

Kittsteiner, Thomas; Ott, Marion; Steinberg, Richard

Article — Accepted Manuscript (Postprint)

Competing Combinatorial Auctions

Information Systems Research

Suggested Citation: Kittsteiner, Thomas; Ott, Marion; Steinberg, Richard (2021) : Competing Combinatorial Auctions, Information Systems Research, ISSN 1526-5536, The Institute for Operations Research and the Management Sciences, Catonsville, MD, Iss. Ahead of Print, <https://doi.org/10.1287/isre.2021.1018>

This Version is available at:

<https://hdl.handle.net/10419/240927>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Competing Combinatorial Auctions^{*†}

Thomas Kittsteiner[‡]

Marion Ott[§]

Richard Steinberg[¶]

March 1, 2021

Abstract

We investigate whether revenue-maximizing auctioneers selling heterogeneous items will allow for combinatorial bidding in the presence of auctioneer competition. We compare the choice of auction format by two competing auctioneers with that of a single auctioneer. Bidders are heterogeneous in their demands, with some having synergies for items. We find that, even if a single auctioneer offers a combinatorial auction, competing auctioneers in a comparable setting will not. Instead, the competing auctioneers will segment the market by restricting allowable package bids in order to increase competition between bidders. This shows that it might not be advantageous for an online market platform to offer combinatorial auctions as a design option to competing auctioneers.

Keywords: Competing auctioneers, combinatorial auction, electronic marketplace, VCG mechanism

1 Introduction

In recent years, we have seen the advent of combinatorial auctions as well as the emergence of online market platforms with competing auctioneers.¹ However, “the combinatorial auction mechanism has yet to become popular in the electronic market place” (Adomavicius et al.

*This is the author accepted manuscript of the article published in *Information Systems Research*, which is available at <https://doi.org/10.1287/isre.2021.1018>.

†The authors thank the Associate Editor and two anonymous reviewers for their especially helpful comments. Financial support from the Deutsche Forschungsgemeinschaft (DFG) under grants KI 1915/1-1 and OT 487/2-1 is gratefully acknowledged by Thomas Kittsteiner and Marion Ott.

‡RWTH Aachen University, e-mail: thomas.kittsteiner@rwth-aachen.de.

§ZEW – Leibniz Centre for European Economic Research, e-mail: marion.ott@zew.de.

¶London School of Economics and Political Science, e-mail: r.steinberg@lse.ac.uk.

¹For combinatorial auctions of a single auctioneer, see Bichler et al. (2006), Cantillon and Pesendorfer (2006), Caplice and Sheffi (2006), Olivares et al. (2012), Ausubel and Baranov (2014), Goossens et al. (2014), Mastropietro et al. (2014). For competing auctioneers, see Bapna et al. (2010), Andersson et al. (2012), Han et al. (2018).

2012). These observations raise two questions: First, why have combinatorial auctions yet to become popular on online auction platforms with competing auctioneers? Second, should combinatorial auctions be offered as a design choice to competing auctioneers on online auction platforms? This paper is, to our knowledge, the first to address these fundamental questions about electronic marketplaces hosting competing auctioneers.

Combinatorial auctions are those auctions that allow bids on packages, that is, subsets of items, thus enabling bidders to express their synergies for items. Consequently, combinatorial auctions avoid the well-known *exposure problem* and the *converse problem*, which we refer to as the *bundle problem*, which is when a bidder needs to bid on a superset of his desired items in order to obtain them. Both problems can have an adverse effect on auctioneer revenue. While combinatorial auctions eliminate these problems, there might exist a concomitant attenuating effect on auction revenue. To explain, consider a bidder β_1 whose bid is complemented by bids of bidders $\beta_2, \beta_3, \dots, \beta_k$ such that together the bids of $\beta_1, \beta_2, \dots, \beta_k$ cover a set of items desired by some competing bidder, γ . Then, bidder β_1 might benefit from the complementing bidders, because β_1 's probability of winning against γ is increasing in each of the bids of $\beta_2, \beta_3, \dots, \beta_k$. Thus, combinatorial auctions create a *free-rider problem*, which is associated with low-revenue non-core outcomes.²

Our main finding shows that, even if a single auctioneer permits bids on all packages of items, competing auctioneers might not. To gain some intuition, first note that it can be beneficial for competing auctioneers to segment bidders. Consider the example from the previous paragraph. If bidders $\beta_1, \beta_2, \dots, \beta_k$ were to bid with a different auctioneer than bidder γ , this *segmentation* would mitigate the free-rider problem. Second, note that if competing auctioneers restrict package bidding, segmentation of bidders can become self-enforcing. Specifically, in order to avoid the bundle problem, bidders $\beta_1, \beta_2, \dots, \beta_k$ would avoid auctioneers who allow bids only on the package desired by bidder γ ; whereas, in order to avoid the exposure problem, bidder γ would avoid auctioneers who allow bids only on subsets of his desired package.

To show this and derive our main finding, we adopt the widely used LLG (local-local-global) framework,³ which is just rich enough to incorporate the free-rider, exposure, and bundle problems. In our benchmark model—the single-auctioneer model—a single auctioneer offers two non-identical items, A and B, to three heterogeneous bidders. One bidder desires only item A and one only item B; these bidders are referred to as *local bidders*. The third bidder,

²An auction outcome is a *non-core outcome* if there is a group of bidders who, together with the auctioneer, can generate a higher payoff for each of them by trading among themselves. The free-rider problem in this context is often referred to as the threshold problem.

³The use of LLG models goes back at least to Krishna and Rosenthal (1996). Other papers employing an LLG model include Erdil and Klemperer (2010), Beck and Ott (2013), Goeree and Lien (2016), Baisa and Burkett (2018), Ausubel and Baranov (2020), Bosshard et al. (2020), Finster (2020).

referred to as a *global bidder*, desires only the package of both items. In the main model—the competing-auctioneers model—two competing auctioneers each offer the two items, A and B, and compete for six bidders; specifically, for two copies of the three heterogeneous bidders. Thus, our model of competing auctioneers is a duplication and then merger of our single-auctioneer model. The timing in our game-theoretic model is as follows: first, the auctioneers announce the packages on which they will accept bids; second, the bidders decide in which auction they will participate; third, bidders submit bids.

We make the following modeling assumptions. First, we concentrate on the most prominent combinatorial auction: the Vickrey-Clarke-Groves (VCG) mechanism (see Ausubel and Milgrom 2002, 2006, Day and Milgrom 2008). This assures that, in most continuation games following the bidders’ participation decisions, each bidder will have a weakly dominant strategy, which is to bid his valuation. The free-rider problem manifests itself indirectly in the VCG mechanism, whereas it may manifest itself directly through low bids, e.g., in pay-as-bid or core-selecting combinatorial auctions.⁴

Second, we parameterize the global bidders’ valuation distribution function in terms of their strength. We employ the uniform distribution, which fulfills many of the technical assumptions often made to obtain tractable models, such as monotonicity of virtual valuations, or to obtain closed-form solutions. From the analysis it follows that, due to continuity and because all inequalities restricting equilibrium behavior hold with slackness, our results remain valid for distribution functions sufficiently close to the uniform.

Third, we assume that the item or package in which a bidder is interested, which we call his *kind*, is common knowledge. However, a bidder’s valuation is private information and independent of the other bidders’ valuations. These are standard assumptions in LLG models. (See the references in Footnote 3.) A bidder’s kind is more apparent and requires less detailed knowledge about the bidder than knowing his valuation. In repeated interactions, knowing the kind of a bidder can be the result of observation and learning. In procurement contexts, production costs are typically private information, whereas information about a firm’s products is not.

Fourth, we allow each bidder to bid with only one auctioneer. This simplification, which is standard and required in models of competing auctioneers of single items to derive closed-form solutions (e.g., Bapna et al. 2010, Kim and Kircher 2015), can be justified by sufficiently high

⁴For combinatorial pay-as-bid auctions, see Baranov (2010), Bosshard et al. (2020); for combinatorial core-selecting auctions, see Sano (2012), Goeree and Lien (2016), Ausubel and Baranov (2020). For the magnitude of the free-rider problem in various auctions see Section 5. An empirical counterfactual analysis for a large scale VCG mechanism found that it performed similarly to the analyzed pay-as-bid combinatorial auction (Kim et al. 2014).

costs of participation, or simply by bidders' physical inability to interact with more than one auctioneer at a time. Furthermore, in an experiment that allowed for bidding in multiple auctions, most bidders indeed decided to bid with only one auctioneer (Bapna et al. 2010).

Fifth, we restrict attention to equilibria in which bidders' participation decisions are valuation independent. This implies that, on the equilibrium path, inferences about other bidders' valuations from their participation decisions are neither necessary nor possible. For this non-empty class of equilibria we provide a full characterization for the relevant parameter range.

The rest of the paper provides a literature review (Section 2), develops our single-auctioneer and competing-auctioneer models (Section 3), presents our results (Section 4), provides additional discussion (Section 5), and concludes (Section 6). The appendix provides detailed proofs.

2 Literature Review

To the best of our knowledge, the restriction of package bidding in combinatorial auctions in a competing-auctioneers setting has not been previously addressed in the literature. This paper is related to four different research streams.

First, there exists a literature on competing single-item auctioneers where each auctioneer offers a single item and bidders are ex-ante symmetric. Consequently, the type of horizontal market segmentation that we find is not possible in these models. These models vary with regard to the assumed market size (e.g., large finite markets, infinite markets, or two-auctioneer markets), bidder information regarding valuations when choosing an auctioneer, the available auction formats, and whether items are sold in sequential, overlapping, or simultaneous auctions (see McAfee 1993, Peters 1997, Peters and Severinov 1997, Burguet and Sakovics 1999, Parlane 2008, Bapna et al. 2009, 2010, Virág 2010, Albrecht et al. 2012, 2014, Kim and Kircher 2015, Truong et al. 2017). For a survey of work in this literature up until 2010, see Pai (2010).

Second, we draw on auction models with bidders facing an exposure problem (see Krishna and Rosenthal 1996, Rosenthal and Wang 1996, Bikhchandani 1999, Szentes and Rosenthal 2003, Szentes 2007, Goeree and Lien 2014). In our framework, we find that bidders respond to the exposure problem by submitting either very low or very high bids and may react to increased bidder competition by reducing their bids (for similar results, see Krishna and Rosenthal 1996).

Third, a single auctioneer's decision regarding the packages on which to allow bids has been investigated for symmetric bidders with additive valuations by Palfrey (1983), Chakraborty (1999), Armstrong (2000) and Jehiel et al. (2007); for symmetric bidders who consider two

items complements or substitutes by Subramaniam and Venkatesh (2009); and for asymmetric bidders with single-item or additive-value multi-item demand for two items by Avery and Hendershott (2000). Allowing bids on all packages can be optimal for a single auctioneer facing symmetric bidders with either additive valuations, substitutes valuations, or strong complements valuations (Jehiel et al. 2007, Subramaniam and Venkatesh 2009). Our benchmark scenario with a single auctioneer has not been addressed in these studies. Furthermore, there exist studies that acknowledge the benefit of reducing complexity for both the bidders and the auctioneer by disallowing bids on some packages (see Rothkopf et al. 1998, Lehmann et al. 2006, Nisan 2006). Our results identify an additional motive for the designer to restrict package bidding or even entirely avoid offering a combinatorial design: the presence of competing auctioneers.

Fourth, the spirit of our result is similar to a central idea from the industrial organization and strategic management literature on the value of product differentiation for competing firms. In order to mitigate competition, firms deliberately differentiate their products from those of other firms. This involves creating a more attractive product for a subset of customers, while at the same time rendering it less attractive for other potential customers. The seminal paper in this large literature is d’Aspremont et al. (1979); Belleflamme and Peitz (2010) provide an introduction to the topic. In contrast to this literature, our paper considers offering different auction formats rather than offering different items for sale.

3 The Two Models

The single-auctioneer model. Consider an auctioneer who offers for sale two non-identical items, A and B, to three bidders.

We use an LLG model, in which bidders are interested in either the one-item package A, the one-item package B, or the two-item package AB. We refer to these three kinds of bidders, respectively, as A-bidders, B-bidders, and AB-bidders. We assume that there is one bidder of each kind. The A- and B-bidders are collectively referred to as local bidders, the AB-bidder is referred to as a global bidder.

Bidder j ’s valuation for any package that contains his preferred package A, B, or AB is given by $v_j \in \mathbb{R}_{\geq 0}$, where v_j is private information.⁵ Valuations for packages that do not contain a bidder’s preferred package are commonly known to be zero. The valuations v_j are drawn independently from the commonly known distribution functions $v_j \sim U[0, 1]$ for a local bidder and $v_j \sim U[0, k]$, $k > 1$, for a global bidder. The parameter k measures the strength of

⁵This implies free disposal.

the global bidder vis-a-vis the local bidders.

Suppose a bidder's valuation for the package he wins is v , where $v = v_j$ or $v = 0$. Then the bidder's payoff is given by $v - p$ where $p \in \mathbb{R}_{\geq 0}$ is the price he pays in the auction. If he wins no package, his payoff is zero. Each bidder maximizes his expected payoff.

The auctions are VCG mechanisms restricted by the allowable bids.⁶ The auctioneer's options can be delineated in terms of three possible auction formats: (i) allowing exclusive bids on A, B, and AB, by offering a VCG mechanism for the two items, denote this option by [A,B]; (ii) allowing non-exclusive bids on A and on B, by offering two separate simultaneous second-price auctions, denote this option by [A][B]; and (iii) allowing bids on AB only, corresponding to a single second-price auction on the package AB, denote this option by [AB]. These three are the only options that need to be considered, because an auctioneer would never want to restrict bids to only on A or only on B, and because options to allow exclusive bids on A and AB, or on B and AB, are equivalent to offering [AB]. (See the proofs in Appendix B.2.4 and B.3.) The auctioneer maximizes his expected revenue. We assume that an auctioneer prefers the combinatorial auction [A,B] over both [AB] and [A][B] only if the combinatorial auction generates a strictly larger expected revenue.⁷

The game consists of the following two stages:

Stage 1: The auctioneer chooses an auction format among [A,B], [A][B], and [AB]. The chosen auction format becomes common knowledge. Thus, each decision by the auctioneer defines a subgame.

Stage 2: The bidders observe their private valuations and then simultaneously and independently submit bids. Items are allocated and payments are made. Ties are broken randomly.

A bidder's strategy consists of his bidding strategy for each auction format. A bidder can submit bids on all permissible packages but has the option not to submit a bid on some or all packages.

