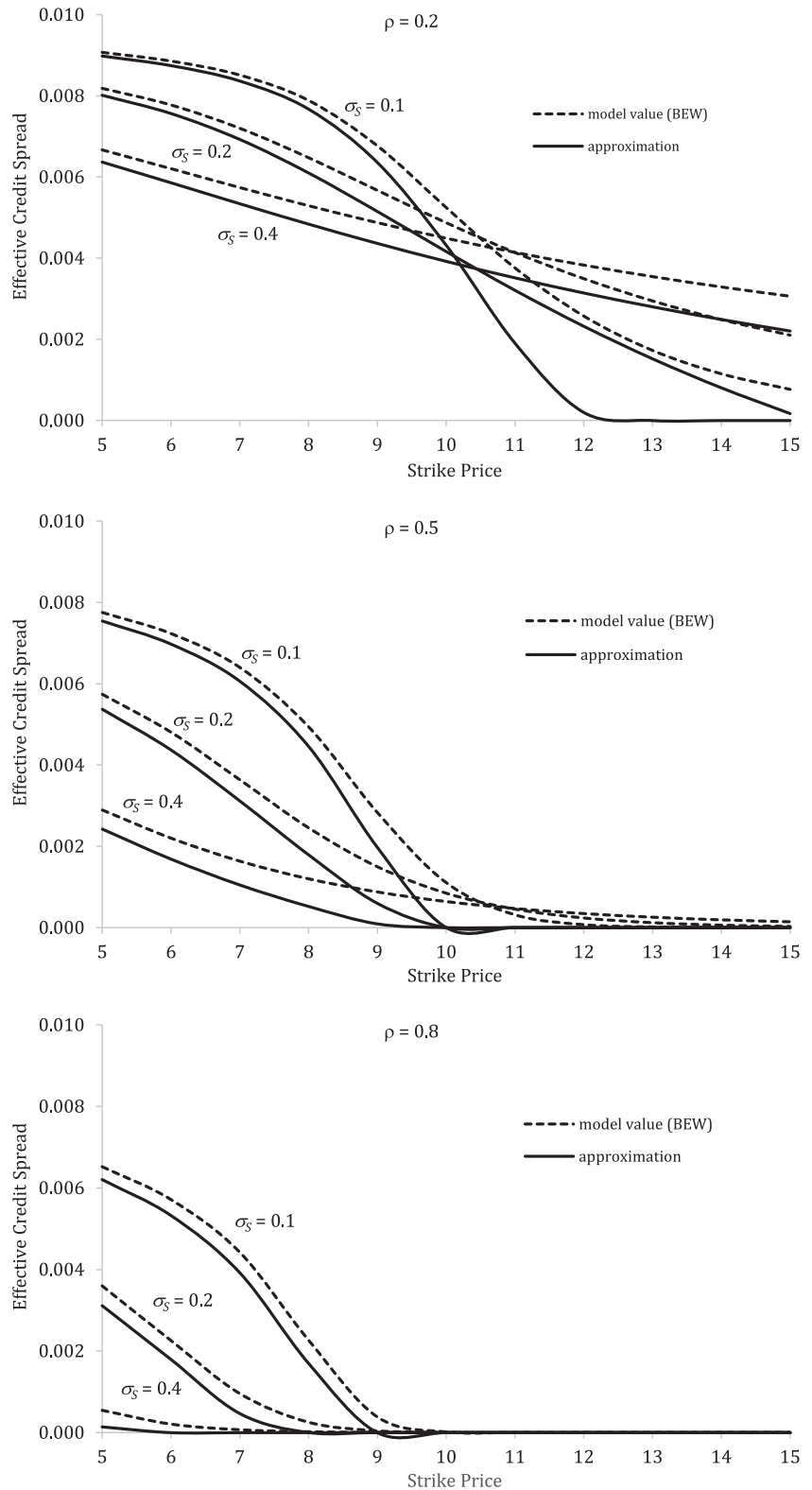


FIGURE 2 Value of a call option (upper graph), put option (middle graph), and discount certificate (lower graph), dependent on the correlation between underlying and issuer asset value. For each derivative, both the underlying level and the strike price are 10, the underlying volatility is 0.2, and the time to maturity is 1 year. The dashed line shows the derivative value based on the Baule et al. (2008; BEW) model, calibrated to a credit spread of 0.01 and a recovery rate of 0.5. The solid line shows the approximation based on Equation (20). The benchmark values of the Black and Scholes (1973) and Hull and White (1995) models are given by the long-dashed lines

among others). This is simply a combination of a zero coupon bond and a short position in a put option (see also the payoff diagram in Figure 1). The construction of the graphs in Figure 5 is analogous to those for call and put options in Figures 3 and 4. For discount certificates, the approximation is nearly identical to the actual model value. For small correlations between underlying and issuer, the effective credit spread is only slightly below the unconditional value. The deviations become larger with increasing correlation, with increasing underlying volatility, and with increasing strike level.

