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# The Purity of Impure Public Goods

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# The Purity of Impure Public Goods

## Abstract

In this paper we demonstrate how the impure public good model can be converted into a pure public good model with satiation of private consumption, which can be handled more easily, by using a variation of the aggregative game approach as devised by Cornes and Hartley (2007). We point out the conditions for impure public good utility functions that allow for this conversion through which the analysis of Nash equilibria can be conducted in a unified way for the impure and the pure public good model and which facilitates comparative statics analysis for impure public goods. Our approach also offers new insights on the determinants for becoming a contributor to the public good in the impure case as well as on the non-neutral effects of income transfers on Nash equilibria when the public good is impure.

JEL-Codes: C720, D640, H410.

Keywords: impure public goods, warm-glow giving, Nash equilibria, aggregative game approach.

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## 1. Introduction

While contributions to a *pure* public good made by some agent only generate benefits for the entire community, a contribution to an *impure* public good in addition entails a private benefit for the contributing agent himself. In the real world examples for such impure public goods are abundant reaching from military alliances (Murdoch and Sandler, 1984) to warm-glow-of-giving (Andreoni, 1989, 1990, and, e.g. Long, 2020), which both have been among the first empirical applications of impure public good theory. In the meantime, other applications as climate policy (Rübelke, 2003), refugee protection (Betts, 2003), environmentally friendly consumption goods (Kotchen, 2005, 2006) and green electricity programs (Kotchen and Moore, 2007; Mitra and Moore, 2018) have attracted attention too.

Seminal works on the theory of impure public goods have been provided by Cornes and Sandler (1984, 1994, 1996) in particular focusing on the voluntary provision of these goods and the comparative static analysis of the ensuing Nash equilibria. However, application of the traditional approach, which uses best response functions for the analysis of Nash equilibria, becomes more challenging in the case of an impure public good model than in the case of a pure public good model. The aim of this paper therefore is to present a novel method by which the impure public good model is interpreted as a slightly modified standard pure public good model. This approach not only facilitates the analysis, but also helps to elucidate the common features and the differences between pure and impure public goods and is, moreover, conducive for doing comparative statics exercises. Moreover, it becomes possible to show on which characteristics of the underlying utility functions the properties of the Nash equilibria and comparative statics effects depend.

The essential tool for our approach is provided by some variation of the aggregative game approach, which has been developed by Cornes, Hartley and Sandler (1999) and Cornes and Hartley (2007) for the case of pure public goods, but which has also been applied to many other fields such as contest theory (Cornes and Hartley, 2003, 2005) and various issues in environmental economics (Cornes, 2016). How useful this approach is for the treatment of general non-cooperative games has been shown by Cornes and Hartley (2012) and Acemoglu and Jensen (2013). A first application to impure public goods has been made by Kotchen (2007) whose approach, however, is different from ours and focuses on a proof of existence and uniqueness of Nash equilibria.

The basic idea underlying the aggregative game approach is that an agent makes her own action dependent on the *aggregate* level instead of the vector of individual choices of the other agents.<sup>1</sup> But in the case of an impure public good some complication arises since here an agent's utility function has three arguments – private consumption, aggregate public good supply and the agent's individual public good contribution. This implies that an additional step is needed to make the aggregative game approach applicable to impure public goods, which specifically means that the initially given impure public good utility function with three arguments has to be transformed into a pure public good utility function with only two arguments. This methodological “trick” lies at the core of the analysis presented in this paper.

The remainder of our paper will be organized as follows. In Section 2 we present a version of the pure public good model in which there is satiation with respect to private consumption so that the marginal utility of private consumption becomes zero (or even negative) for sufficiently high levels of private consumption. Also for this specific class of pure public goods Nash equilibria can be determined in the usual way by means of the aggregative game approach. In Section 3 we then demonstrate how the impure public good model can be transformed into a pure public good public model with satiation and, in addition, pinpoint the conditions on general impure public goods utility functions that make the conversion to the pure public good model possible. Examples of specific impure public good utility functions, which satisfy these conditions, are given in Section 4. In Section 5 some comparative statics exercises are conducted, while in Section 6 a comparison between the impure and the standard pure public good model is made. Section 7 concludes.

## 2. Pure Public Goods with Satiation of Private Consumption

We now assume that there are  $n$  agents  $i = 1, \dots, n$  with preferences  $u^i(c_i, G)$ , where  $c_i$  is the agent's private consumption and  $G$  is the supply level of a pure public good. Agent  $i$ 's individual endowment (“income”) is denoted by  $w_i$ , and both goods' prices are set to one. The

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<sup>1</sup> In the context of oligopoly theory this idea dates back to McManus (1962) and Selten (1970). McGuire (1974) puts forward a similar argument in his analysis of pure public good provision by an arbitrary number of heterogeneous agents. Okuguchi (1993) suggests to apply the approach to different kinds of Cournot models such as oligopoly and pure public good provision.

public good  $G$  is, as in the standard case, produced by a summation technology and the marginal rate of transformation between the private and the public good is equal to one for all agents  $i = 1, \dots, n$ .

The utility function  $u^i(c_i, G)$  of agent  $i$  is assumed to be defined for all  $c_i < \bar{c}_i \leq \infty$  (with  $\bar{c}_i \geq w_i$ ) and all  $G > 0$ , to be twice continuously differentiable in both variables and to be quasi-concave. The first partial derivatives of  $u^i(c_i, G)$  with respect to  $c_i$  and  $G$  are denoted by  $u_1^i(c_i, G)$  and  $u_2^i(c_i, G)$ , respectively, and the second partial derivatives by  $u_{kl}^i(c_i, G)$  for  $k, l = 1, 2$ . While we assume  $u_2^i(c_i, G) > 0$  everywhere, we will allow – in contrast to the standard pure public good model (see, e.g., Cornes and Sandler, 1996, p. 144) – that for any  $G > 0$  there exists a satiation level  $\hat{c}_i(G) < \infty$  for private consumption. This means that we have strictly positive marginal utility of private consumption  $u_1^i(c_i, G) > 0$  only if  $c_i < \hat{c}_i(G)$ , but  $u_1^i(c_i, G) = 0$  if  $c_i \geq \hat{c}_i(G)$ , i.e. after the satiation level  $\hat{c}_i(G)$  has been attained. As a real-world example, we can imagine that increasing the number of privately owned cars will not increase the drivers' well-being (or may even reduce it) if the public good infrastructure in terms of well-built roads is missing.