The competing-auctioneers model. This setup is a duplication and merger of the single-auctioneer setting. Two auctioneers, each offering items A and B, compete for six bidders: two A-bidders, two B-bidders, and two AB-bidders.

The game consists of the following three stages:

⁶For a definition of the VCG mechanism, see e.g., Krishna (2010), ch. 16, or Ausubel and Milgrom (2006).

⁷That is, we break an auctioneer's indifference between [A,B] and [A][B], or between [A,B] and [AB], in favor of the less complex auction format. In the single-auctioneer model, this rule will be applied only for one value of k . In the competing-auctioneers model below, we will need to apply this tie-breaking rule only in cases where the auctions' sets of participants are equal and consist only of local or only of global bidders.

Stage 1: The two auctioneers simultaneously and independently choose an auction format among $[A,B]$, $[A][B]$, and $[AB]$. The chosen auction formats become common knowledge. Thus, each combination of decisions by the auctioneers defines a subgame.

Stage 2: The bidders observe their private valuations and then simultaneously and independently make their participation decisions; each bidder can participate in the auction of only one auctioneer.

Stage 3: The bidders observe all participation decisions and then simultaneously and independently submit bids. Items are allocated and payments are made. Ties are broken randomly, with exceptions where this would not preserve value.⁸

A bidder's strategy consists of his participation decision for each feasible pair of offered auction formats by the auctioneers together with his bidding strategies for each observed bidder partition.

Equilibrium concept. We consider Perfect Bayesian Equilibria in pure strategies with the following three properties.

P1 A bidder's participation decisions are independent of his valuation.

P2 A bidder submits only undominated bids.

P3 If a global bidder participates in $[A][B]$, then bidders play according to an efficient ex-post equilibrium, if it exists.⁹

Formal specifications of these properties are provided in Appendix B.1. It should be emphasized that these properties constitute an equilibrium selection criterion and not a restriction on the strategies available to players. If an equilibrium with these properties exists, then it is selected. In Theorem 2 we show existence for a range of parameter values of k .

Property P1 can be thought of as a simplicity criterion. If an equilibrium can include pooling participation strategies, then these strategies should be selected, because they imply simpler beliefs for the bidders than non-pooling strategies. Property P1 implies that a bidder does not reveal any private information with his participation decision. Thus no bidder has

⁸Specifically, in $[A][B]$, ties among global bidders are broken such that both items are allocated to the same global bidder. If global bidders in $[A][B]$ submit the same bid on one package but not on the other, this tie is broken in favor of the bidder with the higher bid on the other package; if global bidders in $[A][B]$ tie with their bids on both packages, both ties are broken in favour of the same global bidder, which is chosen randomly.

⁹Existence of such an equilibrium in the continuation game is guaranteed if, in addition to the global bidder, another global bidder and/or local bidders of only one kind participate. See Lemma 2 in Appendix B.2.1. As this cannot happen in the single-auctioneer case, property P3 will not appear in Theorem 1.

an incentive to switch auctions after observing the participation decisions of other bidders. In this sense participation decisions are stable, and we do not need to assume that, once executed, participation decisions cannot be revoked. Also, for any equilibrium with Property P1 there is a payoff-equivalent equilibrium in a modified game in which bidders learn their valuation only after the participation decision. Properties P2 and P3 are common refinements.

4 Results

In order to identify and evaluate the relevance of competition for the choice of auction format, we start out by analyzing the cases of a single auctioneer and of competing auctioneers separately, and then compare the two cases to derive our main result. The single auctioneer's optimal decision is given in the following theorem.

Theorem 1. *Consider the single-auctioneer model. There exists a unique \hat{k} such that in any equilibrium in pure strategies with properties P1 and P2:*

- *For all $k > \hat{k}$, the auctioneer offers the combinatorial auction [A,B].*
- *For all $k \leq \hat{k}$, the auctioneer offers [AB].*

The single auctioneer never offers [A][B] because for all k he is better off by offering [A,B], which shields the global bidder from the exposure problem. Whether the auctioneer prefers [A,B] or [AB] depends on the strength of the global bidder. With a strong global bidder, generating competition for the global bidder is more important; with a weak global bidder, generating competition between local bidders is more important. In [A,B] local bidders complement each other against the global bidder, whereas [AB] creates competition between local bidders.

In contrast to the single-auctioneer case, the competing auctioneers' auction format choices determine revenue by affecting the number and kind of participating bidders. The following theorem describes the equilibria of the competing-auctioneers model.

Theorem 2. *Consider the competing-auctioneers model. There exists a unique k^* such that an equilibrium in pure strategies with properties P1, P2, and P3 exists if and only if $k \leq k^*$. For every $k \leq k^*$, the following hold:*

- (a) *There exists an equilibrium in which one auctioneer offers [A][B] and one auctioneer offers [AB]. There does not exist an equilibrium in which both auctioneers offer a combinatorial auction.*
- (b) *In every equilibrium, all local bidders bid for the items of the same auctioneer and all global bidders bid for the items of the other auctioneer.*

If global bidders are weak, specifically if $k \leq \hat{k}$ (as determined in Theorem 1), there may be additional equilibria to the one mentioned in Theorem 2, part (a). These are: (i) both auctioneers offer $[A][B]$ and one attracts the local bidders and the other attracts the global bidders, and (ii) one auctioneer offers $[A,B]$ and attracts the local bidders, and the other auctioneer offers $[A][B]$ or $[AB]$ and attracts the global bidders.

Remarkably, even if the combinatorial auction $[A,B]$ is offered in equilibrium, only local bidders will enter that auction, and thus the auctioneer with the combinatorial auction allows bids on more packages than is necessary to accommodate the heterogeneity of the participating bidders. We discuss this counterintuitive result in Section 5.

In all equilibria, one auctioneer is better off than the other auctioneer (unless $k = 2$). The auctioneer who attracts the local bidders receives an expected revenue of $2/3$, and the auctioneer who attracts the global bidders receives an expected revenue of $k/3$. (See Proposition 1 in Appendix B.2.)

If global bidders are significantly stronger than local bidders, $k > k^*$, then equilibria with properties P1, P2, and P3 fail to exist in the competing-auctioneers setting, because some subgames fail to have equilibria with these properties.

Our main finding, Theorem 3, shows that competing auctioneers might not choose a combinatorial auction, even if a single auctioneer would do so.

Theorem 3. *Let \hat{k} be as in Theorem 1 and k^* as in Theorem 2. It holds that $\hat{k} < k^*$. For every $\hat{k} < k \leq k^*$, in every equilibrium in pure strategies with properties P1, P2, and P3 of the competing-auctioneers model, one auctioneer offers $[A][B]$ and one offers $[AB]$, whereas a single auctioneer offers $[A,B]$.*

Thus, for levels of k for which the single auctioneer offers the combinatorial auction, neither of the competing auctioneers offers a combinatorial auction. This is because a single auctioneer, by choosing an auction format, needs to trade-off the negative effects on revenue caused by the exposure problem (in $[A][B]$) or the bundle problem (in $[AB]$) with that of the free-rider problem (in $[A,B]$). Competing auctioneers share the market. They have the option to segment the market, which can be achieved by offering non-combinatorial auctions, thereby eliminating the free-rider, exposure, and bundle problems. In order to achieve segmentation, the auctioneers can offer different non-combinatorial auction formats. One auctioneer offers $[A][B]$ and attracts all local bidders; the other auctioneer offers $[AB]$, attracts all global bidders, and has the higher expected revenue.

Example to illustrate the auctioneers' choices. Let $k = 7/3 \in (\hat{k}, k^*]$. (This value of k is in the relevant range for Theorem 3 because, as shown in the Appendix, $\hat{k} = 2$ and

$k^* = 8/3$.) A single auctioneer’s revenue from offering $[A][B]$, $[AB]$, or $[A,B]$ is 0.57, 0.60, or 0.64, respectively. Competing auctioneers’ revenues in Stage 1 are depicted in Table 1 for the nine subgames. (Complete tables for any $k \leq k^*$ are derived in Appendix B.2.) Auctioneers’ equilibrium revenues are in bold.

Table 1: Expected auctioneer revenue for $k = 7/3$.

| | | Auctioneer 2 | | |
|--------------|----------|-----------------------------|-----------------------------|------------|
| | | $[A][B]$ | $[AB]$ | $[A,B]$ |
| Auctioneer 1 | $[A][B]$ | (.57, .57) | (.67 , .78) | (.67, .78) |
| | $[AB]$ | (.78 , .67) | (.60, .60) | (.78, .67) |
| | $[A,B]$ | (.78, .67) | (.67, .78) | (.64, .64) |

A single auctioneer would offer a combinatorial auction and receive an expected revenue of 0.64. This is also the expected revenue of a competing auctioneer if both auctioneers offer $[A,B]$ and attract the same kinds of bidders. However, for every auction format offered by the competitor, an auctioneer has an incentive to offer a different auction format targeted at local ($[A][B]$) or global ($[AB]$) bidders, thereby inducing bidder segmentation and higher expected revenue of 0.67 or 0.78, respectively. Only if one auctioneer offers $[A][B]$ and the other auctioneer offers $[AB]$ does neither auctioneer have an incentive to deviate.

Idea of proof of Theorem 2. We focus on the main drivers of the players’ equilibrium decisions. First we describe bidding behavior in the various possible auctions, then we discuss what drives bidders’ participation decisions, and finally we explain why this makes auctioneers prefer to differentiate.

With the exception of global bidders in $[A][B]$, all bidders—regardless of the auction in which they chose to participate—bid their valuation. A global bidder’s bidding strategy in $[A][B]$ depends on his information about the other bidders in the auction, consisting of the number and kind of the other bidders. In particular, if there is at least one A-bidder and at least one B-bidder, global bidders in $[A][B]$ face an exposure problem. To avoid this problem, a global bidder either submits a bid of zero on either item if his valuation for the package is sufficiently small, or otherwise bids in a way that ensures that he wins against any local bidder.

Bidders’ participation decisions are driven by four considerations. First, global bidders tend to avoid $[A][B]$ with local bidders due to the exposure problem. Second, global bidders tend to avoid the combinatorial auction $[A,B]$ with local bidders, because local bidders who want different items complement each other, which effectively makes them jointly compete

against global bidders. Third, local bidders avoid [AB] with other local (and global) bidders because of a bundle problem, i.e., because they will need to compete against all of the bidders.¹⁰ Fourth, bidders tend to avoid an auctioneer who attracts more or stronger bidders, and thus there is a tendency towards an equal distribution of bidders. Bidder segmentation is mainly a consequence of the first three considerations if auctioneers offer different auction formats.

In general, for $k \leq k^*$, auctioneers collectively benefit from a segmented market and the resulting bidder homogeneity; that is, they benefit from attracting only local or only global bidders. This is because, in each auction of a segmented market, local and global bidders do not compete against each other, and each bidder can bid on his desired package. Therefore, the free-rider, exposure, and bundle problems vanish and low-revenue (non-core) outcomes do not occur. Which market segment—that of the local bidders or that of the global bidders—is more remunerative for an auctioneer depends on the relative strengths of these subsets of bidders, as determined by the value of k . Being able to attract the more remunerative market segment provides the auctioneer with an additional incentive for segmentation.

If global bidders are weak, specifically if $k \leq \hat{k}$, then the competing auctioneer maximizes his expected revenue if he attracts the market segment of the local bidders. If one auctioneer offers [A][B] and the other offers [AB], then bidders will segment and the auctioneers will have no incentive to deviate from their choices. The auctioneer offering [A][B] already has the highest possible expected revenue. The auctioneer offering [AB], by deviating to [A,B], will still attract the global bidders, and by deviating to [A][B] expects a lower revenue due to the exposure problem (as then bidders do not segment). There cannot be an equilibrium where both auctioneers offer the combinatorial auction [A,B], because at least one of them would prefer to attract all global bidders by offering [AB].

If global bidders are stronger, specifically if $\hat{k} < k \leq k^*$ (as in the example above with $k = 7/3$), the highest expected revenue a competing auctioneer can obtain is that from the market segment of the global bidders. Also, attracting all local bidders is more remunerative than attracting one bidder of each kind, which creates a free-rider, exposure, or bundle problem. If only one auctioneer offers [AB], this auctioneer will necessarily (i.e., in any equilibrium of a subgame following such a choice) attract the global bidders, and thus he cannot do better. If one auctioneer offers [AB] and the other offers [A][B], the auctioneer offering [A][B] cannot, by deviating to a different auction, attract the global bidders.

If global bidders are significantly stronger than local bidders, $k > k^*$, then only in the three subgames starting at the auctioneers' choices of ([A][B],[A][B]), ([AB],[AB]), or ([A,B],[A,B])

¹⁰Note that in [A][B] and [A,B] local bidders compete only with local bidders of the same kind (and global bidders) and, in addition, local bidders can free-ride on bids of local bidders of the other kind.

do equilibria with properties P1, P2, and P3 exist. No competing auctioneer will be able to attract both global bidders, because a global bidder with a high valuation ($v \in (k^*, k]$) wants to avoid the other global bidder even if that requires him to compete against all local bidders. In subgames, only equilibria in which bidders split into two symmetric groups are possible. As for $k \leq k^*$, these can occur only if both auctioneers offer the same format.

5 Discussion

Bidders segment in all equilibria satisfying P1, P2, and P3 (Theorem 2, part (b)). This segmentation naturally occurs in equilibria where the auctioneers offer auction formats [AB] and [A][B] tailored to the respective segments. However we can have additional, less intuitive, equilibria if global bidders are sufficiently weak, but each will have a more intuitive counterpart with the same revenues for auctioneers and payoffs for bidders. If $k < \hat{k}$, we can have one auctioneer offering the combinatorial auction [A,B] and attracting the local bidders, and the other offering [A][B] and attracting the global bidders. Notably, one auctioneer allows bids on more packages than is necessary to accommodate the participating bidders' heterogeneity whereas the other auctioneer does not allow the participating global bidders to bid on the package. To understand why these counterintuitive equilibria exist, note that the auctioneer who offers the combinatorial auction [A,B] attracts the more remunerative segment, viz., the local bidders. This is possible because, conditional on segmentation, these two auction formats, [A,B] and [A][B], are payoff-equivalent for all bidders (due to the lack of an exposure problem in [A][B] without local bidders). The auctioneer offering [A,B] would attract a strictly less remunerative set of bidders if he offered a different auction format. If he offered the same format as his competitor then the market would not be segmented, and if he offered [AB] then he would attract the global bidders. The auctioneer who offers [A][B] would attract the same (global) bidders and receive the same expected revenues if he deviated to [AB], or would attract one bidder of each kind if he offered [A,B], which would be less remunerative due to the low-revenue (free-rider) problem of [A,B]. We cannot have that one auctioneer offers [A,B] and the other offers [AB], unless global bidders are almost as weak as local bidders. (For this exceptional case, $k \approx 1$, see Section B.2.3 in the Appendix.) In contrast, if $k \geq \hat{k}$ there is no equilibrium in which an auctioneer offers the combinatorial auction [A,B]. This is because, given the choice between [AB] and [A,B], global bidders will necessarily participate in [AB]. Thus the auctioneer offering [A,B] will get the local bidders and will want to deviate to [A][B].

