

Shinozaki, Hiroki; Kazumura, Tomoya; Serizawa, Shigehiro

**Working Paper**

## Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity

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# Supplementary material for “Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity”

Hiroki Shinozaki, Tomoya Kazumura, and Shigehiro Serizawa

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In this supplementary material, we provide the proofs and the example omitted in the main text (Shinozaki et al., 2020).

## 1 Proofs of Remarks

In this section, we give the proofs of some remarks in the main text.

**Remark 3 (Upward convexity).** Let  $m' \in M$  with  $m' > 0$ . Let  $i \in N$ ,  $R_i \in \mathcal{R}^{ND}$ , and  $z_i \in M \times \mathbb{R}$ . (i) For each  $x_i \in M(m')$ ,  $\frac{x_i}{m'}v_i(m', z_i) \geq v_i(x_i, z_i)$ . (ii) If there is  $x_i \in M(m') \setminus \{0, m'\}$  such that  $\frac{x_i}{m'}v_i(m', z_i) > v_i(x_i, z_i)$ , then for each  $x'_i \in M(m') \setminus \{0, m'\}$ ,  $\frac{x'_i}{m'}v_i(m', z_i) > v_i(x'_i, z_i)$ .

*Proof.* (i) Let  $x_i \in M(m')$ . Then

$$\begin{aligned} m'v_i(x_i, z_i) &= m' \left( \sum_{x=0}^{x_i-1} (v_i(x+1, z_i) - v_i(x, z_i)) \right) \\ &\leq x_i \left( \sum_{x=0}^{m'-1} (v_i(x+1, z_i) - v_i(x, z_i)) \right) && \text{(by } R_i \in \mathcal{R}^{ND}) \\ &= x_i v_i(m', z_i). \end{aligned}$$

(ii) Suppose there is  $x_i \in M(m') \setminus \{0, m'\}$  such that  $\frac{x_i}{m'}v_i(m', z_i) > v_i(x_i, z_i)$ . Then there is  $x \in M(m') \setminus \{0, m'\}$  such that  $v_i(x+1, z_i) - v_i(x, z_i) > v_i(x, z_i) - v_i(x-1, z_i)$ . Let  $x'_i \in M(m') \setminus \{0, m'\}$ . Then the inequality above holds strictly for  $x'_i$ .  $\square$

**Remark 7.** Let  $R_i \in \mathcal{R}^{++}$ . (i) Let  $x_i \in M \setminus \{m\}$  and  $h^+(\cdot; x_i) : \mathbb{R} \rightarrow \mathbb{R}_{++}$  be such that  $h^+(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i$  for each  $t_i \in \mathbb{R}$ . Then  $h^+(\cdot; x_i)$  is strictly decreasing in  $t_i$ . (ii) Let  $x_i \in M \setminus \{0\}$  and  $h^-(\cdot; x_i) : \mathbb{R} \rightarrow \mathbb{R}_{++}$  be such that  $h^-(t_i; x_i) = t_i - V_i(x_i - 1, (x_i, t_i))$  for each  $t_i \in \mathbb{R}$ . Then  $h^-(\cdot; x_i)$  is strictly decreasing in  $t_i$  as well.

*Proof.* (i) Let  $t_i, t'_i \in \mathbb{R}$  be such that  $t'_i < t_i$ . Note that  $(x_i + 1, V_i(x_i + 1, (x_i, t_i))) \succ I_i (x_i, t_i)$ . Thus, by  $t_i - t'_i > 0$ ,  $R_i \in \mathcal{R}^{++}$  implies

$$(x_i + 1, V_i(x_i + 1, (x_i, t_i)) - (t_i - t'_i)) \succ P_i (x_i, t_i - (t_i - t'_i)) = (x_i, t'_i) \succ I_i (x_i + 1, V_i(x_i + 1, (x_i, t'_i))).$$

Thus,  $V_i(x_i + 1, (x_i, t_i)) - (t_i - t'_i) < V_i(x_i + 1, (x_i, t'_i))$ , or  $h^+(t_i; x_i) = V_i(x_i + 1, (x_i, t_i)) - t_i < V_i(x_i + 1, (x_i, t'_i)) - t'_i = h^+(t'_i; x_i)$ .

(ii) We can show (ii) in a symmetric way, and we omit the proof.  $\square$

## 2 Proof of Proposition 3

In this section, we give the proof of Proposition 3.

