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Journal of Risk and Financial Management

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Suggested Citation: Klebanov, Lev B.; Roozegar, Rasool (2018) : Modified Stieltjes transform and generalized convolutions of probability distributions, Journal of Risk and Financial Management, ISSN 1911-8074, MDPI, Basel, Vol. 11, Iss. 1, pp. 1-6, https://doi.org/10.3390/jrfm11010005

Journal of
Risk and Financial Management

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Modified Stieltjes Transform and Generalized Convolutions of Probability Distributions

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Received: 3 November 2017; Accepted: 11 January 2018; Published: 14 January 2018

Abstract: The classical Stieltjes transform is modified in such a way as to generalize both Stieltjes and Fourier transforms. This transform allows the introduction of new classes of commutative and non-commutative generalized convolutions. A particular case of such a convolution for degenerate distributions appears to be the Wigner semicircle distribution.

Keywords: Stieltjes transform; characteristic function; generalized convolution; beta distribution

1. Introduction

Let us begin with definitions of classical and generalized Stieltjes transforms. Although these are usual transforms given on a set of functions, we will consider more convenient for us a case of probability measures or for cumulative distribution functions. Namely, let μ be a probability measure of Borel subsets of real line $\mathbb{R}^1$. Its Stieltjes transform is defined as[1]

$$S(z) = S(z;\mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x-z},$$

where $\mathtt{Im}(z) \neq 0$. Surely, the integral converges in this case. The generalized Stieltjes transform is represented by

$$S_\gamma(z) = S_\gamma(z;\mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x-z)^\gamma}$$

for real $\gamma > 0$. For more examples of the generalized Stieltjes transforms of some probability distributions, see Demni (2016) and references therein.

A modification of generalized Stieltjes transform was proposed in Roozegar and Bazyari (2017). Our aim in this paper is to use this modification of the Stieltjes transform to define a class of generalized stochastic convolutions and give their probability interpretation (see Theorem 1 below) in lines of preprint Klebanov and Roozegar (2016).

2. Preliminary Results

Now we prefer to switch to the modified form, and define the following form of transform:

$$R_\gamma(u) = R_\gamma(u;\mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1-iux)^\gamma}. \tag{1}$$

[1] Sometimes with opposite sign.

Connection to the generalized Stieltjes transform is obvious. It is convenient for us to use this transform for real values of u. It is clear that the limit

$$\lim_{\gamma\to\infty} R_\gamma(u/\gamma) = \int_{-\infty}^{\infty} \exp\{iux\} d\mu(x) \tag{2}$$

represents the Fourier transform (characteristic function) of the measure μ (we used the dominated convergence theorem here to change the order of integration and limit). The uniqueness of a measure recovering from its modified Stieltjes transform follows from the corresponding result for generalized Stieltjes transform.

Relation (2) gives us the limit behavior of the modified Stieltjes transform as $\gamma \to \infty$. Another possibility ($\gamma \to 0$) without any normalization gives trivial limit equal to 1. However, a more proper approach is to calculate the limit $(R_\gamma(u) - 1)/\gamma$ as $\gamma \to 0$. It is easy to see that

$$\lim_{\gamma\to 0} (R_\gamma(u) - 1)/\gamma = \int_{-\infty}^{\infty} \log \frac{1}{1 - iux} d\mu(x). \tag{3}$$

If the measure μ has compact support, it is possible to write series expansion for modified Stieltjes transform:

$$R_\gamma(u) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1 - iux)^\gamma} = \sum_{k=0}^{\infty} (-1)^k i^k \binom{-\gamma}{k} m_k(\mu) x^k,$$

where $m_k(\mu) = \int_{-\infty}^{\infty} x^k d\mu(x)$ is the kth moment of the measure μ.

The modified Stieltjes transform may be interpreted in terms of characteristic functions. Namely, let us consider a gamma distribution with probability density function

$$p(x) = \frac{1}{\lambda^\gamma \Gamma(\gamma)} x^{\gamma-1} \exp(-x/\lambda), \tag{4}$$

for $x > 0$, $\lambda > 0$, and zero in other cases. Note that this distribution is an ordinary gamma distribution for positive λ, and its "mirror reflection" on negative semi-axes for negative λ. Let us now consider λ as a random variable with cumulative distribution function μ. In this case, Relation (1) gives the characteristic function of gamma distribution with such random parameter:

$$f(t) = \int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{(1 - it\lambda)^\gamma}. \tag{5}$$

The Gauss-hypergeometric function ${}_2F_1$, which is defined by the series

$${}_2F_1(c, a; b; z) = \sum_{n=0}^{\infty} \frac{(c)_n (a)_n}{(b)_n n!} z^n,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$, $n \geq 1$, denotes the rising factorial. Gauss-hypergeometric function ${}_2F_1$ has Euler's integral representation of the form

$${}_2F_1(c, a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \frac{t^{a-1}(1-t)^{b-a-1}}{(1-zt)^c} dt. \tag{6}$$

For more details on Gauss-hypergeometric function and its properties, see Abramowitz and Stegun (2012) and also Andrews et al. (1999).