By  $m_i(c_i, G) = \frac{u_1^i(c_i, G)}{u_2^i(c_i, G)}$  we denote the marginal rate of substitution between the private and

the public good at any point  $(c_i, G)$  in the domain of the utility function  $u^i(c_i, G) > 0$ . If  $c_i < \hat{c}_i(G)$ , clearly  $m_i(c_i, G) > 0$  holds, while  $m_i(c_i, G) = 0$  if  $c_i \geq \hat{c}_i(G)$ . We assume that  $m_i(c_i, G)$  is strictly increasing in  $G$  as long as it is positive, which – as shown by a straightforward calculation – follows from normality of the private good. Moreover, we assume that the limit properties  $\lim_{G \rightarrow 0} m_i(c_i, G) = 0$  and  $\lim_{G \rightarrow \infty} m_i(c_i, G) = \infty$  apply for all  $c_i < \bar{c}_i$ .

In a  $c_i$ - $G$ -diagram we now consider three types of curves:

- (i) The *indifference curves*  $G_i^{\bar{u}^i}(c_i)$  along which the agent's utility is constant, i.e. for which  $u^i(c_i, G_i^{\bar{u}^i}(c_i)) = \bar{u}^i$  holds for some given utility level  $\bar{u}^i$ . Quasi-concavity of  $u^i(c_i, G)$  implies that left to  $\hat{c}_i(G)$  these indifference curves are convex and decreasing as in the standard case. To the right of  $\hat{c}_i(G)$  they are flat lines instead.

(ii) The *satiation curve*  $\hat{c}_i(G)$ , which is assumed to be non-decreasing in  $G$ , which means that the satiation level of private consumption increases with public good supply. This relationship becomes apparent if we again consider our example where the use of cars represents the private good while road infrastructure is the public good. Concerning the shape of the  $\hat{c}_i(G)$ -curve we distinguish two types that depend on the agent's preferences  $u^i(c_i, G)$  and her income  $w_i$ .

**Type 1:** There is a public good level  $\hat{G}_i$  at which the satiation curve intersects the vertical line passing through the point  $(w_i, 0)$ , i.e.  $\hat{c}_i(\hat{G}_i) = w_i$  holds. This means that for sufficiently high levels of public good supply no satiation occurs for private consumption below the agent's income.

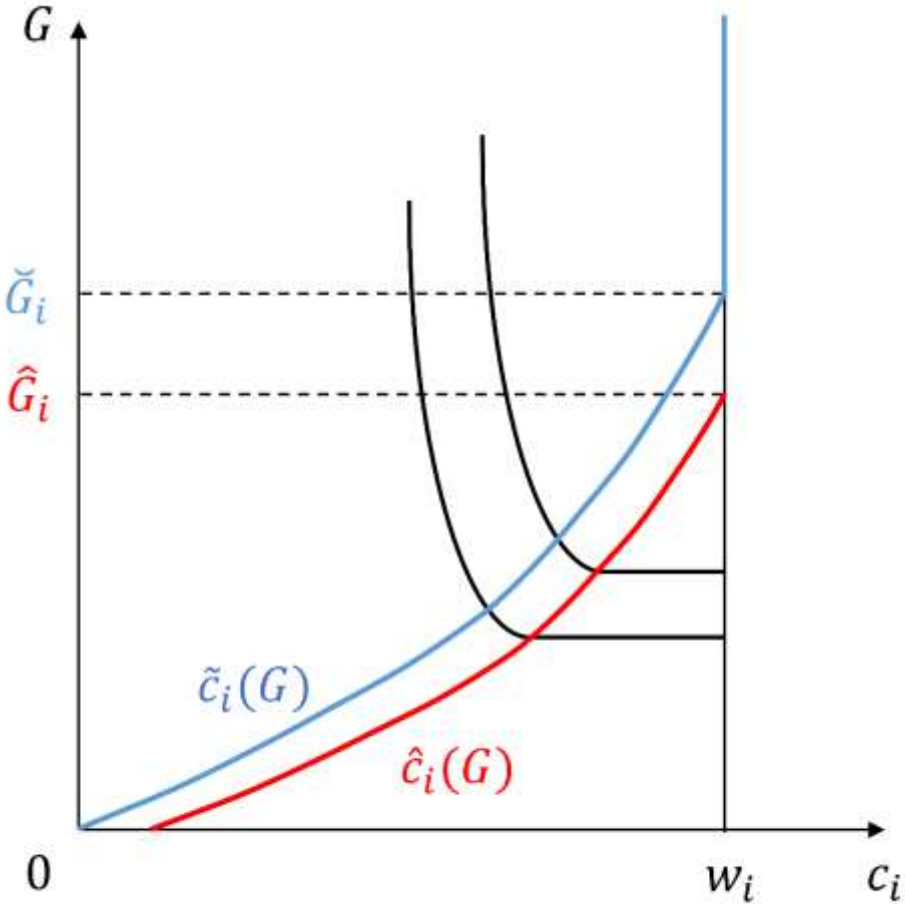
**Type 2:** There is no such  $\hat{G}_i$ , i.e. there exists a  $\hat{c}_i \leq w_i$  so that  $\hat{c}_i(G) < \hat{c}_i$  holds for all  $G > 0$ . In this case we set  $\hat{G}_i = \infty$ .

For preferences of type 2 we have two subcases, i.e. type 2a where  $\hat{c}_i = w_i$  and type 2b where  $\hat{c}_i < w_i$ .