**Proposition 3 (The inverse-demand sets).** Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ . Then (i)  $P(0; R_0) = [V_0(1, \mathbf{0}), \infty)$ , (ii)  $P(1; R_0) = [t^*(1), V_0(1, \mathbf{0})]$ , (iii)  $P(x; R_0) = [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$  for each  $x \in M \setminus \{0, 1, m\}$ , and (iv)  $P(m; R_0) = [0, \frac{t^*(m-1)}{m-1}]$ .

Before proving Proposition 3, we show the following two lemmas.

**Lemma 12.** Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ . (i) Let  $x \in M \setminus \{0, m\}$  and  $p \in P(x; R_0)$ . Then  $p \geq \frac{t^*(x)}{x}$ . (ii) Let  $x \in M \setminus \{0, 1\}$  and  $p \in P(x; R_0)$ . Then  $p \leq \frac{t^*(x-1)}{x-1}$ .

*Proof.* (i) Suppose  $p < \frac{t^*(x)}{x}$ . Then  $px < t^*(x)$ , and by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 10 (i) implies  $V_0(x + 1, (x, px)) - px > \frac{px}{x} = p$ , or  $V_0(x', (x, px)) > px'$ , where  $x' \equiv x + 1$ . This implies  $(x', px') \succ P_0 (x, px)$ , which contradicts  $p \in P(x; R_0)$ .

(ii) We can show (ii) in the same way as (i) by using Lemma 10 (ii) instead of Lemma 10 (i).  $\square$

**Lemma 13.** Let  $R_0 \in \mathcal{R}^D$  and  $p \in \mathbb{R}_+$ . (i) Let  $x \in M \setminus \{0\}$ . If  $p \geq \frac{t^*(x)}{x}$ , then for each  $x' \in M$  with  $x' > x$ ,  $(x, px) \succ R_0 (x', px')$ . (ii) Let  $x \in M \setminus \{m\}$ . If  $p \leq \frac{t^*(x-1)}{x-1}$ , then for each  $x' \in M$  with  $x' < x$ ,  $(x, px) \succ R_0 (x', px')$ .

*Proof.* (i) Suppose  $p \geq \frac{t^*(x)}{x}$ . Then  $px \geq t^*(x)$ . Thus, by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 10 (i)

implies  $V_0(x+1, (x, px)) - px \leq \frac{px}{x} = p$ . Let  $x' \in M$  with  $x' > x$ . Then

$$\begin{aligned}
V_0(x', (x, px)) - px &= \sum_{x''=x}^{x'-1} \left( V_0(x''+1, (x, px)) - V_0(x'', (x, px)) \right) \quad (\text{by Remark 1 (iii)}) \\
&\leq (x' - x) \left( V_0(x+1, (x, px)) - V_0(x, (x, px)) \right) \quad (\text{by } R_0 \in \mathcal{R}^D) \\
&= (x' - x) \left( V_0(x+1, (x, px)) - px \right) \quad (\text{by Remark 1 (iii)}) \\
&\leq (x' - x)p \quad (\text{by } V_0(x+1, (x, px)) - px \leq p) \\
&= px' - px,
\end{aligned}$$

or  $V_0(x', (x, px)) \leq px'$ . This implies  $(x, px) R_0 (x', px')$ .

(ii) By using Lemma 10 (ii) instead of Lemma 10 (i), we can show (ii) in a symmetric way.  $\square$

We now proceed to the proof of Proposition 3.

(i) First, we show  $P(0; R_0) \subseteq [V_0(1, \mathbf{0}), \infty)$ . Let  $p \in P(0; R_0)$ . Then  $\mathbf{0} R_0 (1, p)$ , which implies  $p \geq V_0(1; \mathbf{0})$ .

Next, we show  $[V_0(1, \mathbf{0}), \infty) \subseteq P(0; R_0)$ . Let  $p \in [V_0(1, \mathbf{0}), \infty)$ . Then by  $p \geq V_0(1, \mathbf{0})$ ,  $\mathbf{0} R_0 (1, p)$ . By  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 11 implies that  $p \geq V_0(1, \mathbf{0}) > t^*(1)$ . Thus, by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 14 (i) implies  $(1, p) R_0 (x, px)$  for each  $x \in M$  with  $x > 1$ . Thus, by  $\mathbf{0} R_0 (1, p)$ ,  $\mathbf{0} R_0 (x, px)$  for each  $x \in M$  with  $x > 0$ . Thus,  $p \in P(0; R_0)$ .

