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## Bonacich Measures as Equilibria in Network Models

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# **Bonacich Measures as Equilibria in Network Models**

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## **ABSTRACT**

We investigate the cases when the Bonacich measures of strongly connected directed bipartite networks can be interpreted as a Nash equilibrium of a non-cooperative game. One such case is a two-person game such that the utility functions are bilinear, the matrices of these bilinear forms represent the network, and strategies have norm at most one. Another example is a two-person game with quadratic utility functions. A third example is an  $m + n$  person game with quadratic utility functions, where the matrices representing the network have dimension  $m \times n$ . For connected directed bipartite networks we show that the Bonacich measures are unique and give a recursion formula for the computation of the measures. The Bonacich measures of such networks can be interpreted as a subgame perfect equilibrium path of an extensive form game with almost perfect information.

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## 1. Introduction

We investigate the cases when the Bonacich measures of strongly connected directed bipartite networks can be interpreted as a Nash equilibrium of a non-cooperative game. One such case is a two-person game such that the utility functions are bilinear, the matrices of these bilinear forms represent the network, and strategies have norm at most one. Another example is a two-person game with quadratic utility functions. A third example is an  $m + n$  person game with quadratic utility functions, where the matrices representing the network have dimension  $m \times n$ . For connected directed bipartite networks we show that the Bonacich measures are unique and give a recursion formula for the computation of the measures. The Bonacich measures of such networks can be interpreted as a subgame perfect equilibrium path of an extensive form game with almost perfect information.

In social networks *nodes* are agents who are connected to other agents via *links*. Links could have a weight that indicates how important the connection is. Links could also be directed if two agents are not symmetrically related to each other.

In network theory so called centrality measures attempt to quantify the importance or influence of an agent in a social network. Centrality measures resemble solution concepts of cooperative game theory. The Shapley value, for example, has been applied in both theories (Shapley 1953).

There is large literature about connections between solutions concepts of cooperative game theory and equilibrium concepts of noncooperative game theory. The "Nash programme" was initiated already by John Nash, who thought that noncooperative game theory is more fundamental than the cooperative theory (Nash 1950, Nash 1953). In this view, "reasonable solutions" of cooperative games should be equilibrium outcomes of some (equally reasonable) noncooperative game (see also Binmore 1987, Rubinstein 1982).

In network theory there are now some papers where centrality measures are shown to be equilibria of noncooperative games. Representative papers are Ballester *et.al* (2006) and Cabrales *et.al* (2011), who show that

the Katz-Bonacich centrality measure is a Nash equilibrium (see also Katz 1953, Bonacich 1987, Bonacich and Lloyd 2001, Jackson and Zenou 2014). In their models agents choose a real number that indicates how much an agent invests in network activities. An adjacency matrix gives the link structure of the network, and payoff functions are quadratic.

Salonen (2014) studies noncooperative link formation games in which agents decide how much to invest in relations with every other agent. Strategy sets are standard simplices and payoff functions are of Cobb-Douglas type. Players have a common popularity ordering over their opponents: it tells how valuable it is to be in contact with other players.

The equilibrium strategies are computed as well as the values of some standard centrality measures from the resulting equilibrium networks. The measures analyzed are the eigenvector centrality, indegree, the Katz-Bonacich measure, and the PageRank measure (see Bonacich 1972, Brin and Page 1998, Freeman 1979). Depending on assumptions of the parameters of payoff functions, the centrality measures give the same or the opposite ordering of the players as the common popularity ordering.

To my knowledge the present paper is the first one in which equilibria and centrality measures of bipartite directed networks are compared. In a bipartite network the node set consist of two separate sets of nodes, say  $V_1$  and  $V_2$ , and there cannot be any links within  $V_1$  or  $V_2$ , but all links are between these groups.

Bonacich (1991) initiated the study of such networks in social sciences. In his paper, networks are unweighted and undirected and in this case each connected bipartite network has two Bonacich measures. These measures are defined in the same way for weighted undirected bipartite networks. However, if a bipartite network is directed and *strongly connected*, then the network has *four* Bonacich measures, although they are defined in the similar manner as in Bonacich (1991).

The paper is organized as follows. In Section 2 basic definitions and notation are given. In subsection 2.1 we take a closer look at the bipartite

networks, Bonacich measures, and their potential applications. In Section 3 main results are stated and proven.

## 2. Preliminaries

A *network*  $G = (V, E)$  consists of a finite set  $V$  of *nodes*, and a finite set  $E$  of *links* between nodes. A network is *directed*, if links are represented by a set of ordered pairs  $E = \{(i, j) \mid i, j \in V\}$ . We may also denote  $i \rightarrow j$  if there is a link from  $i$  to  $j$ .

A network  $G$  can be represented by a matrix  $A$  whose rows and columns are indexed by the nodes  $i \in V$ . A network is *weighted*, if entries  $a_{ij}$  can take any nonnegative values, representing the *strength* or intensity of the link from  $i$  to  $j$ . If  $a_{ij} = 0$  then there is no link from  $i$  to  $j$ . In this paper the networks will be directed and weighted.

A network is *bipartite* (sometimes called a *bimodal* or an *affiliation network*), if the node set  $V$  is partitioned into two nonempty subsets  $V_1$  and  $V_2$  such that if two nodes are connected then they cannot be elements of the same partition member  $V_i$ . A bipartite network can be represented by an  $m \times n$  matrix  $A$ , where  $m = |V_1|$  and  $n = |V_2|$ .

A weighted directed bipartite network  $G$  may be represented by a pair  $(A, B)$ , where  $A$  and  $B$  are both  $m \times n$  nonnegative matrices. An entry  $a_{ij}$  of  $A$  would give the weight of the link from  $i \in V_1$  to  $j \in V_2$ , and  $b_{ij}$  would give the the weight of the link from  $j \in V_2$  to  $i \in V_1$ .