A comparison of revenues in the competing-auctioneer and single-auctioneer models reveals that auctioneers benefit from competition if market size, in terms of the number and kind

of buyers, increases proportionally. More precisely, the sum of the competing auctioneers’ expected revenues is larger than twice the expected revenue in the single-auctioneer model. If market segments do not differ too much in their attractiveness to auctioneers, viz., if k is neither too small nor too large, then each auctioneer is better off under competition than in the single-auctioneer setup. This suggests that allowing for or intensifying competition in an electronic marketplace can help attract auctioneers, if the average number of bidders per auctioneer does not decrease.

We have discussed the free-rider problem in the VCG mechanism. This problem is not specific to this auction. For example, for $k = 2$ in our LLG setting, the expected revenue from [A,B] is 0.583, and the expected revenue from the combinatorial pay-as-bid auction and from various bidder-optimal core-selecting auctions has been calculated to values between 0.500 and 0.596 (Baranov 2010, Beck and Ott 2013, Ausubel and Baranov 2020). Specifically, these expected revenues are below the expected total valuations of the losing bidders (0.708), which implies the occurrence of non-core outcomes in these auctions (see, e.g., Ausubel and Milgrom 2006, Day and Milgrom 2008, for the low-revenue problem and the core).

6 Conclusion

Although a single auctioneer may offer a combinatorial auction, competing auctioneers in a comparable setting will not. Instead, the competing auctioneers will segment the market by restricting the packages on which they accept bids, attracting more homogeneous sets of bidders, resulting in increased bidder competition. The advantage to an auctioneer of accommodating bidder heterogeneity via a combinatorial auction is dominated in a competing-auctioneer setting by the greater advantage from attracting homogeneous sets of bidders who will compete more fiercely with each other.

The intuition for our results rests upon the presence of the free-rider problem and the associated low revenues. The intuition does not extend to scenarios without a free-rider problem, for example if all items are substitutes for all bidders. Our model is rich enough to incorporate the free-rider problem while still maintaining tractability. This allows us to identify the free-rider, exposure, and bundle problems as sufficient drivers for beneficial market segmentation. The effect of additional auction design features (e.g., payment rules, reserve prices) on these drivers, and therefore on the auctioneers’ incentives to segment the market, remain unaddressed. We leave this for future research.

According to Adomavicius et al. (2012), the explanation for the scarcity of online combinatorial auctions is “the computational complexity of determining winners in such auctions

and the cognitive complexity of formulating combinatorial bids.” In this paper, we examined combinatorial auctions in online markets where both types of complexity become insignificant, revealing an additional explanation for the scarcity of competitive combinatorial auctions in the electronic marketplace.

Our findings may help to explain why, despite the increased use of combinatorial auctions in markets with a single auctioneer, combinatorial auction formats have largely been absent from online market platforms. These findings have implications for market design and, in particular, show that it might not be advantageous for an online market platform on which different sellers and buyers interact to offer combinatorial auctions as a design option to competing auctioneers.

References

- Adomavicius, G., Curley, S. P., Gupta, A., and Sanyal, P. (2012). Effect of information feedback on bidder behavior in continuous combinatorial auctions. *Management Science*, 58(4):811–830.
- Albrecht, J., Gautier, P., and Vroman, S. (2012). A note on Peters and Severinov, “Competition among sellers who offer auctions instead of prices”. *Journal of Economic Theory*, 147:389–392.
- Albrecht, J., Gautier, P., and Vroman, S. (2014). Efficient entry in competing auctions. *American Economic Review*, 104(10):3288–3296.
- Andersson, T., Andersson, C., and Andersson, F. (2012). An empirical investigation of efficiency and price uniformity in competing auctions. *Economics Letters*, 116:99–101.
- Armstrong, M. (2000). Optimal multi-object auctions. *The Review of Economic Studies*, 67(3):455–481.
- Ausubel, L. M. and Baranov, O. V. (2014). Market design and the evolution of the combinatorial clock auction. *American Economic Review: Papers & Proceedings*, 104(5):446–451.
- Ausubel, L. M. and Baranov, O. V. (2020). Core-selecting auctions with incomplete information. *International Journal of Game Theory*, 49:251–273. <https://doi.org/10.1007/s00182-019-00691-3>.
- Ausubel, L. M. and Milgrom, P. (2006). The lovely but lonely Vickrey auction. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 1, pages 17–40. Cambridge, MA: MIT Press.
- Ausubel, L. M. and Milgrom, P. R. (2002). Ascending auctions with package bidding. *The B.E. Journal of Theoretical Economics*, 1(1):1–44.
- Avery, C. and Hendershott, T. (2000). Bundling and optimal auctions of multiple products. *Review of Economic Studies*, 67(3):483–497.
- Baisa, B. and Burkett, J. (2018). Large multi-unit auctions with a large bidder. *Journal of Economic Theory*, 174:1–15.
- Bapna, R., Chang, S. A., Goes, P., and Gupta, A. (2009). Overlapping online auctions: Empirical characterization of bidder strategies and auction prices. *MIS Quarterly*, 33(4):763–783.
- Bapna, R., Dellarocas, C., and Rice, S. (2010). Vertically differentiated simultaneous Vickrey auctions: Theory and experimental evidence. *Management Science*, 56(7):1074–1092.
- Baranov, O. (2010). Exposure vs. free-riding in auctions with incomplete information. Working Paper, University of Maryland.

- Beck, M. and Ott, M. (2013). Incentives for overbidding in minimum-revenue core-selecting auctions. Annual Conference 2013 (Duesseldorf): Competition Policy and Regulation in a Global Economic Order 79946, Verein für Socialpolitik / German Economic Association.
- Belleflamme, P. and Peitz, M. (2010). *Industrial Organization: Markets and Strategies*. Cambridge University Press.
- Bichler, M., Davenport, A., Hohner, G., and Kalagnanam, J. (2006). Industrial procurement auctions. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 23, pages 593–612. Cambridge, MA: MIT Press.
- Bikhchandani, S. (1999). Auctions of heterogeneous objects. *Games and Economic Behavior*, 26:193–220.
- Bosshard, V., Bünz, B., Lubin, B., and Seuken, S. (2020). Computing Bayes-Nash equilibria in combinatorial auctions with verification. *Journal of Artificial Intelligence Research*, 69:531–570.
- Burguet, R. and Sakovics, J. (1999). Imperfect competition in auction designs. *International Economic Review*, 40(1):231–247.
- Cantillon, E. and Pesendorfer, M. (2006). Auctioning bus routes: The London experience. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 22, pages 573–591. Cambridge, MA: MIT Press.
- Caplice, C. and Sheffi, Y. (2006). Combinatorial auctions for truckload transportation. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 21, pages 539–571. Cambridge, MA: MIT Press.
- Chakraborty, I. (1999). Bundling decisions for selling multiple objects. *Economic Theory*, 13:723–733.
- d’Aspremont, C., Gabszewicz, J. J., and Thisse, J.-F. (1979). On Hotelling’s “Stability in competition”. *Econometrica*, 47(5):1145–1150.
- Day, R. and Milgrom, P. R. (2008). Core-selecting package auctions. *International Journal of Game Theory*, 36(3):393–407.
- Erdil, A. and Klemperer, P. (2010). A new payment rule for core-selecting package auctions. *Journal of the European Economic Association*, 8(2–3):537–547.
- Finster, S. (2020). Strategic bidding in product-mix, sequential, and simultaneous auctions. Economics Papers 2020-W03, Economics Group, Nuffield College, University of Oxford.
- Goeree, J. K. and Lien, Y. (2014). An equilibrium analysis of the simultaneous ascending auction. *Journal of Economic Theory*, 153:506–533.
- Goeree, J. K. and Lien, Y. (2016). On the impossibility of core-selecting auctions. *Theoretical Economics*, 11:41–52.
- Goossens, D. R., Onderstal, S., Pijnacker, J., and Spieksma, F. C. R. (2014). Solids: A combinatorial auction design for real estate. *Interfaces*, 44(4):351–363.
- Han, J., Qiu, C., and Popkowski Leszczyc, P. (2018). The effects of competitive reserve prices in online auctions. *European Journal of Marketing*, 52(7/8):1439–1456.
- Jehiel, P., Meyer-ter-Vehn, M., and Moldovanu, B. (2007). Mixed bundling auctions. *Journal of Economic Theory*, 134:494–512.
- Kim, K. and Kircher, P. (2015). Efficient competition through cheap talk: The case of competing auctions. *Econometrica*, 83(5):1849–1875.
- Kim, S. W., Olivares, M., and Weintraub, G. Y. (2014). Measuring the performance of large-scale combinatorial auctions: A structural estimation approach. *Management Science*, 60(5):1180–1201.

- Krishna, V. (2010). *Auction Theory*. Academic Press, second edition.
- Krishna, V. and Rosenthal, R. W. (1996). Simultaneous auctions with synergies. *Games and Economic Behavior*, 17:1–31.
- Lehmann, D., Müller, R., and Sandholm, T. (2006). The winner determination problem. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 12, pages 297–317. Cambridge, MA: MIT Press.
- Mastropietro, P., Batlle, C., Barroso, L. A., and Rodilla, P. (2014). Electricity auctions in South America: Towards convergence of system adequacy and RES-E support. *Renewable and Sustainable Energy Reviews*, 40:375–385.
- McAfee, R. P. (1993). Mechanism design by competing sellers. *Econometrica*, 61(6):1281–1312.
- Nisan, N. (2006). Bidding languages for combinatorial auctions. In Cramton, P., Shoham, Y., and Steinberg, R., editors, *Combinatorial Auctions*, chapter 9, pages 215–232. Cambridge, MA: MIT Press.
- Olivares, M., Weintraub, G. Y., Epstein, R., and Yung, D. (2012). Combinatorial auctions for procurement: An empirical study of the Chilean school meals auction. *Management Science*, 58(8):1458–1481.
- Pai, M. M. (2010). Competition in mechanisms. *SIGecom Exchanges*, 9(1).
- Palfrey, T. R. (1983). Bundling decisions by a multiproduct monopolist with incomplete information. *Econometrica*, 51(2):463–483.
- Parlane, S. (2008). Auctioning horizontally differentiated items. *Review of Industrial Organization*, 33(2):113–128.
- Peters, M. (1997). A competitive distribution of auctions. *Review of Economic Studies*, 64:97–123.
- Peters, M. and Severinov, S. (1997). Competition among sellers who offer auctions instead of prices. *Journal of Economic Theory*, 75:141–179.
- Rosenthal, R. W. and Wang, R. (1996). Simultaneous auctions with synergies and common values. *Games and Economic Behavior*, 17:32–55.
- Rothkopf, M. H., Pekeč, A., and Harstad, R. M. (1998). Computationally manageable combinatorial auctions. *Management Science*, 44(8):1131–1147.
- Sano, R. (2012). Non-bidding equilibrium in an ascending core-selecting auction. *Games and Economic Behavior*, 74(2):637–650.
- Subramaniam, R. and Venkatesh, R. (2009). Optimal bundling strategies in multiobject auctions of complements or substitutes. *Marketing Science*, 28(2):264–273.
- Szentesi, B. (2007). Two-object two-bidder simultaneous auctions. *International Game Theory Review*, 9(3):483–93.
- Szentesi, B. and Rosenthal, R. W. (2003). Three-object two-bidder simultaneous auctions: chopsticks and tetrahedra. *Games and Economic Behavior*, 44:114–133.
- Truong, H. M., Gupta, A., Ketter, W., and van Heck, E. (2017). Effects of pre-sales posted price channel on sequential B2B Dutch flower auctions. In *ICIS 2017 Proceedings*, volume 14.
- Virág, G. (2010). Competing auctions: Finite markets and convergence. *Theoretical Economics*, 5:241–274.

Appendix

A Proof of Theorem 1

We show that for $k \leq 2$ the auctioneer offers [AB] and for $k > 2$ he offers [A,B], i.e., $\hat{k} = 2$. When an A-bidder with $v_A \sim U[0, 1]$, a B-bidder with $v_B \sim U[0, 1]$, and a global bidder with $v_{AB} \sim U[0, k]$ participate, a single auctioneer's expected revenue if he allows only a bid on A or only a bid on B is zero. The revenue when only exclusive bids on A and AB, or on B and AB, are allowed is the same as from offering [AB]. The expected revenues from the auction formats [A,B], [AB], and [A][B] are

$$\begin{aligned} E[\pi_0]^{[A][B]} &= (k - 1)/k \text{ for } k > 1 \\ E[\pi_0]^{[AB]} &= (4k - 1)/(6k) \text{ for } k > 1 \\ E[\pi_0]^{[A,B]} &= \begin{cases} (k^4 - 8k^3 + 24k^2 - 20k + 6)/(12k) & \text{for } 1 < k < 2 \\ (6k - 5)/(6k) & \text{for } k \geq 2. \end{cases} \end{aligned}$$

In [A,B], all bidders bid their valuation (by Property P2). With these bids, $E[\pi_0]^{[A,B]}$ is calculated for $1 < k < 2$ as $\int_{k-1}^1 \int_0^{k-a} \int_{a+b}^k (a+b)/k \, dc \, db \, da + \int_0^{k-1} \int_0^1 \int_{a+b}^k (a+b)/k \, dc \, db \, da + 2 \int_{k/2}^1 \int_{k-a}^a \int_a^k (2c-a-b)/k \, dc \, db \, da + 2 \int_0^{k-1} \int_a^1 \int_b^{a+b} (2c-a-b)/k \, dc \, db \, da + 2 \int_{k-1}^1 \int_a^{k-a} \int_b^{a+b} (2c-a-b)/k \, dc \, db \, da + 2 \int_0^1 \int_a^1 \int_a^b (c-a)/k \, dc \, db \, da = (k^4 - 8k^3 + 24k^2 - 20k + 6)/(12k)$ and for $k \geq 2$ as $\int_0^1 \int_0^1 \int_{a+b}^k (a+b)/k \, dc \, db \, da + 2 \int_0^1 \int_0^1 \int_a^{a+b} (c-a)/k \, dc \, da \, db = (6k - 5)/(6k)$.