(iii) The *expansion path*  $\tilde{G}_i(c_i)$  along which the slope of the indifference curves is equal to the marginal rate of transformation between the private and the public good, i.e.  $m_i(c_i, \tilde{G}_i(c_i)) = mrt = 1$  holds. Since  $m_i(c_i, G)$  is continuous and monotonically increasing in  $G$  it follows from  $\lim_{G \rightarrow 0} m_i(c_i, G) = 0$  and  $\lim_{G \rightarrow \infty} m_i(c_i, G) = \infty$  by the intermediate value theorem that such a  $\tilde{G}_i(c_i)$  exists for any  $c_i < w_i$  if agent  $i$  is of type 1 and  $c_i < \hat{c}_i$  if agent  $i$  is of type 2. The expansion path is strictly increasing in  $c_i$  if the public good is normal. As  $\tilde{G}_i(c_i)$  is continuous and monotonic, its inverse  $\tilde{c}_i(G)$  exists, which is defined on a finite interval for agents of type 1 and on  $(0, \infty)$  for preferences of type 2. The expansion path  $\tilde{c}_i(G)$  clearly lies above the satiation curve  $\hat{c}_i(G)$  so that the vertical line through  $(\hat{c}_i, 0)$  must be an asymptote to the expansion path if an agent is of type 2. If an agent instead is of type 1, from  $\lim_{G \rightarrow \infty} m_i(w_i, G) = \infty$  we get that the expansion path will intersect the vertical straight line through  $(w_i, 0)$  in the point  $(w_i, \tilde{G}_i)$ .

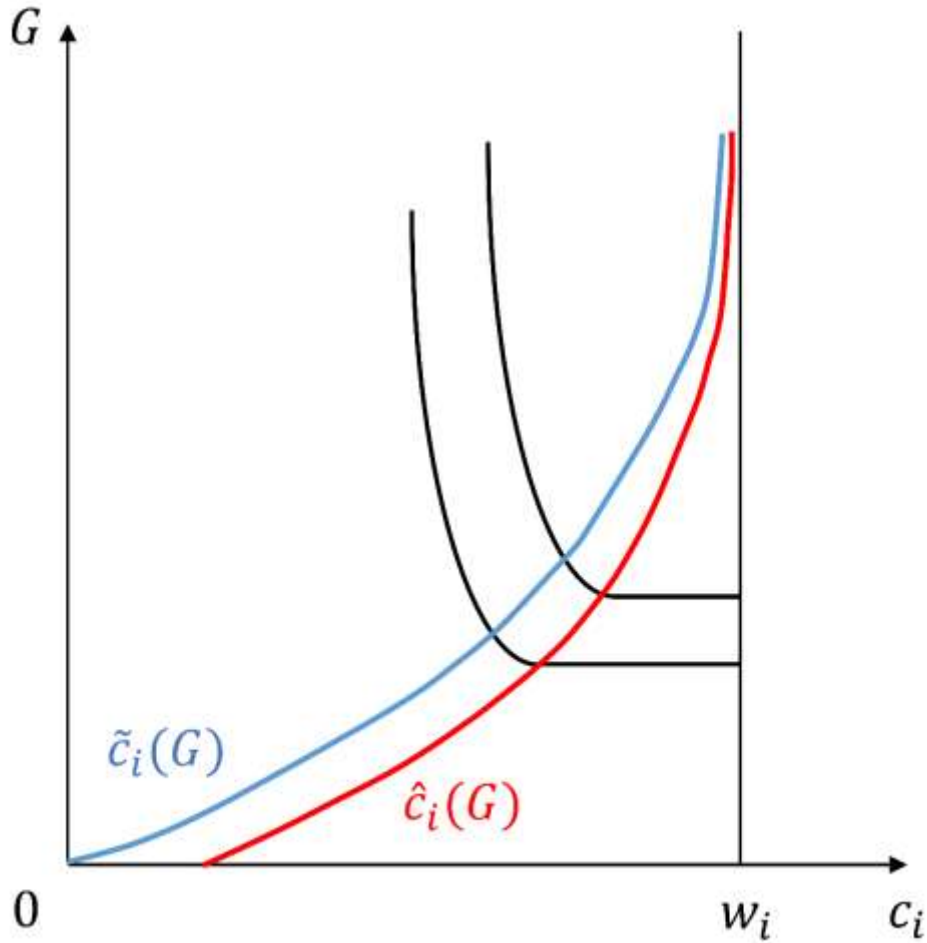


In Figures 1, 2a and 2b the curves described by (i), (ii) and (iii) are depicted in black, red and blue, respectively, for the different types 1, 2a and 2b of agents.



**Figure 1** Indifference curves,  $\hat{c}_i(G)$ -curves and expansion paths  $\tilde{c}_i(G)$  for type 1 agents

Along agent  $i$ 's expansion path  $\tilde{c}_i(G)$  agent  $i$ 's marginal rate of substitution between the private and the public good coincides with the technically given marginal rate of transformation. This implies that  $\tilde{c}_i(G)$  describes the positions agent  $i$  can attain as a contributor to the public good in a Nash equilibrium of non-cooperative public good provision: A small deviation from such a position, by either increasing or decreasing her public good contribution, would make an agent worse off.

