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The Dynamics of Working Hours and Wages Under Implicit Contracts*

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Abstract

In this paper, we explore the dynamics of working hours and wages in a model economy where a firm and its workforce are linked to each other by an implicit contract. Specifically, we develop a deterministic and a stochastic framework in which the firm sets its level of labour utilization by considering that workers' earnings tend to adjust in the direction of a fixed level. Without any uncertainty in firm's profitability, we show that the existence and the properties of stationary solutions rely on factors that usually determine the enforceability of contracts and we provide evidence that wages tend to move counter-cyclically towards the allocation preferred by the firm. Moreover, we show that adding uncertainty does not overturn the counter-cyclical pattern of wages but is helpful in explaining their dynamic behaviour in response to demand shocks as well as their typical stickiness observed at the macro level.

JEL Classification: D86; E24; J41.

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1 Introduction

The theory of implicit contracts starts from the premise that the labour market is far from being a spot market but, on the contrary, workers and firms usually display the tendency to be involved in long-lasting and non-anonymous relationships characterized by a strong degree of customization (cf. Okun, 1981). Consequently, if there is some kind of uncertainty about actual production outcomes and entrepreneurs are more risk-prone than workers, then it may happen that the two parties will consensually rely on informal agreements on labour provisions and wage payments that optimally share the burden of realized labour income fluctuations (cf. Baily, 1974; Gordon, 1974; Azariadis, 1975).

The theoretical literature on implicit contracts collects a number of contributions in which labour market outcomes are determined in a time-less perspective (e.g. Geanakoplos and Ito, 1982; Azariadis and Stiglitz, 1983; Bull, 1983, 1987; Chiari, 1983; Baker et al. 1997). In more recent years, however, after the seminal work by Harris and Holmstrom (1982) in which the terms of long-run implicit contracts follow from the inter-temporal maximization of workers' utility subject to the evolution of the expected profits of their employer, a number of authors spent some effort in the exploration of the dynamic consequences for hours, (un)employment and wages arising from the existence of optimal risk-sharing in labour contracts.

Within the literature on dynamic contracting, Haltiwanger and Maccini (1985) develop an inter-temporal framework in which the existence of implicit labour contracts may lead firms to rely on temporary layoffs and subsequent recalls. Robinson (1999) exploits the theory of repeated games to provide a dynamic model of strikes in which walkouts reduce output and are used by employed workers as punishment mechanisms to enforce implicit contracts in a context of asymmetric information. Gürtler (2006) compares repeated games of implicit contracts with infinite and finite horizon by stressing the importance of discounting for the enforcement of the agreements achieved between workers and firms (cf. Pearce and Stacchetti, 1998). Michelacci and Quadrini (2009) as well as Calmès (2007) reverse the framework by Harris and Holmstrom (1982) and develop dynamic implicit-contract models in which firms maximize their profits by taking into account the evolution of the expected utility of their workers (cf. Spear and Srivastava, 1987). More recently, Wang (2015) explores how the wage dynamics implied by long-term employment relationships may be helpful in mitigating the adverse effects of financial shocks.

In this paper, we aim at contributing to the literature on dynamic implicit-contract models by deriving the smooth out-of-equilibrium dynamics of working hours and wages in a theoretical setting where workers and firms are linked to each other by an implicit contract that tends to stabilize real consumption in a long-run perspective (cf. Beaudry and Pages, 2001; Romer, 2019, Chapter 11). Specifically, we develop a theoretical framework with no information asymmetries in which a representative firm inter-temporally sets its optimal level of labour utilization by taking into account that workers' earnings tend to adjust in the direction of a fixed level set out in the contract that is assumed to coincide with desired long-run consumption (cf. Abowd and Card, 1987).

Our theoretical exploration is split up in two parts. First, we explore the disequilibrium adjustments of working hours and wages in a model economy with no uncertainty in firm's profitability. Thereafter, we consider the optimal trajectories of the two mentioned variables by assuming that the effectiveness of labour is hit by random shocks that systematically alter the profitability of the firm. The former preparatory analysis allows us to discuss the conditions under which the suggested contractual agreement between the firm and its workers conveys meaningful solutions. The latter provides the basis to assessing the cyclical properties of a dynamic implicit-contract economy.

Overall, our analysis provides a number of interesting findings. On the one hand, depending on selected parameter values that usually relate to factors determining the self-enforceability of contracts, the deterministic model may have one, two or no stationary solution. Interestingly, whenever there are two steady-state allocations for hours and wages the resting points of the economy without uncertainty can be ordered according to the preferences of each party. Moreover, in the two-solution case, the local dynamics of the model reveals that wages display the tendency to move in the opposite direction with respect to working hours by converging over time towards the allocation preferred by the firm. This result is consistent with the empirical tests of the implicit contract theory carried out in the US at the micro level by Beaudry and DiNardo (1995); indeed, in their pioneering study – controlling for labour productivity – higher wages appear associated with lower hour provision and vice versa. In addition, the deterministic economy has the property that when the initial contract wage overshoots (undershoots) its long-equilibrium value, workers' earnings remain above (below) the contract fixed level during the whole adjustment process.

On the other hand, simulations of the stochastic model run by targeting the observed volatility of US output reveal that disturbances in firm's profitability does not overturn the counter-cyclical pattern of wages by mirroring the typical macroeconomic effects triggered by aggregate demand shocks (cf. Fleischmann, 1999; Chiarini, 1998). Moreover, we show that the insurance scheme underlying the dynamic implicit contract tend to underestimate the volatility of labour earnings but it has the potential to explain the degree of real-wage stickiness often observed in aggregate real business cycle data (cf. Ravn and Simonelli, 2007; Shimer, 2005).

This paper is arranged as follows. Section 2 describes the theoretical setting. Section 3 analyses the deterministic economy. Section 4 analyses the stochastic economy with uncertainty in the effectiveness of labour. Finally, section 5 concludes by discussing the direction for further developments.

2 Theoretical setting

We consider a model economy in which time is continuous and a representative risk-neutral firm deals with a group of risk-averse identical workers that cannot purchase insurance against fluctuations in the level of their long-run labour income. Within this environment, given the different attitude towards risk, we make the hypothesis that the firm and its workers are linked

to each other by an informal wage contract that seeks to stabilize the level of labour earnings. Assuming the absence of non-labour incomes and savings on the side of workers, this means that the informal agreement between the workers and the firm will tend to stabilize real consumption in a long-run perspective (cf. Abowd and Card, 1987).

On the productive side – similarly to Guerrazzi (2011, 2020, 2021) – we assume that the representative firm is endowed with a quadratic production function so that instantaneous output $Y(t)$ is equal to

$$Y(t) = A(t)L(t) - \frac{1}{2}(L(t))^2 \quad (1)$$

where $A(t) > 0$ is a technology variable and/or a measure of the economy-wide output taken as given by the firm and its workers whereas $L(t)$ is the labour provision of the workers attached to the firm measured in hours.

On the supply side, the variable $A(t)$ in eq. (1) affects the marginal productivity of employed workers in a linear manner and it conveys the actual realization of the state of the world. Specifically, the higher (lower) the value of $A(t)$, the higher (lower) the marginal productivity of employed workers and the better (worse) the realized state of the world. On the demand side, higher (lower) values of $A(t)$ mean that the firm can obtain a higher (lower) relative price for each level of production. In the remainder of the paper, we follow both perspectives and we assume that $A(t)$ might move over time according to an Ornstein-Uhlenbeck process (cf. Cox and Miller, 1967). Formally speaking, this means that

$$\dot{A}(t) = \kappa(\mu_A - A(t)) + \sigma_A \dot{x}(t) \quad (2)$$

where $\mu_A > 0$ is the long-run mean of the process, $\kappa > 0$ is its speed of mean reversion, $\sigma_A > 0$ is its instantaneous standard deviation whereas $\dot{x}(t)$ is a standard Brownian motion with zero drift and unit variance.