In [AB], all bidders bid their valuation (by Property P2). With these bids, the auctioneer's revenue for all $k > 1$ is $E[\pi_0]^{[AB]} = 2 \int_1^k \int_0^1 \int_0^b b/k \, da \, db \, dc + 6 \int_0^1 \int_0^c \int_b^c a/k \, da \, db \, dc = (4k - 1)/(6k)$.

In [A][B], all local bidders bid their valuation (by Property P2). The global bidder's expected utility from bidding $bA \leq 1$ and $bB \leq 1$ for item A and B, respectively, is $v_{AB} bA bB - \int_0^{bA} v_A dv_A - \int_0^{bB} v_B dv_B$, and his utility from bidding more than 1 is the same as from bidding $bA = 1$ and/or $bB = 1$. The first-order conditions, $v_{AB} bB - bA = 0$ and $v_{AB} bA - bB = 0$ for an interior maximum can only be fulfilled for $v_{AB} = 1$, in which case any bids $bA = bB$ are optimal and which occurs with probability zero. His optimal bids for all $v_{AB} < 1$ are $bA = 0$ and $bB = 0$ with a utility of zero, because bidding $(b_A, b_B) \in \{(1, v_{AB}), (v_{AB}, 1), (1, 1)\}$ results in a negative expected utility. For all $v_{AB} \geq 1$, $bA = 1$ and $bB = 1$ are optimal bids, with the strictly positive expected utility $v_{AB} - 1$. With these bids of the three bidders, the auctioneer's revenue for all $k > 1$ is zero if the global bidder bids zero and the local bidders win, which occurs with probability $1/k$, and the revenue is $v_A + v_B$ if the global bidder wins, which occurs

with probability $1 - 1/k$. Thus, $E[\pi_0]^{[A][B]} = (1 - 1/k) \cdot 2 \int_0^1 v_A dv_A = (k - 1)/k$.

Comparing the revenues, we find that, for $k > 2$, we have that $E[\pi_0]^{[A,B]} > \max\{E[\pi_0]^{[A][B]}, E[\pi_0]^{[AB]}\}$ because then $(6k - 5)/(6k) > \max\{(k - 1)/k, (4k - 1)/(6k)\}$ due to $(6k - 5)/(6k) - (k - 1)/k = 1/(6k)$ and $(6k - 5)/(6k) - (4k - 1)/(6k) = 1/3 - 2/(3k)$. For $k = 2$, we have that $E[\pi_0]^{[AB]} = E[\pi_0]^{[A,B]} > E[\pi_0]^{[A][B]}$ because $7/12 = (4k - 1)/(6k) = (6k - 5)/(6k) > (k - 1)/k = 1/2$. For $1 < k < 2$ we have that $E[\pi_0]^{[AB]} > \max\{E[\pi_0]^{[A][B]}, E[\pi_0]^{[A,B]}\}$ because then $(4k - 1)/(6k) > (k^4 - 8k^3 + 24k^2 - 20k + 6)/(12k) > (k - 1)/k$.

Every bidder prefers participating and submitting a bid over not submitting a bid in any of the auction formats $[A][B]$, $[AB]$, and $[A,B]$. Thus, P1 is trivially fulfilled.

B Proof of Theorem 2

To prove Theorem 2, we show that for $k^* = 8/3$ an equilibrium with properties P1, P2, and P3 exists if and only if $k \leq k^*$. Section B.2 characterizes equilibria with properties P1, P2, and P3 for all $k \leq 8/3$, and Section B.3 proves that equilibria with properties P1, P2, and P3 do not exist if $k > 8/3$.

B.1 Notation and Properties P1, P2, and P3

We label the two auctioneers by $i \in \{S1, S2\}$, the two bidders interested in A by A1, A2, the two bidders interested in B by B1, B2, and the two bidders interested in obtaining the package by AB1, AB2. We label bidders by j where $j \in \mathbf{J} := \{A1, A2, B1, B2, AB1, AB2\}$. The valuations v_j are drawn independently from the commonly known distribution functions $v_j \sim U[0, 1]$ for a local bidder $j \in \{A1, A2, B1, B2\}$ and $v_j \sim U[0, k]$, where $k > 1$, for a global bidder $j \in \{AB1, AB2\}$.

The auctioneers simultaneously choose their respective auction format from the strategy set Σ . Auctioneer i 's strategy is denoted by σ_i , i.e., auctioneer i chooses an auction format: $\sigma_i \in \Sigma$, where $i \in \{S1, S2\}$. To simplify the proof, we first restrict the auctioneers' options to $\Sigma = \{[A,B], [A][B], [AB]\}$ and argue (in Section B.2.4) that auctioneers would never prefer to offer any other of the possible restrictions on bidding.

We denote the participation decision of bidder $j \in \mathbf{J}$ by γ_j , where $\gamma_j \in \{S1, S2\}$ denotes the auctioneer in whose auction bidder j participates. We denote the bid of bidder j by a triple $(bA_j, bB_j, bAB_j) \in (\mathbb{R}_{\geq 0} \cup \{out\})^3$, where bA_j is bidder j 's bid on A if $\sigma_{\gamma_j} \in \{[A][B], [A,B]\}$ and is ignored if $\sigma_{\gamma_j} = [AB]$, and where *out* signifies that the bidder opts out of bidding. Equivalently, bB_j constitutes j 's bid on B and bAB_j his bid on AB, and these bids are ignored if the auction does not allow bids on the respective package. Bidder j 's bids depend on his

valuation v_j , the auctioneers' strategies $\sigma = (\sigma_{S1}, \sigma_{S2}) \in \Sigma^2$, and the vector of the participation decisions of the other bidders, denoted by γ_{-j} . Thus, the strategy of bidder $j \in \mathbf{J}$ is a vector $(\gamma_j(v_j, \sigma), bA_j(v_j, \sigma, \gamma_{-j}), bB_j(v_j, \sigma, \gamma_{-j}), bAB_j(v_j, \sigma, \gamma_{-j})) \in \{S1, S2\} \times (\mathbb{R}_{\geq 0} \cup \{out\})^3$.

We denote an equilibrium strategy of auctioneer i by σ_i^* and of bidder j by

$$(\gamma_j^*(v_j, \sigma), bA_j^*(v_j, \sigma, \gamma_{-j}), bB_j^*(v_j, \sigma, \gamma_{-j}), bAB_j^*(v_j, \sigma, \gamma_{-j})).$$

Bidder j 's equilibrium belief at the beginning of Stage 3 regarding the distribution of bidder ℓ 's valuation is given by $\mu_{j\ell}^*(\sigma, \gamma_{-j})$.

Our equilibrium refinement uses the three properties P1, P2, and P3, formalized as:

P1 A bidder's participation decisions are independent of his valuation:

$$\text{For all } \sigma \in \Sigma^2 : \quad \gamma_j^*(v_j, \sigma) = \gamma_j^*(\hat{v}_j, \sigma) \quad \forall v_j, \hat{v}_j.$$

P2 A bidder submits only undominated bids:

- (a) If $\sigma_{\gamma_j} = [A, B]$, a local X-bidder j ($X = A, B$) bids $bX_j^*(v_j, \sigma, \gamma_{-j}) = bAB_j^*(v_j, \sigma, \gamma_{-j}) = v_j$, and opts out of bidding on the other item. A global bidder j bids $bAB_j^*(v_j, \sigma, \gamma_{-j}) = v_j$, and opts out of bidding on both A and B.
- (b) If $\sigma_{\gamma_j} = [AB]$, then $bAB_j^*(v_j, \sigma, \gamma_{-j}) = v_j$.
- (c) If $\sigma_{\gamma_j} = [A][B]$, a local X-bidder j bids $bX_j^*(v_j, \sigma, \gamma_{-j}) = v_j$, and opts out of bidding on the other item.
- (d) If $\sigma_{\gamma_j} = [A][B]$, a global bidder j bids $bA_j^*(v_j, \sigma, \gamma_{-j}) \leq v_j$ and $bB_j^*(v_j, \sigma, \gamma_{-j}) \leq v_j$.

P3 If there is a global bidder j with $\sigma_{\gamma_j} = [A][B]$, then, if an efficient ex-post equilibrium of $[A][B]$ exists, bidders bid according to an efficient ex-post equilibrium.

P2 excludes dominated bids. Bids (other than opting out of bidding) in (a), (b), and (c) are weakly dominant and, thus, are the only undominated bids. In (d), the bid $(bA_j^*(v_j, \sigma, \gamma_{-j}), bB_j^*(v_j, \sigma, \gamma_{-j}))$ with $bA_j^*(v_j, \sigma, \gamma_{-j}) > v_j$ is dominated by $(v_j, bB_j^*(v_j, \sigma, \gamma_{-j}))$ because the marginal value of winning in $[A]$ is at most v_j and bidding $bA_j^*(v_j, \sigma, \gamma_{-j}) > v_j$ wins additionally only in cases where payments in $[A]$ exceed v_j . The same argument holds for $bB_j^*(v_j, \sigma, \gamma_{-j})$.

B.2 Equilibria for $k \leq 8/3$

The following proposition summarizes the auctioneers' equilibrium decisions. Note that the cutoff-values 2 and $8/3$ in the proposition correspond to \hat{k} in Theorem 1 and to k^* in Theorem

2, respectively. Proposition 1 implies that for $k \leq 8/3$ the statements in Theorem 2(a) and (b) hold.

Proposition 1. *Consider the competing-auctioneers model. For all $k \leq 8/3$, an equilibrium in pure strategies with properties P1, P2, and P3 exists. For each such equilibrium:*

- (a1) *If $2 < k \leq 8/3$, then one auctioneer offers $[A][B]$ and receives an expected revenue of $2/3$, and the other auctioneer offers $[AB]$ and receives an expected revenue of $k/3$.*
- (a2) *If $k \leq 2$, one of the following must hold:*
 - (i) *One auctioneer offers $[A][B]$ and receives an expected revenue of $2/3$, and the other auctioneer offers $[A][B]$ or $[AB]$ and receives an expected revenue of $k/3$.*
 - (ii) *One auctioneer offers $[A,B]$ and receives an expected revenue of $2/3$, and the other auctioneer offers $[A][B]$ or $[AB]$ and receives an expected revenue of $k/3$.*
- (b) *If the equilibrium revenue is $2/3$, this is because all local bidders participate in that auction; if it is $k/3$, this is because all global bidders participate in that auction.*

Proposition 1 will be proven in Sections B.2.1–B.2.4. In Section B.2.1 we derive the optimal bids in Stage 3 of the game. In Section B.2.2 we derive the bidders’ optimal participation decisions in Stage 2. In Section B.2.3 we derive the auctioneers’ equilibrium choices in Stage 1. Finally, in Section B.2.4, we show that the results hold if we relax a restriction on the auctioneers’ strategy sets that we had imposed in the previous arguments.

B.2.1 Stage 3: Bidding Decisions

According to P2, all bidders in $[AB]$ and $[A,B]$ and local bidders in $[A][B]$ bid their valuation. The following two lemmas characterize important properties of a bidder’s equilibrium payoffs if he participates in $[A][B]$, for the case where at least one global bidder and at least one other bidder participate.

Lemma 1. *Consider the continuation game given by $[A][B]$ in which $m > 0$ global bidders and at least one A- and one B-bidder participate. Suppose that bidders’ beliefs equal their prior beliefs. Let h be the sum of the expected maximum valuations of A-bidders and the expected maximum valuations of B-bidders in $[A][B]$. For every equilibrium in which bidders choose only undominated bids, the following holds:*

- (a) *Every global bidder with valuation $v \in [0, h]$ has an expected payoff of zero. If $h < k$ and $m = 1$, a global bidder with $v \in (h, k]$ wins both items and his expected payoff is $v - h$.*

If $h < k$ and $m = 2$, there exists a global bidder with valuation $v \in (h, k]$ whose expected payoff is strictly below $v - h$.

- (b) If $m = 1$, a local bidder with valuation v has an expected payoff of $hv^n/(kn)$. If $m = 2$, a local bidder's expected payoff is at least $v^n/(nk^2)$, where n is the number of local bidders participating in his auction.

Proof of Lemma 1. Denote the cdf of the highest valuation of X-bidders, $X \in \{A, B\}$ by F_X (with support $[0, 1]$) with density f_X . Let h be the sum of the expected maximum valuations of A-bidders and B-bidders. Local bidders have the weakly dominant strategy to bid their valuation.

Assume first that $m = 1$. The expected utility of the global bidder with valuation v who bids $bX \leq 1$ on item $X \in \{A, B\}$ (bidding $bX > 1$ gives the same expected utility as $bX = 1$) is given by $U(bA, bB, v) = vF_B(bB)F_A(bA) - \int_0^{bA} x dF_A(x) - \int_0^{bB} x dF_B(x)$. We get $\partial U(bA, bB, v)/\partial bA = (vF_B(bB) - bA)f_A(bA)$ and an interior maximum exists only if $bA = vF_B(bB)$ and $bB = vF_A(bA)$. Thus, for $0 < v < 1$, by $F_A(bA) \leq bA$ and $F_B(bB) \leq bB$ we get that $bA = bB = 0$ and an interior maximum does not exist. If $v = 1$ and $F_A(x) = F_B(x) = x$, $bA = bB$ is another bid with expected utility zero. It cannot be optimal to submit a positive bid on only one item, A or B, which results in a negative expected utility. Thus in equilibrium either the global bidder's payoff is zero or it is optimal to bid $bA \geq 1$ and $bB \geq 1$. The resulting utility $U(bA, bB, v) = v - \int_0^1 x dF_A(x) - \int_0^1 x dF_B(x)$ is negative for $v < h := \int_0^1 x dF_A(x) + \int_0^1 x dF_B(x)$. Therefore, the global bidder's equilibrium payoff is zero if $v \leq h$ and it is $v - h$ if $v > h$. The probability that one of the local bidders wins item X, $X \in \{A, B\}$, is h/k . Assume the number of X-bidders is n . If $n = 1$, an X-bidder's expected payoff is vh/k . If $n = 2$, an X-bidder has an expected payoff of $(h/k) \int_0^v (v - x)(n - 1)x^{n-2} dx = hv^n/(kn)$.

If $h < k$, there must exist a global bidder with $v \in (h, k]$ (i.e., who would submit a positive bid if he was the only global bidder) who is strictly worse off as compared to a situation where he was the only global bidder (the case $m = 1$) and consequently faced less competition. Also a global bidder with $v < 1$ bids $bA = bB = 0$ because he could not obtain a positive expected payoff with a positive bid even if there was no other global bidder. Thus, the probability that an X-bidder wins $X \in \{A, B\}$ is at least $1/k^2$ and an X-bidder's expected payoff is at least $(1/k^2) \int_0^v (v - x)(n - 1)x^{n-2} dx = v^n/(nk^2)$, where n is the number of X-bidders. \square

Lemma 2. Consider a continuation game given by $[A][B]$.