A *path* from a node  $i \in V$  to node  $j \in V$  is a subset of nodes  $i_0, \dots, i_k$  such that  $i = i_0, i_k = j, i_j \rightarrow i_{j+1}$  for all  $j = 0, \dots, k - 1$ , and  $i_n \neq i_m$  for all  $n, m < k, n \neq m$ . A path is a *cycle*, if  $i = j$ .

A subset  $C$  of nodes  $V$  of a directed network is *connected*, if for every  $i, j \in C$  there is a path from  $i$  to  $j$  or a path from  $j$  to  $i$ . A connected subset  $C$  is *strongly connected*, if for every  $i, j \in C$  there is a path from  $i$  to  $j$  and a path from  $j$  to  $i$ .

A (strongly) connected subset  $C$  is a (strongly) connected *component* if no proper superset of  $C$  is (strongly) connected. That is,  $C$  is a maximal



(strongly) connected subset of  $V$ .

If any node  $i \in V$  of a directed network belongs to a strongly connected component, then the node set  $V$  can be partitioned into disjoint strongly connected components. If there are some nodes that do not belong to any strongly connected component, then  $V$  has a partition such that partition members are either strongly connected components or singletons.

A collection  $\mathcal{C}$  of strongly connected components is connected, if the members of the collection can be indexed by numbers  $i$  such that  $\mathcal{C} = \{V^1, \dots, V^t\}$ , and there is a link from  $V^j$  to  $V^{j-1}$ ,  $j = 2, \dots, t$ . Note that there cannot be any links from  $V^k$  to  $V^j$ , if  $j > k$ . Therefore we may say that  $\{V^1, \dots, V^t\}$  is an *ordered collection*.

The node set  $V$  of a directed network can not always be partitioned into connected components.

*Example 1.* Let  $V = \{1, 2, 3, 4, 5, 6\}$ , and let  $\{1, 2\}$ ,  $\{3, 4\}$ , and  $\{5, 6\}$  be the strongly connected components of a directed network. Assume that there are only two other links:  $5 \rightarrow 1$  and  $6 \rightarrow 3$ . Then  $\{1, 2, 5, 6\}$  and  $\{3, 4, 5, 6\}$  are the only connected components, but they do not partition  $V$ .

### 2.1. Bonacich measures and applications

Let  $G = (V_1 \cup V_2, E)$  be a weighted directed bipartite network represented by a pair  $(A, B)$  of  $m \times n$  matrices. We may call rows of the matrices *agents* who are members of some *clubs* that are represented by the columns. For example,  $a_{ij}$  could be the value agent  $i$  gets from club  $j$ , and  $b_{ij}$  could measure the contribution agent  $i$  makes to club  $j$ .

A typical entry  $a_{ij}$  of  $A$  ( $b_{ij}$  of  $B$ ) gives the strength of the link from agent  $i$  to club  $j$  (from club  $j$  to agent  $i$ ). If  $a_{ij} = 0$  ( $b_{ij} = 0$ ), we interpret that there is no link from  $i$  to  $j$  (from  $j$  to  $i$ ).

For example, rows could be researchers and columns could be journals. Value  $a_{ij}$  of a link from  $i \in V_1$  to  $j \in V_2$  could give the number of times author  $i$  cites an article in journal  $j$ . Value  $b_{ij}$  could be number of times author  $i$ 's paper has been cited in journal  $j$ .

As another potential application, let  $V_1$  and  $V_2$  be subsets of teachers and students, respectively, of some university. Entry  $a_{ij}$  could be the grade (quality of teaching) student  $i$  gives to teacher  $j$ , and  $b_{ij}$  could be the grade (of an exam) teacher  $j$  gives to student  $i$ .

Given a pair  $(A, B)$  of matrices representing a bipartite network with a node set  $V_1 \cup V_2$ , we can form unimodal networks  $G_1$  and  $G_2$  with node sets  $V_1$  and  $V_2$  as follows.

Consider the product matrix  $AB^T$ . A typical entry  $(AB^T)_{ij}$  of  $AB^T$  (an  $m \times m$  matrix) is of the form  $\sum_k a_{ik}b_{kj}$ : it is the sum of products  $a_{ik}b_{kj}$  over  $k$ , where this product gives the strength of the directed path  $i \rightarrow k \rightarrow j$  from agent  $i$  to agent  $j$  via club  $k$ . We will denote by  $G_1$  the directed network of agents (nodes in  $V_1$ ) represented by  $AB^T$ .

Product matrix  $B^T A$  (an  $n \times n$  matrix) would give directed network between clubs. Typical entry  $(B^T A)_{ij}$  is of the form  $\sum_k b_{ki}a_{kj}$ , where  $b_{ki}a_{kj}$  is the strength of the path  $i \rightarrow k \rightarrow j$  from club  $i$  to club  $j$  via agent  $k$ . We will denote by  $G_2$  the directed network of clubs (nodes in  $V_2$ ) represented by  $B^T A$ .

The left and right eigenvectors  $y^T B^T A = \lambda y^T$  and  $B^T A x = \lambda x$  (corresponding to the largest eigenvector  $\lambda$  of  $B^T A$ ) measure different things. The left eigenvector  $y_i$  measures the importance of club  $i$  in terms of outgoing links, and  $x_i$  measures the importance of club  $i$  in terms of incoming links. Let's put these into equation

$$y^T B^T A = \lambda y^T \tag{1}$$

$$B^T A x = \lambda x. \tag{2}$$

The eigenvectors  $y$  and  $x$  are called the *Bonacich measures* of a directed network  $G_2$  represented by  $B^T A$ . Bonacich (1991) originally defined these measures for unweighted undirected bipartite networks. Kleinberg (1999) has proposed a measure called HITS for unweighted directed networks that can be represented by square matrices. HITS is calculated in the same way as the Bonacich measure.

We may sometimes call  $y$  the (Bonacich) row measure, and  $x$  the (Bonacich) column measure. Bonacich measures for the network  $G_1$  are similarly defined as the eigenvectors of  $AB^T$ . Note that the left (right) eigenvectors of  $AB^T$  are the right (left) eigenvectors of  $BA^T$ , the transpose of  $AB^T$ .

