

Shinozaki, Hiroki; Kazumura, Tomoya; Serizawa, Shigehiro

Working Paper

Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity

ISER Discussion Paper, No. 1097

Provided in Cooperation with:

The Institute of Social and Economic Research (ISER), Osaka University

Suggested Citation: Shinozaki, Hiroki; Kazumura, Tomoya; Serizawa, Shigehiro (2020) : Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity, ISER Discussion Paper, No. 1097, Osaka University, Institute of Social and Economic Research (ISER), Osaka

This Version is available at:

<https://hdl.handle.net/10419/234910>

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Supplementary material for “Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity”

Hiroki Shinozaki, Tomoya Kazumura, and Shigehiro Serizawa

August 6, 2020

In this supplementary material, we provide the proofs and the example omitted in the main text (Shinozaki et al., 2020).

1 Proofs of Remarks

In this section, we give the proofs of some remarks in the main text.

Remark 3 (Upward convexity). Let $m' \in M$ with $m' > 0$. Let $i \in N$, $R_i \in \mathcal{R}^{ND}$, and $z_i \in M \times \mathbb{R}$. (i) For each $x_i \in M(m')$, $\frac{x_i}{m'}v_i(m', z_i) \geq v_i(x_i, z_i)$. (ii) If there is $x_i \in M(m') \setminus \{0, m'\}$ such that $\frac{x_i}{m'}v_i(m', z_i) > v_i(x_i, z_i)$, then for each $x'_i \in M(m') \setminus \{0, m'\}$, $\frac{x'_i}{m'}v_i(m', z_i) > v_i(x'_i, z_i)$.

Proof. (i) Let $x_i \in M(m')$. Then

$$\begin{aligned} m'v_i(x_i, z_i) &= m' \left(\sum_{x=0}^{x_i-1} (v_i(x+1, z_i) - v_i(x, z_i)) \right) \\ &\leq x_i \left(\sum_{x=0}^{m'-1} (v_i(x+1, z_i) - v_i(x, z_i)) \right) && \text{(by } R_i \in \mathcal{R}^{ND}\text{)} \\ &= x_iv_i(m', z_i). \end{aligned}$$

(ii) Suppose there is $x_i \in M(m') \setminus \{0, m'\}$ such that $\frac{x_i}{m'}v_i(m', z_i) > v_i(x_i, z_i)$. Then there is $x \in M(m') \setminus \{0, m'\}$ such that $v_i(x+1, z_i) - v_i(x, z_i) > v_i(x, z_i) - v_i(x-1, z_i)$. Let $x'_i \in M(m') \setminus \{0, m'\}$. Then the inequality above holds strictly for x'_i . \square

Remark 7. Let $R_i \in \mathcal{R}^{++}$. (i) Let $x_i \in M \setminus \{m\}$ and $h^+(\cdot; x_i) : \mathbb{R} \rightarrow \mathbb{R}_{++}$ be such that $h^+(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$ for each $t_i \in \mathbb{R}$. Then $h^+(\cdot; x_i)$ is strictly decreasing in t_i . (ii) Let $x_i \in M \setminus \{0\}$ and $h^-(\cdot; x_i) : \mathbb{R} \rightarrow \mathbb{R}_{++}$ be such that $h^-(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$ for each $t_i \in \mathbb{R}$. Then $h^-(\cdot; x_i)$ is strictly decreasing in t_i as well.

Proof. (i) Let $t_i, t'_i \in \mathbb{R}$ be such that $t'_i < t_i$. Note that $(x_i + 1, V_i(x_i + 1, (x_i, t_i))) I_i (x_i, t_i)$. Thus, by $t_i - t'_i > 0$, $R_i \in \mathcal{R}^{++}$ implies

$$(x_i + 1, V_i(x_i + 1, (x_i, t_i)) - (t_i - t'_i)) P_i (x_i, t_i - (t_i - t'_i)) = (x_i, t'_i) I_i (x_i + 1, V_i(x_i + 1, (x_i, t'_i))).$$

Thus, $V_i(x_i + 1, (x_i, t_i)) - (t_i - t'_i) < V_i(x_i + 1, (x_i, t'_i))$, or $h^+(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i < V_i(x_i + 1, (x_i, t'_i)) - t'_i = h^+(t'_i; x_i)$.