FIGURE 3 Effective credit spread of a call option. The underlying level is 10, and the time to maturity is 1 year. The dashed line shows the effective credit spread dependent on the strike price, based on the Baule et al. (2008; BEW) model, calibrated to a credit spread of 0.01 and a recovery rate of 0.5. The solid line shows the approximation based on Equation (20). The three graphs represent settings with correlations between underlying and issuer asset value of 0.2 (upper), 0.5 (middle), and 0.8 (lower), each with underlying volatilities of 0.1, 0.2, and 0.4



3.3 | Comparative static

While the effective credit spread allows for a first judgment of the appropriateness of the approach, in this section we consider the ultimate output of the model, the derivative value. We calculate call values, put values, and discount certificate values for a variety of strike prices, with five different models:

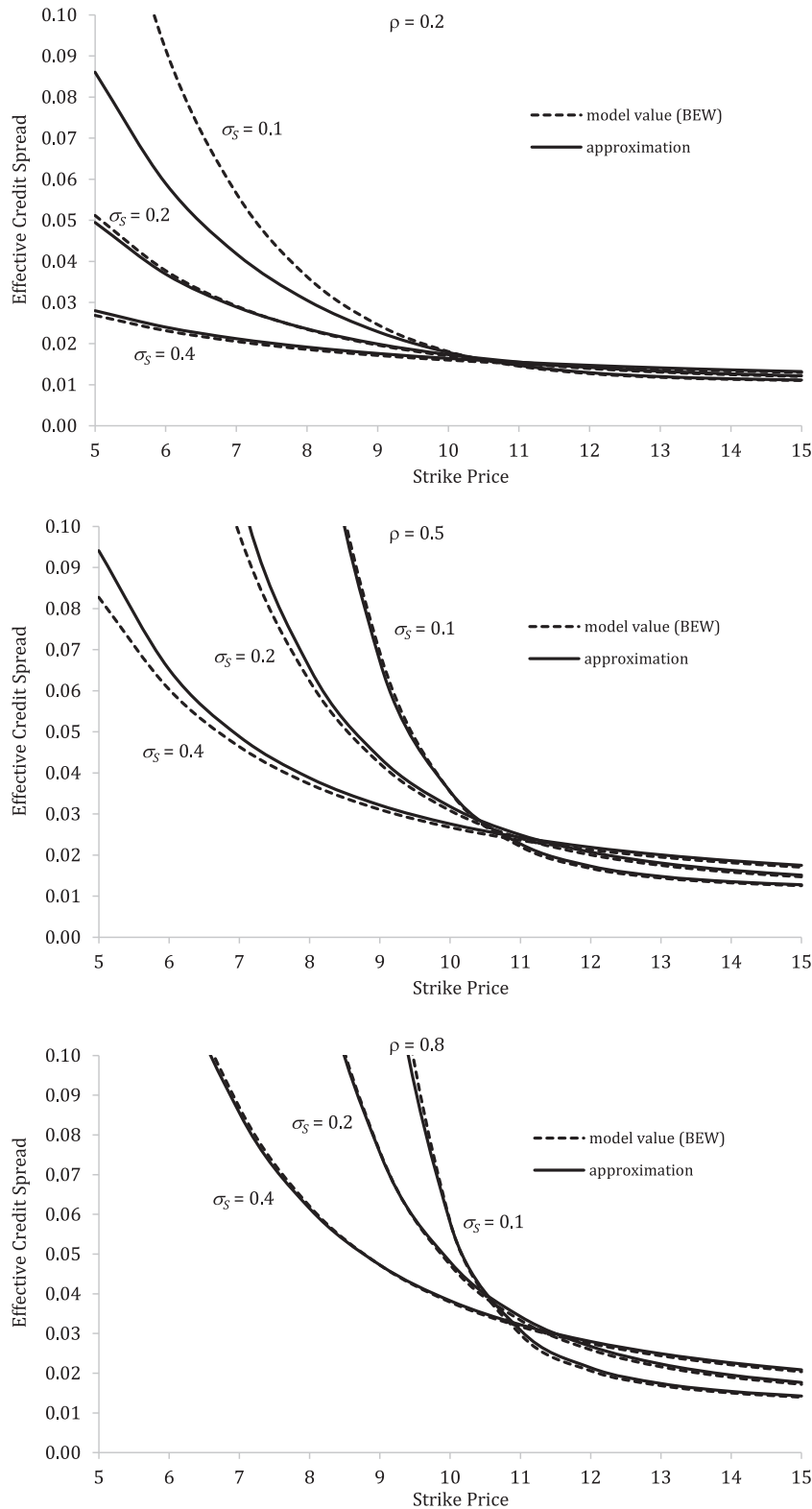
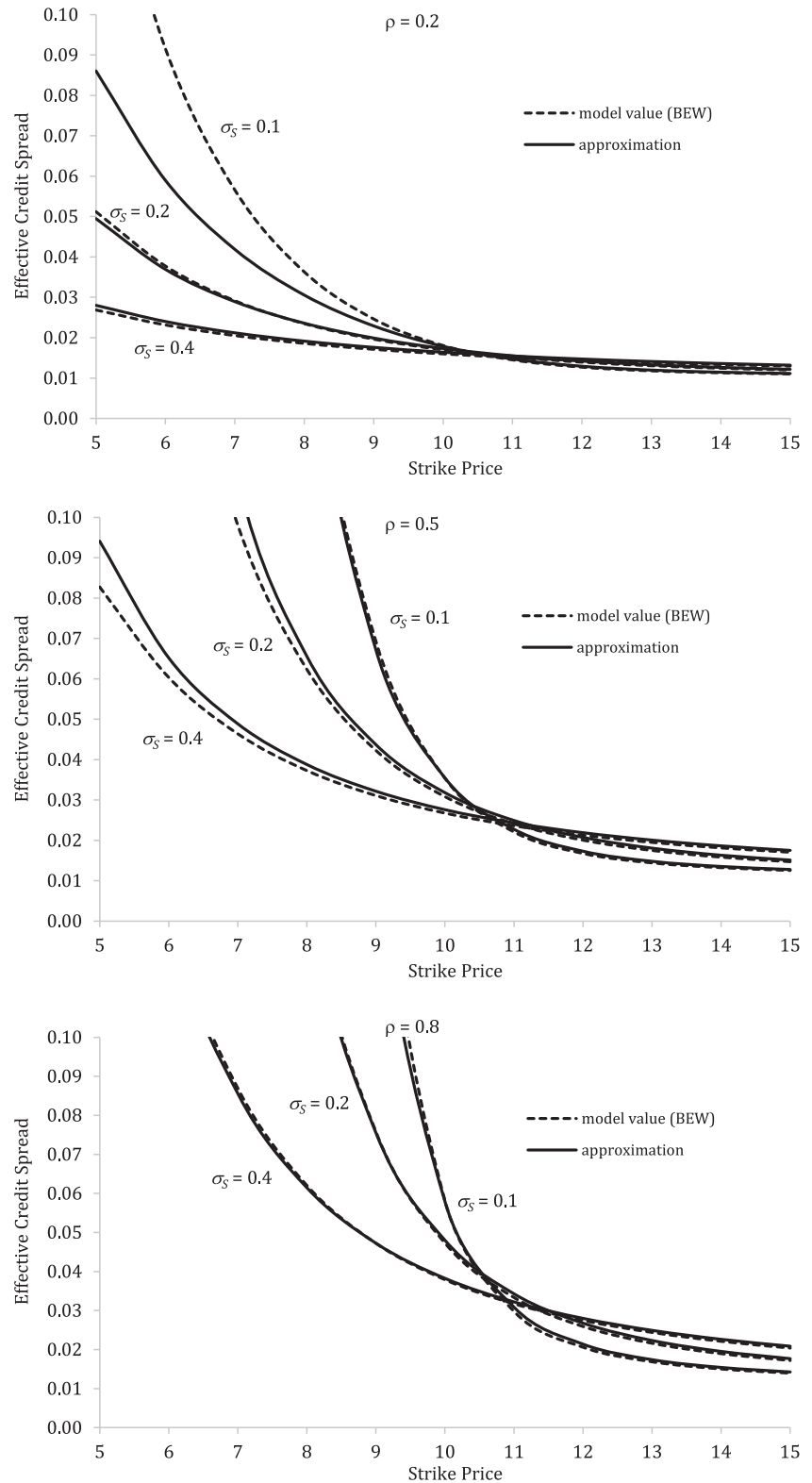


