The Reduced Form of a Block Recursive Model

Hans Schneeweiss
Erich-Otto Maschke
Manfred Pfannes
Department of Statistics,
Ludwig-Maximilians-Universität München,
Akademiestr. 1/I, D-80799 München, Germany
email: Schneeweiss@stat.uni-muenchen.de

Abstract

Various estimators of the reduced form of a block recursive model are investigated and compared to each other. In particular it is shown that the structural reduced form estimator, which results from estimating separately each block of the block recursive model by some efficient method and then solving the system for the endogenous variables, is more efficient than the OLS estimator of the reduced form. Other reduced form estimators derived from OLS or Two Stage LS estimators of a partially reduced form have intermediate efficiency properties. The paper has been published in Schneeweiss et al (2001), but without the appendices.

Key Words: Block recursive model, reduced form, simultaneous equations model.

1 Introduction

The reduced form of a structural simultaneous equations model can be estimated in (at least) two ways: either by estimating it directly by OLS or, indirectly, by estimating first the structural parameters of the model by some efficient method and then solving for the endogenous variables. According to Dhrymes (1973) the indirect estimation method leads to more efficient estimates than the direct method because it utilizes all the information inherent in the structural model. Dhrymes, however, assumed, with one exception, that the covariance matrix of the errors was not restricted in any way. He therefore did not consider to use the information that might be inherent in the error covariance matrix. The exception was the case of a diagonal error covariance matrix, which however did not lead him to design a special estimation method for this case. One particular and quite important case ist the so-called recursive model,which is characterized by the two requirements that the system matrix of the endogenous variables should be triangular and the error covariance matrix diagonal. For this model Schneeweiss (1993), building on earlier work of Kosák(1988), proved a similar result as was given by Dhrymes (1973) in the case of the more traditional model with an unrestricted error covariance matrix.

An interesting feature of the recursive model is that it is identified without any further conditions and that OLS applied to each structural equation already results in efficient estimators, whereas the traditional (limited or fully) "efficient" estimators like Two-Stage Least Squares or Three-Stage Least Squares either do not work or, if they do, are not necessarily efficient. This is shown in some detail in the present paper.

The present paper combines the results of Dhrymes (1973) and Schneeweiss (1993) in that it studies a *block* recursive model. Here the blocks have unrestricted error covariance matrices and can each be estimated by some efficient method, where the endogenous variables that are "explained" in subsequent blocks are considered as exogenous for the particular block to be estimated. Again it can be shown that the direct estimator of the reduced form is less efficient than the indirect one, which is constructed by first estimating each block separately with some efficient method and then solving for the endogenous variables.

The proof of this statement is accomplished by referring to Dhrymes' (1973) corresponding result for each block, thereby reducing the block recursive model to a recursive model and then using the result of Schneeweiss (1993) for the latter model, see also Pfannes (1996). In doing so, it comes as a convenience that in Schneeweiss (1993) the recursive model had already been structured in blocks, although it was not a block recursive model in the usual sense. The proof is only given for the case of a two-blocks model, but can be generalized by induction to an arbitrary number of blocks. The result may seem not too surprising. If the errors are jointly normally distributed and if conditional ML estimators are used for each block (conditional on the endogenous variables of subsequent blocks), then the resulting estimators of the structural parameters of the whole model are ML estimators and so are the derived reduced form estimators. They are therefore more efficient than the direct estimators of the reduced form parameters, as these ar not ML. However, this result holds also true even if the errors are not normally distributed and if other efficient estimators for each block have been constructed which are not ML. For this reason, the solution of an old problem presented in this paper may still be of some interest.

The recursive model has been advocated by Wold (1953, 1964). It was soon found, however, that simultaneous equations models which were not recursive prevailed in practice. An intermediate model type is the block recursive model, which is recursive between the blocks, but has a simultaneity (interdependence) structure within each block. Often models used to describe an economy are "almost" block recursive: only one or two equations spoil the block recursive structure. In small models that describe only sectors of an economy, block recursiveness may occur more often.

Estimation of the reduced form of a model has been considered by several authors, among others by Goldberger et al (1961) and Court (1973). Partially restricted form estimators have been studied by Kakwani and Court (1972), Sant (1978), Nagar and Sahay (1978). Schneeweiss (1996) compared reduced form estimators of recursive models for finite samples. In the present paper only asymptotic results are presented.

In Section 2 we introduce the partially reduced form, discuss the properties of its OLS estimator and compare it with its structural estimator via Dhrymes' theorem. In Section 3 we clarify the relation between estimators of the partially reduced form and of the reduced form. Two important estimators of the reduced form, OLS and RLS, are then related to each other in Section 4. The main result, a comparison of the last two estimators with the structural estimator of the reduced form is presented in Section 5. In Section 6 we discuss Two and Three Stage Least Squares of the partially reduced form and their implications for estimating the reduced form. Section 7 has some concluding remarks. Some details of the proofs are delegated to an appendix.

2 Estimating the partially reduced form

We consider the following two-blocks recursive model

$$Y_1G_1 = \quad Y_2F_1 + \quad X_1F_2 + W_1 \qquad (1)$$
$$Y_2G_2 = \qquad\qquad X_2F_3 + W_2 \qquad (2)$$

where X_i and Y_i are Txp_i and Txq_i matrices, $i = 1, 2$, of exogenous and endogenous variables, respectively, with $p_2 \neq 0$; W_1 and W_2 are independent Txq_i matrices of unobservable error variables with iid rows having expectation zero and unrestricted, but nonsingular, covariance matrices Σ_{w_i} ; the G_i and F_j are parameter matrices endowed with *a priori* restrictions such that each block is identifiable; the G_i are supposed to be non-singular, so that the system can be uniquely solved for Y_1 and Y_2. The exogenous variables are taken to be nonstochastic variables. Let X be the matrix of all exogenous variables such that $X = (X_{11}, X_{12}, X_{22})$, $X_1 = (X_{11}, X_{12})$, $X_2 = (X_{12}, X_{22})$. We introduce selection matrices A_i such that $X_i = XA_i$, $i = 1, 2$. Thus $A_1 = (I, 0)', A_2 = (0, I)'$.

In order to study the asymptotic properties of various estimators we conceive a series of models (1), (2), indexed by T, where the transition from T to $T+1$ is accomplished by the addition of one further row to each of the matrices $X_i, Y_i, W_i, i = 1, 2$. We assume that $X'X$ is nonsingular and that, as $T \to \infty$, $lim \frac{1}{T} X'X = M$ with a nonsingular matrix M.

