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Article

Modified Stieltjes Transform and Generalized Convolutions of Probability Distributions

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Abstract: The classical Stieltjes transform is modified in such a way as to generalize both Stieltjes and Fourier transforms. This transform allows the introduction of new classes of commutative and non-commutative generalized convolutions. A particular case of such a convolution for degenerate distributions appears to be the Wigner semicircle distribution.

Keywords: Stieltjes transform; characteristic function; generalized convolution; beta distribution

1. Introduction

Let us begin with definitions of classical and generalized Stieltjes transforms. Although these are usual transforms given on a set of functions, we will consider more convenient for us a case of probability measures or for cumulative distribution functions. Namely, let μ be a probability measure of Borel subsets of real line \mathbb{R}^1 . Its Stieltjes transform is defined as¹

$$S(z) = S(z; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x - z},$$

where $\text{Im}(z) \neq 0$. Surely, the integral converges in this case. The generalized Stieltjes transform is represented by

$$S_{\gamma}(z) = S_{\gamma}(z; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x - z)^{\gamma}}$$

for real $\gamma > 0$. For more examples of the generalized Stieltjes transforms of some probability distributions, see Demni (2016) and references therein.

A modification of generalized Stieltjes transform was proposed in Roozegar and Bazyari (2017). Our aim in this paper is to use this modification of the Stieltjes transform to define a class of generalized stochastic convolutions and give their probability interpretation (see Theorem 1 below) in lines of preprint Klebanov and Roozegar (2016).

2. Preliminary Results

Now we prefer to switch to the modified form, and define the following form of transform:

$$R_{\gamma}(u) = R_{\gamma}(u; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1 - iux)^{\gamma}}. \quad (1)$$

¹ Sometimes with opposite sign.

Connection to the generalized Stieltjes transform is obvious. It is convenient for us to use this transform for real values of u . It is clear that the limit

$$\lim_{\gamma \rightarrow \infty} R_\gamma(u/\gamma) = \int_{-\infty}^{\infty} \exp\{iux\} d\mu(x) \tag{2}$$

represents the Fourier transform (characteristic function) of the measure μ (we used the dominated convergence theorem here to change the order of integration and limit). The uniqueness of a measure recovering from its modified Stieltjes transform follows from the corresponding result for generalized Stieltjes transform.

Relation (2) gives us the limit behavior of the modified Stieltjes transform as $\gamma \rightarrow \infty$. Another possibility ($\gamma \rightarrow 0$) without any normalization gives trivial limit equal to 1. However, a more proper approach is to calculate the limit $(R_\gamma(u) - 1)/\gamma$ as $\gamma \rightarrow 0$. It is easy to see that

$$\lim_{\gamma \rightarrow 0} (R_\gamma(u) - 1)/\gamma = \int_{-\infty}^{\infty} \log \frac{1}{1 - iux} d\mu(x). \tag{3}$$

If the measure μ has compact support, it is possible to write series expansion for modified Stieltjes transform:

$$R_\gamma(u) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1 - iux)^\gamma} = \sum_{k=0}^{\infty} (-1)^k i^k \binom{-\gamma}{k} m_k(\mu) x^k,$$

where $m_k(\mu) = \int_{-\infty}^{\infty} x^k d\mu(x)$ is the k th moment of the measure μ .

The modified Stieltjes transform may be interpreted in terms of characteristic functions. Namely, let us consider a gamma distribution with probability density function

$$p(x) = \frac{1}{\lambda^\gamma \Gamma(\gamma)} x^{\gamma-1} \exp(-x/\lambda), \tag{4}$$

for $x > 0$, $\lambda > 0$, and zero in other cases. Note that this distribution is an ordinary gamma distribution for positive λ , and its “mirror reflection” on negative semi-axes for negative λ . Let us now consider λ as a random variable with cumulative distribution function μ . In this case, Relation (1) gives the characteristic function of gamma distribution with such random parameter:

$$f(t) = \int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{(1 - it\lambda)^\gamma}. \tag{5}$$

The Gauss-hypergeometric function ${}_2F_1$, which is defined by the series

$${}_2F_1(c, a; b; z) = \sum_{n=0}^{\infty} \frac{(c)_n (a)_n}{(b)_n n!} z^n,$$

where $(a)_0 = 1$ and $(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)$, $n \geq 1$, denotes the rising factorial. Gauss-hypergeometric function ${}_2F_1$ has Euler’s integral representation of the form

$${}_2F_1(c, a; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 \frac{t^{a-1} (1 - t)^{b-a-1}}{(1 - zt)^c} dt. \tag{6}$$

For more details on Gauss-hypergeometric function and its properties, see [Abramowitz and Stegun \(2012\)](#) and also [Andrews et al. \(1999\)](#).