By the Perron-Frobenius theorem the eigenvectors of  $B^T A$  are unique (up to multiplication by positive constants), if the nonnegative square matrix  $B^T A$  is *irreducible*. That is, for any  $i, j$  there exists a strictly positive integer  $t(i, j)$  such that the  $(i, j)$  entry of  $(B^T A)^{t(i, j)}$  is strictly positive, where  $X^t$  denotes the  $t$ 'th power of matrix  $X$ . This means that the directed network  $G_2$  represented by  $B^T A$  is strongly connected.

Note that  $B^T A$  may be irreducible although  $AB^T$  is not irreducible. But if both of these matrices are irreducible, then a directed bipartite network  $G$  that is represented by  $(A, B)$  is strongly connected. In this case  $G$  has *four* Bonacich measures associated with it, since both  $B^T A$  and  $AB^T$  have two eigenvectors each.

We give next another way of getting all four Bonacich measures of a directed bipartite network (this method was used already by Bonacich 1991). Consider the system of equations

$$Bp = \alpha q \tag{3}$$

$$q^T A = \beta p^T. \tag{4}$$

Now  $Bp = \alpha q$  means that  $p^T B^T = \alpha q^T$ . Hence  $q^T = (1/\alpha)p^T B^T$ , and therefore from equation (4) we get that  $(1/\alpha)p^T B^T A = \beta p^T$ . On the other hand, from equation (4) we get that  $p = (1/\beta)A^T q$ , and then equation (3) implies  $BA^T q = \alpha \beta q$ . We have the following equations

$$p^T B^T A = \alpha \beta p^T \tag{5}$$

$$BA^T q = \alpha \beta q. \tag{6}$$

If the networks  $G_1$  and  $G_2$  are strongly connected, the left eigenvectors  $y$  and  $p$  of  $B^T A$  in equations (1) and (5) are the same.

As an example, row  $i$  of  $A$  could consist of the grades student  $i$  gives to schools, and column  $j$  of  $B$  could consist of the grades school  $j$  gives to students. Then vector  $p$  consists of the weights of schools as evaluators, and vector  $q$  consists of the weights of students as evaluators. Student  $i$  gets a high grade as an evaluator, if he has given a high score to those schools that have received high scores from other students on the average.

If we reformulate equations (3) and (4) and look solutions of

$$z^T B = \alpha' w^T \tag{7}$$

$$Aw = \beta' z \tag{8}$$

we get that

$$z^T BA^T = \alpha' \beta' z^T \tag{9}$$

$$B^T Aw = \alpha' \beta' w \tag{10}$$

If both  $G_1$  and  $G_2$  are strongly connected, then the following relations hold. The vector  $z$  in equation (9) is the left eigenvector of  $BA^T$ , and  $q$  in equation (6) is its right eigenvector. The right eigenvector  $w$  of  $B^T A$  in equation (10) is the same as  $x$  in equation (2). The left eigenvector  $p$  of  $B^T A$  is the same as  $y$  in equation (1).

By irreducibility of  $BA^T$  and  $B^T A$ , the eigenvalues  $\lambda$  of equations (1) and (2),  $\alpha\beta$  of equations (5) and (6), and  $\alpha'\beta'$  of equations (9) and (10) are the same. (It holds in general that if  $X$  is an  $m \times n$  matrix and  $Y$  is an  $n \times m$  matrix, then the non-zero eigenvalues of  $XY$  and  $YX$  are the same.)

We can interpret equations (7) and (8) in the example where students and teachers are grading each other. In this case, row  $i$  of  $A$  consists of the grades student  $i$  gets from schools, and column  $j$  of  $B$  consists of the grades school  $j$  gets from the students. Vector  $z$  gives the values of students as their "weighted grade point averages" from schools, and vector  $w$  is a similar measure for schools.

So if we look at the eigenvectors  $y$  and  $x$  of  $B^T A$  in equations (1) and (2), we note that the left eigenvector  $y = p$  gives a measure of schools as

evaluators, and the right eigenvector  $x = w$  measures the quality of schools in the eyes of students. The left and right eigenvectors  $q$  and  $z$  of  $AB^T$  are corresponding measures of students.

### 3. Results

Throughout this section  $(A, B)$  is a pair of nonnegative  $m \times n$  matrices representing a directed bipartite network  $G$ . We assume also *w.l.o.g.* that each node  $v \in V_1 \cup V_2$  has at least one link.

Let  $\|\cdot\|$  denote the Euclidean norm. Let  $B^k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$  be the unit ball of  $\mathbb{R}^k$ , and let  $B_+^k = B^k \cap \mathbb{R}_+^k$  be the intersection of the unit ball with the nonnegative orthant of  $\mathbb{R}^k$ .

Given a directed bipartite network  $G$ , let  $G^*$  denote the following two-person normal form game associated with  $G$ . The strategy set of player 1 ("row player") is  $S_1 = B_+^m$ , and the strategy set of player 2 ("column player") is  $S_2 = B_+^n$ . The utility function of player 1 is  $u_1(x, y) = x^T Ay$ , and the utility function of player 2 is  $u_2(x, y) = x^T By$ , where  $x \in S_1, y \in S_2$ .

**Proposition 1.** *Let  $(A, B)$  be a pair of nonnegative  $m \times n$  matrices representing a strongly connected directed bipartite network  $G$ . Let  $G^*$  be the two-person normal form game associated with  $G$ . If  $z \in S_1$  and  $w \in S_2$  are the eigenvectors given by equations (9) and (10) such that  $\|z\| = \|w\| = 1$ , then  $(z, w)$  is a Nash equilibrium of  $G^*$ . If  $(z, w) \neq (0, 0)$  is a Nash equilibrium of  $G^*$ , then  $z$  and  $w$  are the eigenvectors given by equations (9) and (10) such that  $\|z\| = \|w\| = 1$ .*

*Proof.* Let  $(z, w) \in S_1 \times S_2$  satisfy equations (9) and (10), and assume  $\|z\| = \|w\| = 1$ . Equation (8) implies that  $u_1(x, w) = \beta' x^T z$ . Hence the best reply for player 1 against  $w$  is  $z$ . Similarly, equation (7) implies that  $u_2(z, y) = \alpha' w^T y$ . Hence the best reply for player 2 against  $z$  is  $w$ . Therefore  $(z, w)$  is a Nash equilibrium of  $G^*$ .