According to the text-book treatment of the implicit contract theory offered by Romer (2019, Chapter 11), in a time-less contracting model where firm's profitability is stochastic, the fixed level of consumption granted to workers in all the states of the world can be conveyed as a non-linear combination of all the possible realizations of firm's profitability whose weights are affected by workers preferences, the available productive technology and the probability distribution of the already mentioned shocks to firm's profitability. Consequently, under the assumption that agents are rational and the information on these fundamental factors is costlessly available to all the involved parties of the contract, such a critical level of consumption can be taken as exogenously given without any substantial loss of generality.¹

Along these lines, in what follows we will not specify the form of workers' utility function and we will assume that the long-run consumption granted to employed workers who reached an agreement with the firm is fixed at the constant level $C > 0$. Thereafter, the out-of-equilibrium dynamics of the contract wage $w(t)$ aimed at equalizing the wage bill to C in a long-run perspective will be given by

¹A formal proof for this statement is sketched in Appendix.

$$\dot{w}(t) = \theta \left(\frac{C}{L(t)} - w(t) \right) \quad (3)$$

where $\theta > 0$ is a measure of the attrition between the actual and the long-run real wage that stabilizes consumption.

The expression in eq. (3) represents the evolution of contract wages coming from the informal agreement achieved between the firm and its workers and it implies that in each instant $w(t)$ increases (decreases) whenever it is below (above) the level of long-run consumption per working hour. Such a differential equation can be conceived as a reduced form that binds in a dynamic way the choice of the firm regarding labour intensity by summarizing in a compact manner all the relevant terms of the implicit contract. In detail, its formal specification is not affected by the evolution of $A(t)$ in order to capture the idea that the wage contract is not renegotiated when the state of the world changes (cf. Ham and Reilly, 2013). Furthermore, having in mind the way in which workers' preferences usually affect the terms of implicit contracts, the parameter θ in the RHS of eq. (3) can be taken as a measure of the degree of aversion with respect to situations of under- or overconsumption; indeed, for any given level of L , the higher (lower) the value of θ , the faster w adjusts itself in the direction of C .²

The adherence of the firm and its workers to the trajectories generated by the reduced form that enters model to represent the implicit contract and makes the wage a state variable deserves further explanations. Following the game-theoretical arguments put forward by Bull (1987), the wage trajectory implied by eq. (3) should be thought as the outcome of a Nash equilibrium of a post-hiring trading game whose self-enforceability is supported by intra-firm reputation. In other words, given the preferences of the firm and the ones of its workers, the values of the parameters C and θ have to be selected in order to avoid the existence of any incentive to renege on the contract (cf. Thomas and Worrall, 1988; Pearce and Stacchetti, 1998; Michelacci and Quadrini, 2009). In the remainder of the paper, we will show that the factors that usually drive enforceability in inter-temporal implicit contract models in our theoretical context determine the existence and dynamic properties of stationary solutions. Consequently, since renegeing on the contract means that one of the two parties – or both – has the desire to deviate from the achieved agreement, the enforceability of the wage contract described by eq. (3) will be assimilated to the existence of a stable stationary solution for working hours and wages.

3 The deterministic economy

We begin our analysis by considering what happens in a model economy without uncertainty so that the state of the world is revealed to the firm and its workers at the beginning of time and then it is assumed to remain constant thereafter. Specifically, we initially assume that

²As we show in Appendix, assuming that the real wage increases (decreases) when labour earnings are below (above) the long-run level of consumption specified in the contract complicates the analytical treatment of the model without any substantial modification in the conclusions achieved throughout the paper.

$$A(t) = A > 0 \quad \text{for all } t \quad (4)$$

In each instant, given the values of C , θ , and A , the inter-temporal problem of the representative firm in the model economy described above is to set the optimal labour input aiming at maximizing its profits by tacking into account that the real wage adjusts itself over time according to the differential equation in (3). Formally speaking, taking into account the production function in eq. (1) and the simplifying assumption in (4), the problem of the representative firm is the following:

$$V(w_0) = \max_{\{L(t)\}_{t=0}^{\infty}} \int_{t=0}^{\infty} \exp(-\rho t) \left(AL(t) - \frac{1}{2} (L(t))^2 - w(t) L(t) \right) dt \quad (5)$$

s.to

$$\dot{w}(t) = \theta \left(\frac{C}{L(t)} - w(t) \right)$$

$$w(0) = w_0$$

where $V(\cdot)$ is the value function, $\rho > 0$ is the discount rate of entrepreneurs whereas $w_0 > 0$ is the initial level of the real wage rate specified on the implicit contract.

The solution of the problem in (5) defines a trajectory for working hours and a trajectory for wages that lead to the stabilization of labour income in the direction of the level of the long-run consumption established in the implicit contract.

The first-order conditions (FOCs) of the problem in (5) can be written as

$$A - L(t) - w(t) - \theta C \frac{\Lambda(t)}{(L(t))^2} = 0 \quad (6)$$

$$\dot{\Lambda}(t) = (\rho + \theta) \Lambda(t) + L(t) \quad (7)$$

$$\lim_{t \rightarrow \infty} (-\rho t) \Lambda(t) w(t) = 0 \quad (8)$$

where $\Lambda(t)$ is the costate variable associated to $w(t)$.

Eq. (6) is the FOC with respect to the control variable of the firm, that is, $L(t)$. Moreover, the differential equation in (7) describes the optimal path of $\Lambda(t)$, whereas (8) is the required transversality condition.

After a trivial manipulation, the results in eq.s (6) and (7) allow us to obtaining the following differential equation for the out-of-equilibrium dynamics of working hours:

$$\dot{L}(t) = \frac{(\rho + \theta) L(t) (A - L(t) - w(t)) + \theta (2C - w(t) L(t))}{2(A - L(t) - w(t)) - L(t)} \quad (9)$$

Starting from given initial conditions to be defined, the differential equations in (3) and (9) describe how working hours and wages move over time once an everlasting state of the world is revealed to the firm and its workers. Consequently, eq.s. (3) and (9) convey the dynamics of hours and wages for a given level of firm's profitability.

3.1 Steady-state equilibria

Within the model under investigation, steady-state equilibria are defined as the set of pairs $\{L^*, w^*\}$ such that $\dot{L}(t) = \dot{w}(t) = 0$. Obviously, the elements of that set are given by allocations in which the real wage bill equals the fixed level of consumption specified on the implicit contract on which the firm and its workers reached an agreement.

From a formal point of view, the derivation of $\{L^*, w^*\}$ is straightforward. First, setting $\dot{w}(t) = 0$ in eq. (3) leads to

$$w^* = \frac{C}{L^*} \quad (10)$$

Thereafter, setting $\dot{L}(t) = 0$ in eq. (9) and plugging the result into eq. (10) leads to the following quadratic expression:

$$(L^*)^2 - AL^* + \frac{C\rho}{\rho + \theta} = 0 \quad (11)$$

The parabola in eq. (11) allows us to state the following three plain propositions:

Proposition 1: When $A = 2\sqrt{C\rho/(\rho + \theta)}$, there is only one stationary solution given by $L_0^* \equiv A/2$ and $w_0^* \equiv 2C/A$.

Proposition 2: When $A > 2\sqrt{C\rho/(\rho + \theta)}$, there are two distinct stationary solutions given by $L_1^* \equiv 1/2 \left(A - \sqrt{A^2 - 4C\rho/(\rho + \theta)} \right)$ and $w_1^* \equiv 2C / \left(A - \sqrt{A^2 - 4C\rho/(\rho + \theta)} \right)$ as well as $L_2^* \equiv 1/2 \left(A + \sqrt{A^2 - 4C\rho/(\rho + \theta)} \right)$ and $w_2^* \equiv 2C / \left(A + \sqrt{A^2 - 4C\rho/(\rho + \theta)} \right)$.

