

Occhino, Filippo

Working Paper

Optimal fiscal policy over the business cycle

Working Paper, No. 2005-02

Provided in Cooperation with:

Department of Economics, Rutgers University

Suggested Citation: Occhino, Filippo (2005) : Optimal fiscal policy over the business cycle, Working Paper, No. 2005-02, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:

<https://hdl.handle.net/10419/23239>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Optimal Fiscal Policy over the Business Cycle

Filippo Occhino*

May 2005

Abstract

How should taxes, government expenditures, the primary and fiscal surpluses and government liabilities be set over the business cycle? We assume that the government chooses expenditures and taxes to maximize the utility of a representative household, utility is increasing in government expenditures, only distortionary labor income taxes are available, and the cycle is driven by exogenous technology shocks. We first consider the commitment case, and characterize the Ramsey equilibrium. In the case that the utility function is constant elasticity of substitution between private and public consumption and separable between the composite consumption good and leisure, taxes, government expenditures and the primary surplus should all be constant positive fractions of production, and both government liabilities and the fiscal surplus should be positively correlated with production. Then, we relax the commitment assumption, and we show how to determine numerically whether the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium. We find that, for realistic values of the preferences discount factor, the Ramsey equilibrium is sustainable.

Keywords: Fiscal policy, Commitment, Time-consistency, Ramsey equilibrium, Markov perfect equilibria, Sustainable equilibria.

JEL Classification Number: E62

1 Introduction

How should taxes, government expenditures, the fiscal and primary surpluses and government liabilities be set over the business cycle? We assume that the government chooses expenditures and taxes to maximize the utility of a representative household, utility is increasing in government expenditures, only distortionary labor income taxes are available, and the cycle is driven by exogenous technology shocks. We also assume that a complete set of one-period Arrow securities are available in each period and state, but securities with longer maturities are not.

*I would like to thank Stefania Albanesi, Roberto Chang, Victor Rios-Rull, Stephanie Schmitt-Grohe, Martin Uribe and seminar participants at Duke University for helpful discussions and suggestions. Address: Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick NJ 08901. E-mail: occhino@rutgers.edu. Web: <http://econweb.rutgers.edu/occhino/>. Any errors are my own.

First, as in Lucas and Stokey (1983), Zhu (1992), and Chari, Christiano and Kehoe (1994), we assume that the government can commit, and characterize the optimal competitive equilibrium or Ramsey equilibrium. Following a suggestion by Kydland and Prescott (1980) and Chang (1998), we add the promised value of government liabilities to the list of state variables, and we express the competitive equilibria recursively as time-invariant functions of the natural state variables (technology and government liabilities) and the additional state variable, for any arbitrary initial value of the additional state variable. More conveniently, we distinguish the first period from the following periods, we express the variables in the first period as functions of the natural state variables only, while we express the variables in the following periods recursively in terms of technology and the additional state variable only. We then characterize the Ramsey equilibrium solving a first-period problem and a continuation problem.

Our main result for the commitment case is that, in the benchmark case that the utility function is constant elasticity of substitution between private and public consumption and separable between the composite consumption good and leisure, taxes, government expenditures and the primary surplus should all be positive constant fractions of production, and both government liabilities and the fiscal surplus should be positively correlated with production.

The intuition is the following. Since households and the government obtain utility from the expected discounted value of public consumption, the government should purchase goods in periods and states where the technology shock and the goods supply are high and the intertemporal price of goods is low, so government expenditures should be increasing in production. Also, since the tax distortion increases more than proportionally with the tax rate, in order to smooth the tax distortion across periods and states the government should smooth the tax rate across periods and states, so taxes should be increasing in production. In the benchmark case, both taxes and government expenditures are constant positive fractions of production, so the primary surplus is a constant fraction of production. The fraction must be positive since the sum of the present discounted value of all primary surpluses is equal to the initial government liabilities, which are positive. Since the primary surplus is a positive constant fraction of production, and government liabilities are equal to the present discounted value of current and future primary surpluses, government liabilities are positively correlated with production. We show that this implies that the fiscal surplus is positively correlated with production as well.

We then relax the commitment assumption, and we show how to determine numerically whether the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium. As a first step, following Kydland and Prescott (1977), Klein and Rios-Rull (2003) and Klein, Krusell and Rios-Rull (2004), we define Markov perfect equilibria, and compare them with the Ramsey equilibrium. Then, adapting and modifying tools developed in Stokey (1989), Chari and Kehoe (1990), Stokey (1991), Chang (1998) and Phelan and Stacchetti (2001), we define recursive sustainable equilibria, and we determine numerically whether the continuation of the Ramsey equilibrium from the second period on can be sustained as a recursive sustainable equilibrium by the threat to revert to a Markov perfect equilibrium. If so, it follows immediately that the Ramsey equilibrium is sustainable. We find that, for realistic values of the preferences discount factor, the Ramsey equilibrium is sustainable.

The focus of this study is the optimal setting of taxes, government expenditures, fiscal and primary surpluses and government liabilities over the business cycle, assuming that the cycle is driven by technology shocks. The two features that distinguish it from other studies are that government expenditures are endogenous, and the government period budget constraint does not necessarily balance. The second feature is necessary for the study of the optimal setting of the primary surplus. With regard to the first feature, we consider an open question how important are the exogenous part of government expenditures and the part endogenously chosen by the government. In our opinion, however, the endogenous part is by far the most important, and accounts for the most sizeable changes like wars and new public programs and investments.

Most previous studies model government expenditures exogenously, and answer a public finance question. Among them, Lucas and Stokey (1983) characterize the optimal labor income tax policy with commitment in a model subject to government expenditures shocks. In addition, they show that the optimal policy with commitment is time-consistent if a complete set of Arrow-Debreu securities for all future periods and states is available in each period and state. Chamley (1986) introduces capital and characterizes the optimal labor and capital income tax policy with commitment. Zhu (1992) Chari, Christiano and Kehoe (1994), and Stockman (1998) characterize the optimal tax policy with commitment in a model subject to government expenditures shocks and technology shocks, while Klein and Rios-Rull (2003) characterize Markov perfect equilibria.

Other studies let government expenditures be determined endogenously, but add the strong assumption that the government budget constraints balance in all periods and states. Kydland and Prescott (1980) focus on the optimal recursive competitive equilibrium, Phelan and Stacchetti (2001) on sequential equilibria, while Klein, Krusell and Rios-Rull (2004) on Markov perfect equilibria.

A common finding in the above public finance literature is that taxes should be approximately proportional to production across periods and states. If the only sources of uncertainty are technology shocks, the primary surplus should increase with production since positive technology shocks increase both production and taxes. However, if the only sources of uncertainty are government expenditures shocks, the primary surplus should decrease with production since positive government expenditures shocks increase government expenditures, production and taxes, but have a larger impact on government expenditures than taxes. The overall sign of the correlation between the primary surplus and production depends on the relative importance of technology shocks and government expenditures shocks. From the perspective of this literature, an additional contribution of this paper is showing that, if the only sources of uncertainty are technology shocks, the primary surplus should increase with production even in the case that government expenditures are chosen endogenously. The result is not an obvious one especially since we also show that government expenditures should increase with production.

In what follows, section 2 describes the model and defines competitive equilibria and implementable allocations. Section 3 studies the Pareto optimum, which characterizes the optimal competitive equilibrium in the case that lump-sum taxes are available. Section 4 assumes that only distortionary labor income taxes are available and studies the Ramsey equilibrium. Section 5 considers Markov perfect equilibria, and section 6 focus on the sustainability of the Ramsey equilibrium. Section 7 concludes.