Let  $(z, w) \neq (0, 0)$  be a Nash equilibrium of  $G^*$ . The first order condition

for  $z_i > 0$  of player 1 is

$$\sum_j a_{ij} w_j = \frac{\beta' z_i}{\sqrt{\sum_k z_k^2}},$$

where  $\beta'$  is the Lagrangian multiplier of the constraint  $\|z\| \leq 1$ . Now  $z_i > 0$  implies  $\beta' > 0$  and hence  $\|z\| = \sqrt{\sum_k z_k^2} = 1$ . Therefore the first order conditions corresponding to  $z_i > 0$  of player 1 satisfy the equations

$$Aw = \beta' z.$$

By the same reasoning, the first order conditions corresponding to  $w_j > 0$  of player 2 satisfy

$$z^T B = \alpha' w^T.$$

But these first order conditions are the same as equations (7) and (8), and we are done.  $\square$

The strategy spaces of the players in game  $G^*$  are nonnegative vectors having norm at most one. Let us see what kind of relations there are between Nash equilibria and Bonacich measures if the strategy sets are standard simplices.

Denote by  $G^\Delta$  a two-person normal form game such that the strategy set for players 1 and 2 are  $\Delta^m = \{x \in \mathbb{R}_+^m \mid \sum_i x_i = 1\}$  and  $\Delta^n = \{y \in \mathbb{R}_+^n \mid \sum_i y_i = 1\}$ , respectively. The utility functions of players are  $u_1(x, y) = x^T A y$  and  $u_2(x, y) = x^T B y$ .

**Proposition 2.** *Let  $z, w$  be the Bonacich measures of a strongly connected directed bipartite network  $G$  represented by  $(A, B)$ . Let  $G^\Delta$  the two-person normal form game associated with  $G$ . Then  $(z, w) \neq (0, 0)$  is a Nash equilibrium of  $G^\Delta$ , iff  $z$  and  $w$  are uniform distributions.*

*Proof.* First note that we can normalize Bonacich measures  $z$  and  $w$  so that  $z \in \Delta^m$  and  $w \in \Delta^n$  (that may change eigenvalues but not eigenvector spaces). Since  $G$  is strongly connected, both  $z$  and  $w$  are strictly positive.

Strategy  $z$  is a best reply against  $w$ , if and only if all elements of the vector  $Aw$  are the same. But then  $z = (1/m, \dots, 1/m)$  by equation (8). By the same argument  $w = (1/n, \dots, 1/n)$   $\square$

*Example 2.* Let  $A$  and  $B$  be the following row and column stochastic matrices, respectively

$$A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \frac{1}{3} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Then the matrices  $AB^T$  and  $B^T A$  are the following row stochastic matrices

$$AB^T = \frac{1}{9} \begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 5 \\ 3 & 2 & 4 \end{bmatrix}, \quad B^T A = \frac{1}{9} \begin{bmatrix} 4 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 5 & 1 \end{bmatrix}$$

The right eigenvector of both  $AB^T$  and  $B^T A$  is  $(1/3, 1/3, 1/3)$ . The left eigenvector of  $AB^T$  is  $(24/91, 23/91, 43/91)$ , and the left eigenvector of  $B^T A$  is  $(38/91, 31/91, 22/91)$ .

As a third game consider a two-person game  $G^c$  such that player 1's strategies are vectors  $x \in \mathbb{R}_+^m$  and player 2's strategies are vectors  $y \in \mathbb{R}_+^n$ . Player 1 has utility function  $u_1(x, y) = x^T A y - (c_1/2)(\sum_i x_i^2)$  and player 2 has utility function  $u_2(x, y) = x^T B y - (c_2/2)(\sum_j y_j^2)$ , where  $c_1, c_2 > 0$  are constants.

**Proposition 3.** *Let  $G$  be a strongly connected directed bipartite network represented by  $(A, B)$ . A strategy pair  $(x, y) \neq (0, 0)$  is a Nash equilibrium of  $G^c$ , iff  $x$  and  $y$  are the Bonacich measures of  $G$ .*

*Proof.* Given  $x' \in \mathbb{R}_+^m$  and  $y' \in \mathbb{R}_+^n$ , the best replies  $x$  and  $y$  for players 1 and 2 satisfy the first order conditions

$$\begin{aligned} Ay' &= c_1 x \\ x'^T B &= c_2 y'^T. \end{aligned}$$

These equations resemble equations (7) and (8). A vector  $(x, y)$  is a Nash equilibrium iff  $x = x'$  and  $y = y'$ . Multiplying the first equation by  $\beta'/c_1$  and the second equation by  $\alpha'/c_2$  does not change the equilibria. But then these equations are like equations (7) and (8), except that the matrices are the same up to a positive constant. But such a change in matrices does not change the eigenvector spaces although eigenvalues may change. Hence the Nash equilibria are also Bonacich measures.

The proof to the other direction follows the same logic.  $\square$

Finally, let us study a  $m + n$  player game  $G^{m,n}$  defined as follows. The player set is  $\{1, \dots, m, m + 1, \dots, m + n\}$ . The strategy set of each player  $i$  is  $\mathbb{R}_+$ . Let us denote by  $x = (x_1, \dots, x_m)$  the strategy profile of the first  $m$  players and by  $y = (y_{m+1}, \dots, y_{m+n})$  the strategy profile of the remaining players. The utility function of a player  $i \leq m$  is  $u_i(x_i, x_{-i}, y) = (x_i, x_{-i})^T A y - (c_1/2)x_i^2$ , where  $c_1 > 0$ . The utility function of a player  $i > m$  is  $u_i(x, y_i, y_{-i}) = x^T B(y_i, y_{-i}) - (c_2/2)y_i^2$ , where  $c_2 > 0$ .

