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# Tractable Hedging: An Implementation of Robust Hedging Strategies

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# Tractable Hedging

# An Implementation of Robust Hedging Strategies

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# Tractable Hedging
# An Implementation of Robust Hedging Strategies

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**Abstract**

This paper provides a theoretical and numerical analysis of robust hedging strategies in diffusion–type models including stochastic volatility models. A robust hedging strategy avoids any losses as long as the realised volatility stays within a given interval. We focus on the effects of restricting the set of admissible strategies to tractable strategies which are defined as the sum over Gaussian strategies. Although a trivial Gaussian hedge is either not robust or prohibitively expensive, this is not the case for the cheapest tractable robust hedge which consists of two Gaussian hedges for one long and one short position in convex claims which have to be chosen optimally.

**JEL:** G12, G13

**Keywords:** Stochastic volatility, robust hedging, tractable hedging, model misspecification, incomplete markets

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## 1. Introduction

Risk management can not be thought of without hedging. Under the ideal conditions of a complete market, the risk of any portfolio can be eliminated completely. There exists a self-financing strategy for which after an initial investment no further in- or outflows occur and for which the terminal value is exactly equal to the payoff of the portfolio. But in most cases, the risk manager has to face conditions which are not ideal, and a perfect hedge simply does not exist. This happens if trading is only possible at discrete points in time but the risk factors follow continuous-time processes.[1] This also happens if the market is incomplete, where one of the most prominent examples is a stochastic volatility model. If only the stock and the money market account are traded, then, e.g., a call option on the stock is not attainable and can not be hedged perfectly. [2]

Another problem is that the risk manager has to rely on a model to determine the hedging strategy. But in reality, the true model is not known and the risk manager has to make some (educated) assumptions about the data-generating process. However, strategies that are based on the hedge model and are optimal with respect to some criterion within this model, i.e. a given incompleteness, fail to be effective if the "true" asset price dynamics deviate from the assumed ones.

In this paper, we analyze hedging in a diffusion–type setup. We assume that the investor does not know the true volatility process. Due to incompleteness and model

[1]Option replication in discrete time and the implication of transaction costs, conducted by Leland (1985), is also studied in Bensaid, Lesne, and Scheinkman (1992), Boyle and Vorst (1992), Avellaneda and Parás (1994), Grannan and Swindle (1996) and Toft (1996). Discretely adjusted option hedges are analysed in Boyle and Emanuel (1980) and Bertsimas, Kogan, and Lo (1998).

[2]Hedging strategies in incomplete markets depend on some dynamic risk measure that has to be minimized. For hedging concepts like quantile and expected shortfall–hedging we refer to Cvitanić and Karatzas (1999), Föllmer and Leukert (1999), Föllmer and Leukert (2000) and Schulmerich and Trautmann (2003). For hedging concepts that rely on the variance we refer to the papers by Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Schweizer (1990), Schweizer (1991), Schweizer (1992), Schweizer (1993), Schweizer (1994), Delbaen and Schachermayer (1996), Delbaen, Monat, Schachermayer, Schweizer, and Stricker (1997), Laurent and Pham (1999) as well as Pham, Rheinländer, and Schweizer (1998).

risk, a perfect hedge is not possible. Within this setup, we consider hedging strategies that are characterised by two properties. First, they are based on a simple model, in our case the model of Black Scholes or a Gaussian model. Second, the hedging strategy dominates the claim to be hedged whenever the realized volatility path stays within some given deterministic volatility interval.

A strategy that meets the second criterion is called a *robust hedging strategy*. By definition, a robust hedging strategy is a superhedging strategy within the class of stochastic volatility models with bounded volatility, i.e. models where volatility stays in the given interval. Robust hedging can be thought of as a modification of superhedging, motivated by at least two problems superhedging suffers from when volatility is not bounded. First, even the cheapest superhedging strategy may be prohibitively expensive. For a European call, e.g., the cheapest superhedge consists of buying the underlying stock, irrespective of how large the strike price is and how far the option is out of the money. Second, there are payoffs like a power call with exponent greater than one for which a superhedge simply does not exist. Robust hedging is also one intuitive way to cope with uncertainty about the true data-generating process. To determine a robust hedging strategy the investor only has to determine some volatility interval. He does neither have to specify the stochastic process for volatility nor to find its parameters nor to infer the current volatility. In consequence, a robust hedge depends on considerably fewer assumptions about the true model than a hedge that is determined within some specific stochastic volatility model.

The first criterion restricts the set of strategies to those strategies that can be written as the sum of Gaussian strategies. Each Gaussian strategy is self-financing within a corresponding hedge model where the price of the underlying follows a geometric Brownian motion, i.e. the volatility is at most a deterministic function of time. The most prominent example is the model of Black Scholes where volatility is even constant. However, it is important to note that the use of Gaussian strategies is not based on the assumption that the true-data generating process is a Gaussian process. Rather, a Gaussian model is a simple choice for the hedge model where the hedge model is then used to determine the hedging strategy.

In case of model uncertainty, every hedge model deviates from the true data-generating process almost surely. We can therefore not expect the hedging strategy to be the optimal one. Rather, the crucial question is how good the hedging strategies are under model risk, or, stated differently, how good the hedge model is. For Gaussian strategies, this question is considered in a number of papers under different scenarios, c.f. Avellaneda, Levy, and Parás (1995), Lyons (1995), Bergman, Grundy, and Wiener (1996), El Karoui, Jeanblanc-Picqué, and Shreve (1998), Hobson (1998), Dudenhausen, Schlögl, and Schlögl (1998) and Mahayni (2003). The key result states that a Gaussian strategy is a superhedge for a convex (concave) payoff if the true volatility is bounded and if the claim is hedged at the upper (lower) volatility bound. If the payoff to be hedged is neither convex nor concave, then a simple Gaussian hedge of this claim never gives a superhedge even if volatility is bounded. In this case, Avellaneda, Levy, and Parás (1995) are the first to determine the (cheapest) robust hedging strategy (henceforth ALP-hedge). It can be derived from the upper price bound which solves a Black–Scholes–Barenblatt (BSB) equation, i.e. a non–linear stochastic differential equation.

The fact that the Gaussian hedging strategy for a mixed payoff is no robust hedge does not imply that Gaussian hedges can not be used for robust hedging at all. One solution is to implement a Gaussian hedge for a dominating convex (or concave) payoff at the upper (lower) volatility bound which gives a robust hedge for the original mixed payoff. We show that there is an even cheaper solution if we do not use one Gaussian strategy, but instead the sum of two (or more) Gaussian strategies where the strategies are determined in different Gaussian models. Such strategies are called *tractable*.

Our tractable robust hedging strategies are based on the idea of hedging a portfolio by hedging each component payoff. We decompose a mixed payoff into a portfolio of convex and concave claims. The robust hedging strategies for the component payoffs are Gaussian strategies, and the sum of these Gaussian strategies gives a tractable and robust hedge for the portfolio. We show how to find the cheapest tractable robust hedge. Furthermore, we analyze the performance of this hedge and

compare it to the ALP-hedge. The latter serves as a benchmark for the performance of tractable hedging strategies that are based on simple Gaussian hedge models.

In more detail, the contributions of our paper are as follows: First, we determine the cheapest tractable robust hedge. This strategy is represented by the sum of two Gaussian hedges, i.e. the sum of the cheapest robust hedges for one convex and one concave payoff which, in sum, envelope the payoff to be hedged. The initial capital needed for this hedge depends on the chosen convex and concave payoff, so the main problem is to find the cheapest choice of these two payoffs. Surprisingly, the optimal choice may well be given by two payoffs the portfolio of which strictly dominates the claim to be hedged: The initial capital for the decomposition of this dominating portfolio may be less than the initial capital needed for any decomposition of the claim itself.

Second, we illustrate our results in a numerical example and analyze the performance of the hedging strategies in a stochastic volatility model with unbounded volatility. We compare the initial capital of the cheapest tractable robust hedge to the smaller initial capital of the cheapest robust ALP-hedge. This shows by how much the initial capital increases due to the restriction to tractable strategies. We also compare the effectiveness of the two hedging strategies, in particular the pathwise performance and the distribution of the total costs, including the shortfall probabilities and the expected shortfall. After accounting for the differences in initial capital, the remaining differences are moderate, so that the use of the cheapest tractable strategy does not worsen the hedge performance excessively.

The paper is organized as follows. The next section introduces the probabilistic setup and gives a review of well known but crucially needed results. In particular, we give a short summary of the approach of Avellaneda, Levy, and Parás (1995), including the robustness result of El Karoui, Jeanblanc-Picqué, and Shreve (1998). In section 3, we proceed with a formal definition of *tractable hedging*. We derive the cost process of a hedge and then focus on tractable robust hedges which can be represented as the sum of two Black/Scholes–type strategies. The determination of the cheapest tractable hedge is reduced to a static optimization problem. We solve this problem for some typical payoff–patterns in section 4. Section 5 gives some

illustrative implementation examples. Finally we give some concluding remarks in section 6 as well as an overview of work in progress.

## 2. Model setup

All stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$, which satisfies the usual hypotheses. Trading terminates at time $T^* > 0$. We consider a financial market which consists of two assets $X$ and $B$. $B(t,T)$ is the time $t$ price of a zero coupon bond paying one unit of money at maturity $T \leq T^*$, and $X_t$ is the price of the asset (or index) $X$ at $t$. The forward price process is

$$X_t^* = \frac{X_t}{B(t,T)},$$

and throughout the following analysis, we use the convention that forward prices are denoted with stars.

We assume that the price processes of the underlying assets are strictly positive, continuous semimartingales. If the probability space $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$ supports a $d$–dimensional Brownian motion $W$, we may write without loss of generality

$$d\,(X_t^*)^M = X_t^* \sigma_t^T \, dW_t, \tag{1}$$

where $d\,(X_t^*)^M$ denotes the martingale part of the Doob–Meyer decomposition of the forward price process $X_t^*$, and $\sigma^T$ denotes the volatility of the forward process. Apart from integrability conditions we are not restricting $\sigma^T$ in any way. In particular, $\sigma^T$ at time $t$ may depend on $t$ and $X_t^*$, the entire path of $X^*$ up to time $t$, and also on some other random variables. In general, we are in a stochastic volatility model.

We assume that the financial market model described by equation (1) is arbitrage free, i.e. there exists at least one forward risk adjusted measure $P_T^*$ on $\mathcal{F}_T$ such that $P_T^* \sim P_T$ $(= P|\mathcal{F}_T)$ and $(X_t^*, \mathcal{F}_t)_{0\leq t\leq T}$ is a $P_T^*$–martingale. Note that $P_T^*$ is not uniquely defined unless the volatility structure $\sigma^T$ is a deterministic function of time (i.e. the model is a Gaussian model) or at most a deterministic function of time and of $X_t^*$ (i.e. the model is a deterministic volatility model). Otherwise, the financial market under consideration is incomplete.