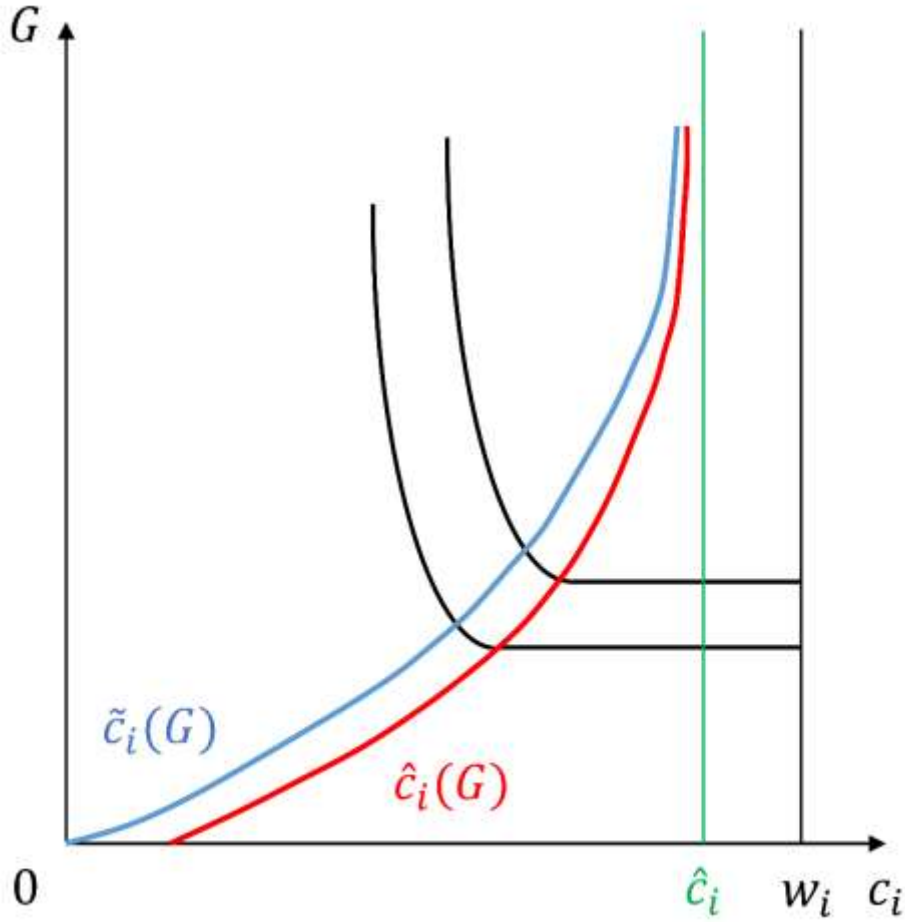


**Figure 2a** Indifference curves,  $\hat{c}_i(G)$ -curves and expansion paths  $\tilde{c}_i(G)$  for type 2a agents

In order to include non-contributors we modify the expansion path when for agents of type 1  $\tilde{c}_i(G)$  intersects the vertical straight line through the point  $(w_i, 0)$  in the “dropout” point  $(w_i, \check{G}_i)$ . Then we cut off the expansion path at  $(w_i, \check{G}_i)$  by letting  $\check{c}_i(G) = \tilde{c}_i(G)$  for  $G \leq \check{G}_i$  and  $\check{c}_i(G) = w_i$  for  $G > \check{G}_i$ .

Using these expansion paths, the aggregative game approach (see, e.g., Cornes and Hartley, 2007) now – in the usual way – allows for a characterization of Nash equilibrium public good supply  $G^N$  through the condition

$$(1) \quad \Phi(G^N) := G^N + \sum_{i=1}^n \check{c}_i(G^N) = \sum_{i=1}^n w_i .$$



**Figure 2b** Indifference curves,  $\hat{c}_i(G)$ -curves and expansion paths  $\tilde{c}_i(G)$  for type 2b agents

Letting  $x_i^N = \tilde{c}_i(G^N)$  the allocation  $(x_1^N, \dots, x_n^N, G^N)$  actually is a Nash equilibrium since each agent being on her expansion path is in a Nash equilibrium position and, at the same time, the aggregate budget constraint is fulfilled. Existence and uniqueness of  $G^N$  as characterized by condition (1) follow by applying the intermediate value theorem since the function  $\Phi(G)$  is

continuous and strictly monotonically increasing in  $G$  with  $\Phi(0) = \sum_{i=1}^n \tilde{c}_i(0) < \sum_{i=1}^n w_i$  and

$$\lim_{G \rightarrow \infty} \Phi(G) = \lim_{G \rightarrow \infty} G = \infty.$$

If  $\tilde{c}_i(G^N) < w_i$ , agent  $i$  contributes  $g_i^N = w_i - \tilde{c}_i(G^N) > 0$  to the public good while agent  $i$  contributes nothing if  $\tilde{c}_i(G^N) = w_i$ .

From this characterization of the Nash equilibrium a crucial difference to the pure public good model without satiation is immediate: If some agent  $i$  is of type 2, she can – in contrast to the standard pure public good model – never become a complete free-rider in the Nash equilibrium because her expansion path always stays to the left of  $w_i$ . Especially for an agent  $i$  of type 2b we have that her public good contributions will be strictly bounded away from zero by  $w_i - \hat{c}_i > 0$  in any possible Nash equilibrium – independent of the other agents' types and the size of the economy. If an agent is of type 2a, her public good contributions in the Nash equilibrium also will always be strictly positive, but they will converge to zero if the aggregate public good contributions of the other agents go to infinity.

### 3. Converting Impure Public Good Preferences

We now show how the impure public good model can be converted into a pure public good model. To this end the agents' impure public good preferences are transformed into pure public good preferences with satiation in the following way: Let agent  $i$  have the impure public good utility function  $U^i(c_i, G, z_i)$  where  $z_i$  is this agent's public good contribution. For the partial derivatives of  $U^i(c_i, G, z_i)$  we assume that  $U_1^i > 0$ ,  $U_2^i > 0$ ,  $U_3^i > 0$  and  $U_{11}^i < 0$  as well as  $U_{22}^i < 0$  and  $U_{33}^i < 0$  hold. Assuming  $U_3^i > 0$  indicates impurity of the public good, i.e. that agent  $i$  derives private co-benefits from her public good contribution  $z_i$  and that these are increasing in  $z_i$ . Given  $w_i$  as agent  $i$ 's income we have  $z_i = w_i - c_i$ . In the case of an impure public good agent  $i$ 's satiation level  $\hat{c}_i(G) < w_i$  of private consumption is given by the first order condition  $U_1^i(\hat{c}_i(G), G, w_i - \hat{c}_i(G)) - U_3^i(\hat{c}_i(G), G, w_i - \hat{c}_i(G)) = 0$ . This means that satiation is attained at that level of private consumption at which marginal utility of the private co-benefits of the agent's public good contribution starts to outweigh her marginal benefits of private consumption.