Proposition 3: When $A < 2\sqrt{C\rho/(\rho + \theta)}$, there are no (real) stationary solutions.

Proposition 1 provides the parameters' combination under which there is a unique steady state (L_0^*, w_0^*) . In that allocation, equilibrium hours are an increasing function of the parameter that conveys the actual state of the world, whereas the equilibrium wage increases (decreases) with the fixed level of consumption granted by the implicit contract (the realized state of the world) virtually signed by the firm and its employees.³ This pattern clearly points out the insurance component of the implicit contract; indeed, workers tend to work more (less) for less (more) in good (bad) states (cf. Romer, 2019, Chapter 11).

By contrast, Proposition 2 reveals the condition under which – similarly to what happens in the dynamic-search model with multiple equilibria developed by Diamond (1982) – there are two different steady states, that is, (L_1^*, w_1^*) and (L_2^*, w_2^*) .⁴ Assuming separability between leisure and consumption in the utility function of workers, the two stationary solutions pointed out in Proposition 2 can be unambiguously ordered according to the preferences of the two parties

³It is worth noticing that the unique stationary solution falls in the concave part of the production function in eq. (1).

⁴Obviously, for (L_1^*, w_1^*) to be feasible it must hold that $A > \sqrt{A^2 - 4C\rho/(\rho + \theta)}$. In the remainder of the paper, we will assume that when the condition pointed out by Proposition 2 is met such an inequality is always fulfilled.

involved in the contract. Specifically, since the implied level of consumption – or the implied labour earnings – is the same in both allocations, (L_1^*, w_1^*) , that is, the stationary solution with low equilibrium hours and high equilibrium wage, is the most preferred by workers because it implies more leisure, whereas (L_2^*, w_2^*) , that is, the stationary solution with high equilibrium hours and low equilibrium wage, is most the preferred by the firm because – everything else being equal – it implies higher profits.

Furthermore, Proposition 3 shows the condition under which a steady state does not exist. For a given value of the state of the world conveyed by A , the impossibility to retrieving a stationary solution for the dynamics of working hours and wages appears alternatively related to an excessive degree of impatience on the side of the firms mirrored in the value taken by ρ , to an excessive fixed level of long-run consumption granted to workers embodied in the actual level of C and/or to a mild rate of mean reversion of contract wages conveyed by the value of θ . Overall, this proposition suggests that in our dynamic implicit-contract model the existence of a stationary solution requires appropriate levels of firm's profitability and workers' risk-aversion combined with not exorbitant discount rates and sober long-run levels of insured labour earnings.

The requirements for the existence of a steady state summarized by Proposition 3 mirror the usual combination of factors that according to the literature reviewed in the introduction should determine the existence and the enforceability of implicit contracts. In detail, a certain degree of risk aversion is the main reason why a group of workers decide to engage in a long-run relationship with a risk-neutral firm (cf. Baily, 1974; Gordon, 1974; Azariadis, 1975). Moreover, the result on discounting recalls the one achieved by Gürtler (2006) in a repeated-game setting where higher values of the discount rate yield a decrease in the future value of firm's profits. Consequently, it becomes less worthwhile for the firm to honour the implicit agreement achieved with its workers since the punishment for renegeing on the contract decreases and in that case the firm may find profitable to withdraw from the agreement (cf. Pearce and Stacchetti, 1998). Furthermore, similar arguments hold for the measure of firm's profitability; indeed, a reduction of output can make it difficult for the firm to respect the terms of the wage contract (cf. Harris and Holmstrom, 1982).

3.2 Local dynamics

Given the stationary solution $\{L^*, w^*\}$, the local dynamics of working hours and wages is described by the following 2×2 linear system:

$$\begin{pmatrix} \dot{L}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{bmatrix} j_{1,1} & j_{1,2} \\ -\frac{\theta C}{(L^*)^2} & -\theta \end{bmatrix} \begin{pmatrix} L(t) - L^* \\ w(t) - w^* \end{pmatrix} \quad (12)$$

where $j_{1,1} \equiv \left. \frac{\partial \dot{L}(t)}{\partial L(t)} \right|_{L(t)=L^*, w(t)=w^*}$ and $j_{1,2} \equiv \left. \frac{\partial \dot{L}(t)}{\partial w(t)} \right|_{L(t)=L^*, w(t)=w^*}$.

In general, the two unspecified elements on the first row of the Jacobian matrix in (12) can be written as

$$j_{1,1} = \frac{((\rho + \theta) \Phi(L^*) - \frac{\theta C}{L^*}) (2\Gamma(L^*) - L^*) + 3((\rho + \theta) (AL^* - (L^*)^2 - C) + \theta C)}{(2\Gamma(L^*) - L^*)^2} \quad (13)$$

$$j_{1,2} = \frac{2((\rho + \theta) (AL^* - (L^*)^2 - C) + \theta C) - (\rho + 2\theta) L^* (2\Gamma(L^*) - L^*)}{(2\Gamma(L^*) - L^*)^2} \quad (14)$$

where $\Phi(L^*) \equiv (AL^* - 2(L^*)^2 - C) / L^*$ and $\Gamma(L^*) \equiv (AL^* - (L^*)^2 - C) / L^*$.

Under the condition pointed out in Proposition 1, that is, when there is only one stationary solution given by (L_0^*, w_0^*) , the Jacobian matrix of the system in (12) merely reduces to

$$\begin{bmatrix} \rho + \theta & \rho \\ -\frac{\theta(\theta + \rho)}{\rho} & -\theta \end{bmatrix} \quad (15)$$

The trace of the matrix in (15) is equal to ρ whereas its determinant is equal to zero. This means that one eigenvalue of the system is zero whereas the other is equal to ρ . Consequently, when the parameters of the deterministic model deliver a unique stationary solution, the out-of-equilibrium dynamics of working hours and wages cannot be assessed; indeed, this characterization represents a degenerate case in which convergence towards the steady state (L_0^*, w_0^*) is possible only if time flows in reverse (cf. Lesovik et al. 2019). From an economic point of view, this result can be rationalized by arguing that when the condition indicated by Proposition 1 is met the agreement achieved between the firm and the workers – described by the problem in (5) – is not self-enforcing. In fact, when there is only one resting point in the system of eq.s (3) and (9), the insurance mechanism provided by the implicit contract becomes pointless. In a forward-looking environment, despite the constancy of labour effectiveness, the actual implementation of the agreement between the firms and its workers requires at least the existence of multiple equilibria. Therefore, when the condition for the uniqueness of the stationary equilibrium is met, the solution of the firm problem is not able to pin down a meaningful out-of-equilibrium dynamics for contract hours and wages.

Under the condition pointed out by Proposition 2, that is, when there are two distinct stationary solutions given by (L_1^*, w_1^*) and (L_2^*, w_2^*) , analytical results are difficult to be derived. Fixing the value of ρ and relying on a computational software, however, it becomes possible to assess – for different values of A , θ and C – the magnitude of the eigenvalues associated to the Jacobian matrix in (12) – say r_1 and r_2 – for each implied stationary solution.⁵ Specifically, setting the value of the discount rate as in Alvarez and Shimer (2011), some sets of numerical solutions are collected in Tables 1, 2 and 3.

⁵All the MATLAB codes used throughout the paper are available from the authors upon request.