(ii) We can show (ii) in a symmetric way, and we omit the proof. \square

2 Proof of Proposition 3

In this section, we give the proof of Proposition 3.

Proposition 3 (The inverse-demand sets). Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. Then (i) $P(0; R_0) = [V_0(1, \mathbf{0}), \infty)$, (ii) $P(1; R_0) = [t^*(1), V_0(1, \mathbf{0})]$, (iii) $P(x; R_0) = [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$ for each $x \in M \setminus \{0, 1, m\}$, and (iv) $P(m; R_0) = [0, \frac{t^*(m-1)}{m-1}]$.

Before proving Proposition 3, we show the following two lemmas.

Lemma 12. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$. (i) Let $x \in M \setminus \{0, m\}$ and $p \in P(x; R_0)$. Then $p \geq \frac{t^*(x)}{x}$. (ii) Let $x \in M \setminus \{0, 1\}$ and $p \in P(x; R_0)$. Then $p \leq \frac{t^*(x-1)}{x-1}$.

Proof. (i) Suppose $p < \frac{t^*(x)}{x}$. Then $px < t^*(x)$, and by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (i) implies $V_0(x + 1, (x, px)) - px > \frac{px}{x} = p$, or $V_0(x', (x, px)) > px'$, where $x' \equiv x + 1$. This implies (x', px') $P_0 (x, px)$, which contradicts $p \in P(x; R_0)$.

(ii) We can show (ii) in the same way as (i) by using Lemma 10 (ii) instead of Lemma 10 (i). \square

Lemma 13. Let $R_0 \in \mathcal{R}^D$ and $p \in \mathbb{R}_+$. (i) Let $x \in M \setminus \{0\}$. If $p \geq \frac{t^*(x)}{x}$, then for each $x' \in M$ with $x' > x$, $(x, px) R_0 (x', px')$. (ii) Let $x \in M \setminus \{m\}$. If $p \leq \frac{t^*(x-1)}{x-1}$, then for each $x' \in M$ with $x' < x$, $(x, px) R_0 (x', px')$.

Proof. (i) Suppose $p \geq \frac{t^*(x)}{x}$. Then $px \geq t^*(x)$. Thus, by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (i)

implies $V_0(x+1, (x, px)) - px \leq \frac{px}{x} = p$. Let $x' \in M$ with $x' > x$. Then

$$\begin{aligned}
V_0(x', (x, px)) - px &= \sum_{x''=x}^{x'-1} \left(V_0(x''+1, (x, px)) - V_0(x'', (x, px)) \right) \quad (\text{by Remark 1 (iii)}) \\
&\leq (x' - x) \left(V_0(x+1, (x, px)) - V_0(x, (x, px)) \right) \quad (\text{by } R_0 \in \mathcal{R}^D) \\
&= (x' - x) \left(V_0(x+1, (x, px)) - px \right) \quad (\text{by Remark 1 (iii)}) \\
&\leq (x' - x)p \quad (\text{by } V_0(x+1, (x, px)) - px \leq p) \\
&= px' - px,
\end{aligned}$$

or $V_0(x', (x, px)) \leq px'$. This implies $(x, px) R_0 (x', px')$.

(ii) By using Lemma 10 (ii) instead of Lemma 10 (i), we can show (ii) in a symmetric way. \square

We now proceed to the proof of Proposition 3.

(i) First, we show $P(0; R_0) \subseteq [V_0(1, \mathbf{0}), \infty)$. Let $p \in P(0; R_0)$. Then $\mathbf{0} R_0 (1, p)$, which implies $p \geq V_0(1; \mathbf{0})$.