A contingent claim $C$ with maturity $T \in [0, T^*]$ is defined by the random payoff received at time $T$, which is described by the $\mathcal{F}_T$-measurable random variable $C_T$. We consider path-independent European contingent claims which are written on the underlying $X$, i.e. where the payoff at time $T$ is given in the form $C_T = h(X_T)$. If the financial market is complete, then there exists a replicating strategy for the claim, and in the absence of arbitrage opportunities, the value of this strategy must be equal to the price $C$ of the claim. If the financial market is incomplete or if we do not know the true data-generating process, then in general we are not able to find such a replicating strategy, which means we cannot determine a perfect hedge.

Instead we consider robust hedging strategies. For a *robust hedging strategy* it holds that the strategy dominates the claim whenever the realized stochastic volatility path stays within some deterministic volatility interval:

Definition 2.1 (Robust hedging strategy). *A trading strategy is called a robust hedging strategy for a claim with payoff $h(X_T)$ if and only if it is a superhedging strategy for this claim in any stochastic volatility model for which the forward price dynamic $X^*$ satisfies equation (1) and for which it holds that $\sigma_t^T(\omega) \in [\sigma_{min}^T(t), \sigma_{max}^T(t)]$ for $\lambda^1 \otimes P$–almost all $(t, \omega) \in [0, T] \times \Omega$.*

For a given claim, the robust hedging strategy only depends on some volatility interval. We neither need to know the true data-generating process, nor the exact parameter values, or the current local volatility. In case of uncertainty about the true data-generating process, robust hedging is therefore one possible hedge criterion which avoids the need to choose some (almost surely wrong) data-generating process. At the same time, it is in general much cheaper than a superhedging strategy which avoids any losses, irrespective of the true process of stochastic volatility.

By definition, a robust hedging strategy is a superhedging strategy in any stochastic volatility model with bounded volatility. To determine the cheapest robust hedging strategy, we use that the superhedging strategy for a claim $C$ with payoff $h(X_T)$ can be derived from the process of the lowest upper price bound $\widehat{U}_t^C$. In case of stochastic volatility models with bounded volatility, the first ones to establish the lowest upper price bound were Avellaneda, Levy, and Parás (1995):

Proposition 2.2 (Price bounds in case of bounded volatility). *The model is a stochastic volatility model with bounded volatility. The volatility bounds are tight in the sense that* $P\left(||\sigma_t^T||^2 > ||\sigma_{max}^T(t)||^2 - \epsilon\right) > 0$ *and* $P\left(||\sigma_t^T||^2 < ||\sigma_{min}^T(t)||^2 + \epsilon\right) > 0$ *for all* $t \in [0,T]$ *and all* $\epsilon > 0$. *Then the lowest upper price bound of an arbitrary payoff function* $h(X_T)$ *is given by the solution of a Black-Scholes-Barenblatt (BSB) equation, i.e. the lowest upper price bound* $\widehat{U}$ *with* $\widehat{U}(t,X_t) = B(t,T)\widehat{U}^*(t,X_t^*)$ *is obtained by solving the final–value problem*

$$\widehat{U}_t^*(t,x^*) + \tfrac{1}{2}\left\|\sigma^T(t,x^*)\right\|^2 (x^*)^2\, \widehat{U}_{xx}^*(t,x^*) = 0 \tag{2}$$

$$\widehat{U}^*(T,x^*) = h(x^*)$$

$$\textit{where} \quad \sigma^T(t,x^*) = \sigma_{max}^T(t)\ 1_{\left\{\widehat{U}_{xx}^*(t,x^*)\geq 0\right\}} + \sigma_{min}^T(t)\ 1_{\left\{\widehat{U}_{xx}^*(t,x^*)<0\right\}}.$$

**Proof.** The proof is given in Avellaneda, Levy, and Parás (1995). Notice that in order to allow for stochastic interest rates as well, the above version is given in terms of forward prices, i.e. in terms of $X_t^*$. □

In general, the price bound $\widehat{U}$ and the corresponding volatility $\sigma^T(t,x^*)$ have to be determined simultaneously. This amounts to solving a stochastic control problem using backward induction (the numerical algorithm proposed by Avellaneda, Levy, and Parás (1995) is analyzed in the appendix of this paper). However, for convex and concave payoffs, the problem simplifies considerably. The cheapest superhedge for a convex (concave) payoff is just the Gaussian hedge for this payoff at the upper (lower) volatility bound, which matches the well known robustness result of El Karoui, Jeanblanc-Picqué, and Shreve (1998).

## 3. Gaussian and Tractable Hedging

In the following, we restrict ourselves to strategies that are based on Gaussian models. In these models, volatility is at most a deterministic function of time, and the price of the underlying is lognormally distributed. In particular, we do not only consider pure Gaussian strategies but also tractable strategies:

Definition 3.1 (Gaussian and tractable strategies). *A Gaussian strategy is given by a terminal value* $g(X_T)$ *and the deterministic volatility function* $\sigma^T(t)$ *which at*

*most depends on time $t$. It is the replicating strategy for the claim $C$ with maturity $T$ and payoff $g(X_T)$ in the Gaussian model with volatility $\sigma^T(t)$, i.e. the number of stocks and bonds is*

$$\phi_t^X := C_x^*(t, X_t^*), \quad \phi_t^B := C^*(t, X_t^*) - C_x^*(t, X_t^*)\, X_t^*$$

*where the forward price $C^*(t, X_t^*)$ of the claim in the Gaussian model solves the pde*

$$C_t^*(t, x^*) + \frac{1}{2}\, C_{xx}^*(t, x^*)||\sigma^T(t)||^2 (x^*)^2 = 0$$

*subject to the terminal condition $C^*(T, X_T^*) = g(X_T)$.*

*A tractable strategy is defined as the sum over $k$ Gaussian strategies. It is given by $k$ functions $g^{(i)}(X_T)$ and by $k$ deterministic volatility functions $\sigma^{(i)T}(t)$, $i = 1, \ldots, k$.*

Both Gaussian and tractable strategies are given in closed form and are thus quite easy to implement. The hedge ratio and the value of the hedge-portfolio are path independent and depend only on time $t$ and on the current value of $X_t^*$.

Is is important to stress that the use of these strategies is not based on the assumption that the true data-generating process is a Gaussian process. Rather, the Gaussian model is one of the most simplest choices in a situation where the true model is not known but some hedge model is needed to implement the hedging strategy. After restricting the set of trading strategies to tractable strategies, the first problem is to find Gaussian and tractable hedging strategies that are robust. The second problem is to find the cheapest strategy which can then be compared to the benchmark solution of Avellaneda, Levy, and Parás (1995). The comparison will show the consequences for the initial capital and the hedging performance which are implied by the restriction to tractable strategies. Besides, it will be the basis for deciding whether Gaussian models are a good choice as hedging models or not.

For convex and concave payoffs, the cheapest robust hedge is already given by a Gaussian strategy so that the problem is trivial. For mixed payoffs, on the other hand, the cheapest robust hedge is no Gaussian hedge, and it is also no tractable hedge, as we will show below in proposition 3.6.

The arguments in the following rely on the cost process which is defined as the difference between the value of the hedge portfolio at time $t$ and the sum of the initial capital plus the accumulated gains and losses up to time $t$.

Definition 3.2 (Cost process). *The hedging strategy is given by the number of stocks $\phi_t^X$ and the number of bonds $\phi_t^B$. The forward value of the hedge portfolio at time $t$ is $\phi_t^X X_t^* + \phi_t^B$. Then the forward cost process is defined as*

$$L_t^* = \left(\phi_t^X X_t^* + \phi_t^B\right) - \left(\phi_0^X X_0^* + \phi_0^B + \int_0^t \phi_u^X dX_u^*\right).$$

If the cost process is identically equal to zero, the strategy is self-financing. If the cost process is decreasing almost surely, the strategy is over-financing, and we can withdraw money. In this case, the strategy is a superhedging strategy for the terminal payoff $\phi_T^X X_T^* + \phi_T^B$ as well as for any lower payoff. If the cost process is decreasing whenever the realized volatility $\sigma_t^T$ is within the interval $[\sigma_{\min}^T(t), \sigma_{\max}^T(t)]$, then the corresponding strategy is a robust hedging strategy for the terminal payoff $\phi_T^X X_T^* + \phi_T^B$ as well as for any lower payoff.

The next lemma gives the cost process for a tractable strategy:

Lemma 3.3 (Cost process for a tractable strategy). *The tractable strategy is given by $k$ functions $g^{(i)}(X_T)$ and by $k$ deterministic volatility functions $\sigma^{(i)T}(t)$, $i = 1, \ldots, k$. Then the forward cost process is given by*

$$L_t^* = \sum_{i=1}^k \underbrace{\frac{1}{2}\int_0^t C_{xx}^{(i)*}(u, X_u^*)\left(||\sigma_u^T||^2 - ||\sigma^{(i)T}(u)||^2\right)(X_u^*)^2 du}_{L_t^{(i)*}}$$

*where $L_t^{(i)*}$ is the forward cost process of the Gaussian strategy $i$.*

**Proof.** For the tractable strategy, the number of stocks at time $t$ is

$$\phi_t^X = \sum_{i=1}^k C_x^{(i)*}(t, X_t^*), \tag{3}$$

and the forward value of the hedge portfolio is

$$C^*(t, X_t^*) = \sum_{i=1}^k C^{(i)*}(t, X_t^*)$$

where $C^{(i)}$ is the price of the payoff $g^{(i)}$ calculated in the Gaussian model given by $\sigma^{(i)T}(t)$. From the definition of $L_t^*$ we get

$$L_t^* \;=\; C^*(t, X_t^*) - \left( C^*(0, X_0^*) + \int_0^t \phi_u^X \, dX_u^* \right)$$

Equation (3) for the hedge ratio and Itôs lemma give

$$dL_t^* \;=\; \sum_{i=1}^{k} \left( C_t^{(i)*}(t, X_t^*)dt + \frac{1}{2} C_{xx}^{(i)*}(t, X_t^*) ||\sigma_t^T||^2 (X_t^*)^2 \, dt \right)$$

Plugging in the pde for $C^{(i)*}$ from definition 3.1 yields the result. □

Lemma 3.3 shows that the cost process depends on the gammas of the claims $g$ in the Gaussian models and on the differences of squared volatilities. It is well known that in a Gaussian model, the sign of gamma depends on the form of the terminal payoff $g$.[3] If the function $g$ is convex, gamma is positive, and if $g$ is concave, gamma is negative. If $g$ is neither convex nor concave, gamma can (and will) change its sign.