2 Model

Let the state of the economy be described by the first-order Markov process $\{s_t\}_{t=0}^\infty$, $s_t \in S$, S finite, with transition probabilities $\pi(s_{t+1}|s_t)$. The initial state s_0 is given. Let $s^t \equiv \{s_j\}_{j=0}^t$ be the history of the state up to period t , and let $\pi_t(s^t|s_0)$ be the probability of s^t conditional on s_0 . Let $\xi(s_t)$ be the technology in period t .

In each period t and history s^t , a representative household is endowed with $\bar{n} > 0$ hours, and chooses to work $n_t(s^t) \in [0, \bar{n}]$ hours. Each hour of work produces $\xi(s_t)$ units of a non-storable consumption good, so aggregate production is $y_t(s^t) \equiv \xi(s_t)n_t(s^t)$. Production can be used for private consumption $c_t(s^t) \geq 0$ or public consumption $g_t(s^t) \geq 0$. The feasibility constraints are then

$$c_t(s^t) + g_t(s^t) \leq \xi(s_t)n_t(s^t), \text{ all } t, s^t \quad (1)$$

Preferences are described by

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t|s_0)$$

where $\beta \in (0, 1)$ is the preferences discount factor, and $u(c, n, g)$ is twice continuously differentiable, strictly increasing in its first and third arguments c and g , strictly decreasing in its second argument n , strictly concave, and satisfies the Inada conditions $\lim_{c \rightarrow 0} u_c(c, n, g) = \infty$ for all n, g , $\lim_{n \rightarrow \bar{n}} u_n(c, n, g) = -\infty$ for all c, g , and $\lim_{g \rightarrow 0} u_g(c, n, g) = \infty$ for all c, n . Alternatively, in place of the second condition, one could assume that $\lim_{n \rightarrow +\infty} u_n(c, n, g) = -\infty$ for all c, g , and that \bar{n} is large enough so that $n < \bar{n}$ is optimal in all the following optimization problems.

A complete set of one-period Arrow securities is available in each period and history, but securities with longer maturities are not. Let $q_t(s^t, s_{t+1}) > 0$ be the price of consumption goods in period $t+1$ and history $\{s^t, s_{t+1}\}$ in terms of consumption goods in period t and history s^t , and let the real interest rate $r_t(s^t)$ be defined by

$$\frac{1}{1 + r_t(s^t)} \equiv \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1})$$

Let $b_t(s^t)$ be the households' real assets equal to the government real liabilities in period t and history s^t , and let $b_0(s_0) \geq 0$ be given. Let $\tau_t(s^t) < 1$ be the labor income tax rate in period t and history s^t . The primary surplus

$$\delta_t^p(s^t) \equiv \tau_t(s^t)\xi(s_t)n_t(s^t) - g_t(s^t)$$

is the difference between taxes and government expenditures, while the fiscal surplus

$$\delta_t^f(s^t) \equiv \delta_t^p(s^t) - \left(1 - \frac{1}{1 + r_t(s^t)}\right) b_t(s^t)$$

is the primary surplus minus the interests on government liabilities.

A competitive equilibrium is an allocation $\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \bar{n}], g_t(s^t) \geq 0, b_{t+1}(s^{t+1})\}_{t=0}^\infty$, and a tax and price system $\{\tau_t(s^t) < 1, q_t(s^t, s_{t+1}) > 0\}_{t=0}^\infty$ such that:

- Given $\{g_t(s^t), \tau_t(s^t), q_t(s^{t+1})\}_{t=0}^\infty, \{c_t(s^t), n_t(s^t), b_{t+1}(s^{t+1})\}_{t=0}^\infty$ solves the representative household's problem:

$$\max_{\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \bar{n}], b_{t+1}(s^{t+1})\}_{t=0}^\infty} \sum_{t=0}^\infty \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t | s_0)$$

$$\text{subject to: } c_t(s^t) - [1 - \tau_t(s^t)] \xi(s_t) n_t(s^t) + \sum_{s^{t+1} \in S} q_t(s^{t+1}) b_{t+1}(s^{t+1}) = b_t(s^t), \text{ all } t, s^t$$

$$\lim_{t \rightarrow \infty} \sum_{s^t} \left(\prod_{j=1}^t q_{j-1}(s^j) \right) b_t(s^t) = 0$$

- The government budget constraints are satisfied

$$b_t(s^t) = \tau_t(s^t) \xi(s_t) n_t(s^t) - g_t(s^t) + \sum_{s^{t+1} \in S} q_t(s^{t+1}) b_{t+1}(s^{t+1}), \text{ all } t, s^t$$

- The market for consumption goods is in equilibrium:

$$c_t(s^t) + g_t(s^t) = \xi(s_t) n_t(s^t), \text{ all } t, s^t$$

By Walras' Law, the representative household's budget constraints and the consumption goods market equilibrium conditions imply that the government budget constraints are satisfied.

From the necessary conditions of the household's problem,

$$\begin{aligned} q_t(s^{t+1}) &= \frac{\beta^{t+1} u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1})) \pi_{t+1}(s^{t+1} | s_0)}{\beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t | s_0)} \\ &= \frac{\beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1})) \pi(s_{t+1} | s_t)}{u_c(c_t(s^t), n_t(s^t), g_t(s^t))}, \text{ all } t, s^t \\ [1 - \tau_t(s^t)] \xi(s_t) &= \frac{-u_n(c_t(s^t), n_t(s^t), g_t(s^t))}{u_c(c_t(s^t), n_t(s^t), g_t(s^t))}, \text{ all } t, s^t \end{aligned}$$

These two conditions, evaluated in equilibrium, express the tax and price system as a function of the allocation, and they define the tax and price system associated with a given allocation.

Substituting the previous expressions for prices and tax rates into the household's budget constraints, we obtain the implementability constraints

$$\begin{aligned} &u_c(c_t(s^t), n_t(s^t), g_t(s^t)) [c_t(s^t) - b_t(s^t)] + u_n(c_t(s^t), n_t(s^t), g_t(s^t)) n_t(s^t) \\ &+ \sum_{s^{t+1} \in S} \beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1})) b_{t+1}(s^{t+1}) \pi(s_{t+1} | s_t) = 0, \text{ all } t, s^t \\ &\lim_{t \rightarrow \infty} \sum_{s^t} \beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t)) b_t(s^t) \pi_t(s^t | s_0) = 0 \end{aligned} \tag{2}$$

Definition 1 (Implementable allocations.) An allocation $\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \bar{n}], g_t(s^t) \geq 0, b_{t+1}(s^{t+1})\}_{t=0}^\infty$ is implementable if it satisfies the feasibility constraints 1 with equality and the implementability constraints 2.

One can show that competitive equilibria are implementable allocations together with their associated tax and price systems $\{\tau_t(s^t) < 1, q_t(s^{t+1}) > 0\}_{t=0}^{\infty}$.

Without loss of generality, we focus on implementable allocations for which $b_t(s^t)$ is the following function of current and future consumption and labor,

$$\begin{aligned} & u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t) \\ &= \sum_{j=t}^{\infty} \sum_{s^j} \beta^{j-t} \{u_c(c_j(s^j), n_j(s^j), g_j(s^j))c_j(s^j) + u_n(c_j(s^j), n_j(s^j), g_j(s^j))n_j(s^j)\} \pi_{t,j}(s^j|s^t) \end{aligned}$$

where $\pi_{t,j}(s^j|s^t)$ is the probability of s^j conditional on s^t .