Another assumption, which we need for the main asymptotic results, is that

$$\sum_{t=1}^{\infty} x_{ti}^2 / t^2 < \infty, \; i = 1, \ldots, p, \qquad \text{(A)}$$

where the x_{ti} are the elements of X. We will make a note whenever a result is based on assumption (A). Note that because of the independence of W_1 and W_2 the variables in Y_2 can be considered as exogenous for the first block in the sense that W_1 is independent of Y_2. One can therefore treat the first block, on the condition that Y_2 is given, as a simultaneous equations model with endogenous variables Y_1 and exogenous variables Y_2 and X_1. The second block is a conventional simultaneous equations model anyway.

Multiplying each block by G_i^{-1}, $i = 1, 2$, we get the following recursive system:

$$
\begin{aligned}
Y_1 = &\quad Y_2 B + \quad X_1 \Gamma_1 + U_1 = Z\Delta + U_1 && (3)\\
Y_2 = &\qquad\qquad\quad X_2 \Gamma_2 + U_2 && (4)
\end{aligned}
$$

with

$$B = F_1 G_1^{-1},\, \Gamma_1 = F_2 G_1^{-1},\, \Gamma_2 = F_3 G_2^{-1}, \qquad (5)$$

$U_1 = W_1 G_1^{-1}$, and $U_2 = W_2 G_2^{-1}$, where the notation was chosen to correspond with the notation in Schneeweiss (1993). In Equation (3) we used the abbreviatons $Z = (Y_2, X_1)$ and $\Delta = (B', \Gamma_1')'$. We denote the covariance matrix of any row of U_i by Σ_i, $i = 1, 2$. Apart from being positive definite Σ_i is not restricted in any way. (Note that $\Sigma_i = G_i^{'-1} \Sigma_{w_i} G_i^{-1}$). We call the system (3), (4) the *partially reduced form* of the structural model (1), (2). Note that equation (4) ist just the reduced form of (2), and equation (3) is the conditonal reduced form of (1) under the condition that Y_2 is given (and therefore considered as exogenous for Y_1).

We consider two estimators for the parameter matrices Δ and Γ_2 of the partially reduced form: The OLS estimators, $\hat{\Delta}$ and $\hat{\Gamma}_2$ are found by regressing Y_1 on Z as in (3) and by regressing Y_2 on X_2 as in (4), respectively. The so-called structural estimators, $\hat{\hat{\Delta}}$ and $\hat{\hat{\Gamma}}_2$, are constructed by first estimating the parameter matrices G_1, F_1, and F_2 of block (1) and G_2 and F_3 of block (2) separately by some efficient estimation procedure like Three Stage Least Squares (TSLS) within each block, taking the *a* *priori* restrictions properly into account and considering Y_2 in block (1) as exogenous, and then computing estimates of B, Γ_1, and Γ_2 via (5).

We will now derive the asymptotic properties of the OLS estimator. To this purpose, we vectorize the parameter matrices. We denote a vectorized matrix by its small letter; thus $\delta = vec\Delta$, $\gamma_2 = vec\Gamma_2$ etc.

Lemma 1:
Denote by $\{y_2\}$ the infinite series of the rows of Y_2 when $T \to \infty$. Then, under (A),

$$\sqrt{T}(\hat{\delta} - \delta)|\{y_2\} \quad \to \quad N(0, V(\hat{\delta})), \text{ a.s.} \tag{6}$$

$$\sqrt{T}(\hat{\gamma}_2 - \gamma_2) \quad \to \quad N(0, V(\hat{\gamma}_2)), \tag{7}$$

where

$$V(\hat{\delta}) = \Sigma_1 \otimes M_Z^{-1} \tag{8}$$

$$V(\hat{\gamma}_2) = \Sigma_2 \otimes M_{22}^{-1} \tag{9}$$

are the asymptotic covariance matrices of $\hat{\delta}$ and $\hat{\gamma}_2$ with

$$M_{22} \quad = \quad lim \frac{1}{T} X_2' X_2 = A_2' M A_2 \tag{10}$$

$$M_Z \quad = \quad lim \frac{1}{T} Z' Z = \Psi' M \Psi + \Sigma_2^*, \text{ a.s} \tag{11}$$

$$\Psi \quad = \quad (A_2 \Gamma_2, A_1)$$

$$\Sigma_2^* \quad = \quad \begin{pmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof: (7) together with (9) is simply the well-known result from OLS theory, and (10) follows from the identity $X_2 = XA_2$. So we need only prove (6) together with (8) and (11).

First note that

$$Z = X\Psi + U_2^* \,, \tag{12}$$

where $U_2^* = (U_2, 0)$. Now, assumption (A) implies $lim \frac{1}{T} X' U_2^* = 0$, a.s. Indeed, let x_t and u_t be the t'th components of any colums of X and U_2, respectively. Then by (A)

$$\sum_1^\infty \frac{1}{t^2} Var(x_t u_t) = \sum_1^\infty \frac{x_t^2}{t^2} \sigma_u^2 < \infty,$$

and by Kolmogorov's criterion, see, e.g., Bauer (1978)§37,

$$lim \frac{1}{T} \sum_1^T x_t u_t = 0, \text{ a.s.}$$

With this result, (11) follows immediately from (12). Thus with probability 1 the series $\{y_2\}$ is such that the limit M_Z exists.

As Σ_2 and the part $A_1' M A_1$ of $\Psi' M \Psi$ are both positive definite, it is easy to see that M_Z is positive definite (see A1) and M_Z^{-1} exists. Therefore, by OLS theory, the conditional distribution of $\sqrt{T}(\hat{\delta} - \delta)$ given $\{y_2\}$ converges, with probability 1, to $N(0, V(\hat{\delta}))$ with $V(\hat{\delta}) = \Sigma_1 \otimes M_Z^{-1}$. This proves (6) and (8).$\diamondsuit$

Remark to the proof: The proof of (11) fills a gap in the proof of Theorem 1 in Schneeweiss (1993), where the almost sure convergence was not proved.

We can now formulate Dhrymes' result as it applies to our model.