FIGURE 4 Effective credit spread of a put option. The underlying level is 10, and the time to maturity is 1 year. The dashed line shows the effective credit spread dependent upon the strike price, based on the Baule et al. (2008; BEW) model, calibrated to a credit spread of 0.01 and a recovery rate of 0.5. The solid line shows the approximation based on Equation (20). The three graphs represent settings with correlations between underlying and issuer asset value of 0.2 (upper), 0.5 (middle), and 0.8 (lower), each with underlying volatilities of 0.1, 0.2, and 0.4

- Baule et al. (2008) model with recovery rate 0.5 (“true value”),
- Baule et al. (2008) model with recovery rate 0.0,
- simplified model using the effective credit spread according to Equation (20),
- Hull and White (1995) model, and
- Black and Scholes (1973) model.

FIGURE 5 Effective credit spread of a discount certificate. The underlying level is 10, and the time to maturity is 1 year. The dashed line shows the effective credit spread dependent upon the strike price (cap), based on the Baule et al. (2008; BEW) model, calibrated to a credit spread of 0.01 and a recovery rate of 0.5. The solid line shows the approximation based on Equation (20). The three graphs represent settings with correlations between underlying and issuer asset value of 0.2 (upper), 0.5 (middle), and 0.8 (lower), each with underlying volatilities of 0.1, 0.2, and 0.4



The derivatives follow the same structure as in Section 3.2. In particular, the initial underlying level is $S_0 = 10$, the time to maturity is $T = 1$ year, and the unconditional credit spread is $s = 0.01$. For the volatility and the correlation we use the intermediate values of $\sigma = 0.2$ and $\rho = 0.5$.