3 Pareto optimum

We begin considering the Pareto optimum, which characterizes the optimal competitive equilibrium in the case that lump-sum taxes are available. A Pareto optimum is a contingent sequence $\{c_t(s^t), n_t(s^t), g_t(s^t)\}_{t=0}^{\infty}$ which solves the following Pareto problem:

$$\begin{aligned} & \max_{\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \bar{n}], g_t(s^t) \geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t|s_0) \\ & \text{subject to: } c_t(s^t) + g_t(s^t) \leq \xi_t(s^t)n_t(s^t), \text{ all } t, s^t \end{aligned}$$

The Pareto optimum solves a sequence of static problems. For comparison with the following sections, we formulate the problem recursively. A recursive Pareto optimum is a set of policy functions $c(s)$, $n(s)$ and $g(s)$ solving the following static optimization problem:

$$\text{For all } s: \max_{\{c \geq 0, n \in [0, \bar{n}], g \geq 0\}} u(c, n, g) \text{ subject to: } c + g \leq \xi(s)n$$

Since u is continuous and strictly concave, and the constrained region is compact and convex, a solution exists and is unique. Since u satisfies the Inada conditions, $c > 0$, $n \in (0, \bar{n})$ and $g > 0$. Since u is strictly increasing in c and g and strictly decreasing in n , the feasibility constraint holds with equality.

Notice that the solution is a time-invariant function of technology $\{c(\xi(s)), n(\xi(s)), g(\xi(s))\}$. Moreover, the previous problem consists in maximizing u subject to a budget constraint where the price of labor in terms of both private and public consumption is $\xi(s)$, and the income is 0. Then, the effect of an increase in technology on consumption and labor is the sum of an income effect and a substitution effect. If private consumption is a normal good, both effects work in the same direction, and private consumption increases, ($dc/d\xi > 0$). The same holds in the case of public consumption ($dg/d\xi > 0$). As a consequence, if both private and public consumption are normal goods, aggregate production, which is equal to the sum of private and public consumption, increases with technology ($dy/d\xi > 0$). Both private and public consumption, then, increase with production. With regard to leisure, however, if leisure is a normal good, the income effect increases leisure, while the substitution effect decreases it. The sign of the total effect depends on the utility function.

The Lagrangian is

$$\mathcal{L} = u(c, n, g) + \mu[\xi(s)n - c - g]$$

where $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraint. The necessary and sufficient conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c} &= u_c(c, n, g) - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial n} &= u_n(c, n, g) + \mu\xi(s) = 0 \\ \frac{\partial \mathcal{L}}{\partial g} &= u_g(c, n, g) - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu} &= \xi(s)n - c - g = 0\end{aligned}$$

which form a system of four equations in the four unknown $\{c, n, g, \mu\}$. For comparison with the next section, notice that, in the Pareto optimum, $u_c = u_g$, $-u_n = \xi u_c$ and $-u_n = \xi u_g$.

The result we are mostly interested is that government expenditures should increase with production. The intuition is that, as long as there is some substitutability between public consumption in different periods and histories, public consumption should be higher in periods and histories where the supply of consumption goods is higher and the intertemporal price of consumption goods is lower.

4 Ramsey equilibrium

We now turn to the analysis of the optimal competitive equilibrium, or Ramsey equilibrium. Following Lucas and Stokey (1983), the allocation of the Ramsey equilibrium maximizes the utility of the representative household among implementable allocations:

$$\max_{\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \bar{n}], g_t(s^t) \geq 0, b_{t+1}(s^{t+1})\}_{t \geq 0}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t | s_0)$$

subject to:

$$\begin{aligned}u_c(c_t(s^t), n_t(s^t), g_t(s^t))[c_t(s^t) - b_t(s^t)] + u_n(c_t(s^t), n_t(s^t), g_t(s^t))n_t(s^t) \\ + \sum_{s_{t+1} \in S} \beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1}))b_{t+1}(s^{t+1})\pi(s_{t+1} | s_t) = 0, \text{ all } t, s^t\end{aligned}$$

$$\lim_{t \rightarrow \infty} \sum_{s^t} \beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t)\pi_t(s^t | s_0) = 0$$

$$c_t(s^t) + g_t(s^t) \leq \xi_t(s^t)n_t(s^t), \text{ all } t, s^t$$

Notice that we write the feasibility constraints with inequality instead of equality. This allows to determine the sign of the multipliers without affecting the solution, since the feasibility constraints are binding at the optimum.

We would like to express the conditions defining implementable allocations (except the limit condition) recursively. We follow a suggestion by Kydland and Prescott (1980) and Chang (1998), and we add the promised value of government liabilities

$$\theta_t(s^t) \equiv u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t), \text{ all } t, s^t$$

to the list of state variables. After substituting the expression for $\theta_t(s^t)$ in the conditions defining implementable allocations, we restrict attention to the implementable allocations which, in the first period, depend on the natural state variables s_t and $b_t(s^t)$, while, in the following periods, depend recursively on the natural state variables s_t and $b_t(s^t)$ as well as the additional state variable $\theta_t(s^t)$. Incidentally, we notice that it would be equivalent (and not more restrictive) to restrict attention to the implementable allocations which, in *all* periods, depend recursively on the natural state variables s_t and $b_t(s^t)$ as well as the additional state variable $\theta_t(s^t)$ for any arbitrary initial value of the additional state variable $\theta_0(s_0)$.

However, for this model, it is possible and convenient to further restrict attention to the following recursive implementable allocations which, from the second period on, do not depend on $b_t(s^t)$:

Definition 2 (Recursive implementable allocations.) *A recursive implementable allocation is a set of first-period functions and continuation functions.*

The first-period functions are $\{c_0(b_0, s_0) \geq 0, n_0(b_0, s_0) \in [0, \bar{n}], g_0(b_0, s_0) \geq 0, \text{ and } \theta_1(b_0, s_0, s_1)\}$ satisfying

$$u_c(c_0, n_0, g_0)b_0(s_0) = u_c(c_0, n_0, g_0)c_0 + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1)\pi(s_1|s_0)$$

$$c_0 + g_0 = \xi(s_0)n_0$$

The continuation functions are allocation functions $\{c(\theta, s) \geq 0, n(\theta, s) \in [0, \bar{n}], g(\theta, s) \geq 0, b(\theta, s)\}$ and a law of motion for the additional state variable $\theta'(\theta, s, s')$ satisfying:

$$\theta = u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta'(s')\pi(s'|s)$$

$$\lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \theta_t(s^t)\pi_t(s^t|s_0) = 0$$

$$c + g = \xi(s)n$$

$$\theta = u_c(c, n, g)b$$

where $\theta_1(s^1) \equiv \theta_1(b_0, s_0, s_1)$, and $\theta_t(s^t)$ is obtained starting from $(\theta_1(s^1), s_1)$ and iterating with the law of motion $\theta'(\theta, s, s')$.

Notice that the first-period functions depend on b and s , while the continuation functions depend on θ and s . Our strategy is to focus on recursive implementable allocations ignoring the limit condition in the definition, characterize the recursive solution, and check that the limit condition is satisfied.

The Ramsey problem, then, can be divided into the following first-period problem and continuation problem.