Proposition 1 (Dhrymes):

Suppose $(\hat{\hat{\delta}}, \hat{\hat{\gamma}}_2)$ are structural estimators and $(\hat{\delta}, \hat{\gamma}_2)$ are the OLS estimators of (δ, γ_2) in the system (3) and (4). Then, under (A), for both types of estimators the statements (6) and (7) hold true, and for their asymptotic covariance matrices we have

$$V(\hat{\hat{\delta}}) \leq V(\hat{\delta}) \tag{13}$$

$$V(\hat{\hat{\gamma}}_2) \leq V(\hat{\gamma}_2) \tag{14}$$

in the sense of the Loewe ordering of symmetric matrices. (For symmetric matrices, $A \leq B$ means that $B - A$ is positive semidefinite).The inequalities become equalities if, and only if, each equation is just identified.

Proof: Let the vector φ_1 comprise all the unknown parameters in G_1, F_1, and F_2 in block (1) and let $\hat{\varphi}_1$ be the TSLS (or any other efficient) estimator of φ_1 with Y_2 taken to be exogenous. We know that, given $\{y_2\}$ and assuming that $\{y_2\}$ is such that M_Z exists as in Lemma 1, the conditional distribution of $\sqrt{T}(\hat{\varphi}_1 - \varphi_1)$ converges to $N(0, V(\varphi_1))$, where $V(\varphi_1)$ depends on M_Z but does not depend on $\{y_2\}$, see Dhrymes (1973). The same then holds true for the derived estimator $\hat{\hat{\delta}}$, i.e., $\hat{\hat{\delta}}$ obeys the same asymptotic law (6) as does $\hat{\delta}$ except that $V(\hat{\hat{\delta}})$ differs from $V(\hat{\delta})$. Similarly $\hat{\hat{\gamma}}_2$ satisfies (7) except that $V(\hat{\hat{\gamma}}_2)$ differs from $V(\hat{\gamma}_2)$.

The inequalities (13) and (14) are now just the statement of Dhrymes' theorem (1973), applied separately to (1), given $\{y_2\}$, and to (2). $\diamondsuit$

3 Estimating the reduced form via the partially reduced form

We now consider the reduced form of the simultaneous equations model (1) and (2). It is the same as the reduced form of the partially reduced form (3) and (4) and is given by

$$Y_1 = X\Pi_1 + V_1 \tag{15}$$

$$Y_2 = X_2\Gamma_2 + U_2 \tag{16}$$

where $\Pi_1 = \Psi\Delta$ and $V_1 = U_2B + U_1$. Note that the reduced form (15), (16) differs from the reduced form as employed in Schneeweiss (1993). There, a matrix W comprising X but not necessarily identical to X was used instead of X and in addition (16) was replaced with $Y_2 = W\Pi_2 + V_2$ or, in the present notation,

$$Y_2 = X\Pi_2 + V_2. \tag{17}$$

Equation (16) looks more natural than (17), and estimating Π_2 by OLS from (17) is less efficient that estimating Γ_2 from (16). Therefore we prefer (16) to (17). But this means that the results of Schneeweiss (1993) cannot simply be taken over but must be modified correspondingly.

Now, the reduced form can again be estimated in (at least) two different ways, either directly by OLS from (15) and (16) separately or by estimating the partially reduced form (3) and (4) by whatever method and then solving this system for Y_1 and Y_2. Of course, the result of the second procedure depends on the estimation method used to estimate (3) and (4) to start with. Let us study this dependency first.

Assume that there are two estimators for the parameters of (3) and (4), denote them again by $(\hat{\delta}, \hat{\gamma}_2)$ and $(\hat{\hat{\delta}}, \hat{\hat{\gamma}}_2)$, respectively. They may but need not be the same estimators as in Proposition 1. For each of these compute reduced form estimates by the indirect method described above. Consider for instance $\hat{\delta} = vec\,\hat{\Delta}$ and $\hat{\gamma}_2 = vec\,\hat{\Gamma}_2$. Then

$$\hat{\Pi}_1 = \hat{\Psi}\hat{\Delta} = (A_2\hat{\Gamma}_2, A_1)(\hat{B}', \hat{\Gamma}_1')'. \tag{18}$$

Let $\pi_1 = vec\,\Pi_1$, $\hat{\pi}_1 = vec\,\hat{\Pi}_1$ and stack π_1 and γ_2 into one vector $\pi = (\pi_1', \gamma_2')'$ and similarly $\hat{\pi} = (\hat{\pi}_1', \hat{\gamma}_2')'$. Do the same for $\hat{\hat{\delta}} = vec\,\hat{\hat{\Delta}}$ and $\hat{\hat{\gamma}}_2 = vec\,\hat{\hat{\Gamma}}_2$ and suppose that $(\hat{\delta}, \hat{\gamma}_2)$ and $(\hat{\hat{\delta}}, \hat{\hat{\gamma}}_2)$ have similar asymptotic properties as the OLS and structural estimators considered in Proposition 1. We then can state the following proposition

Proposition 2:
Let $\hat{\gamma}_2$ and $\hat{\hat{\gamma}}_2$ be two asymptotically normal estimators of γ_2 being constructed from the data X_2 and Y_2 of (4). Let $\hat{\delta}$ and $\hat{\hat{\delta}}$ be two estimators of δ constructed from the data X_1, Y_1 and Y_2 of (3) such that they are a.s. conditonally asymptotically normal on the condition that $\{y_2\}$, a realization of the process generated by the rows of Y_2 ($T \to \infty$), is given. The conditional asymptotic covariance matrices of $\hat{\delta}$ and $\hat{\hat{\delta}}$ are supposed to be independent of $\{y_2\}$. Suppose further that the asymptotic covariance matrices of these estimators satisfy (13) and (14). Then $\hat{\pi}$ and $\hat{\hat{\pi}}$, computed from $(\hat{\delta}, \hat{\gamma}_2)$ and $(\hat{\hat{\delta}}, \hat{\hat{\gamma}}_2)$, respectively, are asymptotically normally distributed with asymptotic covariance matrices that satisfy

$$V(\hat{\hat{\pi}}) \leq V(\hat{\pi}) \,.$$

Suppose (3) is identifiable in the conventional sense (see Section 6), then $V(\hat{\hat{\pi}}) = V(\hat{\pi})$ if, and only if, equality holds in (13) and (14).