The model values are shown in Table 1. For call options, in line with the analysis in Section 3.1, the Black and Scholes (1973) model values do not differ too much from the “true” values. In contrast, the Hull and White (1995)

TABLE 1 Comparative static analysis

Strike	8	9	10	11	12
Call options					
BEW ("true")	2.317	1.541	0.9405	0.5291	0.2766
BEW ($\delta = 0$)	2.318	1.541	0.9408	0.5292	0.2766
This paper	2.318	1.542	0.9413	0.5293	0.2767
HW	2.299	1.528	0.9320	0.5241	0.2739
BS	2.322	1.543	0.9413	0.5293	0.2767
Put options					
BEW ("true")	0.0808	0.2655	0.6262	1.176	1.884
BEW ($\delta = 0$)	0.0799	0.2641	0.6245	1.174	1.882
This paper	0.0804	0.2650	0.6255	1.175	1.883
HW	0.0851	0.2742	0.6394	1.192	1.903
BS	0.0860	0.2769	0.6458	1.204	1.922
Discount certificates					
BEW ("true")	7.606	8.382	8.892	9.393	9.646
BEW ($\delta = 0$)	7.606	8.383	8.983	9.395	9.648
This paper	7.606	8.382	8.983	9.394	9.647
HW	7.601	8.373	8.969	9.376	9.627
BS	7.678	8.457	9.059	9.471	9.723

Note: The table reports values of call options, put options, and discount certificates, for different strike prices obtained with different models: the Baule et al. (2008) (BEW) model with recovery rate 0.5 as the benchmark, the same model but with recovery rate 0.0, the simple approximation proposed in this paper, the Hull and White (1995) (HW) model, and the Black and Scholes (1973) (BS) model. The underlying level is 10, the time to maturity is 1 year, the volatility is 0.2, and the correlation parameter is 0.5.

model values substantially underestimate the true values. The reason is that for call options, the effective credit spread is considerably smaller than the unconditional spread. It is therefore better to go without any adjustment for credit risk than to adjust by an amount which is far too large. The simplified approach suggested in this paper yields a considerable improvement compared to Black and Scholes (1973) for in-the-money calls. The effect of the simple recovery assumption is of similar size as the remaining approximation (linearization of the payoff). The largest error occurs for at-the-money calls, but remains fairly small with 0.0008 in absolute terms and 0.08% in relative terms.

For put options, the situation is reversed: The Hull and White (1995) model underestimates the value of credit risk. Thus, the Black and Scholes (1973) model without any adjustment performs worse. But also the Hull and White approach only goes half way for in-the-money puts and less for out-of-the-money puts. It is therefore also not recommendable to use a model without correlation adjustment for put options. The simplified model is fairly close to the true value, with some deviations in relative value for out-of-the-money puts. Interestingly, the linearization error offsets parts of the error which occurs according to the recovery assumption. The absolute error stays below 0.0005 to 0.001 in all cases, while the largest relative error occurs for out-of-the-money puts with 0.5%. At the same time, the Hull and White model error is 5% for out-of-the-money puts.

For discount certificates, the use of the Hull and White (1995) model leads to less biased results compared to plain-vanilla options. It overestimates the value of credit risk, but its results are far closer to the true value than the Black and Scholes (1973) values. The simplified model yields a further significant improvement. Throughout all strike levels, the absolute error is below 0.001 and the relative error below 0.01%. In contrast, the worst results for the Hull and White approach are obtained for large strike prices, with 0.02 (absolute) and 0.2% (relative).

Table 2 takes a closer look at out-of-the-money calls and puts. As Figures 3 and 4 suggest, larger deviations between the actual effective spread and the proposed approximation occur for out-of-the-money options. However, as these options have small values, discounting with a biased spread might have a negligible impact. Table 2 provides

TABLE 2 Additional comparative static analysis for out-of-the-money options

ρ	0.2	0.8				
σ			0.1	0.4		
T					0.5	2
Call options (Strike = 12)						
BEW ("true")	0.2757	0.2767	0.0300	1.0136	0.0891	0.6559
This paper	0.2760	0.2767	0.0300	1.0139	0.0891	0.6566
HW	0.2739	0.2739	0.0297	1.0038	0.0887	0.6436
BS	0.2767	0.2767	0.0300	1.0139	0.0891	0.6566
Put options (Strike = 8)						
BEW ("true")	0.0840	0.0749	0.0014	0.5368	0.0226	0.1859
This paper	0.0840	0.0746	0.0014	0.5357	0.0225	0.1854
HW	0.0851	0.0851	0.0016	0.5517	0.0237	0.1988
BS	0.0860	0.0860	0.0016	0.5572	0.0238	0.2028

Note: The table reports values of call options and put options with strike price 20% out of the money, with varying correlation parameter, volatility, and time to maturity. In the base case $\rho = 0.5$, $\sigma = 0.2$, $T = 1$, and year. The models applied are the Baule et al. (2008) (BEW) model with recovery rate 0.5 as the benchmark, the simple approximation proposed in this paper, the Hull and White (1995) (HW) model, and the Black and Scholes (1973) (BS) model.

comparative statics of call and put option values with a strike price 20% out of the money, where the correlation, the volatility, and the time to maturity is modified. Basically, the results confirm the findings for the base case ($\rho = 0.5$, $\sigma = 0.2$, $T = 1$). The Hull and White (1995) model overestimates the value of credit risk for call options and underestimates the value for put options, increasingly for larger correlations. The approximation suggested in this paper works fairly well in all situations, with an absolute error below 0.0003 in most cases. Only for put options at higher volatilities or with larger maturities does it reach values up to 0.001 in absolute terms, and the relative error goes up to 0.5%, similar to the base case.