- (a) If two global bidders and no local bidders participate, then an efficient ex-post equilibrium in undominated strategies exists. In every such equilibrium, each bidder bids according to the same strategy $(bA(v), bB(v))$, consisting of weakly increasing bidding functions that

fulfill $bA(v) + bB(v) = v$ for all $v \in [0, k]$, where v denotes the bidder's valuation. If each bidder believes that the other bidder has valuation $\tilde{v} \sim U[0, k]$, then each bidder has an expected payoff of $v^2/(2k)$.

- (b) If $m > 0$ global bidders, $n > 0$ A-bidders, and no B-bidders participate, then an efficient ex-post equilibrium in undominated strategies exists. In every such equilibrium, a global bidder with valuation $v \leq 1$ bids $(bA(v), bB(v)) = (v, 0)$. A global bidder with valuation $v > 1$ bids as in case (a) with the additional requirement that $bA(v) \geq 1$. Suppose that a bidder's belief equals his respective prior belief. Then each global bidder has an expected payoff of $v^{m+n}/((m+n)k^{m-1})$ if $v \in [0, 1]$ and an expected payoff of $((n+m)v^m - n)/(m(m+n)k^{m-1})$ if $v \in (1, k]$, and each local bidder has an expected payoff of $v^{m+n}/((m+n)k^m)$.

Proof of Lemma 2. (a) Ex-post efficiency requires that the same global bidder wins both items and that the bidder with the higher valuation always wins. Thus, in every efficient equilibrium, both bidders must bid according to the same bidding strategies $(bA(v), bB(v))$, where $bA(v)$ and $bB(v)$ increase weakly in v , and $b(v) := bA(v) + bB(v)$ increases strictly.¹¹

Assume for now that $b(v)$ is continuous (which implies that $bA(v)$ and $bB(v)$ are continuous as well). We first show that in every symmetric equilibrium with weakly increasing continuous functions $bA(v)$ and $bB(v)$ such that $b(v)$ is strictly increasing, we have that $b(v) = v$. Assume bidder $j \neq i$ with valuation v_j bids according to the increasing continuous functions $(\hat{b}A(v_j), \hat{b}B(v_j))$ such that $\hat{b}(v_j) := \hat{b}A(v_j) + \hat{b}B(v_j)$ is strictly increasing. We show that bidder i with valuation v has the unique best response $(bA(v), bB(v)) = (\hat{b}A(\hat{v}), \hat{b}B(\hat{v}))$ such that $\hat{b}A(\hat{v}) + \hat{b}B(\hat{v}) = v$, i.e., $(bA(v), bB(v)) = (\hat{b}A(\hat{b}^{-1}(v)), \hat{b}B(\hat{b}^{-1}(v)))$. With this bid, i wins both items if $v > \hat{b}(v_j)$ and does not win any item if $v < \hat{b}(v_j)$. (If $v = \hat{b}(v_j)$, he either wins both or no item due to the tie-breaking rule and is indifferent between winning both items and winning nothing.) Thus, there exists $\tilde{v} = \hat{b}^{-1}(v)$ such that i wins against all $v_j < \tilde{v}$ and loses against all $v_j > \tilde{v}$. His bid does not influence his payment conditional on winning. If he instead bids $(\hat{b}A(\hat{b}^{-1}(v + \alpha)), \hat{b}B(\hat{b}^{-1}(v + \alpha)))$, $\alpha > 0$, in expectation he is strictly worse off because he then additionally wins both items against (a set with positive measure of)

¹¹One can more generally show that $b(v)$ is strictly increasing in every efficient equilibrium for every tie-breaking rule that awards both items to a single bidder. Assume, to the contrary, that in equilibrium there exists β_A and β_B such that $\{v \mid bA(v) = \beta_A\} \cap \{v \mid bB(v) = \beta_B\} =: I$ contains more than one valuation v . The intersection I is an interval because $bA(v)$ and $bB(v)$ are weakly increasing and v has full support on $[0, k]$. Then, if a bidder with a valuation $v \in I$ and $v < \beta_A + \beta_B$ exists who wins with positive probability in a tie among these valuations, this bidder will prefer to reduce his bids marginally to avoid the loss. If such a valuation does not exist, then $v \geq \beta_A + \beta_B$ for all $v \in I$ who win with positive probability in a tie among these valuations. By I being an interval, there is more than one such valuation and more than one bidder with $v > \beta_A + \beta_B$. One of these bidders will want to marginally increase his bid to prevent the tie. This gives a contradiction to our assumption.

valuations $v_j > \tilde{v}$ but makes a loss against each of them. If $v > 0$ and he submits the bid $(\hat{b}A(\hat{b}^{-1}(v - \alpha)), \hat{b}B(\hat{b}^{-1}(v - \alpha)))$, $v \geq \alpha > 0$, in expectation he is strictly worse off because there is a positive measure of valuations $v_j < \tilde{v}$ such that i does not win anything against them and forgoes the positive gains from winning both items against them. If he varies his bids $(bA(v), bB(v))$ in another way, he either is strictly worse off for reasons just given, or he wins against different valuations of opponents with his bids on A and B, which makes him even worse off because he may pay a positive price for one item he values at zero.

We now confirm that $b(v)$ has to be continuous. Assume bidder $j \neq i$ bids according to the weakly increasing functions $(\hat{b}A(v), \hat{b}B(v))$ such that $\hat{b}(v)$ is strictly increasing. Assume that $\hat{b}(v)$ has discontinuities, i.e., there exist q' and q'' such that $q' < q''$ and $\hat{b}^{-1}((q', q'')) = \emptyset$. Consider bidder i with valuation v . Adjusting and applying the argument from the previous paragraph, for $v \in [q', q'']$ in each best response he will choose a bid $b(v) \in [q', q'']$. (He bids to win against bids below v and to avoid winning against bids above v .) Therefore, either $b^{-1}((q', q'')) \neq \emptyset$ or $b^{-1}(q') \cup b^{-1}(q'') = [q', q'']$, a contradiction to $b(v)$ being strictly increasing.

The expected payoff of a bidder with valuation $v \in (0, k]$ if all bidders choose the same strategy $(bA(v), bB(v))$ such that $bA(v) + bB(v) = v$, is $\int_0^v (v - x)/k dx = v^2/2k$.

(b) We restrict the argument to the non-trivial case $m = 2$. A-bidders follow their weakly dominant strategy and bid their respective valuation on A, and ex-post efficiency requires that the global bidders with $v \leq 1$ bid v on A. If both global bidders bid $(bA(v), bB(v)) = (v, 0)$, then, just as in a second-price auction, unilaterally deviating will make them weakly worse off for all $v \leq 1$. For $v > 1$, the analysis from (a) applies as long as global bidders always win against local bidders (which is required by ex-post efficiency). Thus, all global bidders with $v > 1$ follow weakly increasing bidding functions $(bA(v), bB(v))$ such that $bA(v) + bB(v) = v$ and $bA(v) \geq 1$.

We next show that ex-post efficiency requires that global bidders with valuation $v \leq 1$ bid zero on B. Bidding positively on B can occur in an ex-post efficient equilibrium only if all global bidders bid according to the same function $bB(v)$. We argued above that all bidders bid their valuation on A (in every ex-post efficient equilibrium). Consider now a global bidder with valuation v , with $bB(v) > 0$, who bids as if he had a valuation \hat{v} , i.e., he bids according to $(\hat{v}, bB(\hat{v}))$. Denote his probability of winning against a global bidder with a valuation below \hat{v} by $F_g(\hat{v})$ and his probability of winning against all A-bidders with valuations below \hat{v} by $F_\ell(\hat{v})$. His expected payoff is then given by $U(\hat{v}, v) = vF_\ell(\hat{v})F_g(\hat{v}) - \int_0^{\hat{v}} x(f_\ell(x)F_g(x) + F_\ell(x)f_g(x))dx - \int_0^{\hat{v}} bB(x)f_g(x)dx$. We have that $\partial U(v, \hat{v})/\partial \hat{v} = v(f_\ell(\hat{v})F_g(\hat{v}) + F_\ell(\hat{v})f_g(\hat{v})) - \hat{v}((f_\ell(\hat{v})F_g(\hat{v}) + F_\ell(\hat{v})f_g(\hat{v}))) - bB(\hat{v})f_g(\hat{v})$, which is negative for $v = \hat{v}$ and $bB(x) > 0$: $-bB(\hat{v})f_g(\hat{v}) < 0$. Therefore, a bidder with valuation v would not want to bid according to $bB(v)$ if $bB(v) > 0$.

In the equilibrium, the expected payoff of a local bidder with valuation v is $v^{m+n}/((m+n)k^m)$, and that of a global bidder with valuation v with $v \leq 1$ is $v^{m+n}/((m+n)k^{m-1})$ (for the calculations, see the proof of Supporting Lemma 2). The expected payoff of a global bidder with valuation $v \in (1, k]$ is $\int_0^v v(m-1)x^{m-2}/k^{m-1}dx - \int_1^v x(m-1)x^{m-2}/k^{m-1}dx - \int_0^1 x(n+m-1)x^{n+m-2}/k^{m-1}dx = ((n+m)v^m - n)/(m(m+n)k^{m-1})$. (We use the density of the first order statistic of $m-1$ independent draws from $U[0, k]$ and n independent draws from $U[0, 1]$, which is $(n+m-1)x^{n+m-2}/k^{m-1}$ for $x \in [0, 1]$ and $(m-1)x^{m-2}/k^{m-1}$ for $x \in (1, k]$.)

By symmetry, Lemma 2 also applies to the case where all local bidders are B-bidders. \square

B.2.2 Stage 2: Bidders' Participation Decisions

We organize properties and revenues of equilibria of the subgames starting in Stage 2 in six lemmas, lemmas 3–8, one for each subgame. We first state all six lemmas, and then provide the proofs. For each subgame, we characterize bidders' equilibrium participation decisions that fulfill P1 (i.e., participation decisions that are valuation independent). On the equilibrium path, because of P1, bidders' equilibrium beliefs are given by $\mu_{j\ell}^*(\sigma, \gamma_{-j}^*) = U[0, 1]$ for $\ell \in \{A1, A2, B1, B2\}$ and $\mu_{j\ell}^*(\sigma, \gamma_{-j}^*) = U[0, k]$ for $\ell \in \{AB1, AB2\}$, for all $j \in \mathbf{J}$. For the results, we do not impose any restrictions on off-the-equilibrium-path beliefs. Off-the-equilibrium-path beliefs matter only for equilibria of subgames in which at least one auctioneer offers $[A][B]$ and bidding decisions of global bidders who do not play according to an ex-post equilibrium in the continuation game following a deviation from the equilibrium participation decision (and may therefore influence participation decisions of other bidders).

Lemma 3. *Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choices $\sigma = ([A][B], [A][B])$. In an equilibrium with properties P1, P2, and P3, one of the following cases occurs:*

- (a) *One global bidder, one A-bidder, and one B-bidder participate in each $[A][B]$. Each auctioneer's expected revenue is $(k-1)/k$.*
- (b) *All global bidders participate in the auction of one auctioneer and all local bidders in the auction of the other auctioneer. The auctioneer with the global bidders has an expected revenue of $k/3$, and the other auctioneer has an expected revenue of $2/3$.*

Lemma 4. *Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choice $\sigma = ([A][B], [AB])$ or $\sigma = ([AB], [A][B])$. In an equilibrium with properties P1, P2, and P3 one of the following cases occurs:*

- (a) *Both global bidders participate in $[AB]$ and all local bidders participate in $[A][B]$. The auctioneer offering $[AB]$ has an expected revenue of $k/3$, and the auctioneer offering*

$[A][B]$ has an expected revenue of $2/3$.

- (b) Both global bidders and one local bidder participate in $[A][B]$, and the remaining local bidders participate in $[AB]$. The auctioneer offering $[A][B]$ has an expected revenue of $(2k^3 + 2k - 1)/(6k^2)$, and the auctioneer offering $[AB]$ has an expected revenue of $1/2$. This case can occur only if $k \leq 2/\sqrt{3} \approx 1.15$.

Lemma 5. Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choice $\sigma = ([A][B], [A,B])$ or $\sigma = ([A,B], [A][B])$. In an equilibrium with properties P1, P2, and P3, one of the following cases occurs:

- (a) Both global bidders participate in $[A,B]$, and all local bidders participate in $[A][B]$. The auctioneer offering $[A,B]$ has an expected revenue of $k/3$, and the auctioneer offering $[A][B]$ has an expected revenue of $2/3$.
- (b) Both global bidders participate in $[A][B]$, and all local bidders participate in $[A,B]$. The auctioneer offering $[A][B]$ has an expected revenue of $k/3$, and the auctioneer offering $[A,B]$ has an expected revenue of $2/3$.
- (c) One A-bidder and one B-bidder participate in $[A][B]$, and the remaining bidders participate in $[A,B]$. The auctioneer offering $[A][B]$ has an expected revenue of 0, and the auctioneer offering $[A,B]$ has an expected revenue of $(7k^5 - 50k^4 + 140k^3 - 80k^2 - 10k + 16)/(60k^2)$. This case can occur only if $k \in (1, k^\circ]$, where k° is the positive solution to $3k^{\circ 4} - 16k^{\circ 3} + 36k^{\circ 2} - 24k^\circ - 2 = 0$ (i.e., $k^\circ \approx 1.23$).

Lemma 6. Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choice $\sigma = ([AB], [AB])$. In an equilibrium with properties P1, P2, and P3, one of the following cases occurs:

- (a) Two local bidders and one global bidder participate in each $[AB]$. Each auctioneer's expected revenue is $(4k - 1)/(6k)$.
- (b) Three local bidders participate in one $[AB]$, and two global bidders and one local bidder participate in the other $[AB]$. The auctioneer who attracts only local bidders has an expected revenue of $1/2$, and the auctioneer who attracts two global bidder and one local bidder has an expected revenue of $(2k^3 + 2k - 1)/(6k^2)$. This case can occur only if $k \leq 2/\sqrt{3} \approx 1.15$.

Lemma 7. Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choice $\sigma = ([AB], [A,B])$ or $\sigma = ([A,B], [AB])$. In an equilibrium with properties P1, P2, and P3, both global bidders participate in $[AB]$ and all local bidders participate in $[A,B]$. The auctioneer offering $[AB]$ has an expected revenue of $k/3$, and the auctioneer offering $[A,B]$ has an expected revenue of $2/3$.