In a first step, we consider Gaussian strategies. The increment of the cost process is

$$dL_t^* \;=\; \frac{1}{2} C_{xx}^*(t, X_t^*) \left( ||\sigma_t^T||^2 - ||\sigma^T(t)||^2 \right) (X_t^*)^2 dt$$

For a convex claim with payoff $g$, gamma is always positive. The cost process is decreasing if the difference of squared volatilities is negative. This holds for any realized volatility in the volatility interval if the hedge volatility $\sigma^T(t)$ is equal to the upper volatility bound $\sigma_{\max}^T(t)$, so that a Gaussian hedge at the upper volatility bound results in a robust hedge. An analogous argument shows that for a concave payoff, a Gaussian hedge at the lower volatility bound is a robust hedge. If $g$ is neither convex nor concave, a Gaussian strategy is no robust hedge for $g$, irrespective of the choice of the deterministic volatility: For any given Gaussian model with volatility $\sigma^T(t)$, calculate the price of the claim and fix some $t$ such that the gamma of the claim at time $t$ can be both positive or negative, depending on $X_t^*$. As

[3] $C^{(i)^*}$ is a Black/Scholes–type pricing formula. Thus the relationship between the gamma of the claim and the form of the terminal payoff is easy to proof. For a more geneneral setup we refer the interested reader to Bergman, Grundy, and Wiener (1996), El Karoui, Jeanblanc-Picqué, and Shreve (1998) or Hobson (1998).

the sign of the difference between the squared volatilities $||\sigma_t^T||^2$ and $||\sigma^T(t)||^2$ does not depend on $X_t^*$, the increment of the cost process will also be either positive or negative, depending on $X_t^*$. So, irrespective of the choice of $\sigma^T(t)$, there are always some volatility and index paths for which the Gaussian strategy needs additional money even if the realized volatility stays within the given interval. Note that this even holds if the volatility is set equal to the value maximizing volatility.

This does not imply that Gaussian strategies are of no use at all for robust hedging of mixed payoffs. One way to obtain a Gaussian robust hedge for a mixed payoff is to implement a Gaussian robust hedge for a dominating payoff that is either convex or concave. Another possibility is to switch to tractable strategies:

Proposition 3.4 (Tractable robust hedge). *Any two convex functions $g^{(1)}$ and $g^{(2)}$ which satisfy the condition $g^{(1)}(x) - g^{(2)}(x) \geq h(x)$ for all $x \geq 0$ define a tractable robust hedge for the payoff $h$. The hedging strategy is given by the sum of the Gaussian hedging strategies for $g^{(1)}$ and $g^{(2)}$ where $g^{(1)}$ is hedged at the upper volatility bound and $g^{(2)}$ is hedged at the lower volatility bound. The forward value $V^*(t, x^*; g^{(1)}, g^{(2)})$ of the tractable robust hedge is*

$$V^*(t, x^*; g^{(1)}, g^{(2)}) \;=\; C^{(1)^*}\left(t, x^*; g^{(1)}\right) - C^{(2)^*}\left(t, x^*; g^{(2)}\right)$$

*where $C^{(1)^*}$ and $C^{(2)^*}$ are the forward prices of $g^{(1)}$ and $g^{(2)}$ calculated at the upper and lower volatility bound.*

**Proof:** In any stochastic volatility model where volatility is bounded, the Gaussian hedge for $g^{(1)}$ at the upper volatility bound $\sigma_{\max}^T(t)$ is a superhedge. Analogously, the Gaussian hedge for $-g^{(2)}$ at the lower volatility bound $\sigma_{\min}^T(t)$ is a superhedge. As the sum of two superhedges is a superhedge again, the resulting tractable strategy is a superhedge for the payoff $g^{(1)} - g^{(2)}$ and therefore also a superhedge for the dominated payoff $h$. It is therefore a robust hedge with respect to the volatility interval $[\sigma_{\min}^T(t), \sigma_{\max}^T(t)]$. □

The idea of a tractable hedge is to decompose the payoff to be hedged into a long and a short position in convex claims and to hedge each component separately. In contrast to a Gaussian hedging strategy for the payoff $h$, the tractable strategy

that is based on such a decomposition of $h$ gives a robust hedge. In contrast to a Gaussian robust hedge for a dominating payoff, the tractable hedge can reduce the initial capital needed significantly. In the following, we consider the cheapest tractable robust hedge which will then be compared to the ALP-hedge that serves as the benchmark.

The cheapest choice of the functions $g^{(1)}$ and $g^{(2)}$ is given by the optimal envelope :

Definition 3.5 (Optimal envelope). *For $t < T$ and $\sigma_{min}(s) \leq \sigma_{max}(s)$ for all $t \leq s < T$ where strict inequality holds at least for one $s$, the optimal envelope for the payoff $h$ is defined as*

$$\begin{aligned}(\tilde{g}^{(1)}, \tilde{g}^{(2)})(t, X_t^*) \quad = \quad & \arg\min_{g^{(1)}, g^{(2)}} \big\{ V^*(t, X_t^*; g^{(1)}, g^{(2)}) \mid \\ & g^{(1)}(x) - g^{(2)}(x) \geq h(x) \ \forall \ x, g^{(1)}, g^{(2)} \ \textit{convex} \big\}.\end{aligned}$$

*For $t < T$ and $\sigma_{min}(s) = \sigma_{max}(s)$ for all $t \leq s < T$ or $t = T$, any choice of functions $g^{(1)}$ and $g^{(2)}$ with*

$$g^{(1)} - g^{(2)} = h(x)$$

*is optimal.*

Two points are important to keep in mind. First, for different choices of the functions $g^{(1)}$ and $g^{(2)}$ which result in the same portfolio payoff the initial capital needed may be different. We therefore have to think about the cheapest decomposition of the payoff. Second, the portfolio formed out of $g^{(1)}$ and $g^{(2)}$ only has to dominate the payoff to be hedged, but it has not to give exactly this payoff. Indeed, the optimal decomposition for a strictly dominating payoff may be cheaper than the optimal decomposition of the payoff itself. We will now discuss these two points in more detail.

First, the initial capital depends on the decomposition of the payoff to be hedged so that there is a cheapest decomposition. An easy example is given by the zero payoff $h(X_T) = 0$. Obviously the value of the cheapest tractable hedge is zero as well. Now, any choice of convex functions $g^{(1)}, g^{(2)}$ with $g^{(1)}(x) = g^{(2)}(x)$ for all $x \geq 0$ implies a tractable robust hedge for $h$. However, since $g^{(1)}$ is priced at the

upper volatility bound while $g^{(2)}$ is priced at the lower volatility bound, the initial investment is greater than zero.[4] This is shown in figure 1 where we have chosen $g^{(1)}$ to be a call option on $X$ with strike $K$. The figure gives the initial capital needed as a function of the strike price $K$.

The main point of the example is that adding a zero payoff can increase initial capital. If the convex functions $g^{(1)}$ and $g^{(2)}$ represent a tractable robust hedge, then for any convex function $f$, the functions $g^{(1)} + f$ and $g^{(2)} + f$ represent a tractable robust hedge also, and the terminal payoff that is hedged is $g^{(1)} - g^{(2)}$ for both portfolios. But while the initial capital needed for the first robust hedge is

$$C^{(1)^*}\left(t, x^*; g^{(1)}\right) - C^{(2)^*}\left(t, x^*; g^{(2)}\right)$$

the initial capital needed for the second robust hedge is

$$C^{(1)^*}\left(t, x^*; g^{(1)}\right) + C^{(1)^*}\left(t, x^*; f\right) - C^{(2)^*}\left(t, x^*; g^{(2)}\right) - C^{(2)^*}\left(t, x^*; f\right).$$

As the price of a convex claim is increasing in volatility, adding $f$ to both claims increases the initial capital.

So, for the optimal choice of functions $g^{(1)}$ and $g^{(2)}$ there must not exist a convex function $f$ such that $g^{(1)} - f$ and $g^{(2)} - f$ are both convex. Otherwise, the latter two functions would result in a tractable robust hedge that is cheaper than the robust hedge given by $g^{(1)}$ and $g^{(2)}$ themselves.

Second, the decomposition of a dominating payoff may give a tractable robust hedge that needs less initial capital than the one implied by a decomposition of the payoff itself. As an example, consider the payoff of a bullish vertical call spread, i.e.

$$h(X_T) = [X_T - K_1]^+ - [X_T - K_2]^+$$

where $0 < K_1 < K_2$. The payoff is bounded above by $K_2 - K_1$ so one possibility to achieve a tractable robust hedge is simply given by a static hedge with initial investment $B(t,T)(K_2 - K_1)$. Formally, we set $g^{(1)} = K_2 - K_1$, and $g^{(2)} = 0$. The resulting portfolio dominates the bullish vertical spread. We now compare this trivial robust hedge to the robust hedge implied by decomposing the payoff into the

[4]For a convexe payoff, the price is increasing in volatility. This is e.g. shown in Henderson, Hobson, Howison, and Kluge (2003).

difference of two convex functions. The obvious possibility to decompose $h$ exactly into the difference of two convex functions $g^{(1)}$ and $g^{(2)}$ is given by

$$g^{(1)}(x) = [x - K_1]^+, \quad g^{(2)}(x) = [x - K_2]^+.$$

The initial investment $V_0^*$ is

$$V^*(t_0, X_{t_0}^*; g^{(1)}, g^{(2)}) \quad = \quad C^{(1)^*}(t_0, X_{t_0}^*; g^{(1)}) - C^{(2)^*}(t_0, X_{t_0}^*, g^{(1)}).$$

Figure 2 gives the initial capital needed as a function of the current value of $X_{t_0}^*$. For medium and high values of the stock price, the initial capital needed for the hedge that is represented by the two calls is greater than the initial capital needed for the trivial superhedge which dominates the payoff. So, the hedge for the dominating payoff may indeed be cheaper.

In general, a strict inequality for the terminal payoffs

$$g^{(1)} - g^{(2)} \quad \geq \quad f^{(1)} - f^{(2)}$$

does not imply that the initial capital needed is greater for the payoff on the left hand side than for the payoff on the right hand side. Indeed, there can be any relationship for the initial capital.

