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Stability issues in Kaleckian models driven by autonomous demand growth—Harroddian instability and debt dynamics

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Abstract

Sraffian supermultiplier models, as well as Kaleckian distribution and growth models that make use of non-capacity creating autonomous demand growth to cope with Harroddian instability, have paid little attention to the financial side of autonomous demand growth as the driver of the system. Therefore, we link the issue of Harroddian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics. For a simple model with autonomous government expenditure growth, zero interest rates and no consumption out of wealth, we find that adding debt dynamics does not change the results obtained by Skott (2017) based on Lavoie's (2016) model without debt, each published in this journal. Hence, in this simple model, the long-run equilibrium is stable if Harroddian instability is not too strong and the autonomous growth rate does not exceed a maximum given by the long-run equilibrium saving rate. Introducing interest payments on government debt as well as consumption out of wealth into the model, however, changes the stability requirements: First, the autonomous growth rate of government expenditures should not fall short of the exogenous monetary interest rate. Second, this growth rate should not exceed a maximum given by the saving rate in

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the long-run equilibrium net of the propensity to consume out of wealth. Third, Harrodian instability may be almost as strong as in the simple model without violating long-run overall stability, particularly if the propensity to consume out of wealth is low. We claim that irrespective of the relevance or irrelevance of Harrodian instability, it is necessary to introduce financial variables into models driven by non-capacity creating autonomous demand in order to assess the long-run (in-)stability and sustainability of growth.

KEYWORDS

autonomous demand growth, financial (in)stability, Harrodian instability, Kaleckian models, supermultiplier

JEL CLASSIFICATION

E11; E12; E25; E62

1 | INTRODUCTION

Distribution and growth models driven by a non-capacity creating component of aggregate demand have become increasingly popular in heterodox distribution and growth theories, recently generating special issues in *Metroeconomica*, 2019, 70 (2) and in the *Review of Keynesian Economics*, 2020, 8 (3). Initially, a ‘Sraffian supermultiplier’ model driven by autonomous demand was proposed by Serrano (1995a, 1995b), and has further been discussed, developed and applied by Cesaratto (2015), Cesaratto et al. (2003), Dejuan (2005), Fazzari et al. (2013), Fazzari et al. (2020), Freitas and Serrano (2015, 2017), Girardi and Pariboni (2016), Pariboni (2016), among others.

Starting with Allain (2015) and Lavoie (2016), several Kaleckian authors have also turned towards introducing a Sraffian supermultiplier process into their models of distribution and growth in order to defend this approach against the Harrodian and Marxian critique. These critics had argued that the Kaleckian notion of an endogenous rate of capacity utilisation beyond the short run is not sustainable, that Kaleckian models are thus facing the problem of Harrodian instability, and that the Kaleckian results of the paradox of saving and a potential paradox of costs cannot be generally validated beyond the short run.¹ Introducing an autonomous growth rate of a non-capacity creating component of aggregate demand, Kaleckian authors have shown in basic and more elaborate models, which allow for convergence towards a normal or target rate of capacity utilisation in the long run, the following. First, under some weak conditions autonomous demand growth is able to tame Harrodian instability, and, second, the paradox of saving and a potential paradox of costs can be preserved for the long-run growth path (Allain, 2015, 2019; Dutt, 2019, 2020; Lavoie, 2016; Nah & Lavoie, 2017, 2018, 2019a, 2019b; Palley, 2019). In these models, the autonomous growth rate of a non-capacity creating component of aggregate demand, that is, autonomous consumption, residential investment, exports or government expenditures, determines long-run growth, and, under the conditions that Harrodian instability in the investment function is not too strong, provides for a

¹For the Harrodian/Marxian critique see, for example, Duménil and Lévy (1999), Shaikh (2009) and Skott (2010, 2012). For a review of this critique and Kaleckian responses see Hein et al. (2011, 2012).

stable adjustment towards the normal rate of capacity utilisation in the long run. A change in the propensity to save or in the profit share will have no effect on the long-run growth rate, but will affect the traverse and thus the long-run growth path. The paradox of saving and the possibility of a paradox of costs from the short run thus disappear with respect to the long-run growth rate, but they remain valid with respect to the long-run growth path.

Of course, the Sraffian supermultiplier models and the integration of autonomous demand growth into Kaleckian models have been critically discussed (Nikoforos, 2018; Skott, 2019). This critique has addressed the implied full endogeneity of investment with respect to output growth, that is, fully induced investment. Furthermore, the assumption that any expenditure growth is fully autonomous with respect to the variation of income and output in the long run, for which these models have been designed, and the exclusion of financial instability issues have been particularly criticised.

Our contribution is related to the question of long-run autonomy of components of demand from income/output and the issue of financial instability. Not only from a post-Keynesian perspective should it be evident that this autonomy implies that demand can be financed independently of income generated in the long run. This is only possible if those sectors generating autonomous demand growth have wealth they can draw on and/or access to credit—the private household sector in the case of autonomous consumption demand and residential investment, the external sector (i.e., the importing countries) in case of exports, and the government in the case of government expenditures as the growth driver. Empirically, financial dynamics are most important when it comes to the sustainability of autonomous demand-driven growth, as is conceded by some of the proponents of this approach (Fiebiger, 2018; Fiebiger & Lavoie, 2019).