**Proposition 4.** *Let  $G$  be a strongly connected directed bipartite network represented by  $(A, B)$ . A strategy profile  $(x, y) \neq (0, 0)$  is a Nash equilibrium of  $G^{m,n}$ , iff  $x$  and  $y$  are the Bonacich measures of  $G$ .*

*Proof.* The first order conditions of players  $i \leq m$  and  $j > m$  are

$$\begin{aligned} \sum_k a_{ik} y_k &= c_1 x_i, \quad i = 1, \dots, m \\ \sum_t b_{tj} x_t &= c_2 y_j, \quad j = m + 1, \dots, m + n. \end{aligned}$$

The rest of the proof is similar to the proof of Proposition 3.  $\square$

A possible interpretation of Proposition 4 is that the choice variable  $x_i$  of "row agent"  $i$  gives the amount of resources agent  $i$  invests in network activities. The  $i$ 'th row of  $A$  says how much utility  $a_{ij}$  each unit invested returns from activity  $j$ . Column agents' choices  $y_j$  and utilities are interpreted in the same way.



### 3.1. A recursive construction of Bonacich measures for connected networks

Suppose  $(A, B)$  represents a *connected* directed bipartite network with a node set  $V = V_1 \cup V_2$  such that any node is a member of a strongly connected component. Let  $\mathcal{C} = \{V^1, \dots, V^k\}$  be the partition of  $V$  into strongly connected components. Since  $\mathcal{C}$  necessarily is an ordered collection we may choose the indices  $i \in \{1, \dots, k\}$  in such a way that for all  $j < k$  there exist  $v \in V^j$  and  $v' \in V^{j+1}$  such that  $v' \rightarrow v$ . Note that if  $j < t$ , there cannot be any links from  $V^j$  to  $V^t$ , since  $V^j$  and  $V^t$  are strongly connected components.

Each  $V^i \in \mathcal{C}$  is of the form  $V^i = V_1^i \cup V_2^i$ , where  $V_1^i \subset V_1$  and  $V_2^i \subset V_2$ . Since by assumption each  $V^i$  contains at least two nodes,  $V_1^i$  and  $V_2^i$  are nonempty subsets for all  $i$ . We will index nodes in such a way that each  $V_1^i$  consists of consecutive natural numbers from the set  $\{1, \dots, m\}$ , so that  $V_1^1 = \{1, \dots, m_1\}$ ,  $V_1^2 = \{m_1 + 1, \dots, m_2\}$ , and so on. Nodes in  $V_2$  are indexed analogously.

We will construct Bonacich row measure  $z$  and column measure  $w$  for the network  $G$  by starting from the first component  $V^1$ , and solving recursively the measures for components  $V^j$ ,  $j > 1$ .

The first member of the collection  $\mathcal{C}$  is  $V^1$ , and  $V^1$  has no links that extend out of  $V^1$ . That is, if  $v \in V^1$  and  $v \rightarrow y$ , then  $y \in V^1$ . The nodes in  $V_1^1$  correspond to the rows  $1, \dots, m_1$  of matrices  $A$  and  $B$ , and nodes in  $V_2^1$  correspond to the columns  $1, \dots, n_1$  of matrices  $A$  and  $B$ . Let  $A(1)$  and  $B(1)$  be the matrices corresponding to these blocks of  $A$  and  $B$ , respectively.

For any  $i \in V_1^1$  (any  $j \in V_2^1$ ), the entry  $a_{ij}$  ( $b_{ij}$ ) of  $A$  ( $B$ ) is strictly positive only if  $j \leq n_1$  ( $i \leq m_1$ ). On the other hand, the matrices  $A(1)B^T(1)$  and  $B^T(1)A(1)$  are irreducible. Hence the left and right eigenvectors for these matrices exist uniquely. So there exist unique  $z(1)$  and  $w(1)$  satisfying equations (9) and (10), where these vectors have the dimensions  $m_1$  and  $n_1$ , respectively. The coordinates of  $z(1)$  and  $w(1)$  are all strictly positive, and  $\|z(1)\| = \|w(1)\| = 1$  *w.l.o.g.*

Let the rows  $m_1 + 1, \dots, m_2$  correspond to the nodes in  $V_1^2$ , and let the

columns  $n_1 + 1, \dots, n_2$  correspond to the nodes in  $V_2^2$ . Let  $A(2)$  be the matrix consisting of the first  $m_2$  rows of  $A$  and the first  $n_2$  columns of  $A$ . Let  $B(2)$  be the matrix consisting of the first  $m_2$  rows of  $B$  and the first  $n_2$  columns of  $B$ .

Consider the following equations, analogous to equations (7) and (8):

$$z^T B(2) = \alpha w^T \quad (11)$$

$$A(2)w = \beta z \quad (12)$$

where  $z$  is a nonnegative  $m_2$  dimensional vector whose first  $m_1$  coordinates are given by the vector  $z(1)$ , and  $w$  is a nonnegative  $n_2$  dimensional vector whose first  $n_1$  coordinates are given by the vector  $w(1)$ .

Note that both  $A(2)B^T(2)$  and  $B^T(2)A(2)$  cannot be irreducible, since the node set  $V^1 \cup V^2$  is not strongly connected. However, there exists unique vectors  $z(2)$  and  $w(2)$  satisfying these equations, for some  $\alpha, \beta > 0$ . Let us prove this claim next.

First note the matrices  $A(2)$  and  $B(2)$  have the following block form

$$A(2) = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}, \quad B(2) = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix}$$

Pairs  $(A^{11}, B^{11}) = (A(1), B(1))$  and  $(A^{22}, B^{22})$  correspond to the strongly connected components  $V^1$  and  $V^2$

For any  $i \in V_1^2$ , all the entries  $a_{ij}$  such that  $i \leq m_1$  and  $j > n_1$  are zero. For any  $j \in V_2^2$ , all the entries  $b_{ij}$  of  $B$  such that  $i > m_1$  and  $j \leq n_1$  are zero. These constraints hold because there cannot be a link from  $v$  to  $v'$  if  $v \in V^1$  and  $v' \in V^2$ . Therefore  $A^{12}$  and  $B^{21}$  are null matrices.