Next, we show $[V_0(1, \mathbf{0}), \infty) \subseteq P(0; R_0)$. Let $p \in [V_0(1, \mathbf{0}), \infty)$. Then by $p \geq V_0(1, \mathbf{0})$, $\mathbf{0} R_0 (1, p)$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 11 implies that $p \geq V_0(1, \mathbf{0}) > t^*(1)$. Thus, by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 14 (i) implies $(1, p) R_0 (x, px)$ for each $x \in M$ with $x > 1$. Thus, by $\mathbf{0} R_0 (1, p)$, $\mathbf{0} R_0 (x, px)$ for each $x \in M$ with $x > 0$. Thus, $p \in P(0; R_0)$.

(ii) We show $P(1; R_0) = [t^*(1), V_0(1, \mathbf{0})]$. Note that by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 11 implies $t^*(1) < V_0(1, \mathbf{0})$, and the interval is well-defined.

We first show $P(1; R_0) \subseteq [t^*(1), V_0(1, \mathbf{0})]$. Let $p \in P(1; R_0)$. By $(1, p) R_0 \mathbf{0}$, $p \leq V_0(1, \mathbf{0})$. Also, by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 12 (i) implies $p \geq t^*(1)$.

Then, we show $[t^*(1), V_0(1, \mathbf{0})] \subseteq P(1; R_0)$. Let $p \in [t^*(1), V_0(1, \mathbf{0})]$. By $p \leq V_0(1, \mathbf{0})$, $(1, p) R_0 \mathbf{0}$. Also, by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $p \geq t^*(1)$, Lemma 14 (ii) gives $(1, p) P_0 (x, xp)$ for each $x \in M$ with $x > 2$. Thus, $p \in P(1; R_0)$.

(iii) Let $x \in M \setminus \{0, 1, m\}$. We show $P(x; R_0) = [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 11, the interval is well-defined. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 12 (i) and (ii) imply $P(x; R_0) \subseteq [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$, and Lemma 14 (i) and (ii) give $[\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}] \subseteq P(x; R_0)$.

(iv) Finally, we show $P(m; R_0) = [0, \frac{t^*(m-1)}{m-1}]$. Note that by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 9 gives that $\frac{t^*(m-1)}{m-1} > 0$. Thus, the interval is well-defined. By $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 12 (ii) implies that $P(m; R_0) \subseteq [0, \frac{t^*(m-1)}{m-1}]$. Also, by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 14 (ii) implies that $[0, \frac{t^*(m-1)}{m-1}] \subseteq P(m; R_0)$. \blacksquare

3 Proof of Proposition 4

In this section, we provide the proof of Proposition 4.

Proposition 4. *Assume $n = 2$ and m is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ satisfy*

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq p(\beta - 1; R_0), \quad (1)$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=1}^{\beta} p(m - x; R_0)$. Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. An inverse-demand-based generalized Vickrey rule on \mathcal{R}^2 satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Let $g \equiv (x, t)$ be an inverse-demand-based generalized Vickrey rule on \mathcal{R}^2 .

Lemma 14. *Let $R \in \mathcal{R}^2$ and $i, j \in N$ be a pair such that $R_i = R_0$ and $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$. Let $v_j > 0$ be a constant marginal valuation associated with R_j . (i) If $x_i(R) > 0$, then $t_i(R) - V_i(x_i(R) - 1, g_i(R)) \geq v_j$. (ii) If $x_i(R) < m$, then $V_i(x_i(R) + 1, g_i(R)) - t_i(R) \leq v_j$.*

Proof. By the definition of the inverse-demand-based generalized Vickrey rule and $R \in (\mathcal{R}^{NI})^2$, $p(x_i(R); R_i) \leq v_j \leq p(x_i(R) - 1; R_i)$ and $t_i(R) = v_j x_i(R)$.

(i) Suppose $x_i(R) = 1$. Then $v_j \leq p(0; R_i) = V_i(1, \mathbf{0})$, where the inequality follows from $v_j \leq p(x_i(R) - 1; R_i)$. Then we have $g_i(R) = (1, v_j)$, $R_i(1, V_i(1, \mathbf{0})) I_i \mathbf{0}$. This implies $V_i(0, g_i(R)) \leq 0$. Thus, by $t_i(R) = v_j$, $t_i(R) - V_i(0, g_i(R)) \geq v_j$.