Interestingly, however, the dynamics of financial assets and liabilities—and of debt in particular—associated with autonomous demand growth have only found little attention so far. A systematic examination of the potential limits generated from the monetary and financial side to the sustainability of autonomous demand growth would thus seem desirable. The model by Allain (2015) with autonomous government expenditure growth assumes that the tax rate is continuously adjusted to ensure a balanced budget at all times, and thus avoids the discussion of government debt dynamics. Pariboni (2016) has introduced household debt into a supermultiplier growth model-driven by autonomous consumption, but has not carefully studied the dynamic interaction of Harroddian instability and debt dynamics. Brochier and Macedo e Silva (2019) and Vieira Mandarino et al. (2020) have included financial wealth/liabilities into a supermultiplier stock-flow consistent model driven by autonomous consumption growth, but have only numerically simulated the respective dynamics. Hein (2018) has discussed autonomous government expenditure growth financed by credit in a Kaleckian distribution and growth model, and has studied the dynamics of government deficits and government debt. However, he has maintained the Kaleckian assumption of an endogenous rate of capacity utilisation and has thus not addressed the Harroddian instability issue. Dutt (2020), in contrast, has included the attainment of a long-run normal rate of utilisation into a model driven by autonomous government expenditure growth financed by credit and has examined the related debt dynamics. But he has ‘switched off’ Harroddian instability, assuming firms’ expectation about long-run growth are given by the growth rate of government expenditures.² Freitas and Christianes (2020) have presented a model driven by autonomous consumption and autonomous government expenditures and they have included the analysis of government debt dynamics. However, different from our model presented in Section 3, they have excluded important feedback effects of interest income and wealth on consumption, which allows them to treat Harroddian instability and debt dynamics separately, which is different from our ultimate purpose in this paper.

Our current contribution attempts to link the issue of Harroddian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics. We will do

²This assumption can already be found in Serrano (1995, p. 77), as Marc Lavoie has pointed out to us.

this in two steps, building on Lavoie's (2016) basic model, as well as Skott's (2017) modification in the first step, and on Hein's (2018) model in the second step. In the basic model, however, we will replace the autonomous consumption growth proposed by Lavoie (2016) with autonomous government expenditure growth as the driver of growth for two reasons. First, we feel that the 2007–2009 financial and economic crises have shown the limits of credit-financed autonomous consumption, residential investment, and export growth as autonomous long-run drivers of growth, such that government expenditure growth remains the 'realistic' alternative, under certain circumstances. Second, using government expenditures as autonomous growth driver allows for helpful simplifications, such as zero interest rates in the basic version of the model. In Section 2, we will thus provide an extremely simple model in which we study Harrodian instability and debt dynamics, ignoring both interest payments and consumption out of wealth. In order to highlight the role of debt dynamics, we will even 'switch off' Harrodian instability in the first version, but not the notion that firms operate at a normal or target rate of utilisation in the long run, following the procedure suggested by Dutt (2019, 2020). Section 3 will then include a positive interest rate on government debt as well as consumption out of financial wealth, as in Hein (2018). We will start with a model version without Harrodian instability, but with a normal rate of utilisation obtained in the long run, and then we will add a Harrodian instability equation in the second version. In the final Section 4, we will summarise and conclude. Before moving to Section 2, we should stress that the purpose of our paper is conceptual and didactic: We would like to add to the understanding of the (potentially destabilising) role of financial dynamics in models driven by autonomous non-capacity creating expenditures, with or without Harrodian instability. The extremely simple models should thus not be taken to the data without further refinements.³

2 | HARRODIAN INSTABILITY AND DEBT DYNAMICS: A SIMPLE VERSION BASED ON LAVOIE (2016) AND SKOTT (2017)

We assume a closed economy, in which a single good for investment and consumption purposes is produced by a fixed coefficient technology, using a non-depreciating capital stock (K) and direct labour (L). For the latter, there is no supply constraint. The rate of the utilisation of productive capacities is defined as the ratio of output to the capital stock: $u = Y/K$. Income is distributed between capitalists and workers, and the profit share ($h = \pi/Y$) is determined by mark-up pricing of firms in an oligopolistic goods market, with the mark-up being affected by the degree of price competition in the goods market and the bargaining power of workers in the labour market. With given institutional conditions, prices are constant, so that nominal and real variables coincide at a price level $p = 1$. Workers do not save, and only capitalists save a fraction of their profits determined by the propensity to save out of profit income s_π . For the sake of simplicity, we assume that all profits are distributed to the capitalists' households as the owners of the firms. We normalise saving (S) by the capital stock and get the saving rate (σ):

$$\sigma = \frac{S}{K} = s_\pi \frac{\pi}{Y} \frac{Y}{K} = s_\pi h u, s_\pi > 0 \quad (1)$$