A	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
1.3	0.351	2.845	$0.025 + 0.039i$	$0.025 - 0.039i$	0.948	1.054	0.091	-0.041
1.4	0.304	3.287	$0.025 + 0.043i$	$0.025 - 0.043i$	1.095	0.912	0.100	-0.050
1.5	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
1.6	0.246	4.061	$0.025 + 0.048i$	$0.025 - 0.048i$	1.353	0.738	0.112	-0.062
1.7	0.226	4.421	$0.025 + 0.049i$	$0.025 - 0.049i$	1.473	0.678	0.117	-0.067

Table 1: Numerical solutions for different values of A ($\rho = 0.05$, $\theta = 0.10$, $C = 1$)

θ	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
0.08	0.328	3.046	$0.025 + 0.039i$	$0.025 - 0.039i$	1.171	0.853	0.092	-0.042
0.09	0.296	3.368	$0.025 + 0.042i$	$0.025 - 0.042i$	1.203	0.831	0.100	-0.050
0.10	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
0.11	0.250	4	$0.025 + 0.049i$	$0.025 - 0.049i$	1.250	0.800	0.115	-0.065
0.12	0.231	4.311	$0.025 + 0.052i$	$0.025 - 0.052i$	1.268	0.788	0.122	-0.072

Table 2: Numerical solutions for different values of θ ($\rho = 0.05$, $A = 1.5$, $C = 1$)

C	L_1^*	w_1^*	$r_1(L_1^*, w_1^*)$	$r_2(L_1^*, w_1^*)$	L_2^*	w_2^*	$r_1(L_2^*, w_2^*)$	$r_2(L_2^*, w_2^*)$
0.8	0.206	3.881	$0.025 + 0.049i$	$0.025 - 0.049i$	1.293	0.618	0.116	-0.066
0.9	0.237	3.787	$0.025 + 0.047i$	$0.025 - 0.047i$	1.262	0.713	0.112	-0.062
1.0	0.271	3.686	$0.025 + 0.046i$	$0.025 - 0.046i$	1.228	0.813	0.107	-0.057
1.1	0.307	3.577	$0.025 + 0.044i$	$0.025 - 0.044i$	1.192	0.922	0.103	-0.053
1.2	0.346	3.459	$0.025 + 0.042i$	$0.025 - 0.042i$	1.153	1.040	0.098	-0.048

Table 3: Numerical solutions for different values of C ($\rho = 0.05$, $A = 1.5$, $\theta = 0.10$)

The numerical results in Tables 1, 2 and 3 can be summarized in the following proposition:

Proposition 4: When $A > 2\sqrt{C\rho/(\rho + \theta)}$, the stationary solution (L_1^*, w_1^*) defined in Proposition 2 is an unstable source with complex dynamics whereas (L_2^*, w_2^*) is a saddle point.

Proposition 4 reveals that when the condition for multiple stationary solution is met the steady state with low equilibrium hours and high equilibrium wage is unstable whereas the steady state with high equilibrium hours and low equilibrium wage is characterized by saddle-path dynamics. This means that given an initial value for the contract wage – say $w(0) = \bar{w}_0 > 0$ – there is only one trajectory that satisfies the dynamic system in (12) which converges to (L_2^*, w_2^*) while all the others diverge. In other words, the equilibrium path towards the steady-state with high equilibrium hours and low equilibrium wage is locally determinate, that is, taking the contract value of \bar{w}_0 there is only a unique value of the initial hours – $L(0)$ – in

the neighbourhood of L_2^* that generates a trajectory converging to (L_2^*, w_2^*) whereas all the others diverge. Strictly speaking, the value of $L(0)$ should be selected in order to verify the transversality condition in (8) by placing the system in (12) exactly on the stable branch of the saddle point (L_2^*, w_2^*) . For the arguments put forward above, the fact that there is a unique optimal converging trajectory means that the dynamic wage contract described by eq. (3) is self-enforceable; indeed, all the diverging trajectories imply lower profits for the firm and do not allow workers to achieve the insured level of consumption.

An interesting implication of Proposition 4 is that – unless the system rests in (L_1^*, w_1^*) – working hours and wages tend to converge towards (L_2^*, w_2^*) , that is, the allocation that leads to higher profits with respect to (L_1^*, w_1^*) . To some extent, the difference in the levels of profits achieved in these two allocations, that equals $A(L_2^* - L_1^*) - 1/2((L_2^*)^2 - (L_1^*)^2)$, can be taken as a proxy of the equilibrium reward that the firm receives for its insurance service.⁶ Moreover, everything else being equal, the absolute value of the convergent root (r_2) is an increasing function (decreasing) of A and θ (C). Obviously, this means that high levels of firm’s profitability as well as a strong risk-aversion for under- or overconsumption imply a quick convergence towards (L_2^*, w_2^*) . By contrast, high values of the constant level of consumption granted by the implicit contract delay the process of convergence.⁷

Using the baseline calibration indicated in the fourth row of Tables 1, 2 and 3 and assuming that $w(0)$ is 1% below or above w_2^* , the saddle path dynamics of hours, wage and their product – which is assumed to coincide with workers’ consumption stated by the implicit contract – is illustrated in the two panels of Figure 1.

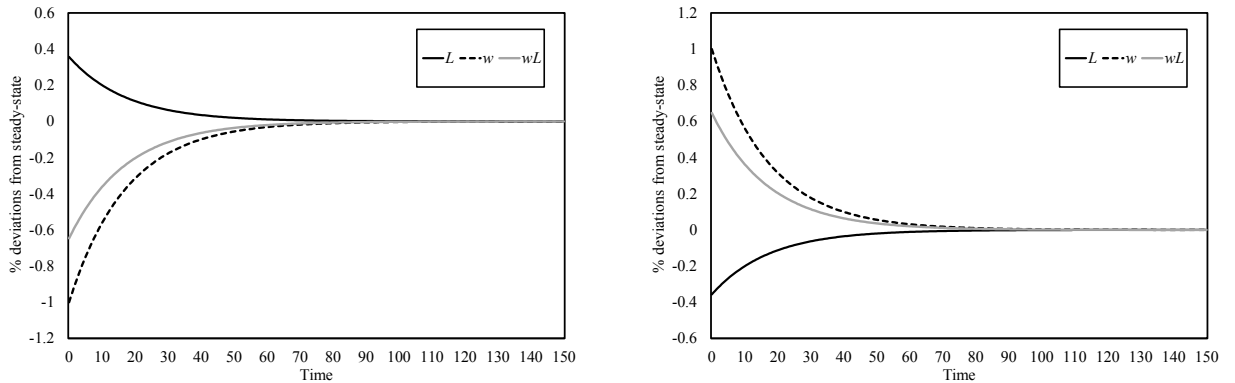


Figure 1: Saddle path adjustments of hours, wages and earnings ($A = 1.5$, $\rho = 0.05$, $\theta = 0.10$, $C = 1$)

The two plots in Figure 1 show that when the starting level of the wage undershoots (overshoots) its stationary reference by 1%, hours overshoot (undershoot) their long-run equilibrium value by 0.35%, whereas earnings undershoot (overshoot) their fixed contractual value

⁶In a similar manner, if $U(C) - V(L)$ is the separable utility function of workers, then the equilibrium cost of the insurance service measured in utils amounts to $V(L_2^*) - V(L_1^*)$.

⁷In addition, it would be possible to show that firm’s impatience works against convergence, indeed, the modulus of r_2 results in being a decreasing function of the value of ρ .

by 0.65%. Thereafter, consistently with the micro-econometric tests of the implicit contract theory, wages move counter-cyclically until (L_2^*, w_2^*) is reached. Moreover, given the absence of savings, the whole adjustment process of hours and wages is characterized by a pattern of under- or overconsumption depending on the initial value of the contract wage.

The dynamic behaviour of hours and wages described above follows in a straightforward manner from the role played by the wage rate in the model economy under investigation. Indeed, taking into account the insurance scheme provided to workers by the self-enforcing implicit contract, the wage does not play any allocative function but it can be thought as a sort of indemnity that the firm corresponds to its workers with the aim of stabilizing their consumption (cf. Barro, 1977; Hall, 1980). On the side of the firm, large (small) indemnities are profitable only when its profitability is high (low) and this happens when the amount of working hours of its employees is low (high). On the side of workers, given the targeted stability of consumption, higher (lower) indemnities will be used to buy additional (sell some) leisure – which is assumed to be a normal good – by leading the insured employees to work for a lower (higher) amount of hours. In other words, consistently with wage equations run in the US at the micro level by controlling for labour productivity and other observable job characteristics, higher (lower) wages have only a positive (negative) income – or endowment – effect that leads workers to work less (more) (cf. Beaudry and DiNardo, 1995).