Suppose instead $x_i(R) > 1$. By $0 < x_i(R) - 1 < m$, $p(x_i(R) - 1; R_i) = \frac{t^*(x_i(R) - 1)}{x_i(R) - 1}$. Thus, by $v_j \leq p(x_i(R) - 1; R_i)$,

$$t_i(R) = v_j x_i(R) \leq p(x_i(R) - 1; R_i) x_i(R) = \frac{x_i(R)}{x_i(R) - 1} t^*(x_i(R) - 1).$$

Thus, by $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (ii), and $t_i(R) = v_j x_i(R)$,

$$t_i(R) - V_i(x_i(R) - 1, g_i(R)) \geq \frac{t_i(R)}{x_i(R)} = v_j.$$

(ii) If $x_i(R) = 0$, then $V_i(x_i(R) + 1, g_i(R)) - t_i(R) = V_i(1, \mathbf{0}) = p(0; R_i) \leq v_j$, where the first equality follows from $g_i(R) = \mathbf{0}$, and the inequality follows from $v_j \geq p(x_i(R); R_i)$.

Suppose instead $x_i(R) > 0$. By $0 < x_i(R) < m$, $p(x_i(R); R_i) = \frac{t^*(x_i(R))}{x_i(R)}$. Thus, by $v_j \geq p(x_i(R); R_i)$,

$$t_i(R) = v_j x_i(R) \geq p(x_i(R); R_i) x_i(R) = t^*(x_i(R)).$$

Thus, by $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$, Lemma 10 (i), and $t_i(R) = v_j x_i(R)$

$$V_i(x_i(R) + 1, g_i(R)) - t_i(R) \leq \frac{t_i(R)}{x_i(R)} = v_j,$$

as desired. \square

We now proceed to the proof of Proposition 4. Since *no subsidy for losers* is immediate from the definition of the rule, we here show the other three properties.

INDIVIDUAL RATIONALITY. Let $R \in \mathcal{R}^2$ and $i, j \in N$ be a distinct pair. If $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$, then $R_i = R_i^{inv}$. Since $g_i(R)$ is the outcome of the Vickrey rule for R^{inv} , $g_i(R) R_i \mathbf{0}$ by *individual rationality* of the Vickrey rule. Thus, we assume $R_i = R_0$. We consider the following two cases.

CASE 1. $R_j = R_0$.

By the definition of the inverse-demand-based generalized Vickrey rule and $R \in (\mathcal{R}^D)^2$, $x_i(R) = \beta$ and

$$t_i(R) = \sum_{x=0}^{m-1} p(x; R_j) - \sum_{x=0}^{\beta-1} p(x; R_j) = \sum_{x=\beta}^{m-1} p(x; R_j) = \sum_{x=1}^{\beta} p(m-x; R_j) = t^*.$$

Then

$$t_i(R) = \sum_{x=1}^{\beta} p(m-x; R_j) \leq \beta p(\beta; R_j) = t^*(\beta; R_j) < V_i(\beta, \mathbf{0}),$$

where the first inequality follows from $R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $\beta = \frac{m}{2}$, and Lemma 11, and the second inequality follows from $R_i = R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 9. Thus, $g_i(R) R_i \mathbf{0}$.

CASE 2. $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$.

By the definition of the inverse-demand-based generalized Vickrey rule, $t_i(R) = v_j x_i(R)$, where $v_j > 0$ is a constant marginal valuation associated with R_j . If $x_i(R) > 0$, then

$$\begin{aligned} t_i(R) - V_i(0, g_i(R)) &= \sum_{x=1}^{x_i(R)} \left(V_i(x, g_i(R)) - V_i(x-1, g_i(R)) \right) && \text{(by Remark 1 (iii))} \\ &\geq x_i(R) \left(V_i(x_i(R), g_i(R)) - V_i(x_i(R)-1, g_i(R)) \right) && \text{(by } R_i \in \mathcal{R}^D) \\ &\geq x_i(R) v_j && \text{(by Lemma 14 (i))} \\ &= t_i(R), \end{aligned}$$

or $V_i(0, g_i(R)) \leq 0$. Thus, $g_i(R) R_i \mathbf{0}$. On the other hand, if $x_i(R) = 0$, then $g_i(R) = \mathbf{0}$.