Firms decide to invest (I) according to the expected trend rate of growth (α). Whenever the actual rate of capacity utilisation (u) falls short of (exceeds) the target or normal rate of utilisation (u_n), they slow down (accelerate) the rate of capital accumulation (g):

³For different views on this and other issues, see Lavoie (2017) and Skott (2017, 2019).

$$g = l/k = \alpha + \beta (u - u_n), \beta > 0 \quad (2)$$

Government consumption expenditure (G) drives our model economy and grows with the rate γ . Governments finance their expenditures by issuing bonds, which are held by the capitalists. In this first version of the model, we ignore interest rates, assuming that bonds are issued at a zero rate or that governments emit money. In addition, wealth effects on capitalists' consumption are ignored for now. Government expenditures are equal to the government deficit and are also normalised by the capital stock such that for the government expenditures-capital stock ratio (b) we get⁴:

$$b = \frac{G}{K} = \frac{G_0 e^{\gamma t}}{K}, \gamma > 0 \quad (3)$$

The short-run goods market equilibrium is given by:

$$\sigma = g + b \quad (4)$$

and the stability condition by:

$$s_\pi h - \beta > 0 \quad (5)$$

Firms adjust output to demand in the short run by means of varying the rate of capacity utilisation. From Equations (1–4) we thus obtain the short-run equilibrium rate of capacity utilisation (u^*) for a given government expenditures-capital ratio and a given government debt-capital ratio ($\lambda = L/K$) inherited from the past:

$$u^* = \frac{\alpha - \beta u_n + b}{s_\pi h - \beta} \quad (6)$$

In the long run, government expenditures grow with the rate γ , and the government expenditures-capital ratio changes according to:

$$\dot{b} = (\gamma - g) b = [\gamma - \alpha - \beta (u^* - u_n)] b \quad (7)$$

with variables with a dot denoting time rates of change. Likewise, in the long run, we have an endogenous rate of change in the government debt-capital ratio, which can be expressed as in equation (8) since $\hat{\lambda} = \hat{L} - \hat{K}$ and $\dot{L} = G$.

$$\dot{\lambda} = b - g\lambda = b - [\alpha - \beta (u^* - u_n)] \lambda \quad (8)$$

⁴This paper largely uses the notation from Hein (2018). For the purposes of comparison, note that in Lavoie (2016) the ratio of autonomous demand to the capital stock (b) is instead denoted z , its growth rate (herein γ) is g_z , the profit share (h) is π , the expected trend rate of growth (α) is γ , and, lastly, the responsiveness of investment to deviations of capacity utilisation from its normal rate (β) is designated by γ_u .

Before we consider Harroddian instability and its effects, we begin by assuming with Dutt (2019, 2020) that firms may have ‘rational’—or better ‘reasonable’—expectations about the trend rate of growth given by government expenditure growth γ .⁵

$$\alpha = \gamma \quad (9)$$

Hence, for the time being and for the purposes of comparison, we begin with our simplest model where Harroddian instability is ‘switched off’.

To analyse the stability of long-run equilibrium values, we must first determine our long-run equilibrium (**), setting each of our dynamic equations equal to zero and making use of the short-run equilibrium rate of capacity utilisation from Equation (6):

$$u^{**} = u_n \quad (10)$$

$$g^{**} = \alpha = \gamma \quad (11)$$

$$b^{**} = s_\pi h u_n - \gamma \quad (12)$$

$$\lambda^{**} = \frac{b^{**}}{g^{**}} = \frac{s_\pi h u_n - \gamma}{\gamma} \quad (13)$$

Since α does not change endogenously, because we have ‘switched-off’ Harroddian instability, there will only be two dynamic equations in the long run, equation (7) for the growth rate of the government expenditures-capital ratio and equation (8) for the government debt-capital ratio. The corresponding Jacobian matrix is given in equation (14a) and is evaluated at the long-run equilibrium values in (14b).⁶

$$J = \begin{bmatrix} \frac{\partial \dot{b}}{\partial \lambda} & \frac{\partial \dot{b}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial \lambda} & \frac{\partial \dot{\lambda}}{\partial \lambda} \\ \frac{\partial \dot{b}}{\partial b} & \frac{\partial \dot{b}}{\partial \lambda} \end{bmatrix} \quad (14a)$$

$$J^{**} = \begin{bmatrix} \frac{-\beta b^{**}}{s_\pi h - \beta} & 0 \\ s_\pi h (\gamma - \beta u_n) & \\ \frac{\gamma (s_\pi h - \beta)}{\gamma (s_\pi h - \beta)} & -\gamma \end{bmatrix} \quad (14b)$$

⁵To quote Dutt (2019, FN 5): ‘Assuming “rational” expectations does not require that firms know the entire structure of the model as in new classical macroeconomic models, but only that the expected long-run growth is equal to the known rate of growth of the exogenous component ...’.

⁶In Lavoie (2016) as well as a working paper version of this paper (Hein & Woodgate, 2020), the Jacobian is defined in terms of the derivatives of growth rates rather than for the time rates of change. Skott (2017, p. 189) has corrected Lavoie's results, and we are following this more adequate version here.