Proof: The proof is almost identical to the proof of the Lemma in Schneeweiss (1993), except that here we replace π_2 with γ_2 and we consider and compare different estimators for δ *and* γ_2, not only for δ. We consider, see (18),

$$\hat{\Pi}_1 - \Pi_1 = \Psi(\hat{\Delta} - \Delta) + (\hat{\Psi} - \Psi)\Delta + R \,,$$

where $R = (\hat{\Psi} - \Psi)(\hat{\Delta} - \Delta)$ turns out to be $o_p(T^{-\frac{1}{2}})$.Applying the vectorization operator and using the fact that $(\hat{\Psi} - \Psi)\Delta =$

$A_2(\hat{\Gamma}_2 - \Gamma_2)B$, we find that

$$\hat{\pi}_1 - \pi_1 = (I \otimes \Psi)(\hat{\delta} - \delta) + (B' \otimes A_2)(\hat{\gamma}_2 - \gamma_2) + o_p(T^{-\frac{1}{2}}),$$

and finally for $\pi = (\pi_1', \gamma_2')'$:

$$\sqrt{T}(\hat{\pi} - \pi) = F\sqrt{T}(\hat{\delta} - \delta) + G\sqrt{T}(\hat{\gamma}_2 - \gamma_2) + o_p(1) \qquad (19)$$

with $F = (I \otimes \Psi', 0)'$, $G = (B \otimes A_2', I)'$. By an argument expounded in the appendix of Schneeweiss (1993) it can be shown that the assumptions of the proposition imply that the joint distribution of $\sqrt{T}(\hat{\delta} - \delta)$ and $\sqrt{T}(\hat{\gamma}_2 - \gamma_2)$ converges to a multivariate normal distribution with (asymptotic) covariance matrices $V(\hat{\delta})$ and $V(\hat{\gamma}_2)$ and (asymptotic) covariance $Cov(\hat{\delta}, \hat{\gamma}_2) = 0$. Therefore $\sqrt{T}(\hat{\pi} - \pi) \rightarrow N(0, V(\hat{\pi}))$ with

$$V(\hat{\pi}) = FV(\hat{\delta})F' + GV(\hat{\gamma}_2)G'. \qquad (20)$$

The same is true for $\hat{\hat{\delta}}$ and $\hat{\hat{\gamma}}_2$. The main assertion of the proposition now follows immediately from (13), (14), and (20).If $V(\hat{\hat{\pi}}) = V(\hat{\pi})$, then (20) implies firstly $V(\hat{\hat{\gamma}}_2) = V(\hat{\gamma}_2)$ because of the special form of G and secondly $V(\hat{\hat{\delta}}) = V(\hat{\delta})$ if Ψ is assumed to have full column rank, see A2. This last condition is equivalent to (3) being identifiable in the sense of Section 6. $\diamondsuit$

4 Estimating the reduced form by direct and recursive OLS

The last step in the line of arguments is a comparison of the reduced form estimator $\hat{\Pi}_1$ resulting indirectly from the OLS estimators $\hat{\Gamma}_2$ and $\hat{\Delta}$ via (18) and the direct OLS estimator of the reduced form, i.e., $\tilde{\Pi}_1 = (X'X)^{-1}X'Y_1$. As before, let $\hat{\pi} = (\hat{\pi}_1', \hat{\gamma}_2')'$ and $\tilde{\pi} = (\tilde{\pi}_1', \hat{\gamma}_2')'$, where $\tilde{\pi}_1 = vec\,\tilde{\Pi}_1$. We call $\hat{\pi}$ the Recursive Least Squares (RLS) estimator and $\tilde{\pi}$ the (direct) OLS estimator of π. Both estimators are asymptotically normal with asymptotic covariance matrices $V(\hat{\pi})$ and $V(\tilde{\pi})$, which we will now derive.

Lemma 2:
The direct OLS estimator $\tilde{\pi}$ of the reduced form (15), (16) is asymptotically normal with an asymptotic covariance matrix given by

$$V(\tilde{\pi}) = \begin{pmatrix} V(\tilde{\pi}_1) & B'\Sigma_2 \otimes A_2 M_{22}^{-1} \\ \Sigma_2 B \otimes M_{22}^{-1} A_2' & \Sigma_2 \otimes M_{22}^{-1} \end{pmatrix}$$

with

$$V(\tilde{\pi}_1) = (\Sigma_1 + B'\Sigma_2 B) \otimes M^{-1}.$$

Proof:
Starting from (15) and (16) we get after vectorization

$$\begin{aligned} y_1 &= (I \otimes X)\pi_1 + v_1 \\ y_2 &= (I \otimes X_2)\gamma_2 + u_2 \end{aligned}$$

with $y_i = vec\,Y_i$, $i = 1, 2$, $v_1 = vec\,V_1$, and $u_2 = vec\,U_2$. Stacking the vectors of these two equations, we obtain

$$y = \tilde{X}\pi + v \tag{21}$$

with $y = (y_1', y_2')'$, $v = (v_1', u_2')'$ and

$$\tilde{X} = \begin{pmatrix} I \otimes X & 0 \\ 0 & I \otimes X_2 \end{pmatrix}.$$

The OLS estimator from (21) is $\tilde{\pi} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'y$. Its covariance matrix (for finite T) is

$$V_T(\tilde{\pi}) = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'(\Sigma_v \otimes I)\tilde{X}(\tilde{X}'\tilde{X})^{-1}$$

with Σ_v being the covariance matrix of a row of (V_1, U_2). As $V_1 = U_2B + U_1$, Σ_v can be partitioned into

$$\Sigma_v = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} B'\Sigma_2 B + \Sigma_1 & B'\Sigma_2 \\ \Sigma_2 B & \Sigma_2 \end{pmatrix}. \tag{22}$$

After some algebraic manipulations we find

$$V_T(\tilde{\pi}) = \begin{pmatrix} \Sigma_{11} \otimes (X'X)^{-1} & \Sigma_{12} \otimes A_2(X_2'X_2)^{-1} \\ \Sigma_{21}(X_2'X_2)^{-1}A_2' & \Sigma_{22} \otimes (X_2'X_2)^{-1} \end{pmatrix},$$

where the identity $(X'X)^{-1}X'X_2 = A_2$ was used. Multiplying by T, going to the limit $T \to \infty$, and substituting (22), we obtain the asymptotic covariance matrix of Lemma 2. The asymptotic normality is simply a consequence of OLS theory.$\diamondsuit$
Remark: Note that $\tilde{\pi}$ here is derived as the OLS estimator computed from (15) and (16) separately and not, as in Schneeweiss

1993, from the multivariate regression system (15), (17). The new $\tilde{\pi}$ of the present paper is more efficient than the $\tilde{\pi}$ of the previous paper.