The original impure public good utility function  $U^i(c_i, G, z_i)$  with its three arguments then is transformed into an auxiliary pure public good utility function  $u^i(c_i, G)$  of the type as considered in the previous section by letting

$$(2) \quad u^i(c_i, G) = U^i(c_i, G, w_i - c_i) \quad \text{if } c_i < \hat{c}_i(G) \\ = U^i(\hat{c}_i(G), G, w_i - \hat{c}_i(G)) \quad \text{if } c_i \geq \hat{c}_i(G).$$

As before, only the part of the indifference curve that lies to the left of the satiation curve  $\hat{c}_i(G)$  is relevant for the determination of Nash equilibria.<sup>2</sup>

We now show which properties of the originally given impure public good utility function  $U^i(c_i, G, z_i)$  ensure that the auxiliary utility function  $u^i(c_i, G)$  exhibits the properties that we have assumed in the previous section for a pure public good utility function with satiation, i.e. that it is quasi-concave, that the marginal rate of substitution  $m_i(c_i, G)$  is increasing in  $G$  (and has some limit properties) and that both the satiation path  $\hat{c}_i(G)$  and the expansion path  $\tilde{c}_i(G)$  are strictly monotonically increasing. If these conditions are fulfilled, the conversion of the impure into a pure public good model with satiation is possible, which in particular implies that the proofs for the existence and uniqueness of the Nash equilibria being based on eq. (1) can be unified for both types of public goods.

It is shown in the Appendix that  $u^i(c_i, G)$  has all these properties if for  $c_i < \hat{c}_i(G)$  we have  $u_{11}^i < 0$ ,  $u_{22}^i < 0$  and  $u_{12}^i = u_{21}^i \geq 0$  for the second partial derivatives. Given  $U^i(c_i, G, z_i)$  and the corresponding auxiliary utility function  $u^i(c_i, G)$  these partial derivatives are  $u_{11}^i = U_{11}^i + U_{33}^i - 2U_{13}^i$ ,  $u_{12}^i = u_{21}^i = U_{12}^i - U_{23}^i$  and  $u_{22}^i = U_{22}^i$ . Hence,  $u^i(c_i, G)$  has the required properties if  $U_{kk}^i < 0$  for  $k = 1, 2, 3$ ,  $U_{12}^i \geq 0$ ,  $U_{31}^i \geq 0$  and  $U_{32}^i \leq 0$ .<sup>3</sup> These signs of the cross derivatives of  $U^i(c_i, G, z_i)$ , which allow the conversion of the impure public good preferences into pure public good preferences with satiation, can be motivated as follows:

- $U_{21}^i = U_{12}^i \geq 0$  means that a higher level of public good supply (e.g., a better infrastructure or an improvement of environmental quality) increases marginal utility of private

<sup>2</sup> Note that to the right of the satiation curve  $\hat{c}_i(G)$  the complete indifference curves given by  $U^i(c_i, G, w_i - c_i)$  would be increasing as described in Brumme, Buchholz and Rübbecke (2020). For the determination and the properties of Nash equilibria this upwards-moving section of the indifference curves is of no relevance.

<sup>3</sup> Clearly,  $U_{12}^i \geq 0$ ,  $U_{31}^i \geq 0$  and  $U_{32}^i \leq 0$  is not necessary for getting an auxiliary utility function  $u^i(c_i, G)$  with the needed properties. This is also obtained if  $U_{12}^i < 0$ ,  $U_{31}^i < 0$  and  $U_{32}^i > 0$ , but are close to zero.

consumption. This is an appealing assumption as, e.g., the quality of living is enhanced through better public transport facilities and less air pollution.

- $U_{31}^i = U_{13}^i \geq 0$  holds if there is some complementarity between utility from private consumption and the co-benefits of the public good contributions. This means in the warm-glow-of-giving version of the impure public good model that a higher private consumption, i.e. an improved material well-being, will increase an agent's marginal joy from giving, which also is a reasonable assumption. "First food, then morals" seems to be a common trait of human behavior.
- $U_{32}^i = U_{23}^i \leq 0$  means that a higher supply of the public good will decrease or at least not increase the individual co-benefits. In the case of warm-glow-of-giving this can be explained by an agent's desire to stand out from the others in her contribution to the public good (see also Yildirim, 2014, p. 102): The more the other agents already contribute, the less the agent appreciates her own contribution because she can no longer consider herself as a moral pioneer. An alternative interpretation may be that after trespassing some threshold an agent's contribution may become less urgent for providing enough of the public good that is, e.g., required for avoiding the danger of a climate catastrophe.

Concerning the required limit properties of  $m_i(c_i, G)$  we need

$$\lim_{G \rightarrow \infty} m_i(c_i, G) = \lim_{G \rightarrow \infty} \frac{U_1^i(c_i, G, w_i - c_i) - U_3^i(c_i, G, w_i - c_i)}{U_2^i(c_i, G, w_i - c_i)} = \infty. \text{ This condition is satisfied if for the}$$

denominator  $\lim_{G \rightarrow \infty} U_2^i(c_i, G, w_i - c_i) = 0$  holds while the numerator is bounded strictly away from zero. The latter condition is ensured also if  $U_{12}^i \geq 0$  and  $U_{32}^i \leq 0$ . An analogous argument applies to  $\lim_{G \rightarrow 0} m_i(c_i, G) = 0$ .

It is also straightforward to infer from the first derivatives of  $U^i(c_i, G, z_i)$  of which type the agent  $i$  is. If  $U_1^i(w_i, G, 0) - U_3^i(w_i, G, 0) \geq 0$  holds for some  $G$  (and consequently for all levels of  $G$  large enough), the agent will be of type 1. An agent is of type 2 instead if  $U_1^i(w_i, G, 0) - U_3^i(w_i, G, 0) < 0$  for all  $G > 0$ . In case of warm-glow this motive thus must be rather strong as

compared to the utility of private consumption. If then  $\sup_{G>0} \hat{c}_i(G) = w_i$ , agent  $i$  is of type 2a.

Otherwise, if  $\sup_{G>0} \hat{c}_i(G) < w_i$ , she is of type 2b.

#### 4. Special Classes of Utility Functions

We now consider specific classes of impure public good utility functions which satisfy the conditions that have been presented in the section before and which allow the conversion of the impure public good model into a pure one.