Finally, very different arguments hold when the condition indicated by Proposition 3 is met, that is, when the stationarity loci for hours and wages do not intersect to each other. In this case, the dynamics of $L(t)$ and $w(t)$ are still described – respectively – by eq.s (9) and (3). However, a stationary solution does not exist and hours (wages) tend to implode (explode). Obviously, this pattern cannot be optimal since it violates the transversality condition in (8).

4 The stochastic economy

Now we deal with the more realistic case in which the variable that conveys the realized state of the world and the firm's profitability is not constant but it follows instead the stochastic process collected in eq. (2). In this case, the firm problem becomes the following:

$$\begin{aligned}
 V(A_0, w_0) = \max_{\{L(t)\}_{t=0}^{\infty}} E_0 \left[\int_0^{\infty} \exp(-\rho t) \left(A(t) L(t) - \frac{1}{2} (L(t))^2 - w(t) L(t) \right) dt \right] \\
 \text{s.to} \\
 \dot{w}(t) = \theta \left(\frac{C}{L(t)} - w(t) \right) \\
 \dot{A}(t) = \kappa (\mu_A - A(t)) + \sigma_A \dot{x}(t) \\
 w(0) = w_0, A(0) = A_0
 \end{aligned} \tag{16}$$

where $E[\cdot]$ is the expectation operator whereas $A_0 > 0$ is the initial value of the state of the world.

Denoting by Q and S , respectively, the set in which are defined all the eligible functions for the control variable L and the set in which are defined all the eligible functions for the state variables A and w , the Hamilton-Jacobi-Bellman (HJB) equation for the firm problem can be written as

$$\begin{aligned} \rho V(A_0, w_0) = \max_{L \in Q} \{ & AL - \frac{1}{2}L^2 - wL \\ & + \theta \left(\frac{C}{L} - w \right) \frac{\partial V(A_0, w_0)}{\partial w} + \kappa (\mu_A - A) \frac{\partial V(A_0, w_0)}{\partial A} + \frac{1}{2} \sigma_A^2 \frac{\partial^2 V(A_0, w_0)}{\partial A^2} \} \end{aligned} \quad (17)$$

where $Q \subseteq \mathbb{R}_+$ whereas $(A, w) \in S \subseteq \mathbb{R}_+^2$.

Obviously, $AL - \frac{1}{2}L^2 - wL$ will be a function defined in $S \times Q$ which returns non-negative values.

The FOC for L requires that along an optimal path it must hold that

$$\frac{\partial V(A_0, w_0)}{\partial w} = \frac{L^2(A - L - w)}{C\theta} \quad (18)$$

It is worth noting that the expression for $\partial V/\partial w$ in eq. (18) is equal to the expression for Λ implied by eq. (6). Moreover, the envelope condition for w is given by

$$\begin{aligned} (\rho + \theta) \frac{\partial V(A_0, w_0)}{\partial w} = \theta \left(\frac{C}{L} - w \right) \frac{\partial^2 V(A_0, w_0)}{\partial w^2} \\ + \kappa (\mu_A - A) \frac{\partial^2 V(A_0, w_0)}{\partial A \partial w} + \frac{1}{2} \sigma_A^2 \frac{\partial^3 V(A_0, w_0)}{\partial A^2 \partial w} + \sigma_A^2 \frac{\partial^2 V(A_0, w_0)}{\partial A^2} \end{aligned} \quad (19)$$

Despite the simplicity of the stochastic process used to describe the evolution of firm's profitability, analytical results for the dynamics of working hours and wages may be difficult to derive. Nevertheless, after a careful calibration, the solution of the stochastic model can be retrieved by using numerical techniques aimed at approximating the value function over a given grid (cf. Kushner and Dupuis, 1992). In what follows, we provide the values of all the model's parameters as well as the output of some simulations grounded on a Markov decision chain approximation.⁸

4.1 Calibration

The stochastic model is simulated in order to match the volatility of the log-deviations of US GDP from its long-run level as reported by Ravn and Simonelli (2007). In other words, we calibrate the model with the aim of replicating the volatility of the observed output fluctuations. To this end, the baseline calibration indicated in the fourth row of Tables 1, 2 and 3 is integrated by calibrating the stochastic process in eq. (2) in the following manner. First, the long-run mean of the stochastic process that conveys the profitability of the firms (μ_A) is set at the same value exploited for the deterministic simulations whose outcome is illustrated in Figure 1. Second, in order to avoid exogenous modifications of the convergence properties of the model, the speed of mean reversion of the stochastic profitability of the firm (κ) is fixed at the value of the convergent root of implied by the already mentioned baseline calibration. Moreover, the

⁸An extensive review of the implemented computational tool is given in Appendix.

volatility of the profitability of the firm (σ_A) is tuned in order to achieve the targeted value of the standard deviation of output.⁹ The whole set of parameters, their description and the respective values are collected in Table 4.

PARAMETER	DESCRIPTION	VALUE
C	<i>Long-run consumption</i>	1.000000
ρ	<i>Discount rate</i>	0.050000
θ	<i>Attrition of the contract wage</i>	0.100000
μ_A	<i>Long-run profitability</i>	1.500000
κ	<i>Attrition of profitability</i>	0.057000
σ_A	<i>Standard deviation of profitability</i>	0.004225

Table 4: Calibration

4.2 Simulation results

Given the parameters' value in Table 4, the theoretical values implied by the model economy are obtained by replicating the typical steps followed in business cycles contributions (cf. Shimer, 2005). Specifically, we first generate 1,200 theoretical observations. Throwing away the first 1,000 in order to mitigate the possible butterfly effect, we remain with 200 observations that represent the corresponding quarterly figures of the typical 50-year horizon covered by business cycle analyses. For each variable of interest, we take the standard deviation and the correlation matrix of the log deviations from the corresponding deterministic long-run reference. Thereafter, such a procedure is repeated for 10,000 times and theoretical values are obtained by averaging the outcomes of each replication. Defining \bar{z} as $\ln z - \ln z^*$, where z^* is the stable stationary solution for the variable z , the simulation results for a set of selected variables are collected in Table 5 (observed values in parenthesis).

VARIABLE		\bar{Y}	\overline{wL}	\bar{L}	\bar{w}
STANDARD DEVIATION (%)		1.56 (1.56)	0.57 (1.01)	0.92 (0.51)	0.69 (0.86)
CORRELATION MATRIX	\bar{Y}	1	0.63	0.94 (0.67)	-0.71 (0.18)
	\overline{wL}	-	1	0.66	-0.04
	\bar{L}	-	-	1	-0.76 (0.01)
	\bar{w}	-	-	-	1

Table 5: Simulation results

⁹The calibration is completed by fixing $w_0 = 0.81$, $A_0 = 1.51$ and setting the time-step of simulation to 0.004.

The figures in Table 5 suggest the following broad conclusions. First, the stochastic model understates the volatility of labour earnings and wages but it overstates the one of working hours. According to simulated figures, earnings should be the variable with the smaller volatility while in real data the lowest dispersion around the mean is observed instead for the hours. Interpreting earnings as a measure of consumption, the figure of volatility is still understated though to a lower extent; indeed, the observed standard deviation of consumption amounts to 0.86% which is definitely higher than 0.57%. An explanation for this pattern is that our theoretical framework does not account for the consumption of unemployed workers which is usually more volatile than the consumption of the employed ones (cf. Pissarides, 2004).