(ii) We show  $P(1; R_0) = [t^*(1), V_0(1, \mathbf{0})]$ . Note that by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 11 implies  $t^*(1) < V_0(1, \mathbf{0})$ , and the interval is well-defined.

We first show  $P(1; R_0) \subseteq [t^*(1), V_0(1, \mathbf{0})]$ . Let  $p \in P(1; R_0)$ . By  $(1, p) R_0 \mathbf{0}$ ,  $p \leq V_0(1, \mathbf{0})$ . Also, by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 12 (i) implies  $p \geq t^*(1)$ .

Then, we show  $[t^*(1), V_0(1, \mathbf{0})] \subseteq P(1; R_0)$ . Let  $p \in [t^*(1), V_0(1, \mathbf{0})]$ . By  $p \leq V_0(1, \mathbf{0})$ ,  $(1, p) R_0 \mathbf{0}$ . Also, by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and  $p \geq t^*(1)$ , Lemma 14 (ii) gives  $(1, p) P_0 (x, xp)$  for each  $x \in M$  with  $x > 2$ . Thus,  $p \in P(1; R_0)$ .

(iii) Let  $x \in M \setminus \{0, 1, m\}$ . We show  $P(x; R_0) = [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$ . By  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and Lemma 11, the interval is well-defined. By  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 12 (i) and (ii) imply  $P(x; R_0) \subseteq [\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}]$ , and Lemma 14 (i) and (ii) give  $[\frac{t^*(x)}{x}, \frac{t^*(x-1)}{x-1}] \subseteq P(x; R_0)$ .

(iv) Finally, we show  $P(m; R_0) = [0, \frac{t^*(m-1)}{m-1}]$ . Note that by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 9 gives that  $\frac{t^*(m-1)}{m-1} > 0$ . Thus, the interval is well-defined. By  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 12 (ii) implies that  $P(m; R_0) \subseteq [0, \frac{t^*(m-1)}{m-1}]$ . Also, by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 14 (ii) implies that  $[0, \frac{t^*(m-1)}{m-1}] \subseteq P(m; R_0)$ .  $\blacksquare$

### 3 Proof of Proposition 4

In this section, we provide the proof of Proposition 4.

**Proposition 4.** *Assume  $n = 2$  and  $m$  is even. Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  satisfy*

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq p(\beta - 1; R_0), \quad (1)$$

where  $\beta \equiv \frac{m}{2}$  and  $t^* \equiv \sum_{x=1}^{\beta} p(m - x; R_0)$ . Let  $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ . An inverse-demand-based generalized Vickrey rule on  $\mathcal{R}^2$  satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Let  $g \equiv (x, t)$  be an inverse-demand-based generalized Vickrey rule on  $\mathcal{R}^2$ .

**Lemma 14.** *Let  $R \in \mathcal{R}^2$  and  $i, j \in N$  be a pair such that  $R_i = R_0$  and  $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ . Let  $v_j > 0$  be a constant marginal valuation associated with  $R_j$ . (i) If  $x_i(R) > 0$ , then  $t_i(R) - V_i(x_i(R) - 1, g_i(R)) \geq v_j$ . (ii) If  $x_i(R) < m$ , then  $V_i(x_i(R) + 1, g_i(R)) - t_i(R) \leq v_j$ .*

*Proof.* By the definition of the inverse-demand-based generalized Vickrey rule and  $R \in (\mathcal{R}^{NI})^2$ ,  $p(x_i(R); R_i) \leq v_j \leq p(x_i(R) - 1; R_i)$  and  $t_i(R) = v_j x_i(R)$ .

(i) Suppose  $x_i(R) = 1$ . Then  $v_j \leq p(0; R_i) = V_i(1, \mathbf{0})$ , where the inequality follows from  $v_j \leq p(x_i(R) - 1; R_i)$ . Then we have  $g_i(R) = (1, v_j)$ .  $R_i(1, V_i(1, \mathbf{0})) I_i \mathbf{0}$ . This implies  $V_i(0, g_i(R)) \leq 0$ . Thus, by  $t_i(R) = v_j$ ,  $t_i(R) - V_i(0, g_i(R)) \geq v_j$ .