The first class is given by completely separable utility functions, for which  $U^i(c_i, G, z_i) = f_i(c_i) + g_i(G) + h_i(z_i)$  with  $U_1^i = f_i' > 0$ ,  $U_2^i = g_i' > 0$ ,  $U_3^i = h_i' > 0$ ,  $U_{11}^i = f_i'' < 0$ ,  $U_{22}^i = g_i'' < 0$  and  $U_{33}^i = h_i'' < 0$  holds. In this case all cross partial derivatives  $U_{kl}^i$  are zero, so that the respective conditions listed in the previous section clearly are all fulfilled and the condition  $\lim_{G \rightarrow \infty} U_2^i(c_i, G, w_i - c_i) = 0$  simply boils down to  $\lim_{G \rightarrow \infty} g_i'(G) = 0$ . Moreover,  $\hat{c}_i(G)$  is constant for all  $G$ : If  $f_i'(w_i) > h_i'(w_i - w_i) = h_i'(0)$ , we have  $\hat{c}_i(G) = w_i$ , and agent  $i$  is of type 1. If instead  $f_i'(w_i) \leq h_i'(0)$ , agent  $i$  is of type 2 and  $\hat{c}_i = \hat{c}_i(G) \leq w_i$  is characterized by the first order condition  $f_i'(\hat{c}_i) = h_i'(w_i - \hat{c}_i)$ . The expansion path then is asymptotical to the vertical line passing through  $(\hat{c}_i, 0)$ . Agent  $i$  is of type 2a, i.e.  $\hat{c}_i = w_i$  holds, if and only if  $f_i'(w_i) = h_i'(0)$ .

The second class of utility functions is given by partially separable utility functions of the type  $U^i(c_i, G, z_i) = v^i(c_i, G) + h_i(z_i)$ , where  $v^i(c_i, G)$  is a pure public good utility function with the usual properties and  $h_i(z_i)$  is monotonically increasing and concave. Such a utility function  $U^i(c_i, G, z_i)$  satisfies the required conditions on the cross partial derivatives if  $U_{12}^i = v_{12}^i > 0$ . Unlike in the case with complete separability, the  $\hat{c}_i(G)$ -curve now is no longer a vertical line, but a strictly increasing function with slope  $\hat{c}_i'(G) = -\frac{v_{12}^i}{v_{11}^i + h_i''} > 0$ . If and only if  $v_1^i(w_i, G) < h_i'(0)$  for all  $G > 0$ , the agent is of type 2 and thus always is a contributor to the public good in a Nash equilibrium.

A completely analogous reasoning can be applied to partially separable utility functions of the alternative type  $U^i(c_i, G, z_i) = f_i(c_i) + \psi^i(G, z_i)$ , where in particular  $U_{23}^i = \psi_{12}^i \leq 0$  is needed

for satisfying the conditions imposed on  $U^i(c_i, G, z_i)$ . A special sub-case of such utility functions is given by  $\psi^i(G, z_i) = \varphi_i(\alpha_i G + \beta_i z_i)$ . Here  $\psi_{12}^i = \alpha_i \beta_i \varphi_i'' < 0$  holds if the function  $\psi^i$  is concave. This class of partially separable utility functions provides a generalization of the utility functions used by Yildirim (2014, p. 103) where  $f_i(c_i) = \ln c_i$  and  $\psi^i(G, z_i) = \ln(\alpha_i G + \beta_i z_i)$ . Then both the satiation curve  $\hat{c}_i(G)$  and the expansion path  $\tilde{c}_i(G)$  are monotonically increasing straight lines, i.e.  $\hat{c}_i(G) = \frac{\alpha_i}{2\beta_i}G + \frac{w_i}{2}$  and  $\tilde{c}_i(G) = \frac{\alpha_i}{\alpha_i + 2\beta_i}G + \frac{\beta_i w_i}{\alpha_i + 2\beta_i}$ , which intersect the vertical line through  $(w_i, 0)$ . Hence, agent  $i$  with such a utility function is of type 1.

## 5. Some Comparative Statics

The application of the aggregative game approach also makes it possible to conduct comparative statics exercises for the impure public good case by the same method as in the pure public good case, i.e. by assessing changes of the Nash equilibrium by considering changes of the expansion paths. We start with a general treatment for which we assume that the expansion path of some agent  $j$  is dependent on some – provisionally unspecified – parameter  $\rho$ . For getting a general approach, which reflects that in the case of an impure public good the expansion paths also depend on individual income, a change of  $\rho$  may also affect the aggregate endowment, i.e. we have  $\tilde{c}_j(G, \rho)$  and  $W(\rho)$ . The partial derivative of  $\tilde{c}_j(G, \rho)$  with respect to  $G$  is denoted by  $\tilde{c}'_j$ , which under the conditions stated in Section 3 is positive, while its partial derivative with respect to  $\rho$  is denoted by  $\tilde{c}_{j\rho} := \frac{\partial \tilde{c}_j}{\partial \rho}$ . (Variables of the functions will now be omitted again.) If  $\tilde{c}_{j\rho} > 0$  ( $\tilde{c}_{j\rho} < 0$ ) holds, then agent  $j$ 's expansion path is shifted to the right (to the left) by a marginal increase of the parameter  $\rho$ .

To simplify the exposition, we assume that the Nash equilibrium with public good supply  $G^N$  is an interior solution so that all agents attain a position on their respective expansion paths. Taking the derivative of condition (1) which characterizes the Nash equilibrium by means of the aggregative game approach gives



$$(3) \quad \frac{dG^N}{d\rho} = \frac{W_\rho - \tilde{c}_{j\rho}}{\Phi'}$$

where  $\Phi' = 1 + \sum_{i=1}^n \tilde{c}'_i$ . Since  $\Phi' > 0$ , condition (3) entails that public good supply in the Nash

equilibrium is increased through a marginal change of the parameter  $\rho$ , i.e.  $\frac{dG^N}{d\rho} > 0$ , if and

only if  $\tilde{c}_{j\rho} < W_\rho := \frac{dW}{d\rho}$ . Private consumption of all agents  $i \neq j$  then changes by

$\frac{dc_i^N}{d\rho} = \tilde{c}'_i \cdot \frac{dG^N}{d\rho}$  which – due to  $\tilde{c}'_i > 0$  – always has the same sign as  $\frac{dG^N}{d\rho}$ . This implies that

the utility of all agents whose expansion paths are not affected by the change of  $\rho$  will uniformly go up or down. The change of agent  $j$ 's private consumption instead is

$\frac{dc_j^N}{d\rho} = \tilde{c}'_j \cdot \frac{dG^N}{d\rho} + \tilde{c}_{j\rho}$ . If  $\tilde{c}_{j\rho} < 0$ , agent  $j$ 's private consumption in the Nash equilibrium thus

might decrease even if private consumption of all other agents increases.