Next we derive the asymptotic covariance matrix of $\hat{\pi}$.

Lemma 3:
Under assumption (A), the RLS estimator $\hat{\pi}$ of the reduced form (15), (16) is asymptotically normal with an asymptotic covariance matrix given by

$$V(\hat{\pi}) = \begin{pmatrix} V(\hat{\pi}_1) & B'\Sigma_2 \otimes A_2 M_{22}^{-1} \\ \Sigma_2 B \otimes M_{22}^{-1} A_2' & \Sigma_2 \otimes M_{22}^{-1} \end{pmatrix}$$

with

$$V(\hat{\pi}_1) = \Sigma_1 \otimes \Psi M_Z^{-1} \Psi' + B'\Sigma_2 B \otimes A_2 M_{22}^{-1} A_2'.$$

Proof: By Lemma 1, the assumptions of Proposition 2 hold true for the OLS estimators $\hat{\delta}$ and $\hat{\gamma}_2$. From the proof of Proposition 2 it follows that $\hat{\pi}$ is asymptotically normal with $V(\hat{\pi})$ as given by (20). Substituting (8) and (9) into (20) we obtain, with some algebra, the desired result. $\diamondsuit$

Remark: Because we took (15), (16) as our reduced form and not (15), (17), $\hat{\pi}$ differs, albeit only slightly, from the $\hat{\pi}$ of Theorem 1 in Schneeweiss (1993). It is for this reason that we had to compute $V(\hat{\pi})$ anew. It is possible to derive the result of Lemma 3 when assumption (A) is replaced with the assumption that $E(u^{2+\varepsilon}) < \infty$ for some $\varepsilon > 0$.

We now compare $V(\hat{\pi})$ to $V(\tilde{\pi})$.

Proposition 3:
For the asymptotic covariance matrices of the reduced form estimators $\hat{\pi}$ and $\tilde{\pi}$ the inequality $V(\hat{\pi}) \leq V(\tilde{\pi})$ holds, with equality if, and only if, $X_1 = X_2 = X$.

Proof: Due to Lemmas 2 and 3 we obviously need only compare $V(\tilde{\pi}_1)$ and $V(\hat{\pi}_1)$. Substituting M_Z from (11) and M_{22} from (10) we see that the assertion of the proposition is proved if we can show that

$$\Psi(\Psi' M \Psi + \Sigma_2^*)^{-1} \Psi' \leq M^{-1} \tag{23}$$

$$A_2(A_2' M A_2)^{-1} A_2' \leq M^{-1} \tag{24}$$

To prove (24) define the $p \times p_2$ matrix $A = M^{\frac{1}{2}} A_2$. Then (24) is equivalent to $A(A'A)^{-1}A' \leq I$, which is obviously true because this is a projection matrix. We have equality if, and only if, A is a square matrix, which means that $p_2 = p$ and thus $X_2 = X$. To prove (23) define the $p \times (q_2 + p_1)$ matrix $H = M^{\frac{1}{2}} \Psi$ and the $q_2 \times (q_2 + p_1)$ matrix $G = (\Sigma_2^{\frac{1}{2}}, 0)$ with rank $G = q_2$. Then (23) is equivalent to

$$R := H(H'H + G'G)^{-1} H' \leq I.$$

Let $L = (H', G')'$, then R is the upper left block of the projection matrix $P := L(L'L)^{-1}L'$ and therefore $R \leq I$.

Now suppose $R = I$. Then the upper right block of P is zero:

$$H(L'L)^{-1}G' = 0.$$

Let $K = (L'L)^{-1}G'$. K is a $(q_2 + p_1) \times q_2$ matrix and rank K =rank $G = q_2$. From $HK = 0$ it now follows that rank $H \leq (q_2 + p_1) - q_2 = p_1$. On the other hand, $R = I$ implies rank $H = P$. Hence $p \leq p_1$. But as also $p_1 \leq p$, therefore $p_1 = p$ and consequently $X_1 = X$.

Conversely, $p = p_1$ implies that L is a square matrix and therefore $P = I$ and hence $R = I$.$\diamondsuit$

5 Structural and direct estimates of the reduced form

Collecting the results of Propositions 1 to 3, we finally arrive at our main result:

Theorem:
Consider the following three estimators of the reduced form of the block recursive model (1), (2) with assumption (A):

- The structural estimator $\hat{\hat{\pi}}$ is found by computing efficient estimates (like TSLS) for the structural parameters of (1) and (2) separately, treating Y_2 in (1) as exogenous, and solving for Y_1 and Y_2.
- The RLS estimator $\hat{\pi}$ is found by applying OLS separately to (3) and (4) and solving for Y_1 and Y_2.

- The (direct) OLS estimator $\tilde{\pi}$ is found by applying OLS to (15) and (16).

The asymptotic covariance matrices of these estimators then satisfy the following chain of inequalities:

$$V(\hat{\hat{\pi}}) \leq V(\hat{\pi}) \leq V(\tilde{\pi}).$$

Proof: The second inequality is just a restatement of Proposition 3. According to Proposition 1 the assumptions of Proposition 2 are satisfied for the structural and the OLS estimators of the parameters of the partially reduced form (3), (4). In particular (13) and (14) hold true. As $\hat{\hat{\pi}}$ and $\hat{\pi}$ are the implied reduced form parameter estimates, the first inequality of the theorem follows from Proposition 2. $\diamondsuit$

6 Two and Three Stage Least Squares

In this section we consider only the partially reduced form model (3), (4). Its parameter matrices Δ and Γ_2 are now assumed to be unrestricted. In this case OLS is the estimation method of first choice. As explained in the introduction, the more traditional Two and Three Stage LS procedures often do not work. They can be applied and lead to consistent estimators if, and only if, the model is identified in the conventional sense, i.e., identified from the reduced form parameter matrix Π and without regard to any restrictions in the error covariance matrix.