Summing up, the determination of the optimal tractable hedge can be described heuristically as follows: Assume that the payoff–function $h$ can be dominated by convex functions. We start with a long position in one of these dominating functions, i.e. we set $g^{(1)}$ equal to one of these convex functions. For this $g^{(1)}$, we determine a convex function $g^{(2)}$ such that

$$g^{(2)}(x) \leq g^{(1)}(x) - h(x).$$

As the price of $g^{(2)}$ with respect to the lower volatility bound is subtracted from the initial capital, it has to achieve the maximum possible price under $\sigma_{\min}$. As the optimal envelope is given by the pair $(g^{(1)}, g^{(2)})$ which altogether implies the lowest initial capital, there must not exist any convex function $f$ such that $g^{(1)} - f$ and $g^{(2)} - f$ are both convex. Therefore, the optimisation problem can be restricted to the set of functions $g^{(1)}$ which are tangent to the payoff $h$ in at least one point.

Once we have determined the optimal envelope, we can compare the tractable robust hedge implied by this optimal envelope to the cheapest robust ALP-hedge. For the comparison, we focus on mixed payoffs, as the ALP-hedge and our strategy would coincide for concave and convex payoffs.

First, the initial capital needed for the tractable robust hedge is always greater than the initial capital needed for the ALP-hedge. Whenever the payoff $h$ is neither convex nor concave, there is simply no choice of functions $g^{(1)}$ and $g^{(2)}$ such that the tractable strategy coincides with the ALP-hedge:

Proposition 3.6 (Comparison of price bounds). *Assume that the volatility process is non–vanishing and that the payoff $h$ is neither convex nor concave. Let the optimal envelope be given by the convex functions $g^{(1)}$ and $g^{(2)}$ and let $\widehat{U}^h$ denote the initial capital needed for the ALP-hedge of the payoff $h$. The initial capital needed for the tractable robust hedge is $C^{(1)}\left(t,x;g^{(1)}\right) - C^{(2)}\left(t,x;g^{(2)}\right)$. Then the following inequality holds:*

$$\widehat{U}^h(t,x) \;<\; C^{(1)}\left(t,x;g^{(1)}\right) - C^{(2)}\left(t,x;g^{(2)}\right) \quad \forall\ t \in [0,T[. \tag{4}$$

**Proof.** The initial capital for the ALP-hedge follows from proposition 2.2, the initial capital for the tractable superhedge follows from definition 3.4.

If $h(x) < g^{(1)}(x) - g^{(2)}(x)$ for at least some $x \geq 0$, then relationship (4) for the initial capital follows from

$$\begin{aligned}
\widehat{U}^{h^*}(t,x^*) &= \sup_{P_T^*} E^{P_T^*}\left[h(X_T^*)|\mathcal{F}_t\right] \\
&< \sup_{P_T^*} E^{P_T^*}\left[g^{(1)}(X_T^*) - g^{(2)}(X_T^*)|\mathcal{F}_t\right] \\
&\leq \sup_{P_T^*} E^{P_T^*}\left[g^{(1)}(X_T^*)|\mathcal{F}_t\right] - \inf_{P_T^*} E^{P_T^*}\left[g^{(2)}(X_T^*)|\mathcal{F}_t\right] \\
&= C^{(1)^*}\left(t,x,g^{(1)}\right) - C^{(2)^*}\left(t,x,g^{(2)}\right)
\end{aligned}$$

where we have used that the smallest initial capital for a superhedge is the maximal price over all forward-risk adjusted measures $P_T^*$.

If $h \equiv g^{(1)} - g^{(2)}$, then both $g^{(1)}$ and $g^{(2)}$ are strictly convex functions. Assume that $\widehat{U}^{h^*}(t,x) = C^{(1)^*}\left(t,x^*;g^{(1)}\right) - C^{(2)^*}\left(t,x^*;g^{(2)}\right)$. The lowest upper price bound $U^*$

solves the BSB-equation (2) in proposition 2.2

$$\widehat{U}_t^*(t,x^*) + \tfrac{1}{2}\|\sigma^T(t,x^*)\|^2\,(x^*)^2\,\widehat{U}_{xx}^*(t,x^*) = 0$$

$$\text{where} \quad \sigma^T(t,x^*) = \sigma_{\max}^T(t)\;1_{\left\{\widehat{U}_{xx}^*(t,x^*)\geq 0\right\}} + \sigma_{\min}^T(t)\;1_{\left\{\widehat{U}_{xx}^*(t,x^*)<0\right\}}.$$

Replacing $\widehat{U}^*$ with the difference between $C^{(1)*}$ and $C^{(2)*}$ gives

$$C_t^{(1)*}(t,x^*) - C_t^{(2)*}(t,x^*) + \tfrac{1}{2}\,\|\sigma^T(t,x^*)\|^2\,(x^*)^2\left(C_{xx}^{(1)*}(t,x^*) - C_{xx}^{(2)*}(t,x^*)\right) = 0 \quad (5)$$

$$\text{where} \quad \sigma^T(t,x^*) = \sigma_{\max}^T(t)1_{\left\{C_{xx}^{(1)*}(t,x^*)-C_{xx}^{(2)*}(t,x^*)\geq 0\right\}} + \sigma_{\min}^T(t)1_{\left\{C_{xx}^{(1)*}(t,x^*)-C_{xx}^{(2)*}(t,x^*)<0\right\}}.$$

The prices $C^{(1)*}$ and $C^{(2)*}$ in the Gaussian models solve the pdes

$$C_t^{(i)*}(t,x^*) + \frac{1}{2}\,C_{xx}^{(i)*}(t,x^*)||\sigma^{(i)T}(t)||^2(x^*)^2 \;=\; 0, \qquad i=1,2$$

where $\sigma^{(1)T}(t) = \sigma_{\max}(t,T)$ and $\sigma^{(2)T}(t) = \sigma_{\min}(t,T)$. Plugging these pdes, $\sigma^{(1)T}(t)$, and $\sigma^{(2)T}(t)$ into (5) gives

$$\left(C_{xx}^{(1)^*}(t,x^*) - C_{xx}^{(2)^*}(t,x^*)\right)||\sigma^T(t,x^*)||^2 \;=\; C_{xx}^{(1)^*}(t,x^*)||\sigma_{\max}^T(t)\|^2 - C_{xx}^{(2)^*}(t,x^*)||\sigma_{\min}^T(t)\|^2.$$

For $C_{xx}^{(1)^*}(t,x^*) - C_{xx}^{(2)^*}(t,x^*) \geq 0$, this implies

$$C_{xx}^{(2)^*}(t,x^*)||\sigma_{\max}^T(t)\|^2 \;=\; C_{xx}^{(2)^*}(t,x^*)||\sigma_{\min}^T(t)\|^2$$

which can only hold for $C_{xx}^{(2)^*}(t,x^*) = 0$. But in a Gaussian model, the gamma of a convex payoff can not vanish for $t < T$. So the assumption must be wrong, and inequality (4) holds. □

As a second point, we compare the local behaviour of the ALP-hedge and the optimal tractable hedge. They differ with respect to the conditions under which there is a local loss at time $t$ and in the size of local gains and losses. In the next proposition, we identify the conditions under which we have to inject money at time $t$:

Proposition 3.7 (Conditions for local losses). *For the ALP-hedge with forward value process $U^*$, there is a loss at time $t$ if one of the following conditions holds:*

(a) $\widehat{U}_{xx}^*(t,x^*) > 0$ *and* $||\sigma_t^T||^2 > ||\sigma_{max}^T(t)||^2$
(b) $\widehat{U}_{xx}^*(t,x^*) < 0$ *and* $||\sigma_t^T||^2 < ||\sigma_{min}^T(t)||^2$

*For the tractable robust hedge given by $g^{(1)}$ and $g^{(2)}$, there is a loss at time $t$ if one of the following conditions holds:*

(a) $C_{xx}^{(1)*}(t,x^*;g^{(1)}) - C_{xx}^{(2)*}(t,x^*;g^{(2)}) > 0$ *and* $||\sigma_t^T||^2 > ||\sigma_{crit}^T(t,x^*)||^2$

(b) $C_{xx}^{(1)*}(t,x^*;g^{(1)}) - C_{xx}^{(2)*}(t,x^*;g^{(2)}) < 0$ *and* $||\sigma_t^T||^2 < ||\sigma_{crit}^T(t,x^*)||^2$

*where the norm of the critical volatility is given by*

$$||\sigma_{crit}^T(t,x^*)||^2 = \frac{C_{xx}^{(1)*}(t,x^*;g^{(1)})||\sigma_{max}^T(t)||^2 - C_{xx}^{(2)*}(t,x^*;g^{(2)})||\sigma_{min}^T(t)||^2}{C_{xx}^{(1)*}(t,x^*;g^{(1)}) - C_{xx}^{(2)*}(t,x^*;g^{(2)})}$$

*If the denominator is zero or if the term on the right hand side is negative, we set the critical volatility equal to zero.*

**Proof:** A loss at time $t$ occurs if the increment $dL_t^*$ of the forward cost process is positive. For the ALP-hedge with forward value process $U^*$, the increment of the cost process is given by

$$dL_t^* = \widehat{U}_{xx}^*(t,X_t^*)\left(||\sigma_t^T||^2 - ||\sigma_{\min}^T(t)||^2 1_{\widehat{U}_{xx}^*(t,X_t^*)<0} - ||\sigma_{\max}^T(t)||^2 1_{\widehat{U}_{xx}^*(t,X_t^*)\geq 0}\right)(X_t^*)^2 dt$$

where $U^*$ is the forward price process of the hedge portfolio from proposition 2.2. For $U_{xx}^*(t,X_t^*) > 0$, the increment is positive if $||\sigma_t^T||^2 > ||\sigma_{\max}^T(t)||^2$. For $U_{xx}^*(t,X_t^*) < 0$, the increment is positive if $||\sigma_t^T||^2 < ||\sigma_{\min}^T(t)||^2$.