Second, as opposed to what is shown by the deterministic model, the stochastic model displays a sound degree of real-wage stickiness; indeed, the standard deviation of simulated wages is more than double with respect to the one of output (cf. Shimer, 2005). In comparison with actual data, however, our theoretical model tends to exacerbate the cyclical correlation of working hours with respect to output. Moreover, the stochastic model replicates in a strong manner the counter-cyclicality of wages that also characterizes the saddle-path trajectories of the deterministic model. Obviously, this means that the insurance scheme implied the dynamics of hours and wages is unable to explain the mild pro-cyclicality of wages observed at the macro level (cf. Harris and Holmstrom, 1982; Calmès, 2007). Considering A as a measure of the economy-wide output, however, this finding can be reconciled with a more refined empirical evidence that shows the counter-cyclical pattern of US wages in response to demand shocks. Indeed, Fleischman (1999) estimates that the correlation of real wages and output in response to an aggregate demand shock amounts to -0.49 . An example of a typical trajectory of hours, wages and labour earnings is illustrated in Figure 2.

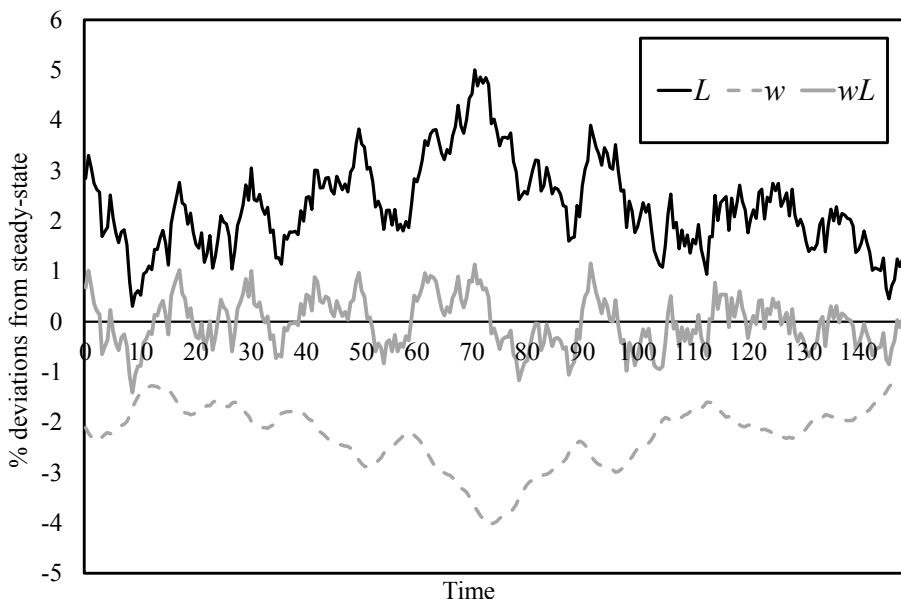


Figure 2: Stochastic adjustments of hours, wages and earnings

The plot in Figure 2 clearly shows the distinct consumption smoothing operated by the dynamic implicit contract via the dynamics of labour earnings as well as the counter-cyclical behaviour of wages; indeed, working hours (wages) are always above (below) their stable long-run references. Such a pattern reveals the existence of a strong amplification mechanism of the shocks to firm's profitability inside the stochastic model coming from the rigidity of wages. Although the negative correlation between hours and wage appear at odds with the available macro evidence, that kind of dynamic behaviour is a direct consequence of the insurance scheme described above and is also consistent with the empirical tests on the implicit contract theory carried out with micro data on hours and wages as well as with macro-econometric assessments of wage cyclicality in response to demand shocks performed even outside the US (cf. Bellou and Kaymak, 2012; Chiarini, 1998).

5 Concluding remarks

In this paper, we developed a dynamic implicit-contract model grounded on optimal control. Specifically, we explored the out-of-equilibrium dynamics of working hours and wages in a model economy where a risk-neutral representative firm and its risk-averse workers are linked to each other by an implicit contract that is assumed to smooth labour earnings and consumption in a long-run perspective. In detail, we build a theoretical framework in which the firm intertemporally sets its optimal level of labour utilization by taking into account that the implied wage bill tends to adjusting in the direction of a fixed level that seeks to stabilizing workers' equilibrium consumption (cf. Beaudry and Pages, 2001; Romer, 2019, Chapter 11).

On the one hand, ignoring uncertainty in firm's profitability revealed that our theoretical setting may have one, two or no stationary solution depending on factors traditionally related the enforceability of contracts. The out-of-equilibrium dynamics of the deterministic economy, however, can be assessed only in the two-solution case and it reveals that wages tend to moving in the opposite direction with respect to working hours by converging towards the allocation in which firm's profit is relatively higher than the corresponding workers' utility. This result corroborated the micro-econometric evidence on the implicit contract theory obtained by regressing wage on hours by controlling for productivity (cf. Beaudry and DiNardo, 1995). Moreover, we showed that when the initial value of the contract wage falls above (below) its long-run equilibrium value, the pattern of workers' consumption is characterized by overconsumption (underconsumption).

On the other hand, adding uncertainty in firm's profitability with the aim of replicating the magnitude of observed output fluctuations revealed the potential of the model to mimicking the real wage stickiness conveyed by macro data (cf. Ravn and Simonelli, 2017; Shimer, 2005). The insurance mechanism provided by our dynamic implicit contract, however, understates the volatility of labour earnings and confirms the counter-cyclicity of wages observed in micro data and in the macroeconomic response to aggregate demand shocks detected in a number of developed countries (cf. Bellou and Kaymak, 2012; Chiarini, 1998).

In the absence of any substitution effect on workers' labour provision, the failure of our model to predicting a pro-cyclical pattern of wages in response to supply shocks is easily attributable also to the lack of adjustments on the extensive margin of the labour input. If positive shocks to the effectiveness of labour lead the firm to hire additional workers and the path of contract wages is given, then the marginal productivity of working hours does not necessarily moves in the same direction of the effectiveness of labour because not only its vertical intercept, but also its slope will be affected by the level of employment. Obviously, this may open the door to a positive co-movement of hours, employment and wages as observed in real macro data where there is no distinction between supply and demand shocks. Furthermore, there might be too much symmetry in our model economy. For instance, it is quite likely that the firm would be willing to lower the wages when earnings are above long-run consumption, but it may be much more reluctant to raise them when it holds the opposite. This kind of asymmetric behaviour may have important cyclical consequences both on hours and wages. The implied extensions of the model, however, are left to further developments.

Appendix: The fixed level of consumption

Consider the production function in eq. (1). For sake of simplicity, suppose that there are only two states of the world and the economy is static. This means that the parameter A can take only two values so that it is equal to A_1 (A_2) with probability p ($1 - p$). Moreover, assume that workers-consumers are endowed with a utility function separable in consumption and leisure where both components – in analogy with the production possibilities of the firm – are given by distinct quadratic expressions. Formally speaking, this amounts to positing that the utility function of workers can be written as

$$U(C_i, L_i) = ZC_i - \frac{1}{2}C_i^2 - VL_i + \frac{1}{2}L_i^2 \quad Z > 0, V > 0, i = 1, 2 \quad (\text{A1})$$

where C_i is consumption, L_i is labour provision measured in hours whereas Z and V are positive parameters.

In a time-less economy described by the production function in eq. (1) and the preferences in (A1), the optimal implicit contract that smooths workers' consumption in all the possible states of the world is found through the solution of the following problem:

$$\max_{\{C_i, L_i\}_{i=1,2}} p \left(A_1 L_1 - \frac{1}{2} L_1^2 - C_1 \right) + (1 - p) \left(A_2 L_2 - \frac{1}{2} L_2^2 - C_2 \right) \quad (\text{A2})$$

s.to

$$p \left(ZC_1 - \frac{1}{2} C_1^2 - VL_1 + \frac{1}{2} L_1^2 \right) + (1 - p) \left(ZC_2 - \frac{1}{2} C_2^2 - VL_2 + \frac{1}{2} L_2^2 \right) \geq u_0 \quad (\text{A3})$$

where u_0 is a fixed level of the fallback utility level of the representative worker (cf. Romer, 2019, Chapter 11).