Suppose instead  $x_i(R) > 1$ . By  $0 < x_i(R) - 1 < m$ ,  $p(x_i(R) - 1; R_i) = \frac{t^*(x_i(R) - 1)}{x_i(R) - 1}$ . Thus, by  $v_j \leq p(x_i(R) - 1; R_i)$ ,

$$t_i(R) = v_j x_i(R) \leq p(x_i(R) - 1; R_i) x_i(R) = \frac{x_i(R)}{x_i(R) - 1} t^*(x_i(R) - 1).$$

Thus, by  $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 10 (ii), and  $t_i(R) = v_j x_i(R)$ ,

$$t_i(R) - V_i(x_i(R) - 1, g_i(R)) \geq \frac{t_i(R)}{x_i(R)} = v_j.$$

(ii) If  $x_i(R) = 0$ , then  $V_i(x_i(R) + 1, g_i(R)) - t_i(R) = V_i(1, \mathbf{0}) = p(0; R_i) \leq v_j$ , where the first equality follows from  $g_i(R) = \mathbf{0}$ , and the inequality follows from  $v_j \geq p(x_i(R); R_i)$ .

Suppose instead  $x_i(R) > 0$ . By  $0 < x_i(R) < m$ ,  $p(x_i(R); R_i) = \frac{t^*(x_i(R))}{x_i(R)}$ . Thus, by  $v_j \geq p(x_i(R); R_i)$ ,

$$t_i(R) = v_j x_i(R) \geq p(x_i(R); R_i) x_i(R) = t^*(x_i(R)).$$

Thus, by  $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$ , Lemma 10 (i), and  $t_i(R) = v_j x_i(R)$

$$V_i(x_i(R) + 1, g_i(R)) - t_i(R) \leq \frac{t_i(R)}{x_i(R)} = v_j,$$

as desired.  $\square$

We now proceed to the proof of Proposition 4. Since *no subsidy for losers* is immediate from the definition of the rule, we here show the other three properties.

INDIVIDUAL RATIONALITY. Let  $R \in \mathcal{R}^2$  and  $i, j \in N$  be a distinct pair. If  $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ , then  $R_i = R_i^{inv}$ . Since  $g_i(R)$  is the outcome of the Vickrey rule for  $R^{inv}$ ,  $g_i(R) R_i \mathbf{0}$  by *individual rationality* of the Vickrey rule. Thus, we assume  $R_i = R_0$ . We consider the following two cases.

CASE 1.  $R_j = R_0$ .

By the definition of the inverse-demand-based generalized Vickrey rule and  $R \in (\mathcal{R}^D)^2$ ,  $x_i(R) = \beta$  and

$$t_i(R) = \sum_{x=0}^{m-1} p(x; R_j) - \sum_{x=0}^{\beta-1} p(x; R_j) = \sum_{x=\beta}^{m-1} p(x; R_j) = \sum_{x=1}^{\beta} p(m-x; R_j) = t^*.$$

Then

$$t_i(R) = \sum_{x=1}^{\beta} p(m-x; R_j) \leq \beta p(\beta; R_j) = t^*(\beta; R_j) < V_i(\beta, \mathbf{0}),$$

where the first inequality follows from  $R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$ ,  $\beta = \frac{m}{2}$ , and Lemma 11, and the second inequality follows from  $R_i = R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and Lemma 9. Thus,  $g_i(R) R_i \mathbf{0}$ .

CASE 2.  $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ .

By the definition of the inverse-demand-based generalized Vickrey rule,  $t_i(R) = v_j x_i(R)$ , where  $v_j > 0$  is a constant marginal valuation associated with  $R_j$ . If  $x_i(R) > 0$ , then

$$\begin{aligned} t_i(R) - V_i(0, g_i(R)) &= \sum_{x=1}^{x_i(R)} \left( V_i(x, g_i(R)) - V_i(x-1, g_i(R)) \right) && \text{(by Remark 1 (iii))} \\ &\geq x_i(R) \left( V_i(x_i(R), g_i(R)) - V_i(x_i(R)-1, g_i(R)) \right) && \text{(by } R_i \in \mathcal{R}^D) \\ &\geq x_i(R) v_j && \text{(by Lemma 14 (i))} \\ &= t_i(R), \end{aligned}$$

or  $V_i(0, g_i(R)) \leq 0$ . Thus,  $g_i(R) R_i \mathbf{0}$ . On the other hand, if  $x_i(R) = 0$ , then  $g_i(R) = \mathbf{0}$ .