For the tractable hedge, the increment of the cost process is given by

$$\begin{aligned} dL_t^* &= \Big\{C_{xx}^{(1)*}(t,X_t^*)\left(||\sigma_t^T||^2 - ||\sigma_{\max}^T(t)||^2\right)(X_t^*)^2 \\ &\qquad - C_{xx}^{(2)*}(t,X_t^*)\left(||\sigma_t^T||^2 - ||\sigma_{\min}^T(t)||^2\right)(X_t^*)^2\Big\}dt \\ &= \left\{C_{xx}^{(1)*}(t,X_t^*) - C_{xx}^{(2)*}(t,X_t^*)\right\} \\ &\quad \cdot \left\{||\sigma_t^T||^2 - \frac{C_{xx}^{(1)*}(t,X_t^*)||\sigma_{\max}^T(t)||^2 - C_{xx}^{(2)*}(t,X_t^*)||\sigma_{\min}^T(t)||^2}{C_{xx}^{(1)*}(t,X_t^*) - C_{xx}^{(2)*}(t,X_t^*)}\right\}(X_t^*)^2 dt \\ &= \left(C_{xx}^{(1)*}(t,X_t^*) - C_{xx}^{(2)*}(t,X_t^*)\right)\left(||\sigma_t^T||^2 - ||\sigma_{\text{crit}}^T(t,x^*)||^2\right)(X_t^*)^2 dt \end{aligned}$$

For $C_{xx}^{(1)*}(t,X_t^*) - C_{xx}^{(2)*}(t,X_t^*) > 0$, this term is positive if $||\sigma_t^T||^2 > ||\sigma_{\text{crit}}^T(t,x^*)||^2$. For $C_{xx}^{(1)*}(t,X_t^*) - C_{xx}^{(2)*}(t,X_t^*) < 0$, this term is positive if $||\sigma_t^T||^2 < ||\sigma_{\text{crit}}^T(t,x^*)||^2$.
□

The critical volatility depends on the current time $t$ and on the current forward price $x^*$. Given $t$ and $x^*$, the critical volatility for the ALP-hedge coincides with one of the volatility bounds. For the tractable hedge, the critical volatility either coincides with one of the volatility bounds, or it is outside the given volatility interval. In

the latter case, there is an additional robustness of the tractable hedge which still avoids losses when the ALP-hedge already suffers from local losses.

## 4. Optimal Tractable Hedge

In this section, we determine the optimal tractable robust hedge for two types of claims. The first claim is a generalisation of power claims. It can be interpreted as a power option with power less than one. The piecewise convexity (respectively concavity) allows the explicit determination of the cheapest possible tractable superhedge.[5]

Proposition 4.1. *Consider a European contingent claim where the payoff at $T$ is*

$$h(X_T) = f(X_T)1_{\{X_T \geq y\}}, \quad y > 0.$$

*The function $f$ is twice continuously differentiable, increasing and concave. Then, for each $t$ ($0 \leq t < T$) and each forward asset price $X_t^* = z$ ($z \geq 0$), the optimal envelope $\tilde{g}^{(1)}, \tilde{g}^{(2)}$ for the payoff $h$ is given by*

$$\tilde{g}^{(1)}(x) = g^{(1)}(x, \hat{x}^*) \qquad \tilde{g}^{(2)}(x) = g^{(2)}(x, \hat{x}^*)$$

*where the functions $g^{(1)}$ and $g^{(2)}$ are defined by*

$$\begin{aligned} g^{(1)}(x, x^*) &= f'(\hat{x}^*)\left[x - \left(\hat{x}^* - \frac{f(\hat{x}^*)}{f'(\hat{x}^*)}\right)\right]^+ \\ g^{(2)}(x, x^*) &= [f(x^*) - f'(x^*)x^*]^+ (x^* - x)^+ + f'(\hat{x}^*)[x - \hat{x}^*]^+ - [f(x) - f(\hat{x}^*)]^+ \end{aligned}$$

*and where*

$$\hat{x}^* = \arg\min_{x^* \geq y} \left\{ C^{(1)^*}\left(t, z; g^{(1)}(\cdot, x^*)\right) - C^{(2)^*}\left(t, z; g^{(2)}(\cdot, x^*)\right) \right\}$$

**Proof.** We follow the heuristics explained in the last section. First, we dominate the payoff $h$ by a convex function which is then priced at the upper volatility bound. We restrict ourselves to those dominating functions $g^{(1)}$ that are the tightest upper

[5] A power option with power less than one can be observed as so–called embedded option in insurance linked products, i.e. contracts which pay out a guaranteed minimum rate of return and a positive excess rate, which is specified on the basis of a benchmark portfolio. Mahayni and Schlögl (2003) analyse robust risk management strategies for these products including an application of the following proposition.

dominations, i.e. there must not exist any convex functions that dominates $h$ as well, but that is strictly lower than $g^{(1)}$. Each $g^{(1)}$ that meets these conditions is a call option with strike $K$. The point where $g^{(1)}$ and $h$ are tangent is denoted by $(x^*, f(x^*))$ where $x^* > y$. This implies that $g^{(1)}$ is a position in $f'(x^*)$ call options with strike

$$K(x^*) = x^* - \frac{f(x^*)}{f'(x^*)}$$

so that

$$g^{(1)}(x, x^*) = f'(x^*) \left[ x - \left( x^* - \frac{f(x^*)}{f'(x^*)} \right) \right]^+$$

$g^{(1)}$ dominates $h$ by the amount $R(x, x^*) = g^{(1)}(x, x^*) - h(x)$, i.e.

$$\begin{aligned} R(x, x^*) &= R_0(x, x^*) + R_1(x, x^*) \\ \text{where } R_0(x, x^*) &= \left( g^{(1)}(x, x^*) - h(x) \right) 1_{\{0 \le x \le x^*\}} \\ \text{and } R_1(x, x^*) &= \left( g^{(1)}(x, x^*) - h(x) \right) 1_{\{x \ge x^*\}} \end{aligned}$$

We now choose $g^{(2)}$ to be the highest convex function that is dominated by $R(x, x^*)$. This gives

$$\begin{aligned} g^{(2)}(x, x^*) &= R_0(0, x^*)^+ (x^* - x)^+ + R_1(x, x^*) \\ &= [f(x^*) - f'(x^*)x^*]^+ (x^* - x)^+ + f'(x^*) (x - x^*)^+ - [f(x) - f(x^*)]^+ \end{aligned}$$

To determine the cheapest tractable hedge, we have to choose the optimal $\hat{x}^*$ which is for any $t$ and any $X_t^* = z$ given as

$$\hat{x}^* = \arg\min_{x^* \ge y} \left\{ C^{(1)^*} \left( t, z; g^{(1)}(\cdot, x^*) \right) - C^{(2)^*} \left( t, z; g^{(2)}(\cdot, x^*) \right) \right\}$$

where $C^{(1)^*}$ and $C^{(2)^*}$ are defined as in proposition 3.4. The cheapest tractable superhedge is then represented by $\tilde{g}^{(1)}, \tilde{g}^{(2)}$ where $\tilde{g}^{(1)}(x) = g^{(1)}(x, \hat{x}^*)$, $\tilde{g}^{(2)}(x) = g^{(2)}(x, \hat{x}^*)$. □

Normally the interesting payoffs have kinks which necessitate a special treatment. To show the optimal decomposition in this case, we consider a bullish vertical call spread:

Lemma 4.2. *The payoff at time $T$ of the bullish vertical spread is given by*

$$h(X_T) = [X_T - K_1]^+ - [X_T - K_2]^+, \quad K_1 < K_2$$

*For each time $t$ $(0 \leq t < T)$ and each forward asset price $X_t^* = x^*$ $(x^* \geq 0)$, the optimal envelope $\tilde{g}^{(1)}, \tilde{g}^{(2)}$ for the payoff $h$ is given by*

$$\tilde{g}^{(1)}(x) = f^{(1)}(x, \widehat{K}_0) 1_A + (K_2 - K_1)(1 - 1_A) \quad \text{and} \quad \tilde{g}^{(2)}(x) = f^{(2)}(x, \widehat{K}_0) 1_A$$

*where the functions $f^{(1)}(x, K_0)$ and $f^{(2)}(x, K_0)$ are defined by*

$$f^{(1)}(x, K_0) = \frac{K_2 - K_1}{K_2 - K_0} [x - K_0]^+ \quad \text{and} \quad f^{(2)}(x, K_0) = \frac{K_2 - K_1}{K_2 - K_0} [x - K_2]^+ .$$

*The optimal $\widehat{K}_0 = \widehat{K}_0(t, x^*) \geq 0$ is given by*

$$\widehat{K}_0 \;=\; \arg \min_{0 \leq K_0 < y} \left[ C^{(1)^*}\left(t, x^*; f^{(1)}(\cdot, K_0)\right) - C^{(2)^*}\left(t, x^*; f^{(2)}(\cdot, K_0)\right) \right]$$

*and the set $A$ is given by*

$$A \;=\; \left\{ C^{(1)^*}\left(t, x^*; f^{(1)}(\cdot, \widehat{K}_0)\right) - C^{(2)^*}\left(t, x^*; f^{(2)}(\cdot, \widehat{K}_0)\right) < K_2 - K_1 \right\}.$$

**Proof.** We use the same reasoning as in the proof of proposition 4.1. Each tightest $g^{(1)}$ coincides with $h$ either for $x = K_2$ or for some $x > K_2$. In the first case, we get that $g^{(1)}$ is some position in call options with strike $K_0 \leq K_1$

$$g^{(1)}(x, K_0) = \frac{K_2 - K_1}{K_2 - K_0} [x - K_0]^+$$

and the corresponding (highest) $g^{(2)}$ is just

$$g^{(2)}(x, K_0) = \frac{K_2 - K_1}{K_2 - K_0} [x - K_2]^+ .$$

In the second case, we get the trivial choice $g^{(1)}(x) = K_2 - K_1$ and $g^{(2)}(x) = 0$. The minimization problem is now straightforward. If we are in the set $A$, then the optimal tractable hedge is given by a decomposition into two options. Otherwise, the optimal tractable hedge is just the trivial choice. □

In the next section, we explore the effectiveness of the tractable hedging strategy which is implied by the optimal decomposition and compare it to the ALP-hedge. This section concludes with an illustration of the initial capital needed for the trivial

robust hedge, the tractable robust hedge represented by the naive decomposition, and the tractable robust hedges for dominating payoffs.

We consider a bullish vertical spread with $K_1 = 90$, $K_2 = 100$ and maturity $T = 0.5$. The upper bound on volatility is 0.4, the lower bound is 0.1, and the interest rate is 5%. The left graphic in figure 3 illustrates the pricing bounds that correspond to the naive decomposition of the claim into one call with strike 90 long and one call with strike 100 short, i.e. $K_0 = K_1 = 90$. The long position is priced at the upper volatility bound, the short position is priced at the lower volatility bound. Besides, the trivial pricing bound $10e^{-0.5 \cdot 0.05}$ is plotted. Obviously, the initial capital needed for the naive tractable hedge is not compatible with the trivial pricing bound: for $X_0 = 90$ e.g., the naive upper price bound is lower.

The right graphic in figure 3 highlights the technique of the envelope based tractable hedge where the different curves correspond to different choices of the lower strike price $K_0$. The lower envelope of the graphs is the initial capital for the cheapest tractable robust hedge. Of course, the payoff under consideration is neither convex nor concave which implies that the initial capital which is needed for the optimal tractable robust hedge is still higher than the initial capital needed for the ALP-hedge. The resulting *overpricing* against the ALP-hedge and its implications for hedging are discussed in the next section.