STRATEGY-PROOFNESS. Let $R \in \mathcal{R}^2$ and $i, j \in N$ be a distinct pair. If $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$,

then since $g(R)$ and $g(R'_i, R_j)$ are both the outcomes of the Vickrey rule, *strategy-proofness* of the Vickrey rule implies that $g_i(R) R_i g_i(R'_i, R_j)$. Thus, we assume $R_i = R_0$.

Let $R'_i \in \mathcal{R}$. Let $x_i \equiv x_i(R'_i, R_{-i})$. We show $g_i(R) R_i (x_i, t_i(R'_i, R_j))$. If $x_i = 0$, then $t_i(R'_i, R_j) = 0$ by Lemma 2. Thus, by *individual rationality* of g , suppose $x_i > 0$.

CASE 1. $R_j = R_0$.

By the definition of the inverse-demand-based generalized Vickrey rule and $R \in (\mathcal{R}^D)^2$, $g_i(R) = (\beta, t^*)$ and $t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j)$.

First, suppose $x_i > x_i(R)$. Then

$$t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j) = t^* + \sum_{x=\beta+1}^{x_i} p(m-x; R_j). \quad (2)$$

For each $x \in M$ with $x \geq \beta+1$, we have

$$\begin{aligned} p(m-x; R_j) &\geq p(\beta-1; R_i) \\ &\geq V_i(\beta+1, g_i(R)) - t^* \\ &= V_i(\beta+1, g_i(R)) - V_i(\beta, g_i(R)) \\ &\geq V_i(x, g_i(R)) - V_i(x-1, g_i(R)), \end{aligned} \quad (3)$$

where the first inequality follows from $R_i = R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x \geq \beta+1$, and Lemma 11, the second one follows from (1), $R_i = R_0$, and $g_i(R) = (\beta, t^*)$, the equality follows from $t^* = t_i(R)$ and Remark 1 (iii), and the last inequality comes from $R_i \in \mathcal{R}^D$ and $x \geq \beta+1$.

Then

$$\begin{aligned} t_i(R'_i, R_j) &= t^* + \sum_{x=\beta+1}^{x_i} p(m-x; R_j) && \text{(by (2))} \\ &\geq t^* + \sum_{x=\beta+1}^{x_i} \left(V_i(x, g_i(R)) - V_i(x-1, g_i(R)) \right) && \text{(by (3))} \\ &= V_i(x_i(R), g_i(R)) + \left(V_i(x_i, g_i(R)) - V_i(x_i(R), g_i(R)) \right) \\ & && \text{(by } g_i(R) = (\beta, t^*) \text{ and Remark 1 (iii))} \\ &= V_i(x_i, g_i(R)), \end{aligned}$$

which implies $g_i(R) R_i (x_i, t_i(R'_i, R_j))$.

Suppose instead $x_i < x_i(R)$. Then

$$t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j) = t^* - \sum_{x=x_i}^{\beta-1} p(m-x-1; R_j). \quad (4)$$

Note that by $\beta = x_i(R) > x_i > 0$, $\beta - 1 > 0$. Thus, by $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 9, $\frac{\beta}{\beta-1}t^*(\beta-1; R_i) = V_i(\beta, (\beta-1, t^*(\beta-1; R_i)))$. This implies

$$\left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right) = \left(\beta, V_i(\beta, (\beta-1, t^*(\beta-1; R_i)))\right) I_i(\beta-1, t^*(\beta-1; R_i)).$$

Thus, by Remark 1 (i) and (iii),

$$V_i\left(\beta-1, \left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right)\right) = V_i(\beta-1, (\beta-1, t^*(\beta-1; R_i))) = t^*(\beta-1; R_i). \quad (5)$$

By $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $\beta = \frac{m}{2}$, Lemma 11 implies

$$t^* = \sum_{x=1}^{\beta} p(m-x; R_i) < \beta p(\beta-1; R_i) = \frac{\beta}{\beta-1}t^*(\beta-1; R_i). \quad (6)$$