3. A Family of Commutative Generalized Convolutions

Using the modified Stieltjes transform, we can introduce a family of commutative generalized convolutions. The main idea for this is the following. Let μ_1 and μ_2 be two probability measures. Take positive γ and consider the product of the modified Stieltjes transforms of these measures $R_\gamma(u, \mu_1)R_\gamma(u, \mu_2)$. We would like to represent this product as a modified Stieltjes transform of a measure. Typically, the product is not a modified Stieltjes transform with the same index γ . However, it can be represented as a modified Stieltjes transform with index $\rho > \gamma$ of a measure ν , which is called a generalized (more precisely “ (γ, ρ) ”) convolution of the measures μ_1 and μ_2 . Let us mention that the indexes ρ and γ are not arbitrary, but there are infinitely many suitable pairs of indexes. Clearly, the measure ν —if it exists—depends on μ_1, μ_2 , and on indexes γ, ρ .

Unfortunately, we cannot describe all pairs γ, ρ for which corresponding generalized convolution ν of measures μ_1 and μ_2 exists. However, we shall show that the pairs of the form $c, 2c$ (where c is positive, but not necessarily integer number) possess this property.

Theorem 1. *Let μ_1, μ_2 be two probability measures on σ -field Borel subsets of a real line. For arbitrary real $c > 0$ there exists “ $(c, 2c)$ ” convolution ν of μ_1 and μ_2 . In other words, for real $c > 0$ and measures μ_1 and μ_2 , there exists a measure ν such that*

$$R_{2c}(u; \nu) = R_c(u; \mu_1)R_c(u; \mu_2). \tag{7}$$

Proof. Because convex combination of probability measures is a probability measure again, and each probability on real line can be considered as a limit in weak-* topology of sequence of measures concentrated in finite number of points each, it is sufficient to prove the statement for Dirac δ -measures only.

Suppose now that the measures μ_1 and μ_2 are concentrated in points a and b correspondingly. We have to prove that there is a measure ν depending on a, b and c such that

$$\int_{-\infty}^{\infty} \frac{d\nu(x)}{(1 - iux)^{2c}} = \frac{1}{(1 - iua)^c} \cdot \frac{1}{(1 - iub)^c}. \tag{8}$$

Of course, it is enough to find the measure ν with compact support². Therefore, we must have for $a > 0$ and $b > 0$

$$m_k = \sum_{j=0}^k \frac{c(c+1) \cdots (c+j-1)}{j!} \cdot \frac{c(c+1) \cdots (c+k-j-1)}{(k-j)!} a^j b^{k-j} / \binom{-2c}{k}, \tag{9}$$

where $m_k = m_k(\nu)$ is the k th moment of ν . It remains to be shown that the left hand side of (9) really defines for $k = 0, 1, \dots$ moments of a distribution.

Let us denote $\lambda = a/b$ and suppose that $|\lambda| < 1$ (the case $|\lambda| = 1$ may be obtained as a limit case). Then, k_m can be rewritten in the form

$$k_m = (-1)^m b^m \sum_{k=0}^m \binom{m}{k} \frac{(c)_k (c)_{m-k}}{(2c)_m} \lambda^k,$$

² Another approach may be based on the expression of the right hand side of (8) through the Lauricella’s fourth function and its integral representation (for close results see [Van Laarhoven and Kalker \(1988\)](#)). However, it is out of the scope of this paper.

where $(s)_j = s \cdots (s + j - 1)$ is the Pochhammer symbol. Simple calculations allow us to obtain from previous equality that

$$k_m = \frac{b^m (c)_m {}_2F_1(-m, c, 1 - m - c, a/b)}{(2c)_m}. \tag{10}$$

Let us consider a random variable X having Beta distribution with equal parameters c and c ; that is, with probability density function

$$p_X(x) = (1 - x)^{c-1} x^{c-1} 2^{2c-1} \Gamma(c + 1/2) / (\sqrt{\pi} \Gamma(c)),$$

for $x \in (0, 1)$, and zero for $x \notin (0, 1)$. It is not difficult to calculate that

$$\mathbb{E} \left(aX + b(1 - X) \right)^m = b^m {}_2F_1(-m, c, 2c, 1 - a/b),$$

which coincides with (10) for non-negative integer m and real $c > 0$. \square

Theorem 1 allows us to define a family of generalized convolutions $\nu = \mu_1 \star_c \mu_2$ depending on c , which is equivalent to the relation (7). Obviously, this operation is commutative. However, it is not associative, which can be easily verified by comparing the convolutions $(\delta_1 \star_c \delta_2) \star_c \delta_3$ and $\delta_1 \star_c (\delta_2 \star_c \delta_3)$, where δ_a denotes Dirac measure at point a . It is easy to verify that $\mu_1 \star_c \mu_2(2A) \xrightarrow{c \rightarrow \infty} \mu_1 * \mu_2(A)$, where $*$ denotes ordinary convolution of measures. We have $2A$ in the left hand side because $\mathbb{E}X = 1/2$. This generalized convolution may be written through independent random variables U and V in the form

$$W = UX + V(1 - X),$$

where X is a random variable independent of (U, V) and having Beta distribution with parameters (n, n) , and the distribution of W is exactly a generalized convolution of distributions of U and V .