3.4 | Error analysis

In addition to the comparative static analysis, this section provides a more comprehensive error analysis. To this end, we construct a random sample of 1000 call options, put options, and discount certificates each. The derivatives follow the same structure as in Section 3.2. In particular, the initial underlying level is $S_0 = 10$, the time to maturity is $T = 1$ year, and the unconditional credit spread is $s = 0.01$. The sample is drawn randomly with the following specifications:

- strike price between 8 and 12, uniformly distributed,
- stock price volatility between 0.1 and 0.4, uniformly distributed,
- correlation parameter between 0.2 and 0.8, uniformly distributed.

All random draws are independent.

For an error decomposition, we separately consider the recovery assumption, the linear approximation, and the first-order Taylor series expansion. Similar to the previous section, we calculate different model values:

- Baule et al. (2008) model value with recovery rate 0.5 ("true value"),
- Baule et al. (2008) model value with recovery rate 0.0,
- simplified model value using the effective credit spread according to Equation (19) without Taylor series approximation, and
- simplified model value using the effective credit spread according to Equation (20) with Taylor series.

TABLE 3 Error analysis

	Simple model				SM(Δ)	HW	BS
	RR	Linear	Taylor	Total			
Call options ($\cdot 10^{-3}$)							
ME	0.30	0.31	0.02	0.63	1.26	−10.16	2.03
RMSE	0.42	0.40	0.03	0.80	1.82	11.57	3.39
MaxAE	1.09	0.95	0.07	1.54	5.77	27.30	15.33
Put options ($\cdot 10^{-3}$)							
ME	−1.75	0.71	0.22	−0.82	4.35	15.52	25.11
RMSE	1.93	0.79	0.28	0.95	5.59	18.13	28.97
MaxAE	3.29	1.77	0.77	1.82	15.47	41.49	65.54
Discount certificates ($\cdot 10^{-3}$)							
ME	1.75	−0.71	−0.01	1.03	−4.47	−15.53	71.79
RMSE	1.93	0.79	0.01	1.18	5.49	18.14	72.53
MaxAE	3.29	1.77	0.03	2.30	15.34	41.50	94.23

Note: The table reports mean errors (ME, biases), root mean squared errors (RMSE), and maximum absolute errors (MaxAE) for a sample of call options, put options, and discount certificates. The sample consists of 1000 random draws each, with the strike price uniformly distributed between 8 and 12, the stock price volatility uniformly distributed between 0.1 and 0.4, and the correlation uniformly distributed between 0.2 and 0.8. The initial underlying level is 10, the risk-free rate is 0.03, and the unconditional credit spread is 0.01. The total model error is the difference between the output of the proposed simplified model and the benchmark Baule et al. (2008) model. It is additively decomposed into recovery assumption error (RR), linearization error (linear), and Taylor series approximation error (Taylor). Errors of the simple model with standard delta, the Hull and White (1995) model, and the Black and Scholes (1973) model are also given. For comparison, the average derivative values in the sample are 1.2 (call options), 0.9 (put options), and 8.7 (discount certificates).

The difference (b) – (a) is the recovery assumption error; the difference (c) – (b) is the linearization error; and the difference (d) – (c) is the Taylor approximation error. The error components add up to the total error (d) – (a).

Additionally, we calculate

- (e) simplified model value with standard delta instead of downside delta,
- (f) Hull and White (1995) model value, and
- (g) Black and Scholes (1973) model value.

Table 3 reports the mean error (bias) of the sample, the root mean squared error, and the maximum absolute error for the 1000 derivatives. The smallest error components occur for call options. The reason is that the effective credit spread s^{eff} for call options is always lower than the unconditional spread s (as long as the correlation is positive). With respect to the Hull and White (1995) approach, the average errors confirm the comparative static analysis: Since the effective credit spread for call options tends to be nearer to zero than to the unconditional spread, adjusting the Black and Scholes (1973) value with the Hull and White approach is very often worse than with no adjustment made at all.

For put options and discount certificates, recovery assumption error and linearization error are mirror images. This observation can be explained by the payoff profiles of put options and discount certificates, which are also mirror images of each other. For both types of derivatives, recovery assumption error and linearization error are negatively correlated; hence, the total error is smaller than the sum of the two single errors. Both the bias and the root mean squared error are similar in size to call options and are below 0.001. (The average call option value in the sample is 1.2, the average put option value 0.9, and the average certificate value 8.7). Compared to Hull and White (1995) and Black and Scholes (1973), this is an improvement by a factor of about 15–70. The maximum absolute error values further show that there are no larger outliers. The proposed simple model can therefore be used for a large range of derivatives without risking excessively large approximation errors.