STRATEGY-PROOFNESS. Let  $R \in \mathcal{R}^2$  and  $i, j \in N$  be a distinct pair. If  $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$ ,

then since  $g(R)$  and  $g(R'_i, R_j)$  are both the outcomes of the Vickrey rule, *strategy-proofness* of the Vickrey rule implies that  $g_i(R) \succeq_i g_i(R'_i, R_j)$ . Thus, we assume  $R_i = R_0$ .

Let  $R'_i \in \mathcal{R}$ . Let  $x_i \equiv x_i(R'_i, R_{-i})$ . We show  $g_i(R) \succeq_i (x_i, t_i(R'_i, R_j))$ . If  $x_i = 0$ , then  $t_i(R'_i, R_j) = 0$  by Lemma 2. Thus, by *individual rationality* of  $g$ , suppose  $x_i > 0$ .

CASE 1.  $R_j = R_0$ .

By the definition of the inverse-demand-based generalized Vickrey rule and  $R \in (\mathcal{R}^D)^2$ ,  $g_i(R) = (\beta, t^*)$  and  $t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j)$ .

First, suppose  $x_i > x_i(R)$ . Then

$$t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j) = t^* + \sum_{x=\beta+1}^{x_i} p(m-x; R_j). \quad (2)$$

For each  $x \in M$  with  $x \geq \beta+1$ , we have

$$\begin{aligned} p(m-x; R_j) &\geq p(\beta-1; R_i) \\ &\geq V_i(\beta+1, g_i(R)) - t^* \\ &= V_i(\beta+1, g_i(R)) - V_i(\beta, g_i(R)) \\ &\geq V_i(x, g_i(R)) - V_i(x-1, g_i(R)), \end{aligned} \quad (3)$$

where the first inequality follows from  $R_i = R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$ ,  $x \geq \beta+1$ , and Lemma 11, the second one follows from (1),  $R_i = R_0$ , and  $g_i(R) = (\beta, t^*)$ , the equality follows from  $t^* = t_i(R)$  and Remark 1 (iii), and the last inequality comes from  $R_i \in \mathcal{R}^D$  and  $x \geq \beta+1$ . Then

$$\begin{aligned} t_i(R'_i, R_j) &= t^* + \sum_{x=\beta+1}^{x_i} p(m-x; R_j) && \text{(by (2))} \\ &\geq t^* + \sum_{x=\beta+1}^{x_i} \left( V_i(x, g_i(R)) - V_i(x-1, g_i(R)) \right) && \text{(by (3))} \\ &= V_i(x_i(R), g_i(R)) + \left( V_i(x_i, g_i(R)) - V_i(x_i(R), g_i(R)) \right) \\ &\quad \text{(by } g_i(R) = (\beta, t^*) \text{ and Remark 1 (iii))} \\ &= V_i(x_i, g_i(R)), \end{aligned}$$

which implies  $g_i(R) \succeq_i (x_i, t_i(R'_i, R_j))$ .

Suppose instead  $x_i < x_i(R)$ . Then

$$t_i(R'_i, R_j) = \sum_{x=1}^{x_i} p(m-x; R_j) = t^* - \sum_{x=x_i}^{\beta-1} p(m-x-1; R_j). \quad (4)$$

Note that by  $\beta = x_i(R) > x_i > 0$ ,  $\beta - 1 > 0$ . Thus, by  $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and Lemma 9,  $\frac{\beta}{\beta-1}t^*(\beta-1; R_i) = V_i(\beta, (\beta-1, t^*(\beta-1; R_i)))$ . This implies

$$\left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right) = \left(\beta, V_i(\beta, (\beta-1, t^*(\beta-1; R_i)))\right) I_i(\beta-1, t^*(\beta-1; R_i)).$$

Thus, by Remark 1 (i) and (iii),

$$V_i\left(\beta-1, \left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right)\right) = V_i(\beta-1, (\beta-1, t^*(\beta-1; R_i))) = t^*(\beta-1; R_i). \quad (5)$$

By  $R_i \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and  $\beta = \frac{m}{2}$ , Lemma 11 implies

$$t^* = \sum_{x=1}^{\beta} p(m-x; R_i) < \beta p(\beta-1; R_i) = \frac{\beta}{\beta-1}t^*(\beta-1; R_i). \quad (6)$$