For local stability in this 2×2 system, the trace of the Jacobian must be negative and the determinant must be positive. The trace and the determinant are given in equations (15) and (16), respectively.

$$Tr(J^{**}) = \frac{-\beta b^{**} - \gamma(s_{\pi}h - \beta)}{s_{\pi}h - \beta} \quad (15)$$

$$Det(J^{**}) = \frac{\beta b^{**} \gamma}{s_{\pi}h - \beta} \quad (16)$$

It follows that $Det(J^{**}) > 0$ and $Tr(J^{**}) < 0$ for any feasible—that is, positive—value of the long-run equilibrium of the government expenditure-capital stock ratio, b^{**} . With reference to equation (13), this implies that the system exhibits local stability around its steady-state values so long as $s_{\pi}hu_n - \gamma > 0$. Hence, even without Harroddian instability, the inclusion of deficit and debt dynamics imposes an upper limit on the growth rate of autonomous government expenditures for stability purposes. This limit is given by the saving rate at normal capacity utilisation: $\gamma < s_{\pi}hu_n$.

Let us now consider the same model, but with Harroddian instability ‘switched on’. Firms’ assessment of the trend rate of growth will now also change, if the goods market equilibrium rate of capacity utilisation deviates (persistently) from the target or normal rate of utilisation. We thus get the following Harroddian equation, which replaces Equation (9), with μ denoting the Harroddian instability parameter:

$$\dot{\alpha} = \mu\beta(u^* - u_n)\alpha, \mu > 0 \quad (17)$$

There has been some disagreement in the literature over whether the equation of the motion of the trend rate of growth should be modelled in terms of its growth rate as in equation (17) or its time rate of change. The former has been employed by Lavoie (2016), Dutt (2019) and others, while Skott (2017) has argued for the latter. In this paper, we opt for growth rates of change, but also report our results given the alternative specification in footnotes.

Our simple model thus now has three dynamic equations in (7), (8) and (17), to be examined for long-run equilibrium stability. Our long-run equilibrium values are the same as before, given by equations (10)–(13). To analyse the stability of the long-run equilibrium, we find the Jacobian matrix (18a) of the dynamic system in equations (7), (8) and (17), which is then evaluated at the long-run equilibrium (18b):

$$J = \begin{bmatrix} \frac{\partial \dot{b}}{\partial b} & \frac{\partial \dot{b}}{\partial \alpha} & \frac{\partial \dot{b}}{\partial \lambda} \\ \frac{\partial \dot{\alpha}}{\partial b} & \frac{\partial \dot{\alpha}}{\partial \alpha} & \frac{\partial \dot{\alpha}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial b} & \frac{\partial \dot{\lambda}}{\partial \alpha} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{bmatrix} \quad (18a)$$

$$J^{**} = \begin{bmatrix} \frac{-\beta b^{**}}{s_{\pi}h - \beta} & \frac{-s_{\pi}hb^{**}}{s_{\pi}h - \beta} & 0 \\ \frac{\mu\beta\gamma}{s_{\pi}h - \beta} & \frac{\mu\beta\gamma}{s_{\pi}h - \beta} & 0 \\ \frac{s_{\pi}h(\gamma - \beta u_n)}{\gamma(s_{\pi}h - \beta)} & \frac{-s_{\pi}hb^{**}}{\gamma(s_{\pi}h - \beta)} & -\gamma \end{bmatrix} \quad (18b)$$

For the local stability of this 3x3 system, the following three Routh-Hurwitz (R-H) conditions must hold

1. $Det(J^{**}) < 0$,
2. $Tr(J^{**}) < 0$, and
3. $-Tr(J^{**}) [Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) > 0$,

where $Det(J_1^{**})$, $Det(J_2^{**})$, $Det(J_3^{**})$ are the determinants of the three second-order principal submatrices of the Jacobian evaluated at its long-run equilibrium values. A fourth R-H condition, namely that $Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**}) > 0$, is frequently mentioned in the related literature. However, it should be stressed that this criterion is in fact redundant, as it will always hold if the three criteria listed above are fulfilled.⁷ Hence, checking the three listed R-H conditions suffices to demonstrate that the eigenvalues of the Jacobian have negative real parts and so that the system is stable near its steady state.

In order to check these three conditions for our dynamic system, we obtain from the Jacobian matrix in equation (18b) the following results:

$$Det(J^{**}) = \frac{-\mu\beta\gamma^2 b^{**}}{s_\pi h - \beta} < 0 \quad (19)$$

$$Tr(J^{**}) = \frac{\gamma\beta\mu - \beta b^{**} - \gamma(s_\pi h - \beta)}{s_\pi h - \beta} < 0 \quad (20)$$

$$\begin{aligned} & -Tr(J^{**}) [Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) \\ &= \frac{-\beta\gamma [\mu^2\gamma\beta\rho + \mu(\gamma q - \beta b^{**}\rho) - qb^{**}]}{(s_\pi h - \beta)^2} > 0 \end{aligned} \quad (21)$$

where $p \equiv b^{**} - \gamma$ and $q \equiv \beta b^{**} + \gamma(s_\pi h - \beta)$.