The continuation problem consists in finding a value function $w(\theta, s)$, and policy functions $\{c(\theta, s) \geq 0, n(\theta, s) \in [0, \bar{n}], g(\theta, s) \geq 0, \theta'(\theta, s, s')\}$ solving the following Bellman equation:

$$\begin{aligned} \text{For all } \theta, s: w(\theta, s) &= \max_{\{c \geq 0, n \in [0, \bar{n}], g \geq 0, \theta'(s')\}} \left\{ u(c, n, g) + \sum_{s' \in S} \beta w(\theta'(s'), s') \pi(s'|s) \right\} \\ \text{subject to: } \theta &= u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta'(s') \pi(s'|s) \\ c + g &\leq \xi(s)n \end{aligned}$$

Once the continuation problem has been solved, government liabilities $b(\theta, s)$ are determined by

$$b(\theta, s) \equiv u_c(c(\theta, s), n(\theta, s), g(\theta, s))/\theta$$

In the numerical examples considered below, for fixed s , the function $b(\theta, s)$ is strictly increasing for values of θ smaller than a threshold (the values corresponding to the good part of the Laffer curve), and strictly decreasing for higher values. We focus on the values of θ smaller than the threshold, we invert $b(\theta, s)$, and we express θ as a function of b and s — Let $\theta(b, s)$ denote the function. Then, we determine the continuation value of the Ramsey equilibrium as a function of b and s by

$$\text{For all } b, s: v^C(b, s) \equiv w(\theta(b, s), s)$$

Given the value function $w(\theta, s)$, the first-period Ramsey problem is

$$\begin{aligned} v^R(b_0(s_0), s_0) &= \max_{\{c_0 \geq 0, n_0 \in [0, \bar{n}], g_0 \geq 0, \theta_1(s_1)\}} \left\{ u(c_0, n_0, g_0) + \sum_{s_1 \in S} \beta w(\theta_1(s_1), s') \pi(s_1|s_0) \right\} \\ \text{subject to: } u_c(c_0, n_0, g_0)b_0(s_0) &= u_c(c_0, n_0, g_0)c_0 + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1) \pi(s_1|s_0) \\ c_0 + g_0 &\leq \xi(s_0)n_0 \end{aligned}$$

4.1 Results

The Lagrangian for the continuation problem is

$$\begin{aligned} \mathcal{L} &= u(c, n, g) + \sum_{s' \in S} \beta w(\theta'(s'), s') \pi(s'|s) \\ &+ \lambda \left[u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta'(s') \pi(s'|s) - \theta \right] + \mu [\xi(s)n - c - g] \end{aligned}$$

where λ is the Lagrange multiplier associated with the implementability constraint, and $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c > 0$, $n \in (0, \bar{n})$ and $g > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= (1 + \lambda)u_c(c, n, g) + \lambda u_{cc}(c, n, g)c + \lambda u_{nc}(c, n, g)n - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial n} &= (1 + \lambda)u_n(c, n, g) + \lambda u_{cn}(c, n, g)c + \lambda u_{nn}(c, n, g)n + \mu \xi(s) = 0 \\
\frac{\partial \mathcal{L}}{\partial g} &= u_g(c, n, g) + \lambda u_{cg}(c, n, g)c + \lambda u_{ng}(c, n, g)n - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \theta'(s')} &= \beta w_\theta(\theta'(s'), s')\pi(s'|s) + \beta \lambda \pi(s'|s) = 0, \text{ all } s' \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta'(s')\pi(s'|s) - \theta = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \xi(s)n - c - g = 0 \\
w_\theta(\theta, s) &= \frac{\partial \mathcal{L}}{\partial \theta} = -\lambda
\end{aligned}$$

The Lagrangian for the first-period problem is

$$\begin{aligned}
\mathcal{L}_0 &= u(c_0, n_0, g_0) + \sum_{s_1 \in S} \beta w(\theta_1(s_1), s')\pi(s_1|s_0) \\
&+ \lambda_0 \left[u_c(c_0, n_0, g_0)[c_0 - b_0(s_0)] + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1)\pi(s_1|s_0) \right] \\
&+ \mu_0[\xi(s_0)n_0 - c_0 - g_0]
\end{aligned}$$

where λ_0 is the Lagrange multiplier associated with the implementability constraint, and $\mu_0 \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c_0 > 0$, $n_0 \in (0, \bar{n})$ and $g_0 > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_0}{\partial c_0} &= (1 + \lambda_0)u_c(c_0, n_0, g_0) + \lambda_0 u_{cc}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{nc}(c_0, n_0, g_0)n_0 - \mu_0 = 0 \\
\frac{\partial \mathcal{L}_0}{\partial n_0} &= (1 + \lambda_0)u_n(c_0, n_0, g_0) + \lambda_0 u_{cn}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{nn}(c_0, n_0, g_0)n_0 + \mu_0 \xi(s_0) = 0 \\
\frac{\partial \mathcal{L}_0}{\partial g_0} &= u_g(c_0, n_0, g_0) + \lambda_0 u_{cg}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{ng}(c_0, n_0, g_0)n_0 - \mu_0 = 0 \\
\frac{\partial \mathcal{L}_i}{\partial \theta_1(s_1)} &= \beta w_\theta(\theta_1(s_1), s_1)\pi(s_1|s_0) + \beta \lambda_0 \pi(s_1|s_0) = 0, \text{ all } s_1 \\
\frac{\partial \mathcal{L}_0}{\partial \lambda_0} &= u_c(c_0, n_0, g_0)[c_0 - b_0(s_0)] + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1)\pi(s_1|s_0) = 0 \\
\frac{\partial \mathcal{L}_0}{\partial \mu_0} &= \xi(s_0)n_0 - c_0 - g_0 = 0 \\
v_b^R(b_0(s_0), s_0) &= \frac{\partial \mathcal{L}}{\partial b_0(s_0)} = -\lambda_0 u_c(c_0, n_0, g_0)
\end{aligned}$$

From the necessary conditions of the first-period problem,

$$-\lambda_0(b_0(s_0), s_0) = w_\theta(\theta_1(s_1), s_1), \text{ all } s_1$$

so $w_\theta(\theta_1(s_1), s_1)$ is constant and equal to λ_0 for all s_1 . From the necessary conditions of the continuation problem,

$$w_\theta(\theta, s) = -\lambda(\theta, s) = w_\theta(\theta'(\theta, s, s'), s'), \text{ all } s'$$

It follows that $w_\theta(\theta, s)$ is constant along the equilibrium path, i.e. for all the values of (θ, s) which are reached starting from $(\theta_1(s_1), s_1)$, all s_1 , and iterating on the law of motion $\theta'(\theta, s, s')$ for all s' . Also, $\lambda(\theta, s)$ is constant and equal to λ_0 along the equilibrium path as well. Then, recalling that $b_0(s_0) \geq 0$ and $g_t(s^t) > 0$ all t and s^t , one can show that $\lambda > 0$ following the same argument as in Lucas and Stokey (1983).

Also, along the equilibrium path, $\theta'(\theta, s, s')$ is not a function of θ or s . It follows that, along the equilibrium path and from the second period on, it is possible to express the current value of θ as a function of the current value of s — Let $\theta(s)$ denote the function. The limit condition then becomes

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t)) b_t(s^t) \pi_t(s^t | s_0) &= 0 \\ \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \theta_t(s^t) \pi_t(s^t | s_0) &= 0 \\ \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \theta(s_t) \pi_t(s^t | s_0) &= 0 \end{aligned}$$

and is satisfied since $\theta(s_t)$ takes a finite set of values and is therefore bounded, while β^t converges to zero.

Comparing the necessary conditions of the first-period and continuation problems, and recalling that $\lambda = \lambda_0$, it follows that period 0 is special whenever $b_0(s_0) > 0$. The reason is that the real government liabilities in period 0 are given and are financed through distortionary taxes in other periods and histories. To minimize the tax distortions, the government has an incentive to decrease the price of consumption goods in period 0 relative to other periods and histories. It also follows that the continuation of the Ramsey equilibrium is not the Ramsey equilibrium for the continuation economy, so the Ramsey problem is not recursive. In particular, $v^R(b, s) > v^C(b, s)$. Then, in the economy with commitment, the optimal policy is not time-consistent. Below, we show that the Ramsey equilibrium can be associated with an optimal (time-consistent) policy in the economy without commitment.