Lemma 8. *Assume $k \in (1, 8/3]$. Consider the subgame that follows the auctioneers' choice $\sigma = ([A,B], [A,B])$. In an equilibrium with properties P1, P2, and P3, one of the following cases occurs:*

- (a) *One A-bidder, one B-bidder, and one global bidder participate in either $[A,B]$. Each auctioneer's expected revenue is $(k^4 - 8k^3 + 24k^2 - 20k + 6)/(12k)$ if $1 < k < 2$ and $(6k - 5)/(6k)$ if $k \geq 2$.*
- (b) *Both global bidders participate in one $[A,B]$, and all local bidders participate in the other $[A,B]$. The auctioneer who attracts the global bidders has an expected revenue of $k/3$, and the other auctioneer receives an expected revenue of $2/3$.*

The proofs of lemmas 3 to 8 all follow the same basic line of argument and make use of the following two supporting lemmas.

Supporting Lemma 1. *Suppose that bidders' beliefs equal their respective priors. For all $k \in [1, 8/3)$, a global bidder with valuation $v \in (0, k]$ is strictly better off competing against another global bidder in either of $[A][B]$, $[AB]$ or $[A,B]$, than competing against all four local bidders in $[A][B]$. For $v = 0$ and for $k = 8/3 = v$, he is indifferent.*

Proof of Supporting Lemma 1. A global bidder with a valuation of v competing against another global bidder in either of $[A][B]$, $[AB]$ or $[A,B]$, receives an expected revenue of $v^2/(2k)$ (see Lemma 2). A global bidder with a valuation of v competing against all four local bidders in $[A][B]$ receives an expected payoff of $v - 4/3$ if $v > 4/3$, and otherwise an expected payoff of 0 by Lemma 1. For $v = 0$ and $v = 8/3 = k$ he is indifferent, for $0 < v \leq 4/3$ we have $v^2/(2k) > 0$, and for $k \geq v > 4/3$ we have $v^2/(2k) > v - 4/3$ if $k < 8/3$. \square

Supporting Lemma 2. *Suppose that bidders' beliefs equal their respective priors. Consider $[AB]$ with n local bidders and m global bidders. Then the expected payoff of a local bidder with valuation v is $\frac{v^{n+m}}{(n+m)k^m}$ and the expected payoff of a global bidder with a valuation $v \leq 1$ is $\frac{v^{n+m}}{(n+m)k^{m-1}}$.*

Proof of Supporting Lemma 2. First, consider a local bidder with a valuation of v . Using the density of the first order statistic of m independent draws from $U[0, k]$ and $n - 1$ independent draws from $U[0, 1]$ which is $f_l(x) := (n - 1 + m) \frac{x^{n+m-2}}{k^m}$, we have that his expected payoff is $\int_0^v (v - x) f_l(x) dx = \frac{v^{n+m}}{(n+m)k^m}$. Second, consider a global bidder with a valuation of $v \leq 1$. Using the density of the first order statistic of $m - 1$ independent draws from $U[0, k]$ and n independent draws from $U[0, 1]$ which is $f_g(x) := (n - 1 + m) \frac{x^{n+m-2}}{k^{m-1}}$, we have that his expected payoff is $\int_0^v (v - x) f_g(x) dx = \frac{v^{n+m}}{(n+m)k^{m-1}}$. \square

Proof of Lemma 3. As we consider only equilibria with P1, P2, and P3, the belief of every bidder on the equilibrium path has to be equal to his respective prior. Based on the number of global bidders in the first [A][B] we distinguish two cases:

- Assume that in an equilibrium (with properties P1, P2, and P3) there are two global bidders in the first [A][B].

If in such an equilibrium there is only one local bidder in the first [A][B], this local bidder competes with two global bidders who bid their valuations and wants to deviate to the second [A][B] to compete only with one local bidder of his own kind.

If in such an equilibrium both A-bidders participate in the first [A][B] and both B-bidders participate in the second [A][B], then an A-bidder would want to deviate to obtain the item for free in the second [A][B].

If there is exactly one A- and one B-bidder in the first [A][B], at least one global bidder with a valuation $v > 1$ is better off deviating because he faces strictly less competition in the second [A][B].

If there are three or four local bidders in the first [A][B], at least one local bidder does not face any competition in the second [A][B] and deviates to obtain the item for free.

It is possible in equilibrium that all local bidders are in the second [A][B] because no local bidder has an incentive to deviate to the first [A][B], where he would compete against two global bidders who are stronger than one local bidder because $k > 1$ (see Lemma 2), whereas in [A][B] he competes against one local bidder. According to Supporting Lemma 1, a global bidder does not want to deviate either.

- Assume that in an equilibrium there is one global bidder in each [A][B].

There exists an equilibrium in which exactly one of each kind of local bidder (A and B) is in each [A][B] because a deviating bidder would face a more competitive situation.

If there is at most one local bidder in one of the [A][B], the global bidder with a low valuation in the other [A][B] has zero payoff due to the exposure problem and enters to have a positive expected payoff (by P3 and Lemma 2) (and therefore this cannot be an equilibrium).

If both A-bidders are in the same [A][B] and both B-bidders are in the other [A][B], a local bidder deviates even if the global bidder in the auction that he enters places his most aggressive bid v (by P2(d), see page 19) on his desired item, because before the deviation he faces a local and a global bidder who bid their valuations, and after the deviation he faces only a global bidder who bids his valuation.

In sum, the only allocations of bidders to auctions that are consistent with equilibrium with properties P1, P2, and P3 are that either exactly one of each kind of bidder (A, B, and global)

is in each $[A][B]$, or all local bidders are in one $[A][B]$ and all global bidders are in the other $[A][B]$. If one of each kind of bidder (A, B, and global) is in each $[A][B]$, the expected revenue (for each auctioneer) is $1/k \cdot 0 + (1 - 1/k)2 \int_0^1 x dx = (k - 1)/k$. If all local bidders are in one $[A][B]$ and all global bidders are in the other $[A][B]$ the expected revenue of the auctioneer with the local bidders is $2 \int_0^1 \int_y^1 x dx dy = 2/3$ and the expected revenue of the auctioneer with the global bidders is $\int_0^k \int_y^k x/k^2 dx dy = k/3$. \square

Proof of Lemma 4. Based on the number of global bidders in $[AB]$ we distinguish three cases:

- Assume that there exists an equilibrium (with properties P1, P2 and P3) in which there are two global bidders in $[AB]$.

If in such an equilibrium there is a local bidder in $[AB]$, this local bidder competes with at least two global bidders who bid their valuations and deviates to compete against at most one local bidder of his kind.

Inversely, if all local bidders are in $[A][B]$, then no local bidder has an incentive to deviate to $[AB]$ and by Supporting Lemma 1 no global bidder has an incentive to deviate from $[AB]$.

- Assume there is exactly one global bidder in $[AB]$.

If there is at least one A- and one B-bidder in $[A][B]$, a global bidder in $[A][B]$ with $0 < v < 1$ receives an expected payoff of zero (by Lemma 1) and deviates to receive a strictly positive expected payoff in $[AB]$.

If in $[A][B]$ there are no local bidders for one of the items but there are two local bidders for the other item, a local bidder in $[AB]$ deviates even if the global bidder in the single-item auction that he enters places his most aggressive bid v (by P2(d), see page 19), because before the deviation he faces a local and a global bidder who bid their valuations, and after the deviation he faces only a global bidder who bids his valuation.

If in $[A][B]$ there are no local bidders for one of the items (say for B) but there is one local bidder for the other item, then the A-bidder in $[AB]$ deviates to $[A][B]$ to face less competition (by P3 and Lemma 2).

If there is no local bidder in $[A][B]$, any local bidder competes with one global and three local bidders in $[AB]$ and deviates to compete only with one global bidder who bids his valuation (see Lemma 2).

- Assume there is no global bidder in $[AB]$.

If there is a local bidder for each of the two items in $[A][B]$, a global bidder with low valuation receives an expected payoff of zero (by Lemma 1) and deviates to $[AB]$ to receive a strictly positive payoff.

If there are two local bidders for one of the items in $[A][B]$ and no local bidder for the other item, a global bidder deviates to $[AB]$ to face less competition and to increase his

payoff from $v^4/(4k)$ (see Lemma 2) to $v^3/3$ (see Supporting Lemma 2).

If there is exactly one local bidder in $[A][B]$ and $k > 2/\sqrt{3} \approx 1.15$, then the local bidder deviates, but if $k \leq 2/\sqrt{3}$, then there are off-the-equilibrium-path beliefs and corresponding bids, such that no bidder wants to deviate. The local bidder in $[A][B]$, say this is an A-bidder, has the payoff $v^3/(3k^2)$ (see Lemma 2). If he deviates, he gets $v^4/4$ (see Supporting Lemma 2). Deviating is profitable for $v > 4/(3k^2)$. Thus, the valuation for which a bidder is most prone to deviating is $v = 1$, and he deviates if and only if $k > 2/\sqrt{3}$. A global bidder has the payoff $v^3/(3k)$ if $0 \leq v \leq 1$ and $(3v^2 - 1)/(6k)$ if $v > 1$ (see Lemma 2). If he deviates to $[AB]$, his payoff is $v^4/4$ if $0 \leq v \leq 1$ and $(4v - 3)/4$ if $1 \leq v \leq k$ and deviating does not pay for $k \leq 2/\sqrt{3}$. The A-bidder in $[AB]$ will not deviate to $[A][B]$ because there he faces more competition (by P3 and Lemma 2). The B-bidders will not deviate to $[A][B]$ if a global bidder then bids most aggressively, i.e., his valuation by P2(d), page 19, in $[B]$. The global bidder may do so if his beliefs about the A-bidder and the other global bidder in $[A][B]$ are that their valuation (and bid) is zero with probability one.

If there is no local bidder in $[A][B]$, a local bidder with valuation v deviates if $v^4/4 < v^3/(3k^2)$, that is, if $0 < v < 4/(3k^2)$. (For the payoffs see Supporting Lemma 2.)

If in equilibrium all local bidders are in $[A][B]$ and all global bidders are in $[AB]$ then the expected revenue of the auctioneer offering $[A][B]$ is $2/3$ (as in Lemma 3) and the expected revenue of the auctioneer offering $[AB]$ is $\int_0^k \int_y^k x/k^2 dx dy = k/3$. This is the only allocation of bidders to auctions consistent with equilibrium with properties P1, P2, and P3 if $k > 2/\sqrt{3}$.

If $k \leq 2/\sqrt{3}$ and two global bidders and one local bidder participate in $[A][B]$ and the remaining local bidders participate in $[AB]$ then the expected revenue of the auctioneer with $[A][B]$ is $(2k^3 + 2k - 1)/(6k^2)$ and the expected revenue of the auctioneer with $[AB]$ is $1/2$. This is the second allocation of bidders to auctions consistent with equilibrium with properties P1, P2, and P3 if $k \leq 2/\sqrt{3}$.

All other allocations of bidders to auctions are not consistent with an equilibrium that fulfills P1, P2, and P3. \square

Proof of Lemma 5. Based on the number of global bidders in $[A,B]$ we distinguish three cases:

- Assume that in an equilibrium (with properties P1, P2, and P3) there are two global bidders in $[A,B]$.

We first show that if there is exactly one local bidder for each item in $[A][B]$ than no bidder has an incentive to deviate if and only if $k \leq k^\circ$ where k° is the positive solution to $3k^{\circ 4} - 16k^{\circ 3} + 36k^{\circ 2} - 24k^\circ - 2 = 0$ (we have that $k^\circ \approx 1.23$). A local bidder with a valuation of v who deviates from $[A,B]$ receives an expected payoff of $v^2/2$ by Supporting Lemma 2. If $k \geq 2$, a local bidder with valuation $v = 1$ would deviate because for him

$\int_0^1 \left[\int_0^{b+v} v \frac{2c}{k^2} dc - \int_b^{b+v} (c-b) \frac{2c}{k^2} dc \right] db = (2v^3 + 3v^2 + 2v)/(6k^2) < v^2/2$. If $k \in (1, 2)$, his expected payoff from not deviating is $(2v^3 + 3v^2 + 2v)/(6k^2)$ if $v \in [0, k-1]$ and is $\int_0^{k-v} \int_b^{v+b} (v-c+b) \frac{2c}{k^2} dc db + \int_{k-v}^1 \int_b^k (v-c+b) \frac{2c}{k^2} dc db + \int_0^1 \int_0^b v \frac{2c}{k^2} dc db = (-v^4 + 6k^2v^2 - (8k^3 - 12k^2)v + 3k^4 - 8k^3 + 6k^2 - 1)/(12k^2)$ if $v \in [k-1, 1]$. The maximum gain from deviating, that is, the maximum difference in expected payoffs, if $v \in [0, k-1]$ is achieved at $v = k-1$ and is strictly positive for all $k > 1.59$, the solution to $3k^4 - 8k^3 + 6k^2 - 2k + 1 = 0$ for $k \in (1, 2)$. The maximum difference if $v \in [k-1, 1]$ and $k \geq 3/2$ is achieved at $v = 1$ and is strictly positive. The maximum difference if $v \in [k-1, 1]$ and $k < 3/2$ is at $v = k-1$ or $v = 1$ and is strictly positive for $k > 1.48$ (the solution to $3k^4 - 16k^3 + 18k^2 - 2 = 0$ for $k \in (1, 3/2)$). A local bidder with $v = 1$ would deviate for all $k > 1.48$. A global bidder with valuation $v \in [1, k]$ would deviate if $(v-1) - (-3v^4 + 16v^3 - 12v^2 + 2)/(24k) > 0$. The left hand side expression takes its maximum at $v = k$ and is positive for $v = k$ for $k > k^\circ$ with $k^\circ \approx 1.23$, the positive solution to $3k^{\circ 4} - 16k^{\circ 3} + 36k^{\circ 2} - 24k^\circ - 2 = 0$. In addition, local bidders in [A][B] have no incentive to deviate as they obtain the object for free.

If there are exactly three local bidders in [A][B], the local bidder in [A,B] competes with two global bidders who bid their valuations and would deviate to [A][B] to compete against at most one local bidder of his kind.

If there is no A-bidder, no B-bidder, or no local bidder at all in [A][B], then there is a local bidder who faces competition in [A,B] and deviates to [A][B] to get the item for free.

If all local bidders are in [A][B], then no local bidder has an incentive to deviate to [A,B] and by Supporting Lemma 1 no global bidder has an incentive to deviate from [A,B].

- Assume that in an equilibrium there is one global bidder in [A,B].