The Taylor approximation error is the smallest component in all cases. This observation justifies its use, which is rather a convenience than a necessity.⁵

Finally, the table demonstrates the usefulness of the downside delta approach: When the simple model is applied with the standard delta, the model still outperforms the Hull and White (1995) approach, but the resulting errors are considerably larger than those with the suggested downside delta.

3.5 | Correlation estimation

While the other model parameters are usually observable, the correlation parameter must be estimated. In the case of exchange-traded issuers, Baule et al. (2008) suggest estimating correlations from time series of the issuer's stock price as a proxy for its asset value. This approach is also common for credit portfolio modeling, for example, with the widely used industry model CreditMetrics (e.g., Crouhy et al., 2001). Duellmann et al. (2010) show that estimating asset correlations from stock price time series leads to less biased results than other approaches. In this section we analyze correlation estimation error and its impact on the model output.

In doing so, we use a similar simulation setup as Duellmann et al. (2010). We simulate time series of the underlying stock price and the issuer asset value according to Equations (1)–(3).⁶ The issuer parameters are the same as in Section 3.2, namely $D/V_0 = 0.8$ and $\sigma_V = 0.1197$. From the issuer asset value we calculate its stock price with the Black and Scholes (1973) model, following Duellmann et al. (2010) in assuming a debt maturity of 1 year. The risk-free rate is 0.03. For the stock price, we draw a random volatility uniformly distributed between 0.1 and 0.4 in each simulation run. We simulate weekly data over 2 years and estimate the correlation between the issuer's observable stock price returns and the underlying, using the standard estimator. This procedure is repeated 1000 times, for correlations of 0.2, 0.5, and 0.8, respectively.

Furthermore, Duellmann et al. (2010) consider stochastic correlations, which follow a mean reversion process:

$$d\rho_t = \kappa (\bar{\rho} - \rho_t) dt + \sigma_\rho dW_t^\rho, \quad (21)$$

with parameters $\kappa = 1$ and $\sigma_\rho = 0.085$ (where the Wiener process W_t^ρ is independent of W_t^S and $W_t^{V'}$). We adapt their approach and also estimate correlations based on simulated time series under this regime, with long-term mean values $\bar{\rho}$ of 0.2, 0.5, and 0.8, respectively.

The estimation errors are given in the first section of Table 4. As in the previous error analysis, the table reports the mean error (bias), the root mean squared error, and the maximum absolute error within the 1000 draws. The bias is slightly negative in all cases. The root mean squared error reaches values of 0.1 and decreases for larger correlation parameters. The estimation of the mean correlation in the stochastic regime is not seriously worse than in the constant correlation case. However, in both cases, some outliers can produce large estimation errors of about 0.3.

The following sections of Table 4 report the impact of correlation estimation error on the final model output, the derivative value. For this analysis, the 1000 random derivatives described in Section 3.4 are revalued with a disturbed correlation parameter, where the disturbance reflects the estimation error as discussed. Looking at the root mean squared error, the impact is largest for low correlations, with a value of about 0.003 for all three derivative types. On the one hand, this error is considerably larger than the model error itself, as analyzed in Table 3. On the other hand, it is still well below the alternative models of Black and Scholes (1973) and Hull and White (1995). Thus, while correlation estimation imputes an unavoidable source of additional error, even a disturbed correlation parameter yields acceptable results in most cases.

⁵The exact solution of Equation (19) is

$$s^{eff} = -\frac{1}{T} \ln \left[\frac{\bar{\Delta}_0 S_0 \cdot \exp(-s^{eff} T) + (f_0 - \bar{\Delta}_0 S_0) \cdot \exp(-sT)}{f_0} \right].$$

⁶The simulations are run under the physical measure with drift rates of 0.12, following Duellmann et al. (2010).

TABLE 4 Correlation estimation

	Constant correlation			Stochastic correlation		
	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.8$	$\bar{\rho} = 0.2$	$\bar{\rho} = 0.5$	$\bar{\rho} = 0.8$
Correlation estimates						
ME	−0.002	−0.016	−0.016	−0.004	−0.012	−0.018
RMSE	0.093	0.076	0.041	0.094	0.078	0.042
MaxAE	0.385	0.345	0.189	0.358	0.358	0.147
Call options ($\cdot 10^{-3}$)						
ME	−0.31	−0.27	−0.01	−0.49	−0.21	−0.02
RMSE	2.80	1.00	0.18	2.90	0.97	0.13
MaxAE	18.87	7.75	1.78	16.27	6.81	1.53
Put options ($\cdot 10^{-3}$)						
ME	−0.03	0.56	0.55	0.12	0.24	0.67
RMSE	3.01	2.85	1.45	3.37	3.03	1.53
MaxAE	11.80	10.59	5.17	12.50	11.45	7.99
Discount certificates ($\cdot 10^{-3}$)						
ME	4.41	−0.58	−0.57	−0.12	−0.24	−0.70
RMSE	3.02	2.91	1.51	3.40	3.09	1.58
MaxAE	11.93	10.70	5.28	12.63	11.60	8.17