In what follows, we restrict attention to the equilibrium path from the second period on. For convenience, we rewrite the following necessary conditions of the continuation problem:

$$\begin{aligned} (1 + \lambda)u_c(c, n, g) + \lambda u_{cc}(c, n, g)c + \lambda u_{nc}(c, n, g)n - \mu &= 0 \\ (1 + \lambda)u_n(c, n, g) + \lambda u_{cn}(c, n, g)c + \lambda u_{nn}(c, n, g)n + \mu \xi(s) &= 0 \\ u_g(c, n, g) + \lambda u_{cg}(c, n, g)c + \lambda u_{ng}(c, n, g)n - \mu &= 0 \\ \xi(s)n - c - g &= 0 \end{aligned} \tag{3}$$

Given the constant λ , the previous conditions form a system of four equations in the four unknown c , n , g and μ .

Assuming that the solution exists and is unique, it is a time-invariant function of technology $\{c(\xi(s)), n(\xi(s)), g(\xi(s)), \mu(\xi(s))\}$, i.e. it does not depend on θ and it depends on s only through technology $\xi(s)$. It follows that the marginal utility of consumption $u_c(\xi(s))$, production $y(\xi(s))$ the tax rate $\tau(\xi(s))$ and the primary surplus $\delta^p(\xi(s))$ are time-invariant functions of technology as well. Prices $q(s, s')$ depend on the current state s as well as the next period state s' , while the real interest rate $r(s)$ depends on the current state s . Also, recalling that $b_t(s^t)$ is a function of current and future consumption and labor, government liabilities $b(s)$ are time-invariant functions of the current state s , and so is the fiscal surplus $\delta^f(s)$. Clearly, if $\xi(s)$ is a one-to-one function, functions of the state can be expressed as functions of technology. We emphasize that, from the second period on, all the variables of interest in this economy are time-invariant functions of the current state s , so they are all perfectly correlated with each other.

It is instructive to compare some properties of the continuation of the Ramsey equilibrium and the Pareto optimum, in the case that u is separable in its three arguments. The necessary conditions 3 imply that, in the Ramsey equilibrium, $-(1+\lambda)u_n - \lambda u_{nn}n = \xi u_g$, so $-u_n < \xi u_g$, while $-u_n = \xi u_g$ in the Pareto optimum. Also, $-(1+\lambda)u_n - \lambda u_{nn}n = (1+\lambda)\xi u_c + \lambda \xi u_{cc}c$, so $-u_n < \xi u_c$, while $-u_n = \xi u_c$ in the Pareto optimum. Then, recalling that $(1-\tau)\xi u_c = -u_n$, the tax rate τ is strictly positive. Intuitively, the distortionary labor income tax tends to discourage labor, production, private consumption and public consumption relative to the Pareto optimum. Finally, $(1+\lambda)u_c + \lambda u_{cc}c = u_g$, so the sign of $u_g - u_c$ is the same as the sign of $u_c + u_{cc}c$, while $u_c = u_g$ in the Pareto optimum.

4.2 Numerical solution

The results so far suggest the following efficient strategy for solving numerically the Ramsey problem.

First, consider the continuation problem. Using the necessary conditions 3, c , n , g and μ are expressed as functions of λ and s — Let $c(\lambda, s)$, $n(\lambda, s)$, $g(\lambda, s)$ and $\mu(\lambda, s)$ denote the four functions. Then, recalling that λ is constant along the equilibrium path, the value ω associated with λ and s is determined by the functional equation

$$\text{For all } \lambda, s: \omega(\lambda, s) = u(c(\lambda, s), n(\lambda, s), g(\lambda, s)) + \sum_{s' \in S} \beta \omega(\lambda, s') \pi(s'|s)$$

Notice that, for fixed λ , the previous is a simple linear system of equations in the unknowns $\omega(\lambda, s)$ for all $s \in S$. Let $\underline{\omega}(\lambda)$ be the column vector of the unknowns $\omega(\lambda, s)$, let $\underline{u}(\lambda)$ be the column vector of the constants $u(c(\lambda, s), n(\lambda, s), g(\lambda, s))$, and let P be the transition matrix of the state. Then,

$$\begin{aligned} \underline{\omega}(\lambda) - \beta P \underline{\omega}(\lambda) &= \underline{u}(\lambda) \\ \underline{\omega}(\lambda) &= (I - \beta P)^{-1} \underline{u}(\lambda) \end{aligned}$$

where I is the identity matrix.

The next step is to determine the function $\lambda(\theta, s)$. However, it is convenient to first determine θ as a function of λ and s . Recall that, along the equilibrium path and from the second period on, the current value of θ is a function of the current value of s . Then, the function $\theta(\lambda, s)$ is determined by the functional equation

$$\begin{aligned} \text{For all } \lambda, s: & u_c(c(\lambda, s), n(\lambda, s), g(\lambda, s))c(\lambda, s) + u_n(c(\lambda, s), n(\lambda, s), g(\lambda, s))n(\lambda, s) \\ & + \sum_{s' \in S} \beta \theta(\lambda, s') \pi(s'|s) - \theta(\lambda, s) = 0 \end{aligned}$$

Here again, for fixed λ , the previous is a simple linear system of equations in the unknowns $\theta(\lambda, s)$ for all $s \in S$. Let $\underline{\theta}(\lambda)$ be the column vector of the unknowns $\theta(\lambda, s)$, let $\underline{u}_c(\lambda)\underline{c}(\lambda) + \underline{u}_n(\lambda)\underline{n}(\lambda)$ be the column vector of the constants, and let P be the transition matrix of the state. Then,

$$\begin{aligned} \underline{\theta}(\lambda) - \beta P \underline{\theta}(\lambda) &= \underline{u}_c(\lambda)\underline{c}(\lambda) + \underline{u}_n(\lambda)\underline{n}(\lambda) \\ \underline{\theta}(\lambda) &= (I - \beta P)^{-1} [\underline{u}_c(\lambda)\underline{c}(\lambda) + \underline{u}_n(\lambda)\underline{n}(\lambda)] \end{aligned}$$

where I is the identity matrix.

Notice that, for fixed s , $\theta(\lambda, s)$ may have the properties of a Laffer curve, i.e. it may be inverted-U shaped, so the correspondence $\lambda(\theta, s)$ may have two values for small values of θ , and no values for large values of θ . This is one reason why it is convenient to first characterize $\theta(\lambda, s)$ rather than $\lambda(\theta, s)$. In the case that $\theta(\lambda, s)$ has the properties of a Laffer curve, we focus on the good part of the Laffer curve, i.e. the part of the curve corresponding to small values of λ until the maximum value of θ is reached. On that part, we invert the function $\theta(\lambda, s)$, and obtain $\lambda(\theta, s)$. Alternatively, for fixed s , $\theta(\lambda, s)$ may be a strictly increasing function for all $\lambda > 0$, as it occurs in the numerical examples considered below. In this case, we invert the function $\theta(\lambda, s)$ for all $\lambda > 0$. Once $\lambda(\theta, s)$ has been obtained, the value function is determined by $w(\theta, s) = \omega(\lambda(\theta, s), s)$, and similarly the associated policy functions.

Government liabilities b as a function of λ and s are determined by

$$b(\lambda, s) = \theta(\lambda, s) / u_c(c(\lambda, s), n(\lambda, s), g(\lambda, s))$$

For fixed s , $b(\lambda, s)$ may have the properties of a Laffer curve, as it occurs in the numerical example considered later in this section. Then, we focus on the good part of the Laffer curve, invert the function $b(\lambda, s)$ on that part, obtain $\lambda(b, s)$, and determine the value function by $v^C(b, s) = \omega(\lambda(b, s), s)$.