Equations (11) and (12) can then be rewritten as

$$z(1)^T B^{12} + z(2)^T B^{22} = \beta w(2)^T \quad (13)$$

$$A^{21}w(1) + A^{22}w(2) = \alpha z(2) \quad (14)$$

where  $z(2)$  has dimension  $m_2 - m_1$  and  $w(2)$  has dimension  $n_2 - n_1$ . Vectors  $z(1)$  and  $w(1)$  are the normalised Bonacich measures of the strongly connected component  $V^1$  satisfying equations (9) and 10.

Solve for  $w(2)$  from equation (13) and insert it into equation (14). Solve also for  $z(2)$  from equation (14) and insert it into equation (13). We get the following

$$A^{22}(B^{22})^T z(2) = \alpha\beta z(2) - \beta A^{21}w(1) - A^{22}(B^{12})^T z(1) \quad (15)$$

$$w(2)^T (A^{22})^T B^{22} = \alpha\beta w(2)^T - \alpha z(1)^T B^{12} - w(1)^T (A^{21})^T B^{22} \quad (16)$$

The following Lemma states that these equations have a unique solution  $z(2), w(2)$  and that these vectors are strictly positive.

*Notation.* Given two vectors  $x = (x_1, \dots, x_t)$  and  $y = (y_{t+1}, \dots, y_m)$ , we denote by  $x \wedge y$  the concatenation of  $x$  and  $y$ :  $x \wedge y = (x_1, \dots, x_t, y_{t+1}, \dots, y_m)$ . Concatenation  $x(1) \wedge \dots \wedge x(k)$  over several vectors  $x(1), \dots, x(k)$  is defined analogously. We may denote by  $\vec{x}(k)$  the concatenation of vectors  $x(1), \dots, x(k)$ :  $\vec{x}(k) = x(1) \wedge \dots \wedge x(k)$ .

**Lemma 1.** *Let  $z(1)$  and  $w(1)$  be the unique eigenvectors having norm 1 and satisfying equations (9) and (10) for the pair  $(A^{11}, B^{11})$  representing a strongly connected network on a node set  $V^1$ . Then equations (15) and (16) have unique solutions  $z(2), w(2)$ , and these vectors are strictly positive.*

*Proof.* There exists at least one solution  $z(2), w(2)$  to equations (15) and (16), for some  $\alpha, \beta > 0$ . To see this, add  $\varepsilon > 0$  to the entries of matrices  $A(2)$  and  $B(2)$  in equations (11) and 12. Then these matrices would correspond a strongly connected bipartite network with node set  $V^1 \cup V^2$ . Using these matrices in equations (11) and (12), strictly positive solutions  $z^\varepsilon, w^\varepsilon$  will exist. These vectors can be chosen to have norm 1. With this normalization,  $z^\varepsilon$  and  $w^\varepsilon$  are unique solutions, for any given  $\varepsilon > 0$ .

Letting  $\varepsilon$  approach zero, all cluster points of  $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon \downarrow 0}$  are nonnegative, and they are solutions of the original equations (11) and (12). Such cluster points exist because the subset of vectors  $z, w$  having norm 1 is a compact set. Note also that while  $\alpha$  and  $\beta$  may depend on  $\varepsilon$ , all values of  $\alpha$  and  $\beta$  stay in a bounded set because  $z^\varepsilon$  and  $w^\varepsilon$  have norm 1.

Equations (15) and (16) must hold approximatively for  $z^\varepsilon$  and  $w^\varepsilon$  when  $\varepsilon$  is small, and the approximation must get better and better as  $\varepsilon$  goes to 0. This holds since matrices  $A^{12}$  and  $B^{21}$  approach null matrices as  $\varepsilon$  goes to 0

Let  $(z^*, w^*)$  be an arbitrary cluster point of  $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon \downarrow 0}$ . Note that  $(z^*, w^*)$  solves also equations (15) and (16), because the first  $m_1$  and  $n_1$  coordinates of  $z^*$  and  $w^*$  must be proportional to vectors  $z(1)$  and  $w(1)$ . Further, since values of  $\alpha$  and  $\beta$  stay in a bounded set, at least some coordinates  $z_i^*$  and  $w_j^*$ , for  $i > m_1, j > n_1$ , must be strictly positive by equations (15) and (16). We show now that  $z^*$  and  $w^*$  are both strictly positive.

From equations (15) and (16) we get the following vector inequalities:

$$A^{22}(B^{22})^T z^*(2) \leq \alpha\beta z^*(2) \quad (17)$$

$$w^*(2)^T (A^{22})^T B^{22} \leq \alpha\beta w^*(2)^T \quad (18)$$

where  $z^*(2) = (z_{m_1+1}^*, \dots, z_{m_2}^*)$  and  $w^*(2) = (w_{n_1+1}^*, \dots, w_{n_2}^*)$ . From inequality (17) we get that there exists  $\alpha'\beta' > 0$  such that (17) holds as equality. But then  $z^*(2)$  is the eigenvector corresponding to the largest eigenvalue  $\alpha'\beta' > 0$  of  $A^{22}(B^{22})^T$ , and hence  $z^*(2)$  is strictly positive. Then  $w^*(2)$  in equation (18) must be the left eigenvector of  $(A^{22})^T B^{22}$  corresponding to  $\alpha'\beta' > 0$ , because  $z^*(2), w^*(2)$  solves equations (9) and (10). Hence  $w^*(2)$  must also be strictly positive.