Then

$$\begin{aligned} t^* - V_i(\beta-1, (\beta, t^*)) &> \frac{\beta}{\beta-1}t^*(\beta-1; R_i) - V_i\left(\beta-1, \left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right)\right) \\ &= \frac{\beta}{\beta-1}t^*(\beta-1; R_i) - t^*(\beta-1; R_i) \\ &= \frac{t^*(\beta-1; R_i)}{\beta-1} \\ &= p(\beta-1; R_j), \end{aligned} \quad (7)$$

where the inequality follows from $R_i \in \mathcal{R}^{++}$, (6), and Remark 7 (ii), the first equality follows from (5), and the last one follows from $R_i = R_j$. Then for each $x \in M$ with $x \leq \beta-1$,

$$\begin{aligned} V_i(x+1, g_i(R)) - V_i(x, g_i(R)) &\geq V_i(\beta, g_i(R)) - V_i(\beta-1, g_i(R)) \\ &= t^* - V_i(\beta-1, (\beta, t^*)) \\ &> p(\beta-1; R_j) \\ &> p(m-x-1; R_j), \end{aligned} \quad (8)$$

where the first inequality follows from $R_i \in \mathcal{R}^D$ and $x \leq \beta-1$, the equality follows from $g_i(R) = (\beta, t^*)$ and Remark 1 (iii), the second inequality follows from (7), and the last

one comes from $R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$, $x \leq \beta - 1$, and Lemma 11. Thus,

$$\begin{aligned}
t_i(R'_i, R_j) &= t^* - \sum_{x=x_i}^{\beta-1} p(m-x-1; R_j) && \text{(by (4))} \\
&> t^* - \sum_{x=x_i}^{\beta-1} \left(V_i(x+1, g_i(R)) - V_i(x, g_i(R)) \right) && \text{(by (8))} \\
&= V_i(x_i(R), g_i(R)) - \left(V_i(x_i(R), g_i(R)) - V_i(x_i, g_i(R)) \right) \\
&\hspace{15em} \text{(by } g_i(R) = (\beta, t^*) \text{ and Remark 1 (iii))} \\
&= V_i(x_i, g_i(R)),
\end{aligned}$$

which implies $g_i(R) P_i(x_i, t_i(R'_i, R_j))$.

CASE 2. $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$.

By the definition of the inverse-demand-based generalized Vickrey rule, $t_i(R) = v_j x_i(R)$ and $t_i(R'_i, R_j) = v_j x_i$, where $v_j > 0$ is a constant marginal valuations associated with R_j .

If $x_i > x_i(R)$, then $x_i(R) < m$, and

$$\begin{aligned}
&V_i(x_i, g_i(R)) - t_i(R) \\
&= \sum_{x=x_i(R)}^{x_i-1} \left(V_i(x+1, g_i(R)) - V_i(x, g_i(R)) \right) && \text{(by Remark 1 (iii))} \\
&\leq (x_i - x_i(R)) \left(V_i(x_i(R)+1, g_i(R)) - V_i(x_i(R), g_i(R)) \right) && \text{(by } R_i \in \mathcal{R}^D) \\
&\leq (x_i - x_i(R)) v_j && \text{(by Lemma 14 (ii))} \\
&= t_i(R'_i, R_j) - t_i(R),
\end{aligned}$$

or $V_i(x_i, g_i(R)) \leq t_i(R'_i, R_j)$. Thus, $g_i(R) R_i(x_i, t_i(R'_i, R_j))$.

The other case can be treated symmetrically.

EFFICIENCY. Let $R \in \mathcal{R}^2$. If $R \in (\mathcal{R}^C \cap \mathcal{R}^Q)^2$, then by *efficiency* of the Vickrey rule, $g(R)$ is efficient for R . Thus, we consider the following two cases.

CASE 1. $R_1 = R_2 = R_0$.