Let us note that the $\star_{3/2}$ -convolution of Dirac measures concentrated at points -1 and 1 gives the well-known Wigner semicircle distribution.

In view of the non-associativity of \star_c -convolution, it does not coincide with K. Urbanik’s generalized convolution (see Urbanik (1964)). At the same time, its non-associativity shows that the expression $\mu_1 \star_c \mu_2 \star_c \mu_3$ has no sense. However, one can define this 3-argument operation by using stochastic linear combinations; that is, linear forms of random variables with random coefficients. Now we define such k -arguments operation. Namely, let U_1, \dots, U_k be independent random variables, and X_1, \dots, X_{k-1} be a random vector having Dirichlet distribution with parameters $(a_1, \dots, a_k) = (c, \dots, c)$. Define

$$W = X_1 U_1 + \dots + X_{k-1} U_{k-1} + \left(1 - \sum_{j=1}^{k-1} X_j \right) U_k. \tag{11}$$

The map from vector U of marginal distributions of (U_1, \dots, U_k) to the distribution of random variable W call k -tuple generalized convolution of the components of U . Clearly, this operation is symmetric with respect to the permutations of coordinates of the vector U . Let us mention that it is probably possible to use Lauricella’s fourth function and its integral representation for the definition of k -tuple generalized convolution. However, we prefer this approach in view of its probabilistic interpretation.

4. Connected Family of Non-Commutative Generalized Convolutions

Let now U_1, \dots, U_k be independent random variables, and X_1, \dots, X_{k-1} be a random vector having Dirichlet distribution with parameters (a_1, \dots, a_k) , possibly different from each other. Using the relation (11), define random variable W . Its distribution will be called a non-commutative generalized convolution of marginal distributions of the vector U . In the particular case of $k = 2$, we obtain a

non-commutative variant of two-tuple generalized convolution, which represents the more general case of (1).

Let us give a property of this generalized convolution. To do so, let us define $\tilde{\text{beta}}_{A,B}$ distribution over interval (A, B) by its probability density function

$$p_{\alpha,\beta}(x) = \begin{cases} \frac{1}{B(\alpha,\beta)(B-A)^{\alpha+\beta-1}}(x-A)^{\alpha-1}(B-x)^{\beta-1}, & \text{if } A < x < B, \\ 0 & \text{otherwise,} \end{cases}$$

for positive α, β . Here $B(\alpha, \beta)$ is beta function.

Theorem 2. Let W_1, W_2 be two independent identical distributed random variables having $\tilde{\text{beta}}_{A,B}(n, n)$ distribution, and μ_1, μ_2 be corresponding probability distributions. Then the measure $\nu = \mu_1 \star_n \mu_2$ corresponds to $\tilde{\text{beta}}_{A,B}(2n, 2n)$ distribution.

Proof. From the proof of Theorem 1 that $W_j \stackrel{d}{=} AX_j + B(1 - X_j)$, where X_1, X_2 are independent identically distributed random variables having $B(n, n)$ distribution. The rest of the proof is just simple calculation. \square

The property given by Theorem 2 is very similar to classical stability definition.

Theorem 3. Let $U_j, j = 1, \dots, k$ be independent random variables having $\tilde{\text{beta}}$ distribution with parameters $\alpha_j = r_j + 1/2, \beta_j = r_j + 1/2$. Let X_1, \dots, X_{k-1} be a random vector having Dirichlet distribution with parameters (r_1, \dots, r_k) . Then, random variable

$$W = X_1U_1 + \dots + X_{k-1}U_{k-1} + \left(1 - \sum_{j=1}^{k-1} X_j\right) U_k$$

has $\tilde{\text{beta}}$ distribution with parameters $\left(\sum_{j=1}^k r_j + 1/2, \sum_{j=1}^k r_j + 1/2\right)$.

Proof. It is sufficient to calculate the modified Stieltjes transform of the distribution of W using some properties of Gauss-hypergeometric function. \square

This property is also similar to the classical stability property, but for the case of k -tuple operation.

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