But then  $(z^*, w^*)$  is the unique cluster point of  $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon \downarrow 0}$ . Choose  $a > 0$  and  $b > 0$  such that  $a(z_1^*, \dots, z_{m_1}^*) = z(1)$  and  $b(w_1^*, \dots, w_{n_1}^*) = w(1)$ , and let  $z(2) = a(z_{m_1+1}^*, \dots, z_{m_2}^*)$  and  $w(2) = b(w_{n_1+1}^*, \dots, w_{n_2}^*)$ . Then  $z(1) \wedge z(2)$  and  $w(1) \wedge w(2)$  are Bonacich measures of the connected network with node set  $V^1 \cup V^2$ , because they are the unique solutions of equations (11) and (12).  $\square$

We can use equations (15) and (16) to solve  $w(2)$  and  $z(2)$  as functions

of  $w(1)$  and  $z(1)$ .

$$z(2) = \frac{\beta_2 A^{21} w(1) + A^{22} (B^{12})^T z(1)}{\alpha_2 \beta_2 - \alpha'_2 \beta'_2} \quad (19)$$

$$w(2) = \frac{\alpha_2 (B^{12})^T z(1) + (B^{22})^T A^{21} w(1)}{\alpha_2 \beta_2 - \alpha'_2 \beta'_2} \quad (20)$$

where  $A^{22} (B^{22})^T z(2) = \alpha'_2 \beta'_2 z(2)$  and  $(B^{22})^T A^{22} w(2) = \alpha'_2 \beta'_2 w(2)$ .

The eigenvalues obtained at the first and second step of recursion are different. To make this point explicit we have indicated the eigenvalues obtained at the second step by subscripts:  $\alpha_2, \beta_2$ . Note also that  $\alpha_2 \beta_2 > \alpha'_2 \beta'_2$ .

We can apply one-to-one the methods used in in the proof of Lemma 1 to construct Bonacich measures for the connected bipartite network with a node set  $V = V^1 \cup \dots \cup V^k$ . Suppose Bonacich measures  $\vec{z}(m-1) = z(1) \wedge \dots \wedge z(m-1)$  and  $\vec{w}(m-1) = w(1) \wedge \dots \wedge w(m-1)$  of the connected network with node set  $V^1 \cup \dots \cup V^{m-1}$  have been found,  $1 < m \leq k$ .

Replace  $A(1)$  and  $B(1)$  by the matrices  $A(m-1)$  and  $B(m-1)$  whose rows correspond to nodes in  $V_1^1 \cup \dots \cup V_1^{m-1}$  and columns correspond to nodes in  $V_2^1 \cup \dots \cup V_2^{m-1}$ . Replace the Bonacich measures  $z(1)$  and  $w(1)$  by the Bonacich measures  $\vec{z}(m-1) = z(1) \wedge \dots \wedge z(m-1)$  and  $\vec{w}(m-1) = w(1) \wedge \dots \wedge w(m-1)$ . Replace  $A(2)$  and  $B(2)$  by the blocks  $A(m)$  and  $B(m)$  of matrices  $A$  and  $B$  corresponding to the strongly connected component  $V^m$ . These matrices have the following block form:

$$A(m) = \begin{bmatrix} A^{m-1,m-1} & A^{m-1,m} \\ A^{m,m-1} & A^{m,m} \end{bmatrix}, \quad B(m) = \begin{bmatrix} B^{m-1,m-1} & B^{m-1,m} \\ B^{m,m-1} & B^{m,m} \end{bmatrix} \quad (21)$$

where  $A^{m-1,m}$  and  $B^{m,m-1}$  are null matrices.

**Proposition 5.** *Let  $z(1)$  and  $w(1)$  be the normalized Bonacich measures of the strongly connected network with nodes set  $V^1$ . Bonacich measures  $z(m)$  and  $w(m)$  of the strongly connected network with node set  $V^m$  satisfy the*

following recursions, for  $1 < m \leq k$ :

$$z(m) = \frac{\beta_m A^{m,m-1} \vec{w}(m-1) + A^{m,m} (B^{m-1,m})^T \vec{z}(m-1)}{\alpha_m \beta_m - \alpha'_m \beta'_m} \quad (22)$$

$$w(m) = \frac{\alpha_m (B^{m-1,m})^T \vec{z}(m-1) + (B^{m,m})^T A^{m,m-1} \vec{w}(m-1)}{\alpha_m \beta_m - \alpha'_m \beta'_m} \quad (23)$$

The Bonacich measures of the connected network with the node set  $V = V^1 \cup \dots \cup V^k$  are given by  $\vec{z}(k) = z(1) \wedge \dots \wedge z(k)$  and  $\vec{w}(k) = w(1) \wedge \dots \wedge w(k)$ .

*Proof.* Apply the proof of Lemma 1.  $\square$

The eigenvalues  $\alpha_m, \beta_m$  and constants  $\alpha'_m, \beta'_m$  are different at different steps of recursion, and therefore these variables have subscripts corresponding to the recursion step.

### 3.2. Bonacich measures as subgame perfect equilibrium paths

Let  $G$  be a directed and connected bipartite network that has an ordered collection  $\mathcal{C} = \{V^1, \dots, V^k\}$  of strongly connected components  $V^m \subset V$ ,  $m = 1, \dots, k$ . We assume that each  $V^m$  contains at least two nodes. Consider the following extensive form game  $\Gamma$  associated with  $G$ .

The player set is  $N = \{1_1, 1_2, \dots, k_1, k_2\}$ . The game has  $k$  stages. At stage  $m$  players  $m_1$  (row player) and  $m_2$  (column player) play a two-person game by choosing simultaneously nonnegative vectors  $z_m$  and  $w_m$ . First players  $1_1$  and  $1_2$  choose their strategies. Players  $m_1$  and  $m_2$  observe all the choices made by players  $i_1, i_2$ , for  $i < m$ , and then they choose their strategies  $z_m$  and  $w_m$  simultaneously.