Note: The first section of the table reports mean errors (ME; biases), root mean squared errors (RMSE), and maximum absolute errors (MaxAE) of correlation estimators. According to the model specification (with drift rate 0.12, asset volatility 0.1197, stock price volatility randomly drawn from a uniform distribution between 0.1 and 0.4), time series of the issuer asset value and the underlying stock price were simulated. From the issuer asset value, its stock price was obtained with the Black and Scholes (1973) model (assuming a debt maturity of 1 year, an initial debt-to-asset ratio of 0.8, and a risk-free rate of 0.03). Correlation estimates were based on 2 years of weekly simulated underlying and issuer stock price returns. The first columns refer to a constant correlation assumption, and the last ones, to stochastic correlations according to Equation (21). The following sections of the table report the impact of a disturbed correlation parameter on the derivative values. The figures are based on random samples as in Table 3, with disturbed correlation parameters according to the estimation method as described.

4 | CONCLUSION

The negative value of default risk inherent in a vulnerable derivative depends on the structure of the derivative when the default of the issuer is not independent of the market risk of the underlying. The effective credit spread, defined as the discount rate to obtain the risk-adjusted value from the default-free value, can have values differing substantially from the unconditional credit spread for straight bonds of the issuer. In particular, for call or put options, but also for other structured derivatives such as discount certificates, using the unconditional credit spread as suggested by Hull and White (1995) can therefore lead to substantially biased results.

We have proposed a simple approach to incorporate correlated credit risk in the valuation of vulnerable derivatives that requires a minimum of data and effort. In addition to the unconditional credit spread, the only necessary input parameter is the correlation between the underlying price and the issuer asset value. Furthermore, the simple calculation requires nothing more than a univariate normal distribution.

However, the correlation parameter is not readily observable and must be estimated. Following Duellmann et al. (2010), among others, we propose estimating this parameter from stock price time series of the issuer and the underlying. When no stock price time series is available for the issuer, an average value of similar issuers—for example, members of a country-specific banking sector index—could be used instead. While such an averaging certainly introduces a further approximation, it is still more desirable than relying on the zero-correlation assumption of Hull and White (1995), especially given the empirical fact that correlations tend to increase with falling stock prices, and thus with increasing default probability (e.g., Longin & Solnik, 2001).

While we have used the simple Black and Scholes (1973) model as a benchmark, the idea of calculating an effective credit spread to discount the default-free value could also be used for other benchmark models. Nonetheless, the effective

credit spread calculation relies upon the assumption of a linear correlation between diffusion processes of the issuer and the underlying. Other dependency structures, such as co-jumps in both processes, are not considered. While there are recent approaches which account for these structures, such as the model of Tian et al. (2014), these models involve both greater complexity and the estimation of further parameters, which is associated with additional estimation error.

In this light, unavoidable model risk remains, since the true dependence structure is not observable. While we have used the Baule et al. (2008) model as a benchmark throughout this paper, this model is of course not perfect. Therefore, deviations from this model, which we have termed “errors,” may be acceptable when they are small relative to the overall impact of correlation. We have shown that this overall impact can be huge: For call options, the Black and Scholes (1973) value is often better than the Hull and White (1995) value; and for put options, the actual value of credit risk can be several times larger than implied by the Hull and White model. It is therefore necessary to account for correlation in the pricing of vulnerable derivatives. Exactly how correlation is incorporated into the model is a second-order question. Given that model risk and parameter estimation risk remain, the simplified model proposed in this paper is as simple as possible and as complicated as necessary.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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REFERENCES

- Ammann, M. (2001). *Credit risk valuation*. Springer.
- Baule, R. (2011). The order flow of discount certificates and issuer pricing behavior. *Journal of Banking & Finance*, 35, 3120–3133.
- Baule, R., & Blonski, P. (2015). The demand for warrants and issuer pricing strategies. *Journal of Futures Markets*, 35, 1195–1219.
- Baule, R., Entrop, O., & Wilkens, M. (2008). Credit risk and bank margins in structured financial products: Evidence from the German secondary market for discount certificates. *Journal of Futures Markets*, 28, 376–397.
- Baule, R., & Tallau, C. (2011). The pricing of path-dependent structured financial retail products: The case of bonus certificates. *Journal of Derivatives*, 18(4), 54–71.
- Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637–654.
- Casella, G., & Berger, R. L. (2002). *Statistical inference* (2nd ed.). Duxbury.
- C  lerier, C., & Vall  e, B. (2017). Catering to investors through security design: Headline rate and complexity. *Quarterly Journal of Economics*, 132, 1469–1508.
- Crouhy, M., Galai, D., & Mark, R. (2001). *Risk management*. McGraw-Hill.
- Duellmann, K., K  ll, J., & Kunisch, M. (2010). Estimating asset correlations from stock prices or default rates—Which method is superior? *Journal of Economic Dynamics & Control*, 34, 2341–2357.
- Egan, M. (2019). Brokers versus retail investors: Conflicting interests and dominated products. *Journal of Finance*, 74, 1217–1260.
- Entrop, O., Fischer, G., McKenzie, M., Wilkens, M., & Winkler, C. (2016). How does pricing affect investors' product choice? Evidence from the market for discount certificates. *Journal of Banking & Finance*, 68, 195–215.
- Henderson, B. J., & Pearson, N. D. (2011). The dark side of financial innovation: A case study of the pricing of a retail financial product. *Journal of Financial Economics*, 100, 227–247.
- Hull, J. C. (2018). *Options, futures, and other derivatives* (10th ed.). Pearson.
- Hull, J. C., & White, A. (1995). The impact of default risk on the prices of options and other derivative securities. *Journal of Banking & Finance*, 19, 299–322.
- Johnson, H., & Stulz, R. (1987). The pricing of options with default risk. *Journal of Finance*, 42, 267–280.
- Klein, P. (1996). Pricing Black-Scholes options with correlated credit risk. *Journal of Banking & Finance*, 20, 1211–1229.
- Longin, F., & Solnik, B. (2001). Extreme correlation of international equity markets. *Journal of Finance*, 56, 649–679.
- Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29, 449–470.
- Pelster, M., & Schertler, A. (2019). Pricing and issuance dependencies in structured financial product portfolios. *Journal of Futures Markets*, 39, 342–365.
- Schertler, A. (2016). Pricing effects when competitors arrive: The case of discount certificates in Germany. *Journal of Banking & Finance*, 68, 84–99.
- Schertler, A., & Stoerch, S. (2015). Issuers' credit risk and pricing of warrants in the recent financial crisis. *Journal of Risk Finance*, 16, 444–462.