Clearly, condition (19) holds for any feasible (i.e. positive) values of b^{**} and μ , given the usual assumption of Keynesian stability. Conditions (20) and (21), however, impose restrictions on the size of μ , the derivation of which we leave to the Appendix. There it is shown that the values of the Harrodian instability parameter which satisfy all three R-H conditions are given by

$$\mu < \frac{s_\pi h u_n - \gamma}{\gamma} \quad (22a)$$

where the right-hand side is, by equation (13), equivalent to the long-run equilibrium value of the government debt-capital ratio, λ^{**} .⁸ Despite having included simplified debt dynamics, the result in condition (22a) is essentially the same as that found in Skott's (2017) modification of Lavoie (2016), with the autonomous growth rate of government expenditures in our model replacing that of autonomous consumption expenditure in the Lavoie model.

⁷For a discussion of whether the number of non-redundant criteria for an $n \times n$ system can be reduced to n , see Fuller (1968).

⁸If we take $\dot{\alpha} = \mu\beta(u^* - u_n)$ instead of equation (17), it can be shown that the restriction on μ that is necessary for local stability becomes $\mu < (s_\pi h u_n - \gamma)$, where the right-hand side is equivalent to the long-run equilibrium value of the government expenditures-capital ratio, b^{**} .

Summing up the conditions for the existence and stability of economically meaningful equilibria, first, from R-H stability condition one, we need the autonomous growth rate of government expenditures to be below the saving rate at normal capacity utilisation, $\gamma < s_{\pi}hu_n$, such that the long-run equilibrium government expenditures-capital stock ratio is positive, that is, $b^{**} > 0$. This is the same as in the model without Harroddian instability. Second, from stability conditions two and three, we obtain that the system will be stable, if the Harroddian instability parameter is below the long-run equilibrium value of the government debt-capital ratio. Rearranging condition (22a) provides an overall condition for stability with respect to the values of the parameters:

$$\gamma < \frac{s_{\pi}hu_n}{1 + \mu} \quad (22b)$$

The corridor for long-run stability is positively affected by the propensity to save out of profits, the profit share and the normal or target rate of utilisation. For a given stability corridor, the values of the Harroddian instability parameter and of the autonomous growth rate of government expenditures are inversely related. The higher (lower) the instability parameter, the lower (higher) the maximum growth rate of expenditures governments can choose without endangering overall stability.

3 | GOVERNMENT DEFICIT AND DEBT DYNAMICS IN A MODEL WITH INTEREST INCOME AND CONSUMPTION OUT OF FINANCIAL AND REAL WEALTH

Including interest payments on government debt and consumption out of real and financial wealth into the model, we follow Hein (2018). Distinguishing between the firm sector and the rentiers' household sector, we assume for simplicity that long-term finance of the real capital stock only consists of equity issued by the firms and held by the rentiers ($K = E$). Firms distribute all profits as dividends to the shareholders, that is, there are no retained earnings. Therefore, rentiers hold the equity issued by the firms, the value of which is equal to the capital stock, and the debt issued by the government. They receive all the profits generated in the production sector (hY) and the interest paid out by the government (iL), the latter determined by the exogenous rate of interest and the stock of government debt. They consume part of their current profit and interest income and save the rest according to their propensity to save (s_R). Furthermore, we assume that rentiers consume part of their wealth according to their propensity to consume out of wealth (c_{WR}). Consumption out of wealth thus reduces saving out of rentiers' current income and our saving function turns to:

$$\sigma = \frac{S}{K} = \frac{s_R(hY + iL)}{K} - \frac{c_{WR}(L + K)}{K} = s_R(hu + i\lambda) - c_{WR}(\lambda + 1), \quad (23)$$

$$s_R > 0, c_{WR} \geq 0$$

From Section 2, we keep the investment function (2) and the function (3) for government expenditures, which rise again with the autonomous growth rate γ .

$$g = I/K = \alpha + \beta(u - u_n), \beta > 0 \quad (2)$$

$$b = \frac{G}{K} = \frac{G_0 e^{\gamma t}}{K}, \gamma > 0 \quad (3)$$

Government expenditures (G) are now equivalent to the primary government deficit. For the total deficit, we have to add government interest payments. The short-run equilibrium condition turns to:

$$\sigma = g + b + i\lambda \quad (24)$$

and the stability condition for the goods market equilibrium becomes:

$$s_R h - \beta > 0 \quad (25)$$

Equations (23), (2), (3) and (24) yield the short-run equilibrium rate of capacity utilisation:

$$u^* = \frac{\alpha - \beta u_n + b + c_{WR} + \lambda [i(1 - s_R) + c_{WR}]}{s_R h - \beta} \quad (26)$$

In order to analyse the long-run dynamic properties related to the government primary deficit- and government debt-capital ratios in our model, and to compare them to the first version of the model from the previous section, we begin again with Harroddian instability 'switched off', as in Equation (9):

$$\alpha = \gamma \quad (9)$$

We thus only obtain two dynamic equations, one for the government expenditures/primary government deficit-capital ratio and one for the government debt-capital ratio:

$$\dot{b} = (\gamma - g)b = [\gamma - \alpha - \beta(u^* - u_n)]b \quad (27)$$

$$\dot{\lambda} = b + (i - g)\lambda = b + [i - \alpha - \beta(u^* - u_n)]\lambda \quad (28)$$