The FOCs of the problem in (A2) – (A3) are given by

$$-1 + \lambda(Z - C_i) = 0 \quad i = 1, 2 \quad (\text{A4})$$

$$A_i - L_i - \lambda(V - L_i) = 0 \quad i = 1, 2 \quad (\text{A5})$$

where λ is the Lagrange multiplier associated to the participation constraint in (A3).

The FOCs in (A4) implies that consumption is constant in all the states of the world so that

$$C_i = C^{**} > 0 \quad i = 1, 2 \quad (\text{A6})$$

Plugging the FOCs in (A4) into the FOCs in (A5) by taking into account the result in (A6) leads to

$$L_i^{**} = \frac{A_i(Z - C^{**}) - V}{Z - C^{**} - 1} \quad i = 1, 2 \quad (\text{A7})$$

The expression in (A7) reveals that – in each state of the world – contract hours are found by equalizing the marginal productivity of labour to the marginal rate of substitution between consumption and leisure pinned down by the fixed level of C . In other words, under the optimal contract the equilibrium provision of hours is determined by the intersection of the conventional labour demand schedule and a constrained labour supply in which the possibilities of substitution between consumption and leisure are bound by the fact that in each state of the world workers have to consume exactly an amount of goods equal to C^{**} .

Plugging the results in (A6) and (A7) into (A3) reveals that C^{**} has to be consistent with the following equation:

$$ZC^{**} + \frac{(L_2^{**})^2 - (C^{**})^2}{2} - VL_2^* + p \left(\frac{V(Z - C^{**})(A_2 - A_1)}{Z - C^{**} - 1} + \frac{(L_1^{**})^2 - (L_2^{**})^2}{2} \right) = u_0 \quad (\text{A8})$$

For reasonable values of A_1 , A_2 , Z , V , and p , the expression on left-hand-side (LHS) of (A8) is monotonically decreasing in C . Consequently, as shown in figure A1, given a positive value of u_0 there exists an unique meaningful value of C^{**} . This corroborates the hypothesis that the value of long-run consumption established in the implicit contract can be taken as given.

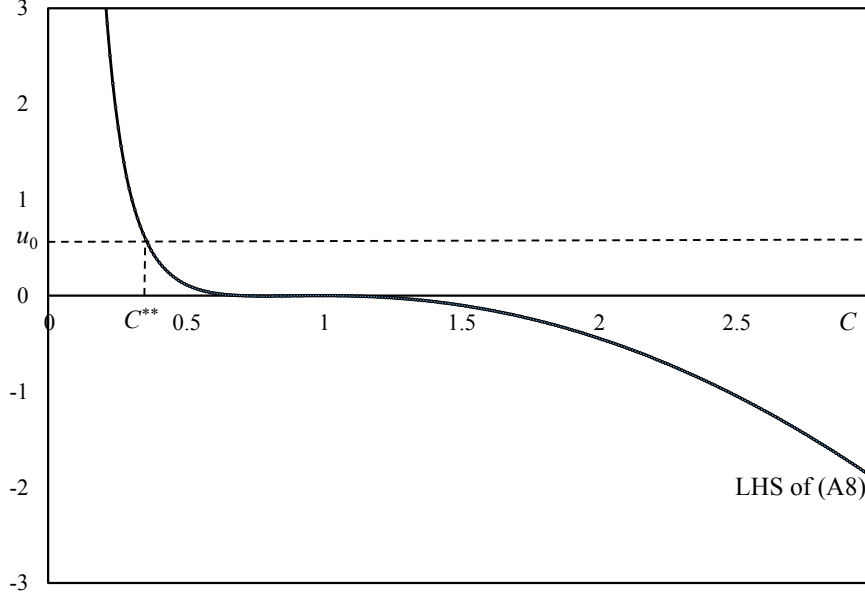


Figure A1: The determination of C^{**} ($A_1 = 1$, $A_2 = 2$, $Z = V = 1$, $p = u_0 = 1/2$)

Appendix: An alternative for the dynamics of contract wages

A sensible alternative for the real wage dynamics fixed by the contract is the one according to which the real wage increases (decreases) when the real wage bill is below (above) the long-run level of consumption. Formally, speaking a sensible alternative for the differential equation in (3) is given by

$$\dot{w}(t) = \theta (C - w(t) L(t)) \quad (\text{B1})$$

In this case, the solution of the firm problem in the deterministic economy leads to following employment dynamics:

$$\dot{L}(t) = \frac{\theta (A - L(t)) (w(t) L(t) - C) - w(t) (A\rho + L(t) (A\theta - \rho) - \theta (L(t))^2 - \rho w(t))}{w(t)} \quad (\text{B2})$$

The differential equations in (B1) and (B2) imply that steady-state level of hours, that is L^* , is consistent with a cubic continuous expression defined as

$$\Psi(L^*) \equiv \theta (L^*)^3 - (L^*)^2 (A\theta - \rho) - A\rho L^* + \rho C = 0 \quad (\text{B3})$$

Straightforward differentiation reveals that the function $\Psi(\cdot)$ has two critical points given by

$$\bar{L} \equiv \frac{A\theta - \rho - \sqrt{A\theta(A\theta + \rho) + \rho^2}}{3\theta} \text{ and } \bar{\bar{L}} \equiv \frac{A\theta - \rho + \sqrt{A\theta(A\theta + \rho) + \rho^2}}{3\theta} \quad (\text{B4})$$

Since $\Psi(0) = \rho C$ and $\lim_{L^* \rightarrow +\infty(-\infty)} \Psi(L^*) = +\infty(-\infty)$, \bar{L} ($\bar{\bar{L}}$) is a maximum (minimum) for $\Psi(\cdot)$. Consequently, it becomes possible to stating the following propositions:

Proposition B1: When $\Psi(\bar{L}) = 0$, $(\bar{L}, C/\bar{L})$ is the only stationary solution to the system of differential equations given by (B1) and (B2).

Proposition B2: When $\Psi(\bar{L}) < 0$, there are two distinct stationary solutions to the system of differential equations given by (B1) and (B2), namely $(L_1^*, C/L_1^*)$ and $(L_2^*, C/L_2^*)$ such that $0 < L_1^* < \bar{L}$ and $L_2^* > \bar{L}$.

Proposition B3: When $\Psi(\bar{L}) > 0$, there are no (real) stationary solutions to the system of differential equations given by (B1) and (B2).

Propositions B1-B3 are qualitatively similar to Propositions 1 – 3. Moreover, using a computational software is possible to show that the set of stationary solutions catalogued by Propositions B1 and B2 have the same dynamic properties of the stationary solutions analysed in the main text. Specifically, for $(\bar{L}, C/\bar{L})$ is not possible to retrieving local dynamics, $(L_1^*, C/L_1^*)$ is an unstable source whereas $(L_2^*, C/L_2^*)$ is a saddle point. Further details are available from the authors upon request.

Appendix: Simulating the stochastic model with a Markov decision chain approximation

Here we examine a mathematical tool that allows to solve numerically the stochastic optimal control problem outlined in Section 4. The approach implemented by the tool is described by Krawczyk and Windsor (1997) and here we provide the principal ideas underlying its solution method.

The first step is the discretization of the state-equation system using the Euler-Maruyama approximation scheme (cf. Kloeden and Platen, 1992). Consider the following general continuous-time form:

$$d\mathbf{X}(t) = f(\mathbf{X}(t), \mathbf{u}(t), t) dt + b(\mathbf{X}(t), \mathbf{u}(t), t) d\mathbf{W}(t) \quad (\text{C4})$$

where \mathbf{X} is the vector of state variables, $\mathbf{u}(t)$ is the vector of control variables whereas $\mathbf{W}(t)$ is a Wiener process.