By the definition of the inverse-demand-based generalized Vickrey rule and $R \in (\mathcal{R}^D)^2$, $g_1(R) = g_2(R) = (\beta, t^*)$. By $g_1(R) = g_2(R)$ and Remark 1 (iii), $R_1 = R_2 \in \mathcal{R}^D$ gives $V_i(x_i(R)+1, g_i(R)) - t_i(R) < t_j(R) - V_j(x_j(R)-1, g_j(R))$ for each pair $i, j \in N$. Thus, by $R \in (\mathcal{R}^{NI})^2$ and Remark 5, $g(R)$ is efficient for R .

CASE 2. $R_i = R_0$ and $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ for some pair $i, j \in N$.

Without loss of generality, let $i = 1$ and $j = 2$. By the definition of the inverse-demand-based generalized Vickrey rule, $t_1(R) = v_2 x_1(R)$, where $v_2 > 0$ is a constant marginal valuation associated with R_2 .

If $x_1(R) = 0$, then by Lemma 14 (ii), $v_2 \geq V_1(1, g_1(R)) - t_1(R)$. If $0 < x_1(R) < m$, then by Lemma 14 (i) and (ii),

$$V_1(x_1(R) + 1, g_1(R)) - t_1(R) \leq v_2 \leq t_1(R) - V_1(x_1(R) - 1, g_1(R)).$$

Finally, if $x_i(R) = m$, then by Lemma 14 (i), $t_1(R) - V_1(m - 1, g_1(R)) \geq v_2$. In either case, by $R \in (\mathcal{R}^{NI})^2$, Remark 5 implies that $g(R)$ is efficient for R . ■

Remark 8. In the proof of Proposition 4, we use the condition (1) only when we prove that an agent has no incentive to misreport his preference at the preference profile $(R_1, R_2) = (R_0, R_0)$. Indeed, all the remaining parts of the proof remain valid without the condition (1) and even the assumption that m is even.

4 Proof of Proposition 5

In this section, we prove Proposition 5.

Proposition 5. *Assume $m \geq 4$ is even. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ violate*

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq p(\beta - 1; R_0), \quad (1)$$

where $\beta \equiv \frac{m}{2}$ and $t^* \equiv \sum_{x=1}^{\beta} p(m - x; R_0)$. Let \mathcal{R} be a class of preferences satisfying $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. No rule on \mathcal{R}^n satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Let $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Suppose by contradiction that there is a rule f on \mathcal{R}^n satisfying efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Note that Steps 1 to 3 in the proof of Theorem 2 depend on neither the number of agents nor the choice of a preference R_0 , and so the discussion is valid here as well. Thus, we hereafter take over the results and the notations in Steps 1 to 3 in the proof of Theorem 2.

As in the proof of Theorem 2, let $R_2 \equiv R_0$. First, we show $z_2^f(R_{-2}; \beta+1) P_2 z_1^f(R_{-2}; \beta)$.

Note that by Step 3, $z_2^f(R_{-2}; \beta) = (\beta, t^*)$. Further,

$$\begin{aligned} t_2^f(R_{-2}; \beta + 1) - t_2^f(R_{-2}; \beta) &= p(m - (\beta + 1); R_1) \\ &= p(\beta - 1; R_1) \\ &< V_2(\beta + 1, z_2^f(R_{-2}; \beta)) - t_2^f(R_{-2}; \beta), \end{aligned}$$

where the first equality follows from Step 3, the second one follows from $\beta = \frac{m}{2}$, and the inequality follows since $R_2 = R_1 = R_0$ violates (1) and $z_2^f(R_{-2}, \beta) = (\beta, t^*)$. Thus, we have $t_2^f(R_{-2}; \beta + 1) < V_2(\beta + 1, z_2^f(R_{-2}; \beta))$. This implies $z_2^f(R_{-2}; \beta + 1) P_2 z_2^f(R_{-2}; \beta)$.