## 5. Tractable Hedging – Some Illustrative Examples

The numerical example in the last section shows that the optimal envelope represents a tractable robust hedge that can well be significantly cheaper than the naive tractable hedge. Nevertheless, the initial capital needed is in general still higher than the initial capital needed for the overall cheapest robust hedge, the ALP-hedge.[6] The restriction from the set of all admissible strategies to the subset of tractable strategies can be interpreted as an additional hedging criterion. A trader might feel more comfortable with a Black/Scholes type hedging recipe yielding a closed form solution instead of a sophisticated numerical tree approximation. However, the question is

[6]Intuitively, it is clear that the lowest initial investment can not decrease if we add the additional constraint that the strategy has to be tractable.

how close the tractable hedge is to the optimal ALP-hedge and by how much the additional restriction worsens the hedge. The aim of this section is to analyze the differences between the two hedges in order to answer this question.

We start with a comparison of the initial capital needed for the tractable hedge and for the ALP-hedge. We have to keep this difference in initial capital in mind when we analyze the path-wise performance of the hedging strategies, that is when we look at asset price paths and their corresponding cost paths. Here, we illustrate that the decomposition based hedge is very similar the ALP-hedge, in particular for short maturities. And we also focus on an additional robustness property of tractable robust hedges which lose money less often than the ALP-hedge in volatility scenarios which violate the imposed volatility bounds.

While the cost path captures the local properties of the hedging strategies, we are also interested in its total success. On this behalf, we calculate the sum of cumulated in– and outflows of the strategies, i.e. the total hedging costs or the negative tracking error. If this term is negative, then a shortfall occurs, otherwise the hedge is successful. We illustrate the distribution of final hedging costs under a stochastic volatility scenario. The resulting cost distributions are quite similar. It is worth mentioning that we do *not* intend to give a justification of tractability, i.e. an answer to the question in which cases the additional investment is reasonable. Instead we want to describe the effects implied by the tractability constraint.

As in the last section, we consider a bullish vertical spread with $K_1 = 90$, $K_2 = 100$, and maturity $T = 0.5$. The volatility interval is $[0.1, 0.4]$, the interest rate is $r = 0$.[7] For the implementation of the tractable hedge, we first determine the optimal strike price $K_0$. The current value of the hedge portfolio and the number of stocks are then the sums of the prices respectively deltas in the two Gaussian models. The tractable hedge is adjusted continuously, i.e. at every point in time, the optimal strike $K_0$ is recalculated. For the ALP-hedge, we rely on a tree approximation in which both the current value of the hedge portfolio and the delta are determined.

[7] $r = 0$ is only assumed in order to facilitate the comparison of in– and outflows of funds to and from the hedge portfolio which occur at different points in time.

The left graph of figure 4 compares the initial capital needed. The thick lines are the initial capital needed for the tractable hedge at time $t = 0.0$ and $t = 0.45$. The thin lines illustrate the initial capital needed for the ALP-hedge. It can be seen that the overpricing increases in time to maturity. The right graph gives the corresponding deltas. The delta for the tractable hedge can be higher or lower than the delta for the ALP-hedge. Again, the difference increases in time to maturity. This explains that for short times to maturity, the tractable hedge performs similiar to the ALP-hedge, as can be seen in the following simulations of the asset and cost paths.

We now turn to the path-wise performance of the hedging strategies. Both strategies are robust. If the realized volatility stays within the deterministic volatility interval, there are no losses at all, and the cost process is strictly decreasing. In particular, this holds if the true model is a Black-Scholes model with volatility $\sigma \in [0.1, 0.4]$. This is illustrated in figure 5 where we assume that the true data-generating process is a geometric Brownian motion with drift $\mu = 0$ and volatility $\sigma = 0.3$.

If the true volatility is outside the volatility interval, it is possible but not necessary that the strategy loses money in which case the cost process is locally increasing. This is illustrated in figure 6 where the true data-generating process is again a geometric Brownian motion with drift $\mu = 0$, but now with volatility $\sigma = 0.6$. Here, both cost processes may increase as well as decrease so that there are local losses as well as local gains. The actual behaviour of the cost process depends on the local gamma of the claim. If gamma is negative (which happens in particular for high asset prices) then the cost process is decreasing, and we can withdraw money. If gamma is positive (which happens in particular for low asset prices) then the ALP-hedge loses money. For the tractable hedge, the behaviour depends also on the critical volatility. Only if gamma is positive and, additionally, the critical volatility is lower than 0.6, the strategy loses money. This may be interpreted as additional robustness of the tractable hedge in comparison to the ALP-hedge. If we look at the simulated cost path, then there are indeed some points in time where the ALP-hedge loses money (the cost path is increasing) while for the tractable hedge we can still withdraw some money (the cost path is decreasing). In this case, either

the gamma of the tractable hedge is negative, or the gamma is positive, but the associated critical volatility is above 0.6.

Besides the path-wise performance, we are also interested in the performance of the hedge over the whole time interval from 0 to $T$. On this behalf, we consider the sum of the inflows minus the sum of the outflows which gives the final costs or the negative of the tracking error. This amount is negative if we can extract more money from the hedge portfolio than we have to inject, i.e. if the hedge is successful. It is positive if we have to inject more money than we can extract, in which case the hedge is not successful. For the example paths given in figures 5 and 6, the final value of the cost process is lower than the starting value, and the hedge is successful.

When we compare the cost distribution of the two hedging strategies, it is important to keep in mind that the initial capital is not equal. So, differences in the distribution can not only be attributed to different hedge ratios, but also to differences in the initial capital which is greater for the tractable hedge than for the ALP-hedge. For $\mu = r$, i.e. under the martingale measure, the expected difference of the costs is just equal to the difference of the initial capital.

To illustrate the cost distribution, we simulate asset price paths under a stochastic volatility model. We choose the same model setup as Avellaneda, Levy, and Parás (1995) which is given by

$$dX_t = \mu X_t dt + exp(Y_t) dW_t^X \tag{6}$$

$$dY_t = \alpha(\gamma - Y_t)\, dt + \rho\, dW_t^Y. \tag{7}$$

The Wiener processes $W^X$ and $W^S$ are assumed to be independent. We use the parameter constellation as given in Avellaneda, Levy, and Parás (1995), i.e. $\alpha = \ln 2$, $\gamma = \ln 0.2$, $\rho = \frac{2\sqrt{\ln 2}}{Z_{95}}$. $Z_{95} = 1.64$ represents the 95% percentil of the standard normal distribution. With these parameter values, the volatility band $0.1 - 0.4$ given above is the centered 90% confidence interval for volatility under the stationary distribution. We assume that the initial value of the index is $X_0 = 90$ and perform a Monte Carlo simulation with 10.000 runs. For each run, we calculate the injections of money and the withdrawals of money for each strategy. As the interest rate is assumed to be zero, the final costs are simply given by the sum of outflows from the

strategies minus the sum of injections. We are interested in the distribution of this cost process at time $T$. Note that this distribution depends on the dynamics of the index and the volatility under the true measure which are given by equations (6) and (7). The drift $\mu$ of the index is not necessarily equal to the risk-free interest rate.

Figure 7 gives the distribution of the final costs for the case $\mu = r = 0$. Thus the mean of the cost difference (given by $-4.27 + 2.63 = -1.64$) of the tractable hedge and the ALP-hedge is simply equal to the difference of initial investments (given by $5.70 - 7.34 = -1.64$). In the right graph, we have set the initial capital of both hedges equal to the initial capital of the tractable hedge so that the means of both distributions coincide. Notice that the distributions are very similar. Besides, it is worth mentioning that both strategies fail in less than 1% of the observations. The above effects, especially the similarity of the cost distributions, are also observed in figure 8 where the drift of the stock is $\mu = 0.1 > r = 0$. For this case, we give the mean and standard deviation of the cost distributions as well as the shortfall probability and expected shortfall in table 1.

## 6. CONCLUSION

The effectiveness of hedging strategies is given in terms of optimality criteria. Most of the criteria which are normally applied explicitly depend on the complete dynamics of the data generating processes. For example, minimizing the shortfall probability or the expected shortfall is only possible if the behaviour of the asset price under the real world probability measure is known. However, in case of model risk, i.e. uncertainty about the true dynamics or one of the parameters, it is no longer possible to derive optimal strategies for these criteria. For this reason, the concept of robust hedging is a very powerful tool as it only depends on the two volatility bounds $\sigma_{\min}$ and $\sigma_{\max}$, but neither on the true process for volatility nor on the parameters of this process or the current value of volatility.

For convex or concave payoffs, the robust hedging strategies are Gaussian strategies, and they can be determined by hedging at the upper or lower volatility bound. This is no longer true if the payoff is neither convex nor concave. In this case, we focus on

tractable strategies that are the sum of two Gaussian strategies. For the naive choice of the tractable hedge the initial capital needed may be much higher than for the ALP-hedge. However, we show that the initial capital can be reduced significantly by the optimal choice of the two Gaussian strategies. After accounting for the remaining difference in the initial capital, the distribution of the terminal hedging error is quite similar for the optimal tractable robust hedge and the ALP-hedge.

## Appendix A. Approximation of BSB solution

A.1. **Trinomial Model.** In order to capture all the difficulties which arise if a trader wants to hedge according to the lowest upper price bound, it is crucial to understand the numerical algorithm used to determine the ALP-hedge. We use the use the trinomial tree proposed by Avellaneda, Levy, and Parás (1995). The forward price process $(S)^n$ either moves up, down, or stays the same in every time step. The changes in the forward price are given by the factors

$$u(n) := \exp\left\{\sigma_{\max}\sqrt{\frac{T}{n}}\right\}; \quad m(n) := 1; \quad d(n) := \exp\left\{-\sigma_{\max}\sqrt{\frac{T}{n}}\right\}. \tag{8}$$

$T$ denotes the maturity of the claim under consideration, the number of time steps is $n$.