Once $w(\theta, s)$ has been obtained, determining the value function $v^R(b, s)$ and the associated policy functions is a simple static optimization problem. We substitute the two constraints into the objective function and use a grid search method. Alternatively, one can use the necessary conditions to determine c , g , n , μ and b as functions of (λ, s) . We find that this second method works only for positive values of b .

4.3 Benchmark economy

We now turn to the case that the utility function is

$$u(c, n, g) \equiv \begin{cases} A \frac{f(c, g)^{1-\sigma} - 1}{1-\sigma} - \Phi \frac{n^{1+\varphi}}{1+\varphi} & \sigma > 0, \sigma \neq 1 \\ A \log(f(c, g)) - \Phi \frac{n^{1+\varphi}}{1+\varphi} & \sigma = 1 \end{cases}$$

$$f(c, g) \equiv \begin{cases} \left(\alpha c^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)g^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} & \epsilon > 0, \epsilon \neq 1 \\ c^\alpha g^{1-\alpha} & \epsilon = 1 \end{cases}$$

where $A > 0$, $\alpha \in (0, 1)$, $\Phi > 0$ and $\varphi > 0$. The utility function is separable between the composite consumption good and labor. The elasticity of substitution between private and public consumption is constant and equal to ϵ . Although $\lim_{n \rightarrow \bar{n}} u_n(c, n, g) = -\Phi \bar{n}^\varphi > -\infty$ we assume that \bar{n} is large enough so that the solution is still described by the necessary conditions 3.

Let us start considering the simpler case $\epsilon = 1$. In this case, $u_{cc}c = [\alpha(1-\sigma) - 1]u_c$, $u_{gc}c = \alpha(1-\sigma)u_g$, and $u_{nn}n = \varphi u_n$, so the necessary conditions 3 become

$$\begin{aligned} [1 + \lambda\alpha(1-\sigma)]u_c - \mu &= 0 \\ (1 + \lambda + \lambda\varphi)u_n + \mu\xi &= 0 \\ [1 + \lambda\alpha(1-\sigma)]u_g - \mu &= 0 \\ \xi(s)n - c - g &= 0 \end{aligned}$$

It is easy to show that λ satisfies $1 + \lambda\alpha(1-\sigma) > 0$.

Notice that $u_c = u_g$ like in the Pareto optimum. However, $-(1 + \lambda + \lambda\varphi)u_n = \xi[1 + \lambda\alpha(1-\sigma)]u_c$, so $-u_n < \xi u_c = \xi u_g$, while $-u_n = \xi u_c = \xi u_g$ in the the Pareto optimum.

The tax rate is determined by

$$1 - \tau = \frac{-u_n}{\xi u_c} = \frac{1 + \lambda\alpha(1-\sigma)}{1 + \lambda + \lambda\varphi} < 1$$

so the tax rate is strictly positive and constant along the equilibrium path. Taxes $\tau\xi n$ are then a strictly positive constant fraction τ of production. Moreover, from $u_c = u_g$, it follows that private consumption c and public consumption g are respectively strictly positive constant fractions α and $1 - \alpha$ of production. The primary surplus δ^p is then a constant fraction $\tau - (1 - \alpha)$ of production.

From $c = \alpha\xi n$ and $g = (1 - \alpha)\xi n$, it follows that u_c is equal to $B(\xi n)^{-\sigma}$, where $B \equiv A(\alpha^\alpha(1-\alpha)^{1-\alpha})^{1-\sigma} > 0$, so the marginal utility of consumption is a strictly decreasing function of production. Then,

$$\begin{aligned} (1 + \lambda + \lambda\varphi)(-u_n) &= [1 + \lambda\alpha(1-\sigma)]u_c\xi \\ (1 + \lambda + \lambda\varphi)\Phi n^\varphi &= [1 + \lambda\alpha(1-\sigma)]B(\xi n)^{-\sigma}\xi \\ (1 + \lambda + \lambda\varphi)\Phi n^{\varphi+\sigma} &= [1 + \lambda\alpha(1-\sigma)]B\xi^{1-\sigma} \end{aligned}$$

so labor n is a strictly increasing (decreasing) function of technology ξ if and only if $1 - \sigma > 0$ ($1 - \sigma < 0$). Also,

$$\begin{aligned} (1 + \lambda + \lambda\varphi)\Phi n^\varphi &= [1 + \lambda\alpha(1-\sigma)]B(\xi n)^{-\sigma}\xi \\ (1 + \lambda + \lambda\varphi)\Phi(\xi n)^{\varphi+\sigma} &= [1 + \lambda\alpha(1-\sigma)]B\xi^{1+\varphi} \end{aligned}$$

so production ξn is a strictly increasing function of technology ξ . Collecting the results so far, production ξn , private consumption c , public consumption g and taxes $\tau \xi n$ are strictly increasing functions of technology ξ , while the marginal utility of consumption u_c is a strictly decreasing function of technology ξ .

We now turn to the sign of the primary surplus δ^p and of government liabilities b . In the simple case that $b_0(s_0) = 0$, period 0 should be treated as all other periods and histories, so the sign of the primary surplus is the same in all periods and histories. Since initial government liabilities $b_0(s_0)$ are equal to the sum of the discounted value of all primary surpluses, the primary surplus is equal to zero in all periods and histories. Then, since government liabilities $b_t(s^t)$ are equal to the sum of the discounted value of all primary surpluses from period t on, government liabilities $b_t(s^t)$ are equal to zero in all periods and histories.

Suppose now that $b_0(s_0) > 0$ implies $b_1(s^1) > 0$ for at least one history s^1 . This is indeed the case in the numerical examples considered below. With arguments analogous to the previous paragraph, since $b_1(s^1)$ is equal to the sum of the discounted value of all primary surpluses from the second period on, and since the sign of the primary surplus is the same from period 1 on, the primary surplus is strictly positive from period 1 on. In turn, this implies that the primary surplus δ^p is a strictly positive fraction of production, so it is a strictly increasing function of technology ξ . Also, since government liabilities $b_t(s^t)$ are equal to the sum of the discounted value of all primary surpluses from period t on, government liabilities $b_t(s^t)$ are strictly positive from period 1 on. In the case that $b_0(s_0) < 0$, the conclusions in this paragraph are reverted.

To determine the real interest rate, government liabilities and the fiscal surplus, we now restrict the transition function assuming that the transition matrix P is a weighted average of the identity matrix I and a stochastic matrix Q with identical rows:

$$P = \rho I + (1 - \rho)Q,$$

where $\rho \in [0, 1)$. Equivalently, letting $\pi_Q(s_{t+1})$ be a probability mass function over the state,

$$\pi(s_{t+1}|s_t) = \rho \chi(s_{t+1} = s_t) + (1 - \rho)\pi_Q(s_{t+1})$$

where χ is an indicator function equal to 1 if $s_{t+1} = s_t$ and 0 otherwise. The process for the state, then, is an average of a perfectly serially correlated process and an i.i.d. process.

With this transition function, the intertemporal price of consumption goods is determined by

$$\begin{aligned} q_t(s^{t+1}) &= \frac{\beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1})) \pi(s_{t+1}|s_t)}{u_c(c_t(s^t), n_t(s^t), g_t(s^t))} \\ q(s_t, s_{t+1}) &= \frac{\beta u_c(\xi(s_{t+1})) \pi(s_{t+1}|s_t)}{u_c(\xi(s_t))} \\ q(s_t, s_{t+1}) &= \beta \left[\rho \chi(s_{t+1} = s_t) + (1 - \rho) \frac{u_c(\xi(s_{t+1})) \pi_Q(s_{t+1})}{u_c(\xi(s_t))} \right] \end{aligned}$$

where $u_c(\xi) = B(\xi n(\xi))^{-\sigma}$.