By the same argument as in the proof of Theorem 2, we can show that for each $x_2 \in M$ with $x_2 < \beta$, $z_2^f(R_{-2}; x_2 + 1) P_2 z_2^f(R_{-2}; x_2)$. Thus, we have $z_2^f(R_{-2}; \beta + 1) P_2 z_2^f(R_{-2}; x_2)$ for each $x_2 \in M$ with $x_2 \leq \beta$, and Lemma 4 implies $x_2(R) \geq \beta + 1$. Since both agents 1 and 2 have the same preferences R_0 , a symmetric argument gives $x_1(R) \geq \beta + 1$. Thus,

$$x_1(R) + x_2(R) \geq 2(\beta + 1) = m + 2,$$

contradicting feasibility. ■

5 Even number of units and more than two agents

In this section, we give an example which demonstrates that when $n \geq 3$ and $m = 6a - 2$ for some $a \in \mathbb{N}$, the inverse-demand-based generalized Vickrey rule violates *strategy-proofness* on $\mathcal{R}^n \equiv ((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$ for any $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$.

Example 5. Let $n \geq 3$ and $m = 6a - 2$, where $a \in \mathbb{N}$ is an arbitrary natural number. Let $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and $\mathcal{R} = (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$. Let $g \equiv (x, t)$ be an inverse-demand-based generalized Vickrey rule on \mathcal{R}^n .

Let $R \in \mathcal{R}^n$ be such that for each $i \in \{1, 2, 3\}$, $R_i = R_0$, and for each $j \in N \setminus \{1, 2, 3\}$, $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $v_j < p(m - 1; R_0)$, where v_j is a constant marginal valuation associated with R_j . By the definition of the inverse-demand-based generalized Vickrey rule and $R_{\{1, 2, 3\}} \in (\mathcal{R}^D)^3$, $x_i(R) = x_j(R) = 2a - 1$ and $x_k(R) = 2a$ for some distinct triple $i, j, k \in \{1, 2, 3\}$, $x_l(R) = 0$ for each $l \in N \setminus \{1, 2, 3\}$,

$$t_i(R) = t_j(R) = p(2a - 1; R_0) + 2 \sum_{x=2a}^{3a-2} p(x; R_0), \quad t_k(R) = 2 \sum_{x=2a-1}^{3a-2} p(x; R_0),$$

and $t_l(R) = 0$ for each $l \in N \setminus \{1, 2, 3\}$.¹ Without loss of generality, let $i = 1$, $j = 2$, and

¹Note that when $a = 1$, i.e., $m = 4$, $t_i(R) = t_j(R) = p(2a - 1; R_0) = p(1; R_0)$. In the subsequent discussion, we consider the case where $a \geq 2$, but the same discussion can be applied to the case where $a = 1$.

$k = 3$. Let $R'_1 \in \mathcal{R}^C \cap \mathcal{R}^Q$ be such that $p(2a; R_0) < v'_1 < p(2a - 1; R_0)$, where v'_1 is a constant marginal valuation associated with R'_1 . By the definition of the inverse-demand-based generalized Vickrey rule and $R_{\{2,3\}} \in (\mathcal{R}^D)^2$, $g_1(R'_1, R_{-1}) = (2a, 2 \sum_{x=2a-1}^{3a-2} p(x; R_0))$. Note that by $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ and Lemma 11,

$$t_1(R) = p(2a - 1; R_0) + 2 \sum_{x=2a}^{3a-2} p(x; R_0) < (2a - 1)p(2a - 1; R_0) = t^*(2a - 1).$$

Thus,

$$\begin{aligned} V_1(2a, g_1(R)) - t_1(R) &> V_1(2a, (2a - 1, t^*(2a - 1))) - t^*(2a - 1) && \text{(by Remark 7 (i))} \\ &= \frac{t^*(2a - 1)}{2a - 1} && \text{(by Lemma 9)} \\ &= p(2a - 1; R_0) \\ &= t_1(R'_1, R_{-1}) - t_1(R), \end{aligned}$$

or $V_1(2a, g_1(R)) > t_1(R'_1, R_{-1})$. This implies $g_1(R'_1, R_{-1}) P_1 g_1(R)$. Thus, g violates *strategy-proofness*.

References

- [1] Shinozaki, H., T. Kazumura, and S. Serizawa (2020), “Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity.” ISER Discussion Paper No. 1097.