According to the Euler-Maruyama scheme, the approximation of (C4) in N partitions is given by

$$\mathbf{Y} = \{\mathbf{Y}_l, l \in \mathbb{N}, 0 \leq l \leq N\} \quad (\text{C5})$$

The expression in (C5) has to be consistent with the following expression:

$$\mathbf{Y}_{l+1} = \mathbf{Y}_l + f(\mathbf{Y}_l, \mathbf{u}_l, \tau_l)(\tau_{l+1} - \tau_l) + b(\mathbf{Y}_l, \mathbf{u}_l, \tau_l)(\mathbf{W}_{\tau_{l+1}} - \mathbf{W}_{\tau_l}) \quad (\text{C6})$$

where $l = 0, \dots, N - 1$ whereas the initial seed equals to $\mathbf{Y}_0 = \mathbf{X}(0)$.

Thereafter, in order to determine the Markov decision process, we have to define a discrete state space, the transition probabilities for each state, and a reward function associated with each transition. The discrete state space for stage l is denoted by \bar{X}_l whereas the extreme values of the state grid are given by $\bar{U}_l = \max \{\bar{X}_l\}$ and $\bar{L}_l = \min \{\bar{X}_l\}$. Consequently, a point $x \in X$ is in the grid \bar{X}_l , if and only if $\bar{L}_l \leq x \leq \bar{U}_l$. Moreover, the set of the discrete state spaces for all stages, formally speaking $\{\bar{X}_l\}_{l=0}^N$, is denoted by \bar{X} . Heuristically, the adopted numerical scheme is able to approximate a generic point of X at stage l by the points of \bar{X}_l which are adjacent to it.

Having defined the discrete state space, we now move to the definition of the transition probabilities. Consider the stochastic process in eq. (C5), i.e., $\mathbf{Y} = \{\mathbf{Y}_l, l = 0, \dots, N\}$, where \mathbf{Y}_l is defined by (C6). This process, although defined at discrete times, can take any real value.

For a given control sequence \mathbf{u}_l and an equidistant discretization time-steps, we can re-write the iterative scheme of (C6) in the following abbreviated form:

$$\mathbf{Y}_{l+1} = \mathbf{Y}_l + \delta f_l + b_l \Delta \mathbf{W}_l \quad (\text{C7})$$

where $f_l = f(\mathbf{Y}_l, \mathbf{u}_l, \tau_l)$, $b_l = b(\mathbf{Y}_l, \mathbf{u}_l, \tau_l)$ whereas $\Delta \mathbf{W}_l = \mathbf{W}_{\tau_{l+1}} - \mathbf{W}_{\tau_l}$.

Suppose that we are at time τ_l , so that $\mathbf{Y}_l = \bar{\mathbf{Y}}_l \in \bar{X}_l$. In a deterministic context (that is, whenever $\Delta \mathbf{W}_l = 0$, $l = 0, \dots, N - 1$), for a given control value \mathbf{u}_l , the process moves to \mathbf{Y}_{l+1} , according to eq. (C7). Consequently,

$$\mathbf{Y}_{l+1} = \bar{\mathbf{Y}}_l + \delta f_l \quad (\text{C8})$$

If \mathbf{Y}_{l+1} has only one state adjacent to it, then the transition probability from \mathbf{Y}_l is equal to 1. By contrast, if there is a pair of states adjacent to \mathbf{Y}_{l+1} , called $(\bar{\mathbf{Y}}_{l+1}^\ominus, \bar{\mathbf{Y}}_{l+1}^\oplus)$, such that $h_l = \bar{\mathbf{Y}}_{l+1}^\oplus - \bar{\mathbf{Y}}_{l+1}^\ominus > 0$, the transition probabilities are determined according to an inverse distance method. Formally speaking, we have

$$\begin{aligned} p(\bar{\mathbf{Y}}_l, \bar{\mathbf{Y}}_{l+1}^\oplus | \mathbf{u}_l) &= \frac{\mathbf{Y}_{l+1} - \bar{\mathbf{Y}}_{l+1}^\ominus}{h_l} \\ p(\bar{\mathbf{Y}}_l, \bar{\mathbf{Y}}_{l+1}^\ominus | \mathbf{u}_l) &= \frac{\bar{\mathbf{Y}}_{l+1}^\oplus - \mathbf{Y}_{l+1}}{h_l} \end{aligned} \quad (\text{C9})$$

In a stochastic context, a Gaussian noise is present in (C7) and, consequently, \mathbf{Y}_{l+1} is no more deterministic. In this case, we can use a partition of the realizations of the Gaussian process $\Delta \mathbf{W}_l$ into M steps. If we choose $M = 3$ and we use these intervals: $(-\infty, -\sqrt{\delta})$, $(-\sqrt{\delta}, +\sqrt{\delta})$, $(+\sqrt{\delta}, +\infty)$, where $\sqrt{\delta}$ is the standard deviation of $\Delta \mathbf{W}_l$, then we can compute the expected values of the noise by using the following expression:

$$\omega = \frac{\sqrt{2\delta}}{\sqrt{\pi \exp(1)} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right)\right)} \quad (\text{C10})$$

where $erf(\cdot)$ is the standard error function defined by $erf(x) = 2/\sqrt{\pi} \int_{t=0}^x \exp(-t^2) dt$.

The transition probabilities for an approximated situation in which the process \mathbf{Y} is perturbed by the discretely valued noise ω_l are defined by:

$$\begin{aligned} P(\omega_l = -\omega) &= p_- \\ P(\omega_l = 0) &= p_0 \\ P(\omega_l = +\omega) &= p_+ \end{aligned} \tag{C11}$$

As a result, if \mathbf{Y}_{l+1} is obtained by (C7) and there is a single adjacent state, then the process reaches $l+1$ with the following probabilities:

$$\begin{aligned} \mathbf{Y}_{l+1}^- &= \mathbf{Y}_{l+1} + b_l \omega_- \text{ with probability } p_- \\ \mathbf{Y}_{l+1}^0 &= \mathbf{Y}_{l+1} \text{ with probability } p_0 \\ \mathbf{Y}_{l+1}^+ &= \mathbf{Y}_{l+1} + b_l \omega_+ \text{ with probability } p_+ \end{aligned} \tag{C12}$$

By contrast, if there are two adjacent states, then it is reasonable to apply the inverse distance method as in (C9) weighted by the proper probabilities defined in (C12). For instance, if we consider \mathbf{Y}_{l+1}^- with the two adjacent states $\overline{\mathbf{Y}}_{l+1}^{-\ominus}$ and $\overline{\mathbf{Y}}_{l+1}^{-\oplus}$, then the transition probabilities are given by

$$\begin{aligned} p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}_{l+1}^{-\oplus} | \mathbf{u}_l\right) &= p_- \frac{\mathbf{Y}_{l+1}^- - \overline{\mathbf{Y}}_{l+1}^{-\ominus}}{h_l} \\ p\left(\overline{\mathbf{Y}}_l, \overline{\mathbf{Y}}_{l+1}^{-\ominus} | \mathbf{u}_l\right) &= p_- \frac{\overline{\mathbf{Y}}_{l+1}^{-\oplus} - \mathbf{Y}_{l+1}^-}{h_l} \end{aligned} \tag{C13}$$

where $h_l = \overline{\mathbf{Y}}_{l+1}^{-\oplus} - \overline{\mathbf{Y}}_{l+1}^{-\ominus} > 0$.

The next phase is to assign the performance function at every transitions of the Markov chain. The objective function to maximize must be the discretised version of the original performance function J on the allowable controls, that is, $\max_{\mathbf{u}} J(0, \overline{x}_0; \mathbf{u})$ subject to eq. (C7). This completes the description of the tool.

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