Vector  $z_m$  ( $w_m$ ) has the same number of coordinates as the matrix  $A^{m,m}$  and  $B^{m,m}$  in equation (21) has rows (columns). Strategies  $z_1$  and  $w_1$  must satisfy the norm bounds  $\|z_1\| \leq 1$  and  $\|w_1\| \leq 1$ . Strategies  $z_m$  and  $w_m$  of

players  $m_1$  and  $m_2$  must satisfy the following norm bounds for  $m > 1$ :

$$\|z_m\| \leq \left\| \frac{\beta A^{m,m-1} \vec{w}_{m-1} + A^{m,m} (B^{m-1,m})^T \vec{z}_{m-1}}{\alpha\beta - \alpha'\beta'} \right\| \quad (24)$$

$$\|w_m\| \leq \left\| \frac{\alpha (B^{m-1,m})^T \vec{z}_{m-1} + (B^{m,m})^T A^{m,m-1} \vec{w}_{m-1}}{\alpha\beta - \alpha'\beta'} \right\| \quad (25)$$

The strategy sets of players at stage  $m$  depend on the choices made at earlier stages in the game. If constraints (24) and (25) cannot be satisfied at some stage then we interpret that the game has ended after the last stage  $t$  such that the strategy sets were still nonempty.

The payoff functions  $u_1^m$  and  $u_2^m$  of players  $m_1$  and  $m_2$  are given by

$$u_1^m(\vec{z}_k, \vec{w}_k) = \vec{z}_m^T A(m) \vec{w}_m, \text{ and } u_2^m(\vec{z}_k, \vec{w}_k) = \vec{z}_m^T B(m) \vec{w}_m, \quad (26)$$

where matrices  $A(m)$  and  $B(m)$  are given by equation (21) for  $m > 1$ , and  $A(1) = A^{1,1}, B(1) = B^{1,1}$  in equation (21).

The Bonacich measures  $z(1) \wedge \dots \wedge z(t+1)$  and  $w(1) \wedge \dots \wedge w(t+1)$  of the connected network with node set  $V^1 \cup \dots \cup V^{t+1}$  is then found by the same way as the above.

**Proposition 6.** *Extensive game  $\Gamma$  has a unique subgame perfect equilibrium path such that the equilibrium strategies of players  $1_1$  and  $1_2$  are nonzero vectors. The equilibrium path is  $((z(1), w(1)), \dots, (z(k), w(k)))$  given by Proposition 5.*

*Proof.* Take any subgame perfect equilibrium such that the equilibrium strategies of players  $1_1$  and  $1_2$  are nonzero vectors. Then in every subgame perfect equilibrium the actions chosen at stage 1 are equal to the Bonacich measures  $z(1)$  and  $w(1)$  by Proposition 1 since the payoffs of players  $1_1$  and  $1_2$  do not depend on actions that will be chosen at stages  $m > 1$ .

At stage  $m > 1$  the payoffs of players do not depend on actions that will be chosen at stages  $t > m$ . Consider the row player  $m_1$ , assume that at stages  $t < m$  actions  $z(t)$  and  $w(t)$  given by Proposition (1). Suppose that

player  $m_1$  expects that player  $m_2$  chooses  $w(m)$ . By equation (26) the payoff from action  $z_m$  to player  $m_1$  is

$$K + z_m^T A^{m,m-1} \vec{w}(m-1) + z_m^T A^{m,m} w(m) = K + \beta z_m^T z(m), \quad (27)$$

where  $\beta > 0$ . In this equation we have replaced the matrix  $A(2)$  in equation (12) by the matrix  $A(m)$  in equation (21). The constant  $K$  is given by  $K = \vec{z}(m-1) A^{m-1,m-1} \vec{w}(m-1)$ . Feasible actions  $z_m$  for player  $m_1$  must satisfy  $\|z_m\| \leq \|z(m)\|$ . Hence the best reply is  $z(m)$ .  $\square$

## References

- Ballester, C., Calvó-Armengol, A. and Zenou, Y. (2006) Who's who in networks: wanted the key player. *Econometrica* **74**: 1403–1417.
- Binmore, K. (1987) Nash bargaining theory I–III. In Binmore, K. and Dasgupta, P. (eds.) "The economics of bargaining". Basil Blackwell, Oxford.
- Bonacich, P. (1972) Factoring and weighting approaches to clique identification. *Journal of Mathematical Sociology* **2**: 113–120.
- Bonacich, P. (1987) Power and centrality: A family of measures. *American Journal of Sociology* **92**: 1170–1182.
- Bonacich, P. (1991) Simultaneous group and individual centralities. *Social Networks* **13**: 155–168.
- Bonacich, P. and Lloyd, p. (2001) Eigenvector-like measures of centrality for asymmetric relations. *Social Networks* **23**: 191–201.
- Brin, S. and Page, L. (1998) The anatomy of a large-scale hypertextual web search engine. *Computer Networks and ISDN Systems* **30**: 107–117.
- Cabrales, A., Calvó-Armengol, A. and Zenou, Y. (2011) Social interactions and spillovers. *Games and Economic Behavior* **72**: 339–360.



- Freeman, L. (1979) Centrality in social networks: Conceptual clarification. *Social Networks* **1**: 215–239.
- Jackson, M.O. and Zenou, Y. (2014) Games on networks. Forthcoming in *Handbook of Game Theory Vol 4*, Young, P. and Zamir, S. (eds.), Elsevier Science, Amsterdam.
- Katz, L. (1953) A new status index derived from sociometric analysis. *Psychometrika* **18**: 39–43.
- Kitti, M. (2012) Axioms for centrality scoring with principal eigenvectors. ACE Discussion Paper 79.
- Kleinberg, J. M. (1999) Authoritative sources in a hyperlinked environment. *Journal of the ACM* **46**: 604–632.
- Nash, J.F. (1950) The bargaining problem. *Econometrica* **18**: 155–162.
- Nash, J.F. (1950) Two-person cooperative games. *Econometrica* **21**: 128–140.
- Rubinstein, A. (1982) Perfect equilibrium in a bargaining model. *Econometrica* **50**: 97–109.
- Salonen, H (2014) Equilibria and centrality in link formation games. ACE Discussion Paper 90.
- Shapley, L.S. (1953) A value for  $n$ -person games. *Annals of Mathematic Studies* **28**: 307–317.

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