Setting equations (27) and (28) equal to zero and using equations (9) and (26), we have for the long-run equilibrium:

$$u^{**} = u_n \quad (29)$$

$$g^{**} = \gamma \quad (30)$$

$$b^{**} = \frac{(\gamma - i)(s_R h u_n - \gamma - c_{WR})}{\gamma + c_{WR} - s_R i} \quad (31)$$

$$\lambda^{**} = \frac{s_R h u_n - \gamma - c_{WR}}{\gamma + c_{WR} - s_R i} \quad (32)$$

In order to analyse the stability properties of the long-run equilibrium, which is now, different from the simple model in Section 2, affected by the rate of interest and the propensity to consume out of wealth, we define:

- $A = s_R h - \beta$,
- $B = i(1 - s_R) + c_{WR}$,
- $C = \gamma + c_{WR} - s_R i$,

- $D = s_R h u_n - \gamma - c_{WR}$,
- $F = \gamma - i$.

Based on our assumptions, we know:

- $A > 0$, because of the assumption of goods market equilibrium stability,
- $B > 0$, because $s_R < 1$,
- $F < C$, because $s_R < 1$ and $c_{WR} > 0$,
- $F + B = C$.

Furthermore, since $\lambda^{**} = D/C$ and $b^{**} = DF/C$, we know that for economically meaningful (i.e., positive) values of these long-run equilibria, it must be the case that⁹:

- $C > 0$,
- $D > 0$
- $F > 0$

The Jacobian matrix of the two-dimensional dynamics from equations (27) and (28), taking into account the goods market equilibrium rate of capacity utilisation from equation (26), to be evaluated at the long-run equilibrium is given by:

$$J = \begin{bmatrix} \frac{\partial \dot{b}}{\partial \lambda} & \frac{\partial \dot{b}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial b} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{bmatrix} \quad (33a)$$

$$J^{**} = \begin{bmatrix} \frac{-\beta DF}{AC - \beta D} & \frac{-\beta BDF}{AC} \\ \frac{AC}{AC} & \frac{-ACF - \beta BD}{AC} \end{bmatrix} \quad (33b)$$

From this, we obtain for the trace and the determinant:

$$Tr(J^{**}) = \frac{-(\beta D + AF)}{A} \quad (34)$$

$$Det(J^{**}) = \frac{\beta DF}{A} \quad (35)$$

Thus for economically meaningful values of λ^{**} and b^{**} , the stability conditions $Det(J^{**}) > 0$ and $Tr(J^{**}) < 0$ are always satisfied. Hence, the existence and stability of a long-run equilibrium are assured when $C > 0$, $D > 0$, and $F > 0$, which in conjunction reduces to the condition where¹⁰:

⁹The alternative constellation where $C < 0$, $D < 0$ and $F > 0$ is not possible since we know that $F < C$.

¹⁰Dutt (2020) has obtained an equivalent result for a model without consumption out wealth but with tax rates on wages and profits.

$$i < \gamma < s_R h u_n - c_{WR} \quad (36)$$

Compared to our simpler models in Section 2, the conditions for stability place a lower as well as an upper limit on the growth rate of autonomous government expenditures. For long-run stability, we need, on the one hand, that the autonomous growth rate of government expenditures, which determines the long-run growth rate of the economy, does not fall short of the interest rate. This is the same as in the model by Hein (2018), which does not include a normal rate of capacity utilisation. On the other, this growth rate should not exceed a maximum given by the saving rate out of profits at the normal rate of utilisation net of the propensity to consume out of wealth.

Finally, we can introduce Harroddian instability back into the model and replace equation (9) with the Harroddian equation (17):

$$\dot{\alpha} = \mu\beta(u^* - u_n)\alpha, \mu > 0 \quad (17)$$

The Jacobian of the three-dimensional dynamics from equations (27), (17) and (28), taking into account the goods market equilibrium from equation (24) is given by:

$$J = \begin{bmatrix} \frac{\partial b}{\partial \alpha} & \frac{\partial b}{\partial \alpha} & \frac{\partial b}{\partial \alpha} \\ \frac{\partial b}{\partial \alpha} & \frac{\partial \alpha}{\partial \alpha} & \frac{\partial \lambda}{\partial \alpha} \\ \frac{\partial b}{\partial \lambda} & \frac{\partial \alpha}{\partial \lambda} & \frac{\partial \lambda}{\partial \lambda} \\ \frac{\partial b}{\partial \lambda} & \frac{\partial \alpha}{\partial \lambda} & \frac{\partial \lambda}{\partial \lambda} \end{bmatrix} \quad (37a)$$

$$J^{**} = \begin{bmatrix} \frac{-\beta DF}{\mu\beta\gamma} & \frac{-s_R h DF}{\mu\beta\gamma} & \frac{-\beta BDF}{\mu\beta\gamma B} \\ \frac{A}{AC - \beta D} & \frac{A}{-s_R h D} & \frac{-ACF - \beta BD}{AC} \end{bmatrix} \quad (37b)$$