If all local bidders are in [A,B], a local bidder with a valuation $v < \min\{1/2, k-1\}$ expects to get $\int_0^1 \int_0^v \int_0^{v+b} v \frac{1}{k} dc da 2b db - \int_0^1 \int_0^v \left[\int_0^{a+b} a \frac{1}{k} dc + \int_{a+b}^{v+b} (c-b) \frac{1}{k} dc \right] da 2b db = v^2(v+1)/(3k)$, where b denotes the highest bid by a local bidder interested in the other item. He deviates profitably to [A][B] to receive an expected payoff of $v^2/(2k) > v^2(v+1)/(3k)$ (by Lemma 1).

If there is at least one A-bidder and at least one B-bidder in [A][B], a global bidder in [A][B] with a valuation below 1 receives an expected payoff of zero (by Lemma 1) and deviates to [A,B] to receive a strictly positive payoff.

If there is exactly one local bidder in [A][B], then the global bidder in [A,B] with $v \leq 1$ has the payoff $v^4/12$ and profitably deviates to [A][B] to get $v^3/(3k)$ (by P3 and Lemma 2).

If there are two A-bidders or two B-bidders in [A][B], then a local bidder in [A,B] faces a local bidder of his kind and a global bidder who bid their valuations and our local bidder profitably deviates to [A][B] even if the global bidder places his most aggressive bid v (by

P2(d), see page 19) in the auction that our local bidder enters.

- Assume that in an equilibrium there is no global bidder in [A,B].

If there are one or more local bidders of each kind in [A][B], a global bidder with $0 < v < 1$ has a payoff of zero and deviates to [A,B] to get a strictly positive expected payoff.

If there is only one local bidder in [A][B], this local bidder competes with two global bidders who bid their valuations (by Lemma 2) and deviates to [A,B] to compete only with one local bidder of his kind.

If there are two local bidders of the same kind in [A][B], then each of them has an incentive to deviate to [A,B] and receive the item for free, being the only local bidder of his kind in [A,B].

It is possible in equilibrium that there are no local bidders in [A][B]. Then a local bidder in [A,B] receives $v^2/2$ and has no incentive to deviate, since in [A][B] he would by Lemma 2 receive $v^3/(3k^2) \leq v^2/2$ because $0 \leq v \leq 1$ and $k \geq 1$. A global bidder in [A][B] receives $v^2/(2k)$ (see Lemma 2). If his valuation $v < 1$ he will not want to deviate because in [A,B] he would receive $\int_0^v \int_0^{v-a} (v-a-b) 2a 2b db da = v^5/30$, which is smaller than $v^2/(2k)$, as $k \leq 8/3$. If $1 \leq v \leq 2$, he will not deviate because in [A,B] he receives $\int_0^{v-1} \int_0^1 (v-a-b) 2a 2b db da + \int_{v-1}^1 \int_0^{v-a} (v-a-b) 2a 2b db da = -v^5/30 + 2v^3/3 - 4v^2/3 + v - 4/15 < v^2/(2k)$ for $k \leq 8/3$ and $1 \leq v \leq \min\{k, 2\}$. If $v > 2$, then a global bidder would receive $v - 4/3$ by deviating to [A,B] and it holds that $v - 4/3 \leq v^2/(2k)$ for $2 < k \leq 8/3$ and $2 < v \leq k$.

For $k \in (1, 8/3]$ it is consistent with an equilibrium with properties P1, P2, and P3 that all global bidders are in [A,B] and all local bidders are in [A][B] or that all local bidders are in [A,B] and all global bidders are in [A][B]. In the former case the expected revenue of the auctioneer with [A,B] is $\int_0^k \int_y^k x/k^2 dx dy = k/3$ and that of the auctioneer with [A][B] is $2 \int_0^1 \int_y^1 x dx dy = 2/3$. In the latter case the expected revenue of the auctioneer with [A,B] is $2/3$ (equivalent to all locals in [A][B]) and that of the auctioneer with [A][B] is $\int_0^k \int_y^k x/k^2 dx dy = k/3$.

If $k \leq k^\circ$ where k° is the positive solution to $3k^{\circ 4} - 16k^{\circ 3} + 36k^{\circ 2} - 24k^\circ - 2 = 0$, it is also possible in equilibrium that both global and exactly one A-bidder and one B-bidder are in [A,B] and the remaining A-bidder and B-bidder are in [A][B]. Then the expected revenue of the auctioneer with [A][B] is zero and that of the auctioneer with [A,B] is $(7k^5 - 50k^4 + 140k^3 - 80k^2 - 10k + 16)/(60k^2)$.

All other allocations of bidders to auctions are not consistent with an equilibrium that fulfills P1, P2, and P3. □

Proof of Lemma 6. Based on the number of global bidders in the first auctioneer's [AB] two cases are relevant:

- Assume that in equilibrium there are two global bidders in the first [AB].

If there is more than one local bidder in the first [AB], such a local bidder competes with two global and at least one local bidder who all bid their valuations and profitably deviates to the second [AB] to compete with at most two local bidders who bid their valuations.

It is possible in equilibrium that there is one local bidder in the first [AB], if $k \leq 2/\sqrt{3} \approx 1.15$, but if $k > 2/\sqrt{3}$ there cannot be one local bidder in the first [AB]. This bidder with valuation v earns $v^3/(3k^2)$ (Supporting Lemma 2), which is less than his expected payoff from deviating to the second [AB] if $v^4/4 > v^3/(3k^2)$. The valuation for which this deviation would be most profitable is $v = 1$. A bidder with this valuation deviates if and only if $k > 2/\sqrt{3}$. A global bidder with $v \leq 1$ deviates if $v^4/4 > v^3/(3k)$. Then, $v^4/4 > v^3/(3k^2)$, and the local bidder also deviates. A global bidder with $v > 1$ deviates if $v - 3/4 > (3v^2 - 1)/(6k)$ or $6kv - 9k/2 - 3v^2 + 1 > 0$. $6kv - 9k/2 - 3v^2 + 1$ takes its maximum at $v = k$ and is positive at $v = k$ for $k > (9 + \sqrt{33})/12 \approx 1.22 > 2/\sqrt{3}$. Thus, for $k > (9 + \sqrt{33})/12$ the global bidder with $v = k$ deviates and for $k \leq (9 + \sqrt{33})/12$ no global bidder with $v \in [1, k]$ deviates.

If there is no local bidder in the first [AB], a local bidder with a valuation $v < 4/(3k^2)$ in the second [AB] earns $v^4/4$ and profitably deviates to the first [AB] to earn $v^3/(3k^2)$.

- Assume in equilibrium there is a global bidder in each [AB].

No bidder deviates if there are two local bidders in each auction, because a deviating local bidder faces more local and the same number of global competitors and a deviating global bidder faces the same number of local bidders but faces a global competitor in addition. To the contrary, if there are more local bidders in one auction than in the other, then local bidders can profitably deviate to the auction with fewer local bidders.

Depending on k , there are two equilibrium allocations of bidders to auctions that are consistent with P1, P2, and P3. If $k \leq 2/\sqrt{3}$, one local and two global bidders are in one [AB] and three local bidders are in the other [AB]. For all k , one global and two local bidders participate in each [AB]. The expected revenue in an auction with one local and two global bidders is $1/k^2(2 \int_1^k \int_1^c \int_0^1 b \text{ dadbdc} + 4 \int_1^k \int_0^1 \int_0^b b \text{ dadbdc} + 6 \int_0^1 \int_0^c \int_b^c a \text{ dadbdc}) = (2k^3 + 2k - 1)/(6k^2)$, with three local bidders it is $6 \int_0^1 \int_0^c \int_b^c a \text{ dadbdc} = 1/2$, and with one global and two local bidders it is $1/k(2 \int_1^k \int_0^1 \int_0^b b \text{ dadbdc} + 6 \int_0^1 \int_0^c \int_b^c a \text{ dadbdc}) = (4k - 1)/(6k)$. All other allocations of bidders to auctions are not consistent with an equilibrium that fulfills P1, P2, and P3. \square

Proof of Lemma 7. Distinguish three cases, depending on the number of global bidders in [A,B].

- Assume that in equilibrium there are two global bidders in [A,B].

If there is at least one A-bidder and one B-bidder in [A,B], the expected payoff of the global bidder with $v \leq 1$ is at most $v^4/(8k)$ (the expected payoff of a global bidder with

$v \leq 1$ if there is exactly one A- and one B-bidder in $[A,B]$. A global bidder with $0 < v \leq 1$ profitably deviates to receive at least $v^3/3$ (the expected payoff of a global bidder with $v \leq 1$ in $[AB]$ if the only other bidders in $[AB]$ are one A- and one B-bidder).

If there is one local bidder in $[A,B]$, say an A-bidder, and no local bidder for the other item, a B-bidder with $v \in (0, k-1]$ in $[AB]$ profitably deviates to $[A,B]$ if $v^3/3 < (2v^3 + 3v^2 + 2v)/(6k^2) = \int_0^1 \int_a^{a+v} (v-c+a) \cdot 2c/k^2 \, dc \, da + \int_0^1 \int_0^a v \cdot 2c/k^2 \, dc \, da$, or $0 < -2(k^2-1)v^2 + 3v + 2$. This condition holds for $k > 1$ if $0 < v < (3 + \sqrt{16k^2 - 7})/(4(k^2 - 1))$, and thus, a B-bidder with low valuation $v \in (0, \min\{k-1, (3 + \sqrt{16k^2 - 7})/(4(k^2 - 1))\})$ deviates to $[A,B]$.

If there are two local bidders for the same item and no local bidder for the other item in $[A,B]$, a global bidder deviates to $[AB]$ to face less competition.

If all local bidders are in $[AB]$, the calculations in the proof of Lemma 6 apply and show that a local bidder would deviate if $v < 4/(3k^2)$.

- Assume that in equilibrium there is one global bidder in $[A,B]$.

If there are three or four local bidders in $[A,B]$, the global bidder with $v \in (0, 1]$ in $[A,B]$ receives an expected payoff of at most $v^4/12$ (his expected payoff if there are exactly three local bidders). He profitably deviates to $[AB]$ if $v \in (0, 4/k)$ because he obtains at least $v^3/3k$ (his payoff if he is in $[AB]$ with one more global and one local bidder).

If there are at most two local bidders in $[A,B]$, a local bidder in $[AB]$ deviates. The case least prone to such a deviation is when there is an A- and a B-bidder in $[A,B]$. The A-bidder with $v < \min\{1, k-1\}$ in $[AB]$ obtains $v^3/(3k)$, whereas in $[A,B]$ he would get $(4v^3 + 3v^2)/(12k)$, which is strictly larger for all $v \in (0, \min\{1, k-1\})$.

- Assume that in equilibrium there is no global bidder in $[A,B]$.

If all local bidders are in $[A,B]$, no bidder wants to deviate. For the global bidders this follows from the corresponding argument in the proof of Lemma 5, and for the local bidders this follows from the argument in the proof of Lemma 4.

If there are one or more local bidders in $[AB]$, such a local bidder deviates if $0 < v < 3k^2/2$ because in $[AB]$ he receives at most $v^3/(3k^2)$ whereas if he deviates he obtains at least $v^2/2$. The only equilibrium allocation of bidders to auctions that is consistent with P1, P2, and P3 is that all local bidders are in $[A,B]$ and all global bidders are in $[AB]$. The expected revenue of the auctioneer with $[A,B]$ is $2/3$ and that of the auctioneer with $[AB]$ is $k/3$. \square

Proof of Lemma 8. We consider two cases, depending on whether in equilibrium both global bidders are in the same $[A,B]$ or not.

- Assume that there are two global bidders in the (w.l.o.g.) first auctioneer's $[A,B]$.

If no local bidder is in the first $[A,B]$, no bidder would want to deviate for the same reasons as in the proof of Lemma 7.

If there is exactly one local bidder in the first [A,B], he deviates to increase his expected payoff from $v^3/(3k^2)$ to $v^2/2$ if $v \in (0, 1]$.

If there are exactly one A-bidder and one B-bidder in the first [A,B], a global bidder with $v \in (0, k]$ profitably deviates to the auction with less competition.

If there are at least two local bidders and at least two of the same kind in the first [A,B], then a local bidder profitably deviates to the second [A,B] to get the item for free.

- Assume that there is one global bidder in each [A,B].

If there is one A-bidder and one B-bidder in each [A,B] then no bidder would deviate as he would face tougher competition.

If there are two local bidders for the same item (w.l.o.g. A-bidders) and zero or one for the other item in the first [A,B], then an A-bidder profitably deviates to the second [A,B] to face less competition and to complement with at least as many B-bidders.

If all local bidders are in the same [A,B], a global bidder with $v \in (0, 1]$ who is with the local bidders in this [A,B] deviates and increases his expected payoff from $v^5/30$ to $v^2/(2k)$. The only equilibrium allocations of bidders to auctions that are consistent with P1, P2, and P3 are that either all local bidders are in one [A,B] and all global bidders are in the other [A,B], or exactly one of each kind of bidder (A, B, and global) is in each [A,B]. The expected revenue from four local bidders is $2/3$ and that from two global bidders is $k/3$ as shown before. The expected revenue from an A-, a B-, and a global bidder is, if $k \geq 2$, $(6k - 5)/(6k)$ and, if $k \in (1, 2)$, $(k^4 - 8k^3 + 24k^2 - 20k + 6)/(12k)$ (calculations are as in the proof of Theorem 1). \square

B.2.3 Stage 1: Auctioneers' Auction-Format Choices

We can now derive the auctioneers' equilibrium choices. Bidders' participation decisions and auctioneers' expected revenues conditional on the auctioneers' choices (σ_1, σ_2) are given by lemmas 3 to 8. We need to show that no auctioneer has an incentive to unilaterally deviate from the equilibrium decision stated in Proposition 1 and that for any other combination of choices $\sigma = (\sigma_1, \sigma_2)$ an auctioneer can profitably deviate.

We will prove case $2 < k \leq 8/3$ of Proposition 1 first. Then, we will prove case $k \leq 2$ in subcases (a) $k \in (2/\sqrt{3}, 2]$ and (b) $k \in (1, 2/\sqrt{3}]$. We will make use of the following inequalities that hold for $k \in (1, 8/3]$: $\frac{k-1}{k} < \frac{k}{3}$, $\frac{k-1}{k} < \frac{2}{3}$, $\frac{4k-1}{6k} < \frac{2}{3}$, and $\frac{6k-5}{6k} < \frac{k}{3}$.

If $k \in (2, 8/3]$, Auctioneer 1's and Auctioneer 2's expected revenues given their choices (σ_1, σ_2) are taken from Lemmas 3 to 8 and summarized in the following table. Multiple entries in a cell for a given (σ_1, σ_2) denote that different expected revenues are possible because the subgame following the auctioneers' choices can have multiple equilibria. The statement follows from comparing, for each pair of revenues in the various cells in the table, an auctioneer's

revenue with the maximum of the respective minimum revenue from the two alternate cells he could reach by a unilateral deviation.