Then, the real interest rate is determined by

$$\frac{1}{1+r_t(s^t)} = \sum_{s_{t+1} \in S} q_t(s^{t+1})$$

$$\frac{1}{1+r(\xi(s_t))} = \beta \left[\rho + (1-\rho) \frac{E_Q(u_c)}{u_c(\xi(s_t))} \right]$$

where $E_Q(u_c) \equiv \sum_{s_{t+1} \in S} u_c(\xi(s_{t+1}))\pi_Q(s_{t+1})$ is a strictly positive constant since the marginal utility of consumption u_c is strictly positive. Since the marginal utility of consumption u_c is a strictly decreasing function of technology ξ , the real interest rate is a strictly decreasing function of technology ξ as well.

Moreover, recalling that government liabilities are time-invariant functions $b(s_t)$ of the state, government liabilities are determined by

$$b_t(s^t) = \tau_t(s^t)\xi(s_t)n_t(s^t) - g_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^{t+1})b_{t+1}(s^{t+1})$$

$$b(s_t) = \delta^p(\xi(s_t)) + \sum_{s_{t+1} \in S} q(s_t, s_{t+1})b(s_{t+1})$$

$$b(s_t) = \delta^p(\xi(s_t)) + \beta \left[\rho b(s_t) + (1-\rho) \frac{E_Q(u_c b)}{u_c(\xi(s_t))} \right]$$

$$(1-\beta\rho)b(\xi(s_t)) = \delta^p(\xi(s_t)) + \beta(1-\rho) \frac{E_Q(u_c b)}{u_c(\xi(s_t))}$$

where $E_Q(u_c b) \equiv \sum_{s_{t+1} \in S} u_c(\xi(s_{t+1}))b(s_{t+1})\pi_Q(s_{t+1})$ is a constant. In the case that $b_0(s_0) > 0$, recall that government liabilities b are strictly positive, so $E_Q(u_c b) > 0$, and the primary surplus δ^p is a strictly increasing function of technology ξ . Then, since the marginal utility of consumption u_c is a strictly decreasing function of technology ξ , government liabilities b are a strictly increasing function of technology ξ and production.

Furthermore, using the definition of fiscal surplus and the government budget constraints, one can derive the following convenient expression for the fiscal surplus:

$$\delta_t^f(s^t) = \sum_{s_{t+1} \in S} q_t(s^{t+1}) (b_t(s^t) - b_{t+1}(s^{t+1}))$$

$$\delta^f(s_t) = \sum_{s_{t+1} \in S} q(s_t, s_{t+1}) (b(\xi(s_t)) - b(\xi(s_{t+1})))$$

$$\delta^f(\xi(s_t)) = \sum_{s_{t+1} \in S} \beta(1-\rho) \frac{u_c(\xi(s_{t+1}))\pi_Q(s_{t+1})}{u_c(\xi(s_t))} (b(\xi(s_t)) - b(\xi(s_{t+1})))$$

$$\delta^f(\xi(s_t)) = \beta(1-\rho) \frac{E_Q(u_c)b(\xi(s_t)) - E_Q(u_c b)}{u_c(\xi(s_t))}$$

Recall that $E_Q(u_c) > 0$ and $E_Q(u_c b) > 0$, government liabilities b are strictly increasing in technology ξ , while the marginal utility of consumption u_c is strictly positive and strictly decreasing in technology ξ . Although the fiscal surplus is not necessarily increasing in technology ξ , it is strictly positive for values of technology above a threshold, and strictly negative

for values of technology below the threshold. This suggests that the fiscal surplus tends to be positively correlated with technology and production. For instance, in the case that there are only two states $s_1 < s_2$, with $\xi(s_1) < \xi(s_2)$, government liabilities take a strictly smaller value when the state is s_1 than when it is s_2 . Then, δ^f is strictly negative when the state is s_1 , and strictly positive when the state is s_2 . In this case, the fiscal surplus is indeed strictly increasing in technology ξ and production.

Most of the previous conclusions hold in the general case of constant elasticity of substitution ϵ . The proof is rather long but straightforward. First, using the necessary conditions, one guesses and verifies that private consumption c and public consumption g are strictly positive constant fractions of production, summing up to one. The fractions are constant in the sense that they do not depend on technology ξ , but they do depend on the constant multiplier λ as well as the parameters of the utility function σ and ϵ . Under the guess, u_c , $u_{cc}c$, u_g and $u_{gg}g$ are all proportional to $y^{-\sigma}$. All the other conclusions, except that u_c is in general different from u_g , follow from the same arguments as in the case of unitary elasticity. Taxes and the primary surplus are strictly positive constant fractions of production. Production is strictly increasing in technology. Government liabilities are strictly increasing in technology and production, and the fiscal surplus tends to be positively correlated with technology and production.

4.4 Numerical example

The following numerical example document the previous conclusions. It has been solved using the numerical strategy described above. The utility function is

$$u(c, n, g) \equiv A[\alpha \log(c) + (1 - \alpha) \log(g)] - \Phi \frac{n^{1+\varphi}}{1 + \varphi}$$

with $A = 1$, $\alpha = 0.75$, $\Phi = 1$ and $\varphi = 1$. Total available hours are $\bar{n} = 4$. The preferences discount factor is $\beta = 0.99$. The state space is $S = \{s_1, s_2\}$, and the transition matrix is $[0.95, 0.05; 0.05, 0.95]$. Technology is $\xi(s_1) = 0.9$ and $\xi(s_2) = 1.1$. The parameter values are chosen so that, approximately, in the deterministic steady state, the ratio of public consumption to private consumption is 1/3, the ratio of labor to total available hours is 1/4, the labor supply elasticity is 1, the real interest rate is 4%, and the first-order serial correlation of technology is 0.9. We chose an unrealistically high value for the standard deviation in order to plot clearer figures. The utility is separable in its three arguments, and the elasticity of substitution between private and public consumption is constant and equal to one.

Let us first focus on the solution of the continuation problem. Figures 1 and 2 plot several functions of λ for each level of s . Recall that λ is constant along the equilibrium path. Then, all the variables of interest are perfectly correlated with each other, and, for each fixed λ , the sign of the correlation can be easily inferred from the figures. Notice that we also consider small values of λ corresponding to negative values of initial government liabilities $b_0(s_0)$.

For this economy, $(1 + \lambda + \lambda\varphi)\Phi n^{\varphi+1} = 1$, so labor n does not depend on s . Aggregate production ξn is the product of technology and labor. Private and public consumption are respectively constant fractions α and $1 - \alpha$ of aggregate production. The multiplier μ is equal to the inverse of aggregate production.

The tax rate τ is constant and equal to $(\lambda + \lambda\varphi)/(1 + \lambda + \lambda\varphi)$. The ratio of the primary surplus δ^p to aggregate production is then constant and equal to $\tau - (1 - \alpha)$. Notice that there is a threshold value λ^* (equal to $1/6$ in this numerical example) such that $\tau - (1 - \alpha)$ is a strictly negative constant for $\lambda < \lambda^*$, and a strictly positive constant for $\lambda > \lambda^*$. Then, for $\lambda < \lambda^*$, the primary surplus is negative and strictly decreasing in production, while for $\lambda > \lambda^*$ it is positive and strictly increasing in production. The same applies to debt b . For $\lambda > \lambda^*$, the fiscal surplus is positive when production is high, and negative when technology and production is low, while the reverse is true for $\lambda < \lambda^*$.