Then

$$\begin{aligned} t^* - V_i(\beta-1, (\beta, t^*)) &> \frac{\beta}{\beta-1}t^*(\beta-1; R_i) - V_i\left(\beta-1, \left(\beta, \frac{\beta}{\beta-1}t^*(\beta-1; R_i)\right)\right) \\ &= \frac{\beta}{\beta-1}t^*(\beta-1; R_i) - t^*(\beta-1; R_i) \\ &= \frac{t^*(\beta-1; R_i)}{\beta-1} \\ &= p(\beta-1; R_j), \end{aligned} \quad (7)$$

where the inequality follows from  $R_i \in \mathcal{R}^{++}$ , (6), and Remark 7 (ii), the first equality follows from (5), and the last one follows from  $R_i = R_j$ . Then for each  $x \in M$  with  $x \leq \beta-1$ ,

$$\begin{aligned} V_i(x+1, g_i(R)) - V_i(x, g_i(R)) &\geq V_i(\beta, g_i(R)) - V_i(\beta-1, g_i(R)) \\ &= t^* - V_i(\beta-1, (\beta, t^*)) \\ &> p(\beta-1; R_j) \\ &> p(m-x-1; R_j), \end{aligned} \quad (8)$$

where the first inequality follows from  $R_i \in \mathcal{R}^D$  and  $x \leq \beta-1$ , the equality follows from  $g_i(R) = (\beta, t^*)$  and Remark 1 (iii), the second inequality follows from (7), and the last



one comes from  $R_j \in \mathcal{R}^D \cap \mathcal{R}^{++}$ ,  $x \leq \beta - 1$ , and Lemma 11. Thus,

$$\begin{aligned}
t_i(R'_i, R_j) &= t^* - \sum_{x=x_i}^{\beta-1} p(m-x-1; R_j) && \text{(by (4))} \\
&> t^* - \sum_{x=x_i}^{\beta-1} \left( V_i(x+1, g_i(R)) - V_i(x, g_i(R)) \right) && \text{(by (8))} \\
&= V_i(x_i(R), g_i(R)) - \left( V_i(x_i(R), g_i(R)) - V_i(x_i, g_i(R)) \right) \\
&\hspace{15em} \text{(by } g_i(R) = (\beta, t^*) \text{ and Remark 1 (iii))} \\
&= V_i(x_i, g_i(R)),
\end{aligned}$$

which implies  $g_i(R) \succ_i (x_i, t_i(R'_i, R_j))$ .

CASE 2.  $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$ .

By the definition of the inverse-demand-based generalized Vickrey rule,  $t_i(R) = v_j x_i(R)$  and  $t_i(R'_i, R_j) = v_j x_i$ , where  $v_j > 0$  is a constant marginal valuations associated with  $R_j$ .

If  $x_i > x_i(R)$ , then  $x_i(R) < m$ , and

$$\begin{aligned}
&V_i(x_i, g_i(R)) - t_i(R) \\
&= \sum_{x=x_i(R)}^{x_i-1} \left( V_i(x+1, g_i(R)) - V_i(x, g_i(R)) \right) && \text{(by Remark 1 (iii))} \\
&\leq (x_i - x_i(R)) \left( V_i(x_i(R)+1, g_i(R)) - V_i(x_i(R), g_i(R)) \right) && \text{(by } R_i \in \mathcal{R}^D) \\
&\leq (x_i - x_i(R)) v_j && \text{(by Lemma 14 (ii))} \\
&= t_i(R'_i, R_j) - t_i(R),
\end{aligned}$$

or  $V_i(x_i, g_i(R)) \leq t_i(R'_i, R_j)$ . Thus,  $g_i(R) \succ_i (x_i, t_i(R'_i, R_j))$ .

The other case can be treated symmetrically.

EFFICIENCY. Let  $R \in \mathcal{R}^2$ . If  $R \in (\mathcal{R}^C \cap \mathcal{R}^Q)^2$ , then by *efficiency* of the Vickrey rule,  $g(R)$  is efficient for  $R$ . Thus, we consider the following two cases.

CASE 1.  $R_1 = R_2 = R_0$ .

By the definition of the inverse-demand-based generalized Vickrey rule and  $R \in (\mathcal{R}^D)^2$ ,  $g_1(R) = g_2(R) = (\beta, t^*)$ . By  $g_1(R) = g_2(R)$  and Remark 1 (iii),  $R_1 = R_2 \in \mathcal{R}^D$  gives  $V_i(x_i(R)+1, g_i(R)) - t_i(R) < t_j(R) - V_j(x_j(R)-1, g_j(R))$  for each pair  $i, j \in N$ . Thus, by  $R \in (\mathcal{R}^N)^2$  and Remark 5,  $g(R)$  is efficient for  $R$ .