$k \in (2, 8/3]$:

| | | Auctioneer 2 | | | |
|--------------|--------|-------------------------------------|---|---------------------------------------|---------------------------------------|
| | | $\sigma_1 \setminus \sigma_2$ | [A][B] | [AB] | [A,B] |
| Auctioneer 1 | [A][B] | | $(\frac{k-1}{k}, \frac{k-1}{k})^{*,**}$ | $(\frac{2}{3}, \frac{k}{3})$ | $(\frac{k}{3}, \frac{2}{3})^{**}$ |
| | | | $(\frac{k}{3}, \frac{2}{3})^{**}$ | | $(\frac{2}{3}, \frac{k}{3})^{*,(**)}$ |
| | | | $(\frac{2}{3}, \frac{k}{3})^*$ | | |
| | [AB] | $(\frac{k}{3}, \frac{2}{3})$ | $(\frac{4k-1}{6k}, \frac{4k-1}{6k})^{*,**}$ | $(\frac{k}{3}, \frac{2}{3})^{(**)}$ | |
| | [A,B] | $(\frac{2}{3}, \frac{k}{3})^*$ | $(\frac{2}{3}, \frac{k}{3})^{(*)}$ | $(\frac{2}{3}, \frac{k}{3})^{*,(**)}$ | |
| | | $(\frac{k}{3}, \frac{2}{3})^{*,**}$ | | $(\frac{k}{3}, \frac{2}{3})^{*,**}$ | |
| | | | $(\frac{6k-5}{6k}, \frac{6k-5}{6k})^{*,**}$ | | |

*/** Auctioneer 1/2 can profitably deviate

(*)/(**) Auctioneer 1/2 prefers a less complex, revenue-equivalent auction

If and only if $\sigma = ([A][B], [AB])$ or $\sigma = ([AB], [A][B])$ no auctioneer can profitably deviate. Unilaterally deviating from $\sigma = ([A][B], [AB])$ or $\sigma = ([AB], [A][B])$ is not profitable because $\frac{k}{3} \geq \max\{\min\{\frac{k-1}{k}, \frac{2}{3}, \frac{k}{3}\}, \min\{\frac{2}{3}, \frac{k}{3}\}\} = \max\{\frac{k-1}{k}, \frac{2}{3}\}$ and $\frac{2}{3} \geq \max\{\frac{4k-1}{6k}, \frac{2}{3}\}$ (and deviating is not profitable independently of the equilibria of the subgames following an auction-format choice). If $\sigma = ([A][B], [A][B])$, at least one auctioneer (with a revenue of $\frac{k-1}{k}$ or $\frac{2}{3}$) can profitably deviate to [AB] to get a revenue of $\frac{k}{3}$. If $\sigma \in \{([A][B], [A,B]), ([A,B], [A][B]), ([A,B], [A,B])\}$, an auctioneer who offers [A,B] can profitably deviate to [AB]. If $\sigma \in \{([AB], [A,B]), ([A,B], [AB])\}$, an auctioneer who offers [A,B] will by tie breaking deviate to [A][B]. If $\sigma = ([AB], [AB])$, an auctioneer can profitably deviate to [A][B].

Consider $k \in (1, 2]$. (a) If $k \in (2/\sqrt{3}, 2]$, the auctioneers' expected revenues given (σ_1, σ_2) are taken from lemmas 3 to 8 and summarized in the following table. For the sake of clarity, let $Y := \frac{7k^5 - 50k^4 + 140k^3 - 80k^2 - 10k + 16}{60k^2}$ and $Z := \frac{k^4 - 8k^3 + 24k^2 - 20k + 6}{12k}$. We have that $\frac{k}{3} < Y < \frac{2}{3}$ for all $1 < k \leq k^\circ \approx 1.23$ and $Z < \frac{k}{3}$ for all $k \in (1, 2]$.

$k \in (2/\sqrt{3}, 2]$:

| | | Auctioneer 2 | | | |
|--------------|--------|-------------------------------|---|---|--|
| | | $\sigma_1 \setminus \sigma_2$ | [A][B] | [AB] | [A,B] |
| Auctioneer 1 | [A][B] | | $(\frac{k-1}{k}, \frac{k-1}{k})^{*,**}$ $(\frac{k}{3}, \frac{2}{3})$ $(\frac{2}{3}, \frac{k}{3})$ | $(\frac{2}{3}, \frac{k}{3})$ | $(\frac{k}{3}, \frac{2}{3})^{(**)}$ for $k=2$ $(\frac{2}{3}, \frac{k}{3})^{(**)}$ $(0, Y)^{\otimes,*}$ |
| | [AB] | | $(\frac{k}{3}, \frac{2}{3})$ | $(\frac{4k-1}{6k}, \frac{4k-1}{6k})^{*,**}$ | $(\frac{k}{3}, \frac{2}{3})^{(**)}$ |
| | [A,B] | | $(\frac{2}{3}, \frac{k}{3})^{(*)}$ for $k=2$ $(\frac{k}{3}, \frac{2}{3})^{(*)}$ $(Y, 0)^{\otimes,**}$ | $(\frac{2}{3}, \frac{k}{3})^{(*)}$ | $(\frac{2}{3}, \frac{k}{3})^{(**)}$ $(\frac{k}{3}, \frac{2}{3})^{(*)}$ $(Z, Z)^{\otimes\otimes,*,**}$ |

\otimes subgame revenues possible only for $k \leq k^\circ \approx 1.23$, $\otimes\otimes$ subgame revenues not possible for $k = 2$

$*/**$ Auctioneer 1/2 can profitably deviate

$(*)/(**)$ Auctioneer 1/2 prefers a less complex, revenue-equivalent auction

No auctioneer can profitably deviate (for appropriate equilibria of the subgames following the auction-format choices) if $\sigma \in \{([A][B],[A][B]),([A][B],[AB]),([AB],[A][B]),([A][B],[A,B]),([A,B],[A][B])\}$ with the following exception. If $\sigma \in \{([A][B],[A,B]),([A,B],[A][B])\}$ and $k = 2$, the auctioneer offering [A,B] will by tie-breaking deviate to [AB]. If $\sigma = ([AB],[AB])$, deviating to [A][B] is profitable. If $\sigma \in \{([AB],[A,B]),([A,B],[AB])\}$ the auctioneer who offers [A,B] will by tie-breaking deviate to [A][B]. If $\sigma = ([A,B],[A,B])$, at least one auctioneer will by tie-breaking deviate to [AB].

(b) If $k \in (1, 2/\sqrt{3}]$, the auctioneers' expected revenues given (σ_1, σ_2) are taken from lemmas 3 to 8 and summarized in the following table, with Y and Z as defined in case (ii.a). We have that $\frac{k-1}{k} < \frac{k}{3} < \frac{1}{2} < \frac{4k-1}{6k} < \frac{2k^3+2k-1}{6k^2} < \frac{2}{3}$ for all $k \in (1, \frac{2}{\sqrt{3}}]$.

$k \in (1, 2/\sqrt{3}]$:

| | | Auctioneer 2 | | | |
|--------------|--------|-------------------------------|---|--|--|
| | | $\sigma_1 \setminus \sigma_2$ | [A][B] | [AB] | [A,B] |
| Auctioneer 1 | [A][B] | | $(\frac{k-1}{k}, \frac{k-1}{k})^{*,**}$ $(\frac{k}{3}, \frac{2}{3})$ $(\frac{2}{3}, \frac{k}{3})$ | $(\frac{2}{3}, \frac{k}{3})$ $(\frac{2k^3+2k-1}{6k^2}, \frac{1}{2})^*$ | $(\frac{k}{3}, \frac{2}{3})$ $(\frac{2}{3}, \frac{k}{3})^{(**)}$ $(0, Y)^*$ |
| | [AB] | | $(\frac{k}{3}, \frac{2}{3})$ $(\frac{1}{2}, \frac{2k^3+2k-1}{6k^2})^{**}$ | $(\frac{4k-1}{6k}, \frac{4k-1}{6k})^{*,**}$ $(\frac{1}{2}, \frac{2k^3+2k-1}{6k^2})^*$ $(\frac{2k^3+2k-1}{6k^2}, \frac{1}{2})^{**}$ | $(\frac{k}{3}, \frac{2}{3})$ |
| | [A,B] | | $(\frac{2}{3}, \frac{k}{3})$ $(\frac{k}{3}, \frac{2}{3})^{(*)}$ $(Y, 0)^{**}$ | $(\frac{2}{3}, \frac{k}{3})$ | $(\frac{2}{3}, \frac{k}{3})^{(**)}$ $(\frac{k}{3}, \frac{2}{3})^{(*)}$ $(Z, Z)^{*,**}$ |

$*/**$ Auctioneer 1/2 can profitably deviate

$(*)/(**)$ Auctioneer 1/2 prefers a less complex, revenue-equivalent auction

No auctioneer can profitably deviate (for appropriate equilibria of the subgames following the auction-format choices) if and only if $\sigma \notin \{([AB],[AB]),([A,B],[A,B])\}$. If $\sigma =([AB],[AB])$, at least one auctioneer can profitably deviate to $[A][B]$. If $\sigma =([A,B],[A,B])$, at least one auctioneer can profitably deviate to $[AB]$.

B.2.4 Allowing for Other Auction Formats

The results in Proposition 1 remain valid if auctioneers are allowed to restrict bidding to any possible set of packages.

Lemma 9. *Allowing bids only on A and AB or only on B and AB is equivalent for bidders and auctioneers to offering [AB]. If auctioneers can decide to accept bids only on A or only on B, then they will not make use of this in an equilibrium with properties P1, P2, and P3 for $k \in (1, 8/3]$.*

Proof of Lemma 9. If bids are allowed only on packages A and AB or on B and AB, only the higher of each bidder's two bids can win and influence payments. The auction format is revenue- and payoff- equivalent to [AB] when each bidder submits only the higher of these two bids.

An auctioneer who allows only bids on A or only bids on B would be better off if he offered [AB] instead. To see this consider w.l.o.g. an auctioneer who allows only bids on A. B-bidders and global bidders will not participate in his auction because they cannot make a positive profit. This auctioneer's expected profit is at most $1/3$ (which occurs if both A-bidders participate in his auction). If the other auctioneer allows only bids on A or only bids on B, our auctioneer can profitably deviate to [AB], thereby attracting two local and both global bidders and strictly increasing his expected revenue. If the other auctioneer offers $[A][B]$, [AB], or [A,B] we can see from the three tables in the proof of Proposition 1 that for all $k \leq 8/3$ our auctioneer can profitably deviate to [AB] to earn $k/3$, $1/2$, $(4k - 1)/(6k)$, or $(2k^3 + 2k - 1)/(6k^2)$, which all are larger than $1/3$. \square

B.2.5 Proof of Proposition 1(b)

The result follows directly from Proposition 1(a) and lemmas 3, 4, 5, and 7.

B.3 Non-Existence of Equilibria for $k > 8/3$

We will show that subgames in which auctioneers offer different auctions do not have equilibria with all three properties P1, P2, and P3 if $k > 8/3$, hence the three-stage game does not have such equilibria.

All profitable deviations considered in the proofs of lemmas 3 to 8 remain profitable if $k > 8/3$. In addition, if $k > 8/3$ a global bidder with valuation $v = k$ is strictly worse off competing against another global bidder in either of $[A][B]$, $[AB]$, or $[A,B]$, than competing against all four local bidders in $[A][B]$. Thus, he deviates even in the worst case of all four local bidders being in the other auction. A global bidder with a valuation of v competing against another global bidder in either of $[A][B]$, $[AB]$ or $[A,B]$, receives an expected payoff of $v^2/(2k)$ (see Lemma 2). A global bidder with a valuation of $v > 4/3$ competing against all four local bidders in $[A][B]$ receives an expected payoff of $v - 4/3$ by Lemma 1. For $k > 8/3$ and $v \in [8/3, k]$ we have $v^2/(2k) - (v - 4/3) < 0$ because for $v = 8/3$ it is $(8/3)^2/(2k) - (8/3 - 4/3) = 4/3(8/(3k) - 1) < 0$ for all $k > 8/3$ and $(\partial/\partial v)(v^2/(2k) - (v - 4/3)) = v/k - 1 \leq 0$ for all $v \leq k$. Therefore, a global bidder in lemmas 3.b, 4.a, 5.a and 5.b, 7, and 8.b has a profitable deviation. For $k > 8/3$, the remaining cases are lemmas 3.a, 6.a, and 8.a with $\sigma = ([A][B], [A][B])$, $\sigma = ([AB], [AB])$, or $\sigma = ([A,B], [A,B])$ and participation of one A-bidder, one B-bidder, and one global bidder. In these remaining cases, no bidder deviates because he would face additional competition in the other auction. Thus, an equilibrium with properties P1, P2, and P3 exists only in the subgames defined by $\sigma = ([A][B], [A][B])$, $\sigma = ([AB], [AB])$, or $\sigma = ([A,B], [A,B])$.

The previous arguments took only auction formats $[A][B]$, $[AB]$, and $[A,B]$ into account. As argued in the proof of Lemma 9, offering an auction format that allows bids only on packages A and AB, or on B and AB, is equivalent for bidders and the auctioneer to offering $[AB]$. An auctioneer that allows bids only on A (or only on B) has an expected profit of at most $1/3$ (which occurs if both appropriate local bidders participate in his auction). If his opponent allowed bids only on A (or only on B), our auctioneer could profitably deviate to $[AB]$, thereby attracting both global and two local bidders who have zero profit in the opponent's auction. If his opponent offered $[A][B]$, $[AB]$, or $[A,B]$, our auctioneer could profitably deviate to offering the same auction format as the opponent. As shown above, this will attract one A-bidder, one B-bidder, and one global bidder. The resulting expected profits (taken from the proof of Theorem 1) of $(k-1)/k$, $(4k-1)/(6k)$, or $(6k-5)/(6k)$, respectively, exceed $1/3$. The previous comparison of profits implies that no auctioneer would want to deviate from $\sigma = ([A][B], [A][B])$, $\sigma = ([AB], [AB])$, or $\sigma = ([A,B], [A,B])$ to allowing only bids on A or only on B.

C Proof of Theorem 3

The statement holds for $\hat{k} = 2$ and $k^* = 8/3$. For the competing-auctioneers model, the statement is shown in Appendix B.2.3 (the case $2 < k \leq 8/3$). The statement concerning the single-auctioneer model is taken from Theorem 1.