In the special case where aggregate endowment  $W$  remains the same after the parameter change for agent  $j$ , i.e. if  $\frac{dW}{d\rho} = 0$  holds, it particularly follows that the direction, in which

Nash equilibrium public good supply, private consumption and the utility of all agents except agent  $j$  change, completely depends on whether agent  $j$ 's expansion path is shifted to the right or to the left through the change of  $\rho$ .

By using the conversion of the impure public good model into a pure one as described above it is now straightforward to use these general results on comparative statics for Nash equilibria with a pure public good also for comparative statics in case of an impure public good. To exemplify this proceeding, we want to determine  $\tilde{c}_{j\rho}$  for two specifications of the parameter  $\rho$ , which are particularly relevant in the case of impure public goods.

In the first scenario the parameter  $\rho$  indicates as in Cornes and Sandler (1994, pp. 414-416, 1996, pp. 267-269) some agent  $j$ 's productivity in generating her private co-benefits or, alternatively, this agent's intensity of joy-of-giving. Letting  $\rho = \kappa$  agent  $j$ 's utility function then

is  $U^j(c_j, G, \kappa z_j) = U^j(c_j, G, \kappa(w_j - c_j))$ . Agent  $j$ 's expansion path  $\tilde{c}_j = \tilde{c}_j(G)$  is defined by the condition

$$(4) \quad m_j(\tilde{c}_j, G) = \frac{U_1^j(\tilde{c}_j, G, \kappa(w_j - \tilde{c}_j)) - \kappa U_3^j(\tilde{c}_j, G, \kappa(w_j - \tilde{c}_j))}{U_2^j(\tilde{c}_j, G, \kappa(w_j - \tilde{c}_j))} = 1 .$$

Differentiating (4) with respect to  $\kappa$ , omitting again all arguments and evaluating the derivative at  $\kappa = 1$  gives

$$(5) \quad \tilde{c}_{j\kappa} = \frac{U_{13}^j - U_{23}^j - (U_3^j + U_{33}^j \cdot (w_j - c_j))}{U_{12}^j + 2U_{13}^j - U_{23}^j - U_{11}^j - U_{33}^j} .$$

Given the assumptions on the utility function  $U^j(c_j, G, \kappa z_j)$  that we have made in Section 3, i.e.  $U_{kk}^j < 0$ ,  $U_{12}^j \geq 0$ ,  $U_{13}^j \geq 0$  and  $U_{23}^j \leq 0$ , the denominator in (5) is positive. The numerator, however, can only be signed unambiguously if  $U_3^j + U_{33}^j \cdot (w_j - c_j)$  is negative, i.e. given

$z_j = w_j - c_j$  if we have  $-\frac{U_{33}^j z_j}{U_3^j} > 1$  for the elasticity of the marginal utility of the private co-

effects of the agent's public good contribution. Then the numerator of (5) also becomes positive so that we get  $\tilde{c}_{j\kappa} > 0 = W_\kappa$ . According to the general comparative statics result presented before, public good supply and private consumption of all other agents except agent  $j$  thus will fall in the Nash equilibrium while private consumption of agent  $j$  (whose productivity rises) will increase. The reason is that in this case agent  $j$  can realize the same level of private

co-benefits making a lower contribution to the public good. If, however,  $-\frac{U_{33}^j z_j}{U_3^j} < 1$ , i.e. if the

elasticity of the marginal co-benefits is low, then it is well possible that  $\tilde{c}_{j\kappa} < 0$  so that in the Nash equilibrium public good supply and private consumption and thus utility of all agents except for agent  $j$  are increasing. Private consumption of agent  $j$  decreases, which makes the utility effect for this agent ambiguous.

That an increase of the productivity parameter  $\kappa$  may in fact increase public good supply in the Nash equilibrium can be seen by considering the case in which agent  $j$  has completely

separable preferences so that only  $U_3 + U_{33} \cdot (w_j - c_j) = h'_j + h''_j \cdot (w_j - c_j)$  appears in the numerator of (5). Hence  $-\frac{U_{33}^j \cdot z_j}{U_3^j} = -\frac{h''_j \cdot z_j}{h'_j} < 1$  not only provides a necessary, but also a sufficient condition for an increase of public good supply (and the ensuing effects on private consumption and utility) when  $\kappa$  changes.

Other changes of the Nash equilibrium, e.g. those that occur when the parameter  $\kappa$  denotes the price of the private consumption good (so that agent  $j$ 's utility becomes  $U^j(\kappa c_j, G, w_j - \kappa c_j)$ ), can be explored in an analogous way.

In a second scenario we explore the effects that a marginal increase of agent  $j$ 's income  $w_j$  has on her expansion path and then on the Nash equilibrium. Letting  $\kappa = 1$  in eq. (4) we get

$$(6) \quad \tilde{c}_{jw_j} = \frac{U_{13}^j - U_{23}^j - U_{33}^j}{U_{12}^j + 2U_{13}^j - U_{23}^j - U_{11}^j - U_{33}^j}.$$

Given again the properties of  $U^j(c_j, G, z_j)$  as described in Section 3,  $\tilde{c}_{jw_j}$  is positive, which means that agent  $j$ 's expansion path is shifted outwards through the increase of her income. Furthermore, as the denominator in eq. (6) is larger than one we get  $\tilde{c}_{jw_j} < W_{w_j} = 1$ . From our general comparative statics results it thus follows that public good supply and private consumption of all agents increase so that also in the case of an impure public good the increase of some agent's income leads to a Pareto improvement.