3. A Family of Commutative Generalized Convolutions

Using the modified Stieltjes transform, we can introduce a family of commutative generalized convolutions. The main idea for this is the following. Let μ_1 and μ_2 be two probability measures. Take positive γ and consider the product of the modified Stieltjes transforms of these measures $R_\gamma(u,\mu_1)R_\gamma(u,\mu_2)$. We would like to represent this product as a modified Stieltjes transform of a measure. Typically, the product is not a modified Stieltjes transform with the same index γ. However, it can be represented as a modified Stieltjes transform with index $\rho > \gamma$ of a measure ν, which is called a generalized (more precisely "(γ,ρ)") convolution of the measures μ_1 and μ_2. Let us mention that the indexes ρ and γ are not arbitrary, but there are infinitely many suitable pairs of indexes. Clearly, the measure ν—if it exists—depends on μ_1, μ_2, and on indexes γ, ρ.

Unfortunately, we cannot describe all pairs γ,ρ for which corresponding generalized convolution ν of measures μ_1 and μ_2 exists. However, we shall show that the pairs of the form $c, 2c$ (where c is positive, but not necessarily integer number) possess this property.

Theorem 1. *Let μ_1, μ_2 be two probability measures on σ-field Borel subsets of a real line. For arbitrary real $c > 0$ there exists "$(c,2c)$" convolution ν of μ_1 and μ_2. In other words, for real $c > 0$ and measures μ_1 and μ_2, there exists a measure ν such that*

$$R_{2c}(u;\nu) = R_c(u;\mu_1)R_c(u;\mu_2). \tag{7}$$

Proof. Because convex combination of probability measures is a probability measure again, and each probability on real line can be considered as a limit in weak-$*$ topology of sequence of measures concentrated in finite number of points each, it is sufficient to prove the statement for Dirac δ-measures only.

Suppose now that the measures μ_1 and μ_2 are concentrated in points a and b correspondingly. We have to prove that there is a measure ν depending on a, b and c such that

$$\int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-iux)^{2c}} = \frac{1}{(1-iua)^c}\cdot\frac{1}{(1-iub)^c}. \tag{8}$$

Of course, it is enough to find the measure ν with compact support[2]. Therefore, we must have for $a > 0$ and $b > 0$

$$m_k = \sum_{j=0}^{k} \frac{c(c+1)\cdots(c+j-1)}{j!}\cdot\frac{c(c+1)\cdots(c+k-j-1)}{(k-j)!}a^j b^{k-j}\Big/\binom{-2c}{k}, \tag{9}$$

where $m_k = m_k(\nu)$ is the kth moment of ν. It remains to be shown that the left hand side of (9) really defines for $k = 0, 1, \ldots$ moments of a distribution.

Let us denote $\lambda = a/b$ and suppose that $|\lambda| < 1$ (the case $|\lambda| = 1$ may be obtained as a limit case). Then, k_m can be rewritten in the form

$$k_m = (-1)^m b^m \sum_{k=0}^{m}\binom{m}{k}\frac{(c)_k(c)_{m-k}}{(2c)_m}\lambda^k,$$

[2] Another approach may be based on the expression of the right hand side of (8) thought the Lauricella's fourth function and its integral representation (for close results see Van Laarhoven and Kalker (1988)). However, it is out of the scope of this paper.

where $(s)_j = s \cdots (s+j-1)$ is the Pochhammer symbol. Simple calculations allow us to obtain from previous equality that

$$k_m = \frac{b^m (c)_m \, {}_2F_1(-m, c, 1-m-c, a/b)}{(2c)_m}. \tag{10}$$

Let us consider a random variable X having Beta distribution with equal parameters c and c; that is, with probability density function

$$p_X(x) = (1-x)^{c-1} x^{c-1} 2^{2c-1} \Gamma(c+1/2) / (\sqrt{\pi}\,\Gamma(c)),$$

for $x \in (0,1)$, and zero for $x \notin (0,1)$. It is not difficult to calculate that

$$\mathbb{E}\Big(aX + b(1-X)\Big)^m = b^m \, {}_2F_1(-m, c, 2c, 1-a/b),$$

which coincides with (10) for non-negative integer m and real $c > 0$. □

Theorem 1 allows us to define a family of generalized convolutions $\nu = \mu_1 \star_c \mu_2$ depending on c, which is equivalent to the relation (7). Obviously, this operation is commutative. However, it is not associative, which can be easily verified by comparing the convolutions $(\delta_1 \star_c \delta_2) \star_c \delta_3$ and $\delta_1 \star_c (\delta_2 \star_c \delta_3)$, where δ_a denotes Dirac measure at point a. It is easy to verify that $\mu_1 \star_c \mu_2(2A) \underset{c\to\infty}{\longrightarrow} \mu_1 * \mu_2(A)$, where $*$ denotes ordinary convolution of measures. We have $2A$ in the left hand side because $\mathbb{E}X = 1/2$. This generalized convolution may be written through independent random variables U and V in the form

$$W = UX + V(1-X),$$

where X is a random variable independent of (U, V) and having Beta distribution with parameters (n, n), and the distribution of W is exactly a generalized convolution of distributions of U and V.