Throughout this section, the concept of identifiability is always to be undestood in this narrow sense. (Note that a recursive model is always identified in the strict sense of the word). The second block (4) is identified without any further conditions. The first block (3) is identified if the well-known rank condition is satisfied. If we write (3) and (4) as

$$\begin{array}{llllll} Y_1 = & Y_2 B + X_{11}\Gamma_{11} + & X_{12}\Gamma_{12} & & +U_1 \\ Y_2 = & & X_{12}\Gamma_{21} & +X_{22}\Gamma_{22} & +U_2\,, \end{array}$$

then the (necessary and sufficient) condition for the identifiability of (3) is that Γ_{22} has full column rank. From the definition of Ψ:

$$\Psi = (A_2\Gamma_2, A_1) = \begin{pmatrix} 0 & I & 0 \\ \Gamma_{21} & 0 & I \\ \Gamma_{22} & 0 & 0 \end{pmatrix}$$

it is seen that this is equivalent to the condition that Ψ has full column rank. It is this "identifiability condition" (I) which we will assume to be satisfied for the rest of this section.

Let us first study Two Stage Least Squares (2SLS). For the second block, (4), 2SLS is identical to OLS: $\hat{\gamma}_{2,2SLS} = \hat{\gamma}_2$. For the first block, (3), the 2SLS estimator is

$$\hat{\Delta}_{2SLS} = (Z'PZ)^{-1}Z'PY_1$$

with $P = X(X'X)^{-1}X'$. The estimator is asymptotically normal:

Lemma 4:
Under the identifiability assumption (I), the 2SLS estimator of δ satisfies

$$\begin{aligned}\sqrt{T}(\hat{\delta}_{2SLS}-\delta) &\rightarrow N(0,V(\hat{\delta}_{2SLS}))\\ V(\hat{\delta}_{2SLS}) &= \Sigma_1 \otimes (\Psi' M \Psi)^{-1},\end{aligned} \tag{25}$$

and $\hat{\delta}_{2SLS}$ and $\hat{\gamma}_2$ are asymptotically independent.

Proof: From the theory of 2SLS we know that $\sqrt{T}(\hat{\delta}_{2SLS}-\delta)$ is asymptotically normal with the (asymptotic) covariance matrix

$$V(\hat{\delta}_{2SLS}) = \Sigma_1 \otimes (M_{ZX} M^{-1} M_{XZ})^{-1}, \tag{26}$$

where $M_{XZ} = plim \frac{1}{T} X'Z$. Because of (12), this limit exists and is given by

$$M_{XZ} = M\Psi. \tag{27}$$

Substituting (27) into (26) gives the desired result (25). Note that the inverse of $\Psi' M \Psi$ exists due to the identifiability condition (I). The independence of $\hat{\delta}_{2SLS}$ and $\hat{\gamma}_2$ is a consequence of the independence of U_1 and U_2.$\diamondsuit$

We can now derive estimates of the reduced form from the 2SLS estimates of Δ and Γ_2 in the same way as we did for the other estimators. Call the resulting estimator the 2SLS reduced form estimator $\hat{\Pi}_{2SLS}$.

Lemma 5:
Suppose the identifiability condition (I) holds, then the 2SLS re-

duced form estimator is asymptotically normal with an asymptotic covariance matrix given by

$$V(\hat{\pi}_{2SLS}) = \begin{pmatrix} V(\hat{\pi}_{1,2SLS}) & B'\Sigma_2 \otimes A_2 M_{22}^{-1} \\ \Sigma_2 B \otimes M_{22} A_2' & \Sigma_2 \otimes M_{22}^{-1} \end{pmatrix}$$

with

$$V(\hat{\pi}_{1,2SLS}) = \Sigma_1 \otimes \Psi(\Psi' M \Psi)^{-1}\Psi' + B'\Sigma_2 B \otimes A_2 M_{22}^{-1} A_2'.$$

Proof: As by Lemma 4 $\hat{\delta}_{2SLS}$ and $\hat{\gamma}_2$ are asymptotically independent and as $\hat{\pi}_{2SLS}$ is derived from these estimators in the same way as $\hat{\pi}$ was derived from $\hat{\delta}$ and $\hat{\gamma}_2$, we can again use (19) from the proof of Proposition 2 to verify the asymptotic normality of $\hat{\pi}_{2SLS}$ and (20) to compute its asymptotic covariance matrix. Just replace $\hat{\delta}$ by $\hat{\delta}_{2SLS}$ and use (25) and (9).$\diamondsuit$

We can now compare the asymptotic covariance matrices of $\hat{\pi}$, $\hat{\pi}_{2SLS}$, and $\tilde{\pi}$.

Proposition 4:
Suppose the identifiability assumption (I) holds true. Then the asymptotic covariance matrices of the reduced form estimators $\hat{\pi}$, $\hat{\pi}_{2SLS}$, and $\tilde{\pi}$ satisfy the following chain of inequalities

$$V(\hat{\pi}) \leq V(\hat{\pi}_{2SLS}) \leq V(\tilde{\pi})$$

with equality in the second inequality if, and only if, $X_2 = X$ and block (3) ist just identified.The first inequality can never become an equality.

Proof: We need only compare the covariance matrices of the estimators of π_1. We have the obvious chain of inequalities

$$\Psi(\Psi' M \Psi + \Sigma_2^*)^{-1}\Psi' \leq \Psi(\Psi' M \Psi)^{-1}\Psi' \leq M^{-1}, \tag{28}$$

and we also have the inequality (24) again. A comparison of the covariance matrices in Lemmas 3, 5 and 2 then immediately implies the assertion of Proposition 4. In particular, equality in $V(\hat{\pi}_{2SLS})$ and $V(\tilde{\pi})$ is equivalent to $A_2 = I$ and Ψ nonsingular, which is equivalent to $X_2 = X$ and block (3) being just identified, respectively, see A3.
Remark: It should be clear that Proposition 3 is not simply implied by Proposition 4. Proposition 3 is not based on the identifiability condition (I), whereas Proposition 4 is. Therefore the inequality chain (28) cannot be used to prove inequality (23). (23) has to be proved without assuming the invertibility of $\Psi' M \Psi$.

With regard to TSLS applied to the partially reduced form (3) and (4), we only remark that in this case it is identical to 2SLS, so long as the independence of the U_i is properly taken into account, see A4. By this we mean that when the joint error covariance matrix of the model is estimated (in preparation to the third stage of TSLS), the covariances between the u-variables of the first block, U_1, and those of the second block, U_2, are not estimated but are taken to be zero a priori. Alternatively, if these covariances are not set to zero but are estimated, without regard to the independence assumption of the model, then TSLS differs from 2SLS. But even in this case, both methods will be asymptotically equivalent.