CASE 2.  $R_i = R_0$  and  $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$  for some pair  $i, j \in N$ .

Without loss of generality, let  $i = 1$  and  $j = 2$ . By the definition of the inverse-demand-based generalized Vickrey rule,  $t_1(R) = v_2 x_1(R)$ , where  $v_2 > 0$  is a constant marginal valuation associated with  $R_2$ .

If  $x_1(R) = 0$ , then by Lemma 14 (ii),  $v_2 \geq V_1(1, g_1(R)) - t_1(R)$ . If  $0 < x_1(R) < m$ , then by Lemma 14 (i) and (ii),

$$V_1(x_1(R) + 1, g_1(R)) - t_1(R) \leq v_2 \leq t_1(R) - V_1(x_1(R) - 1, g_1(R)).$$

Finally, if  $x_i(R) = m$ , then by Lemma 14 (i),  $t_1(R) - V_1(m - 1, g_1(R)) \geq v_2$ . In either case, by  $R \in (\mathcal{R}^{NI})^2$ , Remark 5 implies that  $g(R)$  is efficient for  $R$ . ■

**Remark 8.** In the proof of Proposition 4, we use the condition (1) only when we prove that an agent has no incentive to misreport his preference at the preference profile  $(R_1, R_2) = (R_0, R_0)$ . Indeed, all the remaining parts of the proof remain valid without the condition (1) and even the assumption that  $m$  is even.

## 4 Proof of Proposition 5

In this section, we prove Proposition 5.

**Proposition 5.** *Assume  $m \geq 4$  is even. Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  violate*

$$V_0(\beta + 1, (\beta, t^*)) - t^* \leq p(\beta - 1; R_0), \quad (1)$$

where  $\beta \equiv \frac{m}{2}$  and  $t^* \equiv \sum_{x=1}^{\beta} p(m - x; R_0)$ . Let  $\mathcal{R}$  be a class of preferences satisfying  $\mathcal{R} \supseteq (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ . No rule on  $\mathcal{R}^n$  satisfies efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Let  $\mathcal{R} \equiv (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ . Suppose by contradiction that there is a rule  $f$  on  $\mathcal{R}^n$  satisfying efficiency, strategy-proofness, individual rationality, and no subsidy for losers.

Note that Steps 1 to 3 in the proof of Theorem 2 depend on neither the number of agents nor the choice of a preference  $R_0$ , and so the discussion is valid here as well. Thus, we hereafter take over the results and the notations in Steps 1 to 3 in the proof of Theorem 2.

As in the proof of Theorem 2, let  $R_2 \equiv R_0$ . First, we show  $z_2^f(R_{-2}; \beta+1) P_2 z_1^f(R_{-2}; \beta)$ .

Note that by Step 3,  $z_2^f(R_{-2}; \beta) = (\beta, t^*)$ . Further,

$$\begin{aligned} t_2^f(R_{-2}; \beta + 1) - t_2^f(R_{-2}; \beta) &= p(m - (\beta + 1); R_1) \\ &= p(\beta - 1; R_1) \\ &< V_2(\beta + 1, z_2^f(R_{-2}; \beta)) - t_2^f(R_{-2}; \beta), \end{aligned}$$

where the first equality follows from Step 3, the second one follows from  $\beta = \frac{m}{2}$ , and the inequality follows since  $R_2 = R_1 = R_0$  violates (1) and  $z_2^f(R_{-2}, \beta) = (\beta, t^*)$ . Thus, we have  $t_2^f(R_{-2}; \beta + 1) < V_2(\beta + 1, z_2^f(R_{-2}; \beta))$ . This implies  $z_2^f(R_{-2}; \beta + 1) P_2 z_2^f(R_{-2}; \beta)$ .