Let us note that the $\star_{3/2}$-convolution of Dirac measures concentrated at points -1 and 1 gives the well-known Wigner semicircle distribution.

In view of the non-associativity of $\star_c$-convolution, it does not coincide with K. Urbanik's generalized convolution (see Urbanik (1964)). At the same time, its non-associativity shows that the expression $\mu_1 \star_c \mu_2 \star_c \mu_3$ has no sense. However, one can define this 3-argument operation by using stochastic linear combinations; that is, linear forms of random variables with random coefficients. Now we define such k-arguments operation. Namely, let $U_1, \ldots, U_k$ be independent random variables, and $X_1, \ldots, X_{n-1}$ be a random vector having Dirichlet distribution with parameters $(a_1, \ldots, a_k) = (c, \ldots, c)$. Define

$$W = X_1 U_1 + \ldots + X_{k-1} U_{k-1} + \big(1 - \sum_{j=1}^{k-1} X_j\big) U_k. \tag{11}$$

The map from vector U of marginal distributions of $(U_1, \ldots, U_k)$ to the distribution of random variable W call k-tuple generalized convolution of the components of U. Clearly, this operation is symmetric with respect to the permutations of coordinates of the vector U. Let us mention that it is probably possible to use Lauricella's fourth function and its integral representation for the definition of k-tuple generalized convolution. However, we prefer this approach in view of its probabilistic interpretation.

4. Connected Family of Non-Commutative Generalized Convolutions

Let now $U_1, \ldots, U_k$ be independent random variables, and $X_1, \ldots, X_{n-1}$ be a random vector having Dirichlet distribution with parameters $(a_1, \ldots, a_k)$, possibly different from each other. Using the relation (11), define random variable W. Its distribution will be called a non-commutative generalized convolution of marginal distributions of the vector U. In the particular case of $k = 2$, we obtain a

non-commutative variant of two-tuple generalized convolution, which represents the more general case of (1).

Let us give a property of this generalized convolution. To do so, let us define $\tilde{beta}_{A,B}$ distribution over interval (A, B) by its probability density function

$$p_{\alpha,\beta}(x) = \begin{cases} \frac{1}{B(\alpha,\beta)(B-A)^{\alpha+\beta-1}}(x-A)^{\alpha-1}(B-x)^{\beta-1}, & \text{if } A < x < B, \\ 0 & \text{otherwise,} \end{cases}$$

for positive $\alpha,\ \beta$. Here $B(\alpha, \beta)$ is beta function.

Theorem 2. *Let W_1, W_2 be two independent identical distributed random variables having $\tilde{beta}_{A,B}(n, n)$ distribution, and μ_1, μ_2 be corresponding probability distributions. Then the measure $\nu = \mu_1 \star_n \mu_2$ corresponds to $\tilde{beta}_{A,B}(2n, 2n)$ distribution.*

Proof. From the proof of Theorem 1 that $W_j \stackrel{d}{=} AX_j + B(1 - X_j)$, where X_1, X_2 are independent identically distributed random variables having $B(n, n)$ distribution. The rest of the proof is just simple calculation. □

The property given by Theorem 2 is very similar to classical stability definition.

Theorem 3. *Let U_j, $j = 1, \ldots, k$ be independent random variables having $\tilde{beta}$ distribution with parameters $\alpha_j = r_j + 1/2$, $\beta_j = r_j + 1/2$. Let $X_1, \ldots, X_{k-1}$ be a random vector having Dirichlet distribution with parameters $(r_1, \ldots, r_k)$. Then, random variable*

$$W = X_1U_1 + \ldots + X_{k-1}U_{k-1} + \left(1 - \sum_{j=1}^{k-1} X_j\right) U_k$$

has $\tilde{beta}$ distribution with parameters $\left(\sum_{j=1}^{k} r_j + 1/2,\ \sum_{j=1}^{k} r_j + 1/2\right)$.

Proof. It is sufficient to calculate the modified Stieltjes transform of the distribution of W using some properties of Gauss-hypergeometric function. □

This property is also similar to the classical stability property, but for the case of k-tuple operation.

Acknowledgments: Authors are grateful to the anonymous referees for useful remarks and for drawing our attention to the connection with the Lauricella's fourth function. The work was partially supported by Grant GACR 16-03708S.

Author Contributions: The first author suggested and designed the new subject. The second author contributed analysis tools and wrote the paper with the first author.

Conflicts of Interest: The authors declare no conflicts of interest.

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