- Schertler, A., & Stoerch, S. (2018). Warrant price responses to credit spread changes: Fact or fiction? *Review of Financial Economics*, 36, 206–219.
- Tian, L., Wang, G., Wang, X., & Wang, Y. (2014). Pricing vulnerable options with correlated credit risk under jump-diffusion processes. *Journal of Futures Markets*, 34, 957–979.
- Wallmeier, M., & Diethelm, M. (2009). Market pricing of exotic structured products: The case of multi-asset barrier reverse convertibles in Switzerland. *Journal of Derivatives*, 17(2), 59–72.

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APPENDIX A: CONDITIONAL EXPECTATION

We are interested in the expected value of the underlying level S_T , given that the issuer asset value V_T has reached the default point D . A default in T occurs if and only if $V_T < D$. Under the assumption of a zero recovery rate, the expected payoff of a corporate zero coupon bond with notional 1 under the risk-neutral measure equals the nondefault probability, which is given by e^{-sT} (Hull & White, 1995):

$$e^{-sT} = 1 - P(V_T < D) = 1 - P(R_T^V < \ln D/V_0)$$

with the log return $R_T^V := \ln V_T/V_0$. The log returns of the issuer and the underlying are normally distributed (e.g., Hull, 2018):

$$\begin{aligned} R_T^V &:= \ln V_T/V_0 \sim N\left(\left(r - \sigma_V^2/2\right) T; \sigma_V^2 T\right), \\ R_T^S &:= \ln S_T/S_0 \sim N\left(\left(r - \sigma_S^2/2\right) T; \sigma_S^2 T\right), \end{aligned}$$

with $\text{Corr}(R_T^S, R_T^V) = \rho$. So we have

$$\begin{aligned} e^{-sT} &= 1 - N\left(\frac{\ln D/V_0 - (r - \sigma_V^2/2) T}{\sigma_V \sqrt{T}}\right) = N\left(-\frac{\ln D/V_0 - (r - \sigma_V^2/2) T}{\sigma_V \sqrt{T}}\right) \\ &\Rightarrow N^{-1}(e^{-sT}) = -\frac{\ln D/V_0 - (r - \sigma_V^2/2) T}{\sigma_V \sqrt{T}} \\ &\Rightarrow \ln D/V_0 = (r - \sigma_V^2/2) T - \sigma_V \sqrt{T} N^{-1}(e^{-sT}). \end{aligned}$$

We use the well-know properties of the conditional normal distribution: For two correlated normals $Y_1 \sim N(\mu_1; \sigma_1^2)$, $Y_2 \sim N(\mu_2; \sigma_2^2)$, the distribution of Y_2 conditional on $Y_1 = y$ is also normal with expectation $\mu_2 + (y - \mu_1) \cdot \rho \sigma_2/\sigma_1$ and variance $(1 - \rho^2) \sigma_2^2$ (e.g., Casella & Berger, 2002). Hence, the conditional expectation of R_T^S is given by:

$$E\left[R_T^S \mid R_T^V = \ln D/V_0\right] = (r - \sigma_S^2/2) T - \rho \sigma_S \sqrt{T} N^{-1}(e^{-sT}).$$

For a normal $Y \sim N(\mu; \sigma^2)$, $\exp(Y)$ is log-normally distributed with expectation $\exp(\mu + \sigma^2/2)$ (e.g., Hull, 2018). With the variance of the conditional log return, $(1 - \rho^2) \sigma_S^2 T$, we finally get the conditionally expected stock price:

$$E\left[S_T \mid R_T^V = \ln D/V_0\right] = S_0 \cdot \exp\left(\left(r - \rho^2 \sigma_S^2/2\right) T - \rho \sigma_S \sqrt{T} N^{-1}(e^{-sT})\right).$$