Lemma A.1. *If the probabilities are given by*

$$p^u(n) = q\left(1 - \frac{1}{2}\sigma_{max}\sqrt{\frac{T}{n}}\right); \quad p^m(n) = 1 - 2q; \quad p^d(n) = q\left(1 + \frac{1}{2}\sigma_{max}\sqrt{\frac{T}{n}}\right) \tag{9}$$

*with $p^u(n), p^m(n), p^d(n) \geq 0$ and $p^u(n) + p^m(n) + p^d(n) = 1$, then it holds:*

$$\begin{aligned} \lim_{n\to\infty}\left(p^u(n)u(n) + p^m(n) + p^d(n)d(n)\right) &= 1 \\ \lim_{n\to\infty}\frac{T}{n}\left(p^u(n)(1-u(n))^2 + p^d(n)(1-d(n))^2\right) &= 2q\sigma_{max}^2 \end{aligned}$$

*So, for $n \to \infty$, the model is arbitrage free, i.e. $p = \lim_{n\to\infty} p(n)$ defines a martingale measure. Furthermore, the local variance of the logarithm of asset price increments converges to $2q\sigma_{max}^2$.*

It is worth mentioning that lemma A.1 assures[8] that for a constant parameter $q$, the (forward) asset price process $S^n = \left(S^n_{t^n_k}\right)_{k=0,\ldots,n}$ where $t^n_k = k\frac{T}{n}$ converges for $n \to \infty$ in distribution to $S = (S_t)_{0\leq t\leq T}$ with

$$S_t = S_0 \exp\left\{-q\sigma_{\max}^2 t + \sqrt{2q}\sigma_{\max}W_t\right\}, \quad S_0 := S_0^n$$

In particular, for $q = \frac{1}{2}\frac{\sigma_{\min}^2}{\sigma_{\max}^2}$ the forward asset price dynamic is given by a geometric Brownian motion with volatility $\sigma_{min}$ while for $q = \frac{1}{2}$ the volatility is equal to $\sigma_{\max}$.

[8]In particular, we use a markov chain approximation scheme where the local consistency conditions are satisfied, c.f. for example Bingham and Kiesel (1998).

For any choice of the process $q^n = \left(q^n_{t^n_k}\right)_{k=0,\ldots,n-1}$ with $\frac{1}{2}\frac{\sigma^2_{\min}}{\sigma^2_{\max}} \leq q^n_{t_i} \leq \frac{1}{2}$, we define the process $C^n(t,S,q)$ by backward induction:

$$
\begin{aligned}
C^n(t^n_n, S^n_{t_n}, q^n_{t_n}) &= h(S^n_{t_n}) \\
C^n(t^n_i, S^n_{t_i}, q^n_{t_i}) &= q^n_{t_i}\left(1 - \frac{1}{2}\sigma_{\max}\sqrt{\frac{T}{n}}\right) C^n\left(t^n_{i+1}, u(n)S^n_{t^n_i}, q^n_{t_{i+1}}\right) \\
&\quad + (1-2q^n_{t_i})C^n\left(t^n_{i+1}, S^n_{t^n_i}, q^n_{t_{i+1}}\right) \\
&\quad + q^n_{t_i}\left(1 + \frac{1}{2}\sigma_{\max}\sqrt{\frac{T}{n}}\right) C^n\left(t^n_{i+1}, d(n)S^n_{t^n_i}, q^n_{t_{i+1}}\right)
\end{aligned}
$$

For $n \to \infty$, $C^n$ converges towards the price of the claim with payoff $h$ in a model where the state-dependent volatility at time $t^n_i$ is equal to $\sqrt{2q^n_{t_i}}\sigma_{max}$. Furthermore, for any admissible choice of the process $q^n$, $C^n$ is, in the limit, an arbitrage free forward price process within the uncertain volatility model. Therefore, in the limit, the lowest upper price bound is given by

$$
\hat{U}^n(t^n_i, S^n_{t_i}) = \sup_{\frac{1}{2}\frac{\sigma^2_{\min}}{\sigma^2_{\max}} \leq q^n_{t_j} \leq \frac{1}{2},\ j=i,\ldots,n-1} C^n(t^n_i, S^n_{t_i}, q^n_{t_i})
$$

Define $p^*(n)$ as the probability of a corresponding binomial tree where only the up– and down-movements are possible, i.e. where $q = \frac{1}{2}$ and $p^*(n) = \frac{1}{2} - \frac{1}{4}\sigma_{\max}\sqrt{\frac{T}{n}}$. Then we it is straightforward to show that, in general, the following equation holds:

$$
\begin{aligned}
&\hat{U}^n(t^n_{n-(i+1)}, S^n_{t_{n-(i+1)}}) \\
&= \left[p^*(n)\hat{U}^n(t^n_{n-i}, u(n)S^n_{t_{n-(i+1)}}) + (1-p^*(n))\hat{U}^n(t^n_{n-i}, d(n)S^n_{t_{n-(i+1)}})\right] \\
&\quad 1_{\left\{A^n\left(t^n_{n-(i+1)}, S^n_{t_{n-(i+1)}}\right)\right\}} \\
&\quad + \left[\frac{\sigma^2_{\min}}{\sigma^2_{\max}}\left[p^*(n)\hat{U}^n(t^n_{n-1}, u(n)S^n_{t_{n-2}}) + (1-p^*(n))\hat{U}^n(t^n_{n-1}, d(n)S^n_{t_{n-2}})\right]\right. \\
&\qquad\qquad \left. + \left(1 - 2\frac{\sigma^2_{\min}}{\sigma^2_{\max}}\right)\hat{U}^n(t^n_{n-1}, S^n_{t_{n-2}})\right] 1_{\left\{(A^n)^C\left(t^n_{n-(i+1)}, S^n_{t_{n-(i+1)}}\right)\right\}}
\end{aligned}
$$

where

$$
\begin{aligned}
&A^n\left(t^n_{n-(i+1)}, S^n_{t_{n-(i+1)}}\right) \\
&= \Big\{p^*(n)\hat{U}^n(t^n_{n-i}, u(n)S^n_{t_{n-(i+1)}}) \\
&\quad +(1-p^*(n))\hat{U}^n(t^n_{n-i}, d(n)S^n_{t_{n-(i+1)}}) \geq \hat{U}^n\left(t^n_{n-i}, S^n_{t_{n-(i+1)}}\right)\Big\}
\end{aligned}
$$

Remark A.2. It is worth mentioning that the approximation of the BSB equation, i.e. the determination of the lowest upper price bound of a European claim, is given in terms of pricing an American type option instead of a simple European claim with payoff $h(S_T)$. In particular, for $\sigma_{\min} \to 0$ we have

$$
\begin{aligned}
\hat{U}^n(t^n_{n-(i+1)}, S^n_{t_{n-(i+1)}}) &= \text{Max}\Big[p^*(n)\hat{U}^n(t^n_{i-1}, u(n)S^n_{t_{n-(i+1)}}) \\
&\quad +(1-p^*(n))\hat{U}^n(t^n_{n-i}, d(n)S^n_{t_{n-(i+1)}}), \hat{U}^n(t^n_{i-1}, S^n_{t_{n-(i+1)}})\Big]
\end{aligned}
$$

Proposition A.3. *The probabilites are given by*

$$
\begin{aligned}
P\left(\frac{S_{t^n_{k+1}}}{S^n_{t^n_k}} = u(n) \middle| \mathcal{F}^n_{t^n_k}\right) &:= p^u_n(t^n_k, S_{t^n_k}) = q^n(t^n_k, S_{t^n_k})\left(1 - \frac{1}{2}\sigma_{max}\sqrt{\frac{T}{n}}\right) \\
P\left(\frac{S_{t^n_{k+1}}}{S^n_{t^n_k}} = m(n) \middle| \mathcal{F}^n_{t^n_k}\right) &:= p^m_n(t^n_k, S_{t^n_k}) = 1 - 2q^n(t^n_k, S_{t^n_k}) \\
P\left(\frac{S_{t^n_{k+1}}}{S^n_{t^n_k}} = u(n) \middle| \mathcal{F}^n_{t^n_k}\right) &:= p^d_n(t^n_k, S_{t^n_k}) = q^n(t^n_k, S_{t^n_k})\left(1 + \frac{1}{2}\sigma_{max}\sqrt{\frac{T}{n}}\right)
\end{aligned}
$$

*where*

$$
q^n(t^n_k, S_{t^n_k}) = \frac{1}{2}\left(\frac{\sigma^2_{min}}{\sigma^2_{max}} 1_{\left\{(A^n)^C\left(t^n_k, S^n_{t_k}\right)\right\}} + 1_{\left\{A^n\left(t^n_k, S^n_{t_k}\right)\right\}}\right),
$$

*and where the factors $u(n), m(n), d(n)$ are given by equation (8). $\mathcal{F}^n$ denotes the filtration generated by $S^n$. Then, the (forward) asset price process $S^n = \left(S^n_{t^n_k}\right)_{k=0,\ldots,n}$ where $t^n_k = k\frac{T}{n}$ converges for $n \to \infty$ in distribution to $S = (S_t)_{0 \leq t \leq T}$ with*

$$
S_t = S_0 \exp\left\{-\frac{1}{2}\int_0^t \sigma^2(u, S_u)\, du + \int_0^t \sigma(u, S_u)\, dW_u\right\}, \quad S_0 := S^n_0
$$

*where*

$$
\sigma(t,x) = \sigma_{min} 1_{\left\{\hat{U}_{xx}(t,x)<0\right\}} + \sigma_{max} 1_{\left\{\hat{U}_{xx}(t,x)\geq 0\right\}}
$$

**Proof.** Note that we have (per construction) $\lim_{n\to\infty} \hat{U}^n = U$. It is straightforward to show $\lim_{n\to\infty} 1_{\{A^n\}} = 1_{\left\{\hat{U}_{xx}\geq 0\right\}}$. The rest of the proof is implied by lemma A.1 and the further remarks. $\square$

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| Strategy | mean | std. | loss prob. | exp. shortfall |
|---|---|---|---|---|
| Tractable hedge | -4.07 | 2.29 | 0.006 | 0.014 |
| Avellaneda–hedge (adj. initial capital) | -4.07 | 2.00 | 0.005 | 0.011 |
| Avellaneda–hedge | -2.41 | 2.00 | 0.064 | 0.041 |

TABLE 1. Cost distribution for the bulllish vertical spread when $\mu = 0.1$

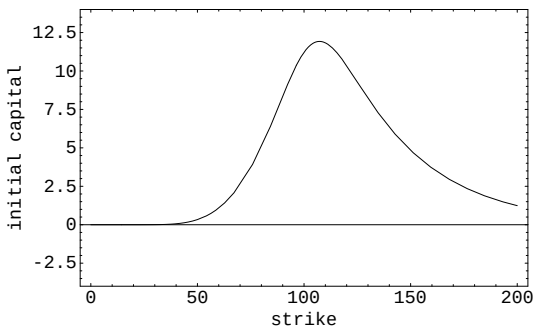


FIGURE 1. Initial capital needed for the (suboptimal) tractable hedge of the zero payoff

The tractable hedge is given by a long and a short position in a call with strike $K$. The current stock price is 100, the time to maturity of the call is one year, the short rate is 5% (continuously compounded) and the volatility interval is given by $[0.1, 0.4]$.