By the same argument as in the proof of Theorem 2, we can show that for each  $x_2 \in M$  with  $x_2 < \beta$ ,  $z_2^f(R_{-2}; x_2 + 1) P_2 z_2^f(R_{-2}; x_2)$ . Thus, we have  $z_2^f(R_{-2}; \beta + 1) P_2 z_2^f(R_{-2}; x_2)$  for each  $x_2 \in M$  with  $x_2 \leq \beta$ , and Lemma 4 implies  $x_2(R) \geq \beta + 1$ . Since both agents 1 and 2 have the same preferences  $R_0$ , a symmetric argument gives  $x_1(R) \geq \beta + 1$ . Thus,

$$x_1(R) + x_2(R) \geq 2(\beta + 1) = m + 2,$$

contradicting feasibility. ■

## 5 Even number of units and more than two agents

In this section, we give an example which demonstrates that when  $n \geq 3$  and  $m = 6a - 2$  for some  $a \in \mathbb{N}$ , the inverse-demand-based generalized Vickrey rule violates *strategy-proofness* on  $\mathcal{R}^n \equiv ((\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\})^n$  for any  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$ .

**Example 5.** Let  $n \geq 3$  and  $m = 6a - 2$ , where  $a \in \mathbb{N}$  is an arbitrary natural number. Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and  $\mathcal{R} = (\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ . Let  $g \equiv (x, t)$  be an inverse-demand-based generalized Vickrey rule on  $\mathcal{R}^n$ .

Let  $R \in \mathcal{R}^n$  be such that for each  $i \in \{1, 2, 3\}$ ,  $R_i = R_0$ , and for each  $j \in N \setminus \{1, 2, 3\}$ ,  $R_j \in \mathcal{R}^C \cap \mathcal{R}^Q$  be such that  $v_j < p(m - 1; R_0)$ , where  $v_j$  is a constant marginal valuation associated with  $R_j$ . By the definition of the inverse-demand-based generalized Vickrey rule and  $R_{\{1, 2, 3\}} \in (\mathcal{R}^D)^3$ ,  $x_i(R) = x_j(R) = 2a - 1$  and  $x_k(R) = 2a$  for some distinct triple  $i, j, k \in \{1, 2, 3\}$ ,  $x_l(R) = 0$  for each  $l \in N \setminus \{1, 2, 3\}$ ,

$$t_i(R) = t_j(R) = p(2a - 1; R_0) + 2 \sum_{x=2a}^{3a-2} p(x; R_0), \quad t_k(R) = 2 \sum_{x=2a-1}^{3a-2} p(x; R_0),$$

and  $t_l(R) = 0$  for each  $l \in N \setminus \{1, 2, 3\}$ .<sup>1</sup> Without loss of generality, let  $i = 1$ ,  $j = 2$ , and

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<sup>1</sup>Note that when  $a = 1$ , i.e.,  $m = 4$ ,  $t_i(R) = t_j(R) = p(2a - 1; R_0) = p(1; R_0)$ . In the subsequent discussion, we consider the case where  $a \geq 2$ , but the same discussion can be applied to the case where  $a = 1$ .

$k = 3$ . Let  $R'_1 \in \mathcal{R}^C \cap \mathcal{R}^Q$  be such that  $p(2a; R_0) < v'_1 < p(2a - 1; R_0)$ , where  $v'_1$  is a constant marginal valuation associated with  $R'_1$ . By the definition of the inverse-demand-based generalized Vickrey rule and  $R_{\{2,3\}} \in (\mathcal{R}^D)^2$ ,  $g_1(R'_1, R_{-1}) = (2a, 2 \sum_{x=2a-1}^{3a-2} p(x; R_0))$ . Note that by  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  and Lemma 11,

$$t_1(R) = p(2a - 1; R_0) + 2 \sum_{x=2a}^{3a-2} p(x; R_0) < (2a - 1)p(2a - 1; R_0) = t^*(2a - 1).$$

Thus,

$$\begin{aligned} V_1(2a, g_1(R)) - t_1(R) &> V_1(2a, (2a - 1, t^*(2a - 1))) - t^*(2a - 1) && \text{(by Remark 7 (i))} \\ &= \frac{t^*(2a - 1)}{2a - 1} && \text{(by Lemma 9)} \\ &= p(2a - 1; R_0) \\ &= t_1(R'_1, R_{-1}) - t_1(R), \end{aligned}$$

or  $V_1(2a, g_1(R)) > t_1(R'_1, R_{-1})$ . This implies  $g_1(R'_1, R_{-1}) P_1 g_1(R)$ . Thus,  $g$  violates *strategy-proofness*.

## References

- [1] Shinozaki, H., T. Kazumura, and S. Serizawa (2020), “Efficient and strategy-proof multi-unit object allocation with money: (Non)decreasing marginal valuations without quasi-linearity.” ISER Discussion Paper No. 1097.