Again, we have to check the three non-redundant Routh-Hurwitz conditions for the local stability of this three-dimensional dynamic system, for which we need:

$$Det(J^{**}) = \frac{-\mu\beta\gamma DF}{A} < 0 \quad (38)$$

$$Tr(J^{**}) = \frac{\mu\beta\gamma - AF - \beta D}{A} < 0 \quad (39)$$

$$\begin{aligned} -Tr(J^{**}) [Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) \\ = -\beta[\mu^2\beta\gamma^2 P + \mu\gamma(FQ - D\beta P) - DFQ]/A^2 > 0 \end{aligned} \quad (40)$$

where $P \equiv D - F$ and $Q \equiv AF + \beta D$.

As before, we will only discuss the long-run stability of economically meaningful, that is, positive long-run equilibrium values for the government primary deficit- and the government debt-capital ratios in equations (31) and (32). Together with the stability condition for the goods market equilibrium

this again implies that A, B, C, D, and F must all be positive. For this to be the case, condition (36) must likewise hold now in our model with Harroldian instability:

$$i < \gamma < s_R h u_n - c_{WR} \quad (36)$$

For the R-H stability conditions, we obtain the following results. The first R-H condition (38) of a negative determinant is fulfilled for any value of the Harroldian instability parameter μ . Restrictions on μ are only imposed by conditions (39) and (40)—again, the exact derivation of which can be found in the Appendix. The result is that for local stability around the model's long-run steady state, it must be the case that¹¹:

$$\mu < \frac{s_R h u_n - \gamma - c_{WR}}{\gamma} \quad (41)$$

Thus, the main novel results from this model with consumption out of wealth, interest payments and Harroldian instability are as follows. The growth rate of autonomous government expenditures, again, should be above the interest rate and below the saving rate at normal capacity utilisation net of the propensity to consume out of wealth. However, now we see from condition (41) that the higher this growth rate is, the lower the degree of Harroldian instability must be in order to ensure the stability of the system. The growth rate of autonomous government expenditures should, therefore, be neither too low nor too high, as the reformulated condition (41) together with the condition (36) show:

$$i < \gamma < \frac{s_R h u_n - c_{WR}}{1 + \mu} \quad (42)$$

The corridor for long-run overall stability becomes broader as the rate of interest and the propensity to consume out of wealth fall, and the propensity to save out of rentiers income, the profit share and the normal or target rate of utilisation rise. And again, for a given stability corridor, the values of the Harroldian instability parameter and of the autonomous growth rate of government expenditures are inversely related. The higher (lower) the instability parameter, the lower (higher) the maximum growth rate of expenditures governments can choose without endangering overall stability.

4 | CONCLUSIONS

We have started from the observation that the Sraffian supermultiplier models, as well as Kaleckian distribution and growth models that incorporate autonomous demand growth in order to cope with Harroldian instability, have paid little attention to the financial side of such growth and to the issue of its stability. Therefore, our attempt has been to link the issue of Harroldian instability in Kaleckian models driven by non-capacity creating autonomous demand growth with the associated financial dynamics.

For this purpose, we have used Lavoie's (2016) model as a starting point and have replaced autonomous consumption growth with government expenditures growth. In the first version with zero interest rates and no consumption out of wealth, we have found that the inclusion of deficit and debt

¹¹If we take $\dot{\alpha} = \mu\beta(u^* - u_n)$ instead of equation (17), the result in condition (41) is instead that $\mu < s_R h u_n - \gamma - c_{WR}$.

dynamics imposes an upper limit on the growth rate of autonomous government expenditures for stability purposes. This limit is given by the saving rate at normal capacity utilisation ($\gamma < s_{\pi} h u_n$). When Harroddian instability is introduced to this simple model, the same limit on the growth rate of autonomous government expenditures is found, and the upper bound on the Harroddian instability parameter is that derived in Skott's (2017) modification of Lavoie (2016).

In the second step, we then introduced interest payments on government debt as well as consumption out of real and financial wealth into the model. Switching off Harroddian instability in the first version, by assuming that firms 'rationally expect' the long-run growth rate of the system given by autonomous government expenditure, has shown that the model may generate stable long-run equilibria, if two conditions are met. First, the autonomous growth rate of government expenditures should not fall short of the exogenous monetary interest rate. Second, this growth rate should not exceed a maximum given by the saving rate out of profits at the normal rate of utilisation in long-run equilibrium minus the propensity to consume out of wealth. This corridor of stability became tighter when Harroddian instability was introduced. The first condition, which defines the lower bound for autonomous government expenditure growth, is also needed when we re-introduced Harroddian instability, but the second condition defining the upper bound has become smaller, that is, the higher the value of the Harroddian stability parameter, the lower the rate of autonomous growth governments can choose without triggering overall instability.