7 Conclusion

The main result of the paper can be summarized in the following diagram:

$$
\begin{array}{ccccccccc}
(A) & \Rightarrow & \hat{\varphi} & & (A) & & (I) & & \\
 & & \downarrow & & \Downarrow & & \Downarrow & & \\
 & & \hat{\hat{\alpha}} & \succ & \hat{\alpha} & \succ & \hat{\alpha}_{2SLS} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \hat{\hat{\pi}} & \succ & \hat{\pi} & \succ & \hat{\pi}_{2SLS} & \succ & \tilde{\pi}
\end{array}
$$

Here φ is the collection of all the unknown parameters in the coefficient matrices G_i and F_j of the block recursive model (1), (2), and $\hat{\varphi}$ is an efficient estimator of φ, where each block has been estimated separately by some efficient method like TSLS. The vector $\alpha = (\delta', \gamma_2')'$ comprises all the coefficients of the partially reduced form (3), (4), and $\hat{\alpha}$ and $\hat{\alpha}_{2SLS}$ are its OLS and 2SLS estimators, respectively, whereas $\hat{\hat{\alpha}}$ is the structural estimator of α derived from $\hat{\varphi}$. The vector π comprises all the reduced form coefficients of (15), (16), and $\tilde{\pi}$ is the OLS estimator of π. $\hat{\hat{\pi}}$, $\hat{\pi}$, and $\hat{\pi}_{2SLS}$ are derived from $\hat{\hat{\alpha}}$, $\hat{\alpha}$, and $\hat{\alpha}_{2SLS}$, respectively, by solving the system (3), (4) for Y_1 and Y_2. The arrows indicate these derivations. The sign $\succ$ is to be read as "is more (or equally) efficient than (as)". The symbols (A) and (I) indicate that the corresponding estimators are based on asssumptions (A) or (I).

The efficiency relation $\hat{\alpha} \succ \hat{\alpha}_{2SLS}$ was not proved explicitly. It follows, however, directly from Lemmas 1 and 4.

We also derived formulas for the various asymptotic covariance matrices. They are given in Lemmas 1 to 5. Finally we stated necessary and sufficient conditions for the equality of the various covariance matrices. It should be noted that in case of equality of the covariance matrices the estimators themselves (except for $\hat{\hat{\alpha}}$ and $\hat{\hat{\pi}}$) are also equal.

Finally we would like to emphasize some of the results which might be considered counter-intuitive.

1. Because $\hat{\alpha}_{2SLS} = \hat{\alpha}_{TSLS}$ and therefore $\hat{\pi}_{2SLS} = \hat{\pi}_{TSLS}$, it would seem to follow from Dhrymes' theorem that $\hat{\pi}_{2SLS}$ was efficient. But, in fact, $\hat{\pi}$ is more efficient. The reason for this is that in a recursive model $\hat{\alpha}_{TSLS}$ is not necessarily efficient any more, in contrast to what one is used in conventional simultaneous equation models. Indeed the OLS estimator $\hat{\alpha}$ is more efficient than $\hat{\alpha}_{TSLS}$.

2. Dhrymes' theorem might also seem to imply that the efficiency relation $\hat{\pi}_{2SLS} \succ \hat{\pi}$ needs no further proof, as it seems to follow directly from this theorem. But as $\hat{\pi}_{2SLS}$ is not efficient in a recursive model, this relation is not so obvious after all.

3. Even though $\hat{\alpha}$ is efficient in a general recursive model, we here have a more efficient estimator, viz., $\hat{\hat{\alpha}}$. But note that $\hat{\alpha}$ is only efficient if the recursive model parameters α are not restricted in any way. If the recursive model is

derived from a block recursive model -as is the case here-, the parameters α will generally be restricted (unless all the equations are just identified). Therefore $\hat{\hat{\alpha}} \succ \hat{\alpha}$ and consequently $\hat{\hat{\pi}} \succ \hat{\pi}$.

References

- Court, R.H. (1973), *Efficient estimation of the reduced form from incomplete econometric models*, Review of Economic Studies 40, 411-417.
- Dhrymes, P.J.(1973), *Restricted and unrestricted reduced forms: Asymptotic distribution and relative efficiency*, Econometrica 41, 119-134.
- Goldberger, A.S., A.L. Nagar and H.S. Odeh (1961), *The covariance matrix of reduced form coefficients and of forecasts for a structural econometric model*, Ecomometrica 29, 556-573.
- Kakwani, N.C. and R.H. Court (1972), *Reduced-form coefficient estimation and forecasting from a simultaneous equation model*, Australian Journal of Statistics 14, 143-160.
- Kosák, J. (1988), *Application of Demidenko's approach to the solution of the prediction problem*, Economicko-Matematický Obzor 24, 68-80.
- Nagar, A.L. and S. Sahay (1978), *The bias and mean squared error of forecasts from partially restricted reduced forms*, Journal of Econometrics 7, 227-244.
- Pfannes, M. (1996), *Die reduzierte Form von block-rekursiven Modellen*, Dissertation, University of Munich.

- Sant, D. (1978), *Partially restricted reduced forms: Asymptotic relative efficiency*, International Economic Review 19, 739-748.

- Schneeweiss, H. (1993), *The reduced form of recursive models: asymptotic properties.* In:Schneeweiss, H. and K.F. Zimmermann (eds.),*Studies in Applied Econometrics*, Heidelberg: Physica, 92-115.

- Schneeweiss, H. (1996), *The reduced form of recursive models: small sample results.* Linear Algebra and its Applications 237/238, Fifth Special Issue on Linear Algebra and Statistics, 277-300.

- Schneeweiss, H. Maschke, E.O. and Pfannes, M. (2001), *The reduced form of a block recusive model.* In:R. Friedmann, L. Knüppel, H. Lütkepohl (Eds):*Econometric Studies, A Festschrift in Honour of Joachim Frohn.* Münster-Hamburg-London, LITVerlag, 31-47.

- Wold, H. in association with Lars Juréen (1953), *Demand Analysis. A Study in Econometrics*, Stockholm: Almquist and Wiksell, New York: John Wiley and Sons.

- Wold, H. (1964), *Econometric Model Building*, Amsterdam: North-Holland Publ. Co.