Based on that it is also possible to infer a central result on the non-neutrality of income redistributions that holds in the impure public good model. Let there be an income transfer  $dT$  from some agent  $k$  that has no private co-benefits from her public good contribution to another agent  $j$  that has such co-benefits (and assume that both agents are contributors to the public good before and after the transfer). Since the expansion path of the donor agent remains invariant after the income transfer, public good supply in the Nash equilibrium changes by  $\frac{dG^N}{dT} = \frac{-\tilde{c}_{jw_j}}{\Phi'} < 0$  according to eq. (3). An analogous result is obtained if the recipient has lower private co-benefits from her public good contribution than the donor (see Andreoni, 1989, 1990).

## 6. A Comparison between the Impure and the Standard Pure Public Good Model

In this paper the structural similarities between the impure and the pure public good model have been emphasized so far. Yet our analysis also sheds light on the differences between impure and pure public goods. A major result in the standard pure public good model is that income redistribution among the contributing agents which leaves the set of contributors unchanged will neither affect the level of public good supply nor the agents' private consumption levels in the Nash equilibrium. It is now easy to see why this famous "Warr neutrality" (see Warr, 1983, Bergstrom, Blume and Varian, 1986, Cornes and Sandler, 1994, pp. 418-420, 1996, and Faias, Moreno-García and Myles, 2020) does no longer hold in the case of an impure public good: For a pure public good the expansion paths of the contributors remain the same after an income redistribution within the contributing group. Hence, condition (1) that characterizes the Nash equilibrium will not change so that invariance of public good supply and private consumption in the Nash equilibrium is immediate. If the public good is impure instead, things are completely different since the converted pure public good utility functions and hence the corresponding expansion paths depend on the agents' individual income levels and thus are usually affected by a redistribution of income. According to eq. (6) agent  $j$ 's expansion path can only be expected to stay the same after a marginal increase of this agent's income if  $U_{13}^j = U_{23}^j = U_{33}^j = 0$  holds. This would bring us back to the case of a pure public good or to a quasi-linear impure public good utility function with linearity in its private co-benefit part.

Another feature of the standard pure public good model is that agents may be non-contributors especially when the number of agents in the economy is large. In an impure public good economy this is also the case if all agents are of type 1. Then – quite analogous to the standard pure public good case without satiation – only agents with the highest  $\hat{G}_i$  and thus with the highest dropout level  $\tilde{G}_i$  will remain contributors if the original public good economy consisting of  $n$  agents is replicated sufficiently often. Yet, due to the satiation phenomenon agents of type 2 will always remain contributors irrespective of how often the economy is replicated or, more generally, how many new agents with high preferences for the public good enter the original economy.<sup>4</sup>

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<sup>4</sup> Similarly, for an impure public good model with government provision, Ribar and Wilhelm (2002) show that crowding-out of private contributions is asymptotically complete if the altruistic motive dominates whereas crowding-out is asymptotically zero if the warm-glow motive dominates.

## 7. Conclusion

In this paper we have shown how the impure public good model can be considered as a special variant of the pure public good model when this is slightly modified by allowing for satiation of private good consumption in the agents' preferences. Tracing back the impure to such a pure public good model with satiation makes it possible to directly apply the aggregative game approach to the analysis of Nash equilibria. This methodological device not only provides the unification of the existence and uniqueness proofs for the Nash equilibria of both types of public goods, but also helps to facilitate both the comparative statics analysis in the impure public good case and the comparison of the impure public good model with the standard public good model without satiation. Moreover, it can be immediately explained by our approach why for impure public goods Warr neutrality cannot be expected a priori and, as a consequence of satiation of private good consumption, agents will in any case be contributors to the public good.

### Appendix: Properties of Utility Functions with two Arguments

In order to simplify the exposition, we omit the index "i" and all arguments in the following analysis of essential properties of a pure public good function  $u(c, G)$ .

*Quasi-concavity* of  $u(c, G)$  holds if  $2\frac{u_1}{u_2}u_{12} - u_{11} - \left(\frac{u_1}{u_2}\right)^2 u_{22} > 0$  holds. This, in particular, is en-

sured if beyond decreasing marginal utility of consumption of the private and the public good, i.e.  $u_{11} < 0$  and  $u_{22} < 0$ , we have  $u_{12} > 0$ , which means that the private and the public good are complements.

The *marginal rate of substitution*  $m(c, G) = \frac{u_1(c, G)}{u_2(c, G)}$  is increasing in  $G$  if  $u_{12}u_2 - u_1u_{22} > 0$

which again is the case if, besides  $u_{22} < 0$ , we have  $u_{12} > 0$ .  $\lim_{G \rightarrow \infty} m(c, G) = \infty$  then holds if

$\lim_{G \rightarrow \infty} u_2(c, G) = 0$  and  $u_1(c, G)$  is strictly bounded away from zero which clearly holds if

$u_{12} \geq 0$ .  $\lim_{G \rightarrow 0} m(c, G) \leq 0$  is only relevant when for some  $c$  we have  $u_1(c, G) > 0$  and thus

$m(c, G) > 0$  for all  $G > 0$ . It is fulfilled if, e.g.,  $\lim_{G \rightarrow 0} u_2(c, G) = \infty$  and  $u_1(c, G)$  is bounded from above.

Concerning the  $\hat{c}(G)$ -curve we get from  $\frac{du_1(\hat{c}(G), G)}{dG} = u_{11}(\hat{c}(G), G)\hat{c}'(G) + u_{12}(\hat{c}(G), G) = 0$

that – given  $u_{11} < 0$  – the curve  $\hat{c}(G)$  is increasing in  $G$  if and only if  $u_{12} > 0$ .

From  $\frac{dm(\tilde{c}(G), G)}{dG} = \frac{(u_{11}\tilde{c}' + u_{12})u_2 - u_1(u_{21}\tilde{c}' + u_{22})}{u_2^2} = 0$  we finally obtain that the expansion

path  $\tilde{c}(G)$  is strictly monotonically increasing if  $u_{12} - u_{11} > 0$  and  $u_{12} - u_{22} > 0$ .

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