Summing up, our exercises in an extremely simple model framework have shown that the financial dynamics, necessarily associated with autonomous demand growth, have an important impact on long-run stability and sustainability of growth driven by autonomous demand. Empirically, and given the experiences from the 2007–2009 financial and economic crises, this should not be surprising. We have provided a starting point for the integration of these concerns into simple analytical distribution and growth models driven by autonomous demand. Further analysis may build on this.

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APPENDIX

RESTRICTIONS IMPLIED BY THE SECOND AND THIRD ROUTH-HURWITZ CONDITIONS IN BOTH 3×3 MODELS

In this appendix, we derive the stability-preserving limits imposed upon the Harroddian instability parameter (μ) by the second and third Routh-Hurwitz (R-H) conditions in both the simple 3×3 model of Section 2 and the 3×3 model with interest payments and consumption out of wealth of Section 3.

A1 | 3×3 model from Section 2

The limit imposed on μ by R-H2 is given by condition (20):

$$Tr(J^{**}) = \frac{\gamma\beta\mu - \beta b^{**} - \gamma(s_{\pi}h - \beta)}{s_{\pi}h - \beta} < 0 \quad (20)$$

With a few simple algebraic manipulations, we see it is:

$$\mu < \frac{b^{**}}{\gamma} + \frac{s_{\pi}h - \beta}{\beta} \quad (A1)$$

The limits imposed on μ by R-H3 is given by condition (21):

$$\begin{aligned} & -Tr(J^{**}) [Det(J_1^{**}) + Det(J_2^{**}) + Det(J_3^{**})] + Det(J^{**}) \\ & = \frac{-\beta\gamma [\mu^2\gamma\beta p + \mu(\gamma q - \beta b^{**} p) - qb^{**}]}{(s_{\pi}h - \beta)^2} > 0 \end{aligned} \quad (21)$$

where $p \equiv b^{**} - \gamma$ and $q \equiv \beta b^{**} + \gamma(s_{\pi}h - \beta)$. Clearly, the values of μ that satisfy condition (21) are those that satisfy the following quadratic:

$$z(\mu) = \mu^2\gamma\beta p + \mu(\gamma q - \beta b^{**} p) - qb^{**} < 0 \quad (A2)$$

The roots of this quadratic are $\mu_1^{\text{root}} = b^{**} / \gamma$ and $\mu_2^{\text{root}} = -q / \beta p$.

The curvature of $z(\mu)$ as well as the roots of $z(\mu)$ depend on the sign of p . In the special case where $p = 0$, $z(\mu)$ is a linear function that is negative and satisfies condition (21) when $\mu < b^{**} / \gamma$. When $p > 0$, $z(\mu)$ is convex with a negative intercept at $-qb^{**}$, one negative root (μ_2^{root}), and one positive (μ_1^{root}). Hence, in this case too, condition (21) is satisfied when $\mu < b^{**} / \gamma$. Lastly, when $p < 0$, $z(\mu)$ is concave with the same negative intercept, but now with two positive roots, where it can be shown that $\mu_2^{\text{root}} > \mu_1^{\text{root}}$. Thus, condition (21) is satisfied for $\mu < b^{**} / \gamma$ or $\mu > \mu_2^{\text{root}} = q / \beta(\gamma - b^{**})$. However, values of μ greater than μ_2^{root} when $p < 0$ are greater than the upper limit imposed by R-H2 condition in (A1), and are thus not compatible with systemic stability. Therefore, regardless of the sign of p , the only values of μ that satisfy both conditions (20) and (21) are:

$$\mu < \frac{s_{\pi} h u_n - \gamma}{\gamma} \quad (22a)$$

A2 | 3 × 3 model from Section 3

The analysis of R-H conditions 2 and 3 for the 3 × 3 model of Section 3 closely mirrors that of the preceding Section 2 model, so we summarise the main steps to avoid repetition. The limit imposed on μ by R-H2 within condition (39) is:

$$\mu < \frac{D}{\gamma} + \frac{AF}{\beta\gamma} \quad (A3)$$

Condition (40) is likewise quadratic in μ . Hence, deducing the implicit limits upon μ is tantamount to solving the following quadratic:

$$Z(\mu) = \mu^2 \beta \gamma^2 P + \mu \gamma (FQ - D\beta P) - DFQ < 0 \quad (A4)$$

where $P \equiv D - F$ and $Q \equiv AF + \beta D$. We can show that the roots of this quadratic are now $\mu_1^{\text{root}'} = D / \gamma$ and $\mu_2^{\text{root}'} = -FQ / \beta \gamma P$. When $P \geq 0$, it follows for the same reasons as in the Section 2 model that condition (39) is satisfied when $\mu < D / \gamma$. When $P < 0$, the quadratic is once again concave, with a negative intercept and two positive roots, where it is easy to show that $\mu_2^{\text{root}'} > \mu_1^{\text{root}'}$. Once again, therefore, two ranges of μ satisfy inequality (A4), but one of these ranges, namely when $\mu > \mu_2^{\text{root}'}$ fails to satisfy condition (A3). Therefore, there is only one range of values of μ that satisfies all R-H conditions and therefore allows for stability around the steady state:

$$\mu < \frac{s_R h u_n - \gamma - c_{WR}}{\gamma} \quad (A5)$$