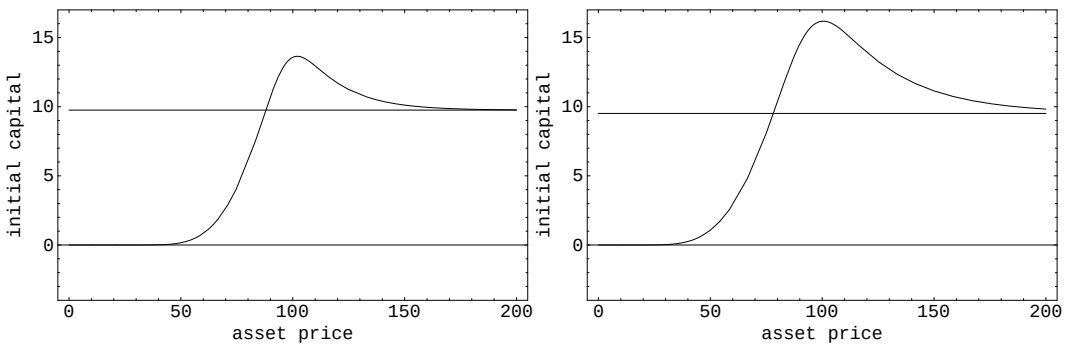


FIGURE 2. Initial capital of decomposition superhedge vs initial capital of trivial superhedge

The figure shows the initial capital needed for a tractable robust hedge of a bullish vertical spread as a function of the current stock price $X_0$. The tractable hedge of the bullish vertical spread is given by a long position in a call with strike $K_1 = 90$ and a short position in a call with strike $K_2 = 100$. The time to maturity of the bullish vertical spread is six month (figure on the left hand), respectively one year (figure on the right hand), the short rate is 5% (continuously compounded), and the volatility interval is given by $[0.1, 0.4]$.

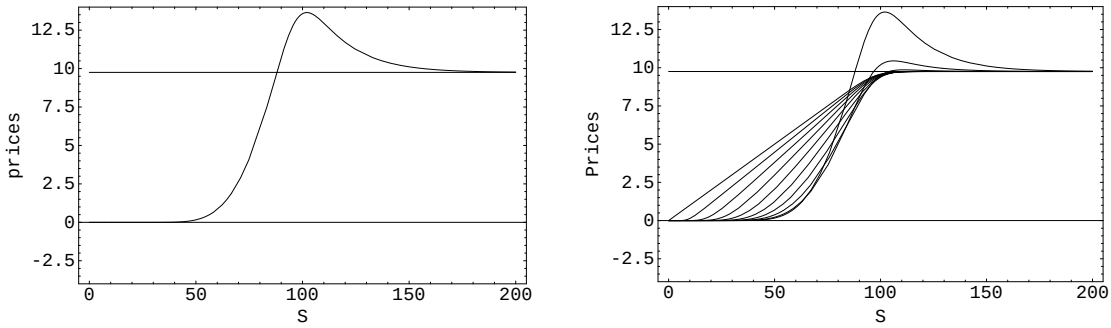


FIGURE 3. Robust hedge of a bullish vertical spread: Comparison of initial capital

The graphs show the initial capital needed for different robust hedging strategies for a bullish vertical spread. The strike prices are $K_1 = 90$ and $K_2 = 100$, the time to maturity is $T = 0.5$, the short rate is $r = 0.05$, and the volatility interval is $[0.1, 0.4]$. The left graph shows the trivial upper price bound $(K_2 - K_1)e^{-rT}$. Furthermore, it shows the initial capital needed for robust hedge implied by the trivial decomposition of the claim into a call with strike $K_1$ long and a call with strike $K_2$ short. The right graph shows the initial capital needed for the robust hedge for decompositions of dominating payoffs. The strike price $K_0$ is $10, 20, \ldots, 90$.

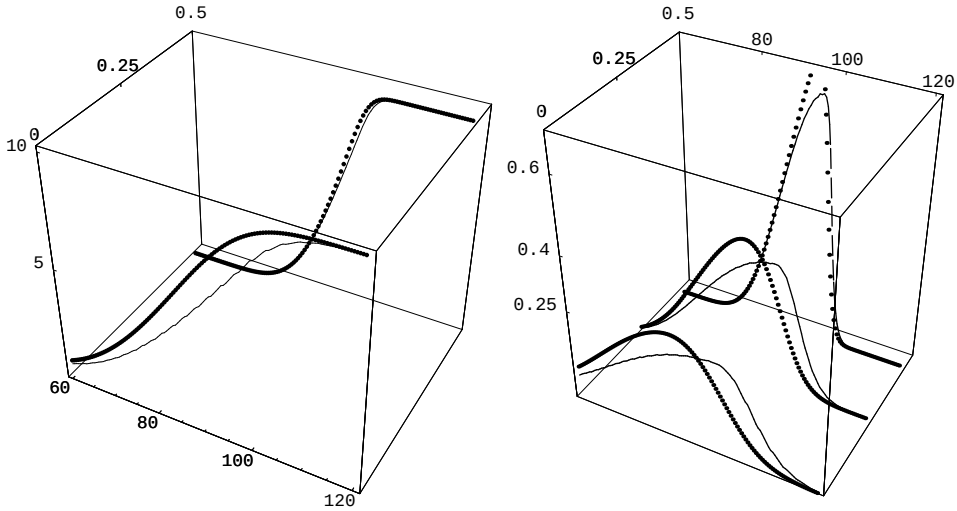


FIGURE 4. Comparison of the Avellenada–hedge and the tractable hedge

The left figure shows the initial capital for the Avellaneda–hedge (thin line) and for the tractable hedge (thick line) as a function of the current asset price and the point in time $t$. The right figure shows the corresponding deltas.

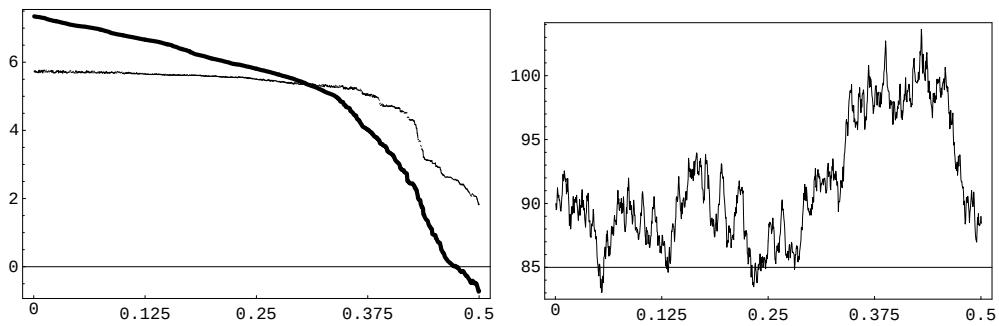


FIGURE 5. Performance of the hedging strategies

The data-generating process is a geometric Brownian motion with $\sigma = 0.3, \mu = 0$. The right figure shows the asset price path, the left figure shows the cost process of the Avellaneda–hedge (thin line) and of the tractable hedge (thick line).

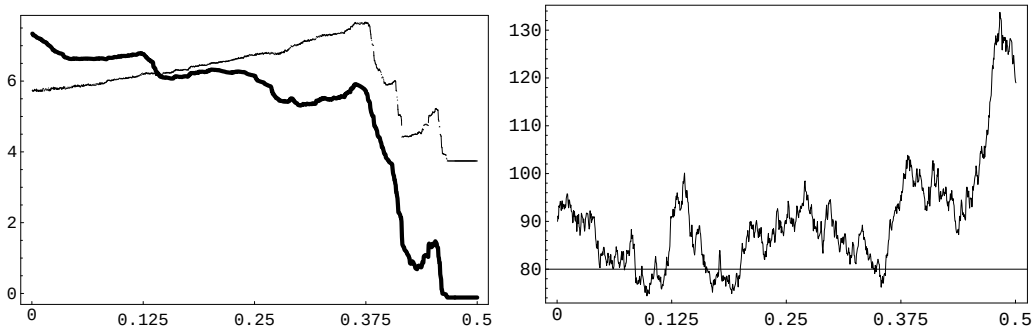


FIGURE 6. Performance of the hedging strategies

The data-generating process is a geometric Brownian motion with $\sigma = 0.6, \mu = 0$. The right figure shows the asset price path, the left figure shows the cost process of the Avellaneda–hedge (thin line) and of the tractable hedge (thick line).

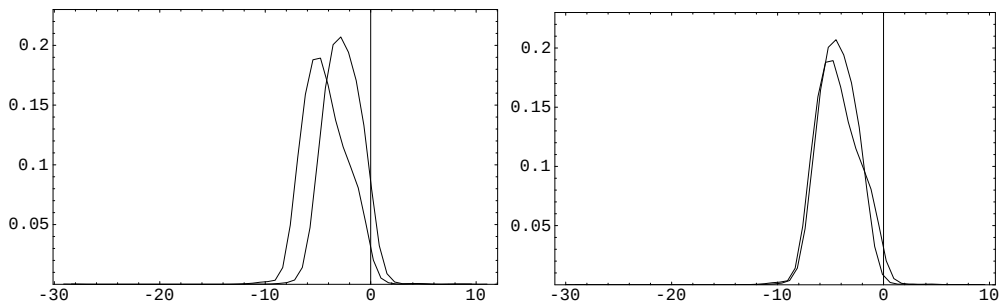


FIGURE 7. Cost Distribution for the bullish vertical spread for $\mu = 0$. The left graph shows the cost distribution for the tractable hedge (left line) and the ALP-hedge (right line). In the right graph, the initial capital of both hedges is set equal to the initial capital of the tractable hedge.

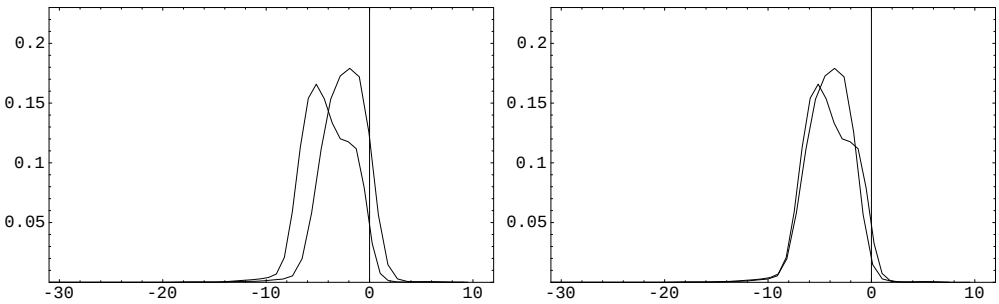


FIGURE 8. Cost Distribution for the bullish vertical spread for $\mu = 0.1$. The left graph shows the cost distribution for the tractable hedge (left line) and the ALP-hedge (right line). In the right graph, the initial capital of both hedges is set equal to the initial capital of the tractable hedge.