Appendix

A1 M_Z is positive definite.

By (11) $M_Z = \Psi^{'} M \Psi + \Sigma_2^*$ with $\Psi = (A_2\Gamma_2, A_1)$. Partition $\Psi^{'} M \Psi$ into four blocks according to the partitioning of Ψ. The lower right block is $A_1{}' M A_1$, which is positive definite (p.d.), because M is p.d. and A_1 has full column rank. Let x be an arbitrary $(q_2 + p_1)$-dimensional vector such that $x' M_Z x = 0$.

Let $x = (x_1{}', x_2{}')'$ with $x_1 \in \mathbb{R}^{q_2}$ and $x_2 \in \mathbb{R}^{p_1}$. Then

$$x' M_Z x = x' \Psi' M \Psi x + x_1{}' \Sigma_2 x_1 = 0. \qquad (A_1)$$

Because Σ_2 is p.d., it follows that $x_1 = 0$, and the first term of (A_1) becomes $x_2' A_1' M A_1 x_2 = 0$, which implies $x_2 = 0$. Hence $x = 0$, and M_Z must be p.d..

A2 The case of equality in Proposition 2

Assume $V(\hat{\pi}) = V(\hat{\hat{\pi}})$. By(20), $V(\hat{\pi}) - V(\hat{\hat{\pi}})$ equals

$$F[V(\hat{\delta}) - V(\hat{\hat{\delta}})]F' + G[V(\hat{\gamma}_2) - V(\hat{\hat{\gamma}}_2)]G' = 0. \qquad \text{(A2)}$$

But since by (13) and (14) both terms of (A2) are positive semi definite, both terms must be zero. With $G' = (B \bigotimes A_2', I)$, the

lower right block of the second term is $V(\hat{\gamma}_2) - V(\hat{\hat{\gamma}}_2)$, which thus is zero. The vanishing of the first term implies, because $F' = (I \bigotimes \Psi', 0)$,
that

$$(I \bigotimes \Psi)[V(\hat{\delta}) - V(\hat{\hat{\delta}})](I \bigotimes \Psi') = 0.$$

If Ψ has full column rank, then $V(\hat{\delta}) - V(\hat{\hat{\delta}}) = 0$. Thus $V(\hat{\pi}) = V(\hat{\hat{\pi}})$ implies $V(\hat{\gamma}_2) = V(\hat{\hat{\gamma}}_2)$ and $V(\hat{\delta}) = V(\hat{\hat{\delta}})$.

A3 The case of equality in Proposition 4

Suppose $V(\hat{\pi}_{2SLS}) = V(\tilde{\pi})$. By Lemmas 5 and 2 this implies

$$\Psi(\Psi' M \Psi)^{-1} \Psi' = M^{-1} \qquad (A3)$$

$$A_2(A_2' M A_2)^{-1} A_2' = M^{-1} \qquad (A4)$$

As in the proof of Proposition 3, we see that (A4) is equivalent to A_2 being square, which is equivalent to $p_2 = p$, and thus $X_2 = X$.

By the same argument, A3 is equivalent to Ψ being square. This means that Y_2 and X_{22} have the same number of columns, which is the condition for (3) to be just identified . Thus

$V(\hat{\pi}_{2SLS}) = V(\tilde{\pi})$, if and only if, $X_2 = X$ and (3) is just identified.

A4 Three Stage Least Squares (TSLS)

In order to derive the TSLS estimator of δ and γ_2, multiply (3) and (4) by the instrumental variable matrix X' from the left:

$$X'Y_1 = X'Z\triangle + X'U_1$$

$$X'Y_2 = X'X_2\Gamma_2 + X'U_2$$

Vectorization yields

$$\begin{aligned} \begin{pmatrix} (I \bigotimes X')y_1 \\ (I \bigotimes X')y_2 \end{pmatrix} &= \begin{pmatrix} I \bigotimes X'Z & 0 \\ 0 & I \bigotimes X'X_2 \end{pmatrix} \begin{pmatrix} \delta \\ \gamma_2 \end{pmatrix} \\ &+ \begin{pmatrix} (I \bigotimes X')u_1 \\ (I \bigotimes X')u_2 \end{pmatrix}, \end{aligned}$$

which we write more shortly as

$$y^+ = X^+\delta^+ + u^+$$

with obvious notations. The covariance matrix of u^+ is given by

$$V^{+} = \begin{pmatrix} \Sigma_1 \bigotimes X'X & 0 \\ 0 & \Sigma_2 \bigotimes X'X) \end{pmatrix}$$

where the independence of u_1 and u_2 was taken into account. The TSLS estimator $\hat{\delta}^{+}$ is then given as the solution to

$$X^{+\prime} V^{+-1} X^{+} \hat{\delta}^{+} = X^{+\prime} V^{+-1} y^{+}.$$

Introducing the projection matrix $P = X(X'X)^{-1}X'$, this system can also be written more explicitly as

$$\begin{pmatrix} \Sigma_1^{-1} \bigotimes Z'PZ & 0 \\ 0 & \Sigma_2^{-1} \bigotimes X_2{}'PX_2) \end{pmatrix} \begin{pmatrix} \hat{\delta} \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \Sigma_1^{-1} \bigotimes Z'Py_1 \\ \Sigma_2^{-1} \bigotimes X_2{}'Py_2 \end{pmatrix}$$

Multiplication by the block diagonal matrix

$$\begin{pmatrix} \Sigma_1 \bigotimes I & 0 \\ 0 & \Sigma_2 \bigotimes I \end{pmatrix}$$

yields

$$\begin{pmatrix} I \bigotimes Z'PZ & 0 \\ 0 & I \bigotimes X_2{}'PX_2) \end{pmatrix} \begin{pmatrix} \hat{\delta} \\ \hat{\gamma}_2 \end{pmatrix} = \begin{pmatrix} I \bigotimes Z'Py_1 \\ I \bigotimes X_2{}'Py_2 \end{pmatrix},$$

which is equivalent to the two equations

$$Z'PZ_1\hat{\triangle} = Z'PY_1$$

$$X_2{}'PX_2\hat{\Gamma}_2 = X_2{}'PY_2.$$

The first one is just the 2SLS estimating equation for $\triangle$, and the second one can be reduced to

$$X_2{}'X_2\hat{\Gamma}_2 = X_2{}'Y_2$$

because $X_2{}'P = X_2{}'$.