One can show that, since utility is logarithmic, the value of debt $\theta(\lambda, s)$ does not depend on s , and is strictly increasing for all $\lambda > 0$, so it does not have the properties of a Laffer curve. However, the last two panels consider a larger domain for λ , and show that both the primary surplus δ^p and the debt b have the properties of a Laffer curve. Focusing on values of λ lower than the value that maximizes the primary surplus (equal to 1.5 in this numerical example) amounts to restricting attention to the good part of the Laffer curve.

Let us now turn to the first-period problem, and compare its optimal policies with the ones of the continuation problem. Figures 3 and 4 plot the relevant variables as functions of debt d for each level of s , for the continuation problem. Figures 5 and 6 plot the relevant variables as functions of debt d for each level of s , for the first-period problem.

There are important qualitative differences. In the continuation problem, the higher government liabilities, the higher taxes and the primary surplus, in accordance with the general principle that the tax distortion should be smoothed across periods and states. Then, the higher the tax rate, the lower labor, production and consumption.

The first-period problem, however, is a one-period problem taking current government liabilities as given. In the case that government liabilities are strictly positive, since current government liabilities must be financed by (current and) future surpluses, the higher real government liabilities b , the stronger the governments incentive to decrease the intertemporal price of current consumption goods in terms of future consumption goods. Hence, the higher government liabilities, the stronger the government's incentive to decrease the tax rate, to increase labor, production and consumption, and to decrease the marginal utility of current consumption. In the numerical example, the incentive outweighs the general principle that the tax distortion should be smoothed across periods and states.

This, however, needs not be true in general. The lower the preferences discount factor β , the more important becomes the general principle of tax smoothing relative to the government's incentive to decrease the intertemporal price of current consumption goods in terms of future consumption goods. For instance, figures 7 and 8 plot the relevant variables as functions of debt d for each level of s , for the first-period problem, for the case that $\beta = 0.5$. In this case, the policies of the first-period problem are qualitative similar to those of the continuation problem. In particular, the higher government liabilities, the higher the tax rate.

5 Markov perfect equilibria

We now turn to the study of the optimal policy without commitment. In each period t , first the state variable s_t , the technology $\xi(s_t)$ and the government liabilities $b_t(s^t)$ are

realized, then the government chooses the tax rate τ_t and public consumption g_t , and finally the households choose private consumption c_t , labor n_t and next-period contingent assets $b_{t+1}(s^{t+1})$.

In this section, we focus on Markov perfect equilibria, studied by Klein and Rios-Rull (2003) and Klein, Krusell and Rios-Rull (2004), and compare them with the Ramsey equilibrium. In the next section, we show how to determine whether the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium.

Borrowing a concept in Kydland and Prescott (1977), the policy associated with a Markov perfect equilibrium is the optimal policy given the past choices of households and the government (given the current situation), and given that the future policy is chosen in the same way. Markov perfect equilibria are sustainable equilibria which depend recursively on the natural state variables only, namely s_t and $b_t(s^t)$. Hence, they can be sustained without any threat to revert to other sustainable equilibria. They are also the limit as the time horizon goes to infinity of the sustainable equilibria of the finite-horizon economy.

A Markov perfect equilibrium is a set of a value function $v(b, s)$, associated policy functions $c(b, s)$, $n(b, s)$, $g(b, s)$, $b'(b, s, s')$, and a *future* policy function $\theta(b, s)$ to be adopted in all future periods, which satisfy

- Given the future policy function $\theta(b, s)$, the value function $v(b, s)$ and its associated policy functions solve the Bellman equation:

$$\text{For all } b, s: v^M(b, s) = \max_{\{c \geq 0, n \in [0, \bar{n}], g \geq 0, b'(s')\}} \left\{ u(c, n, g) + \sum_{s' \in S} \beta v^M(b'(s'), s') \pi(s'|s) \right\}$$

$$\text{subject to: } u_c(c, n, g)b = u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta(b'(s'), s') \pi(s'|s)$$

$$c + g \leq \xi(s)n$$

- The future and the current policy functions are the same:

$$\text{For all } b, s: \theta(b, s) = u_c(c(b, s), n(b, s), g(b, s))b$$

In the Ramsey equilibrium, the government chooses optimally a unique set of policy functions to be adopted both in the current period and in all future periods. In particular, the government chooses the future policy functions taking into full account how the current households' choices depend on the future policy functions. In a Markov perfect equilibrium, however, the government chooses optimally a set of policy functions to be adopted only in the current period, for a given set of policy functions to be adopted in all future periods. Only after the current period optimization problem has been solved, the requirement that current and future policy functions be the same is added. It might help the intuition considering the government in the current period as a different agent from the government in all future periods, so it cannot choose the policy functions to be adopted in all future periods. The future policy functions will be chosen by the future government taking as given the choices in the current period, and ignoring their dependence on the future policy functions. The value of a Markov perfect equilibrium is, then, lower, in general, than the value of the Ramsey equilibrium.

5.1 Results

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= u(c, n, g) + \sum_{s' \in S} \beta v^M(b'(s'), s') \pi(s'|s) \\ &+ \lambda \left[u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta(b'(s'), s') \pi(s'|s) - u_c(c, n, g)b \right] \\ &+ \mu [\xi(s)n - c - g] \end{aligned}$$

where λ is the Lagrange multiplier associated with the implementability constraint, and $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c > 0$, $n \in (0, \bar{n})$ and $g > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= (1 + \lambda)u_c(c, n, g) + \lambda u_{cc}(c, n, g)(c - b) + \lambda u_{nc}(c, n, g)n - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial n} &= (1 + \lambda)u_n(c, n, g) + \lambda u_{cn}(c, n, g)(c - b) + \lambda u_{nn}(c, n, g)n + \mu \xi(s) = 0 \\ \frac{\partial \mathcal{L}}{\partial g} &= u_g(c, n, g) + \lambda u_{cg}(c, n, g)(c - b) + \lambda u_{ng}(c, n, g)n - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial b'(s')} &= \beta v_b^M(b'(s'), s') \pi(s'|s) + \beta \lambda \theta_b(b'(s'), s') \pi(s'|s) = 0 \text{ all } s' \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta(b'(s'), s') \pi(s'|s) - u_c(c, n, g)b = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu} &= \xi(s)n - c - g = 0 \\ v_b^M(b, s) &= \frac{\partial \mathcal{L}}{\partial b} = -\lambda u_c(c, n, g) \end{aligned}$$

Also, from the Markov perfect equilibrium definition,

$$\text{For all } b, s: \theta(b, s) = u_c(c(b, s), n(b, s), g(b, s))b$$

$$\theta_b(b, s) = u_c(c(b, s), n(b, s), g(b, s)) + \frac{du_c(c(b, s), n(b, s), g(b, s))}{db} b$$

Relative to the Ramsey equilibrium, it is harder to characterize how variables co-vary with technology. First, λ is not constant any more but depends on b and s . Also, government liabilities b appear in the necessary conditions. Next-period government liabilities $b'(b, s, s')$ depend on the next-period state s' as well, i.e. government liabilities are state-contingent and vary with technology. Hence, as s and $\xi(s)$ vary, b varies as well, and the full effect of s and $\xi(s)$ on any variable is the sum of its direct effect and its indirect effect through b .

The following observations, however, may be helpful. On one hand, the higher real government liabilities b , the higher the tax rate, the lower labor, production and consumption, the higher the marginal utility of consumption u_c . On the other hand, since current government liabilities b must be financed by (current and) future surpluses, the higher real government liabilities b , the stronger the governments incentive to increase labor and consumption

and decrease the marginal utility of consumption u_c , in order to decrease the intertemporal price of current consumption goods. This incentive is stronger when government liabilities are higher, and disappears when they are equal to zero. Hence, the sign of the term $du_c = db$ in the last equation is positive, at least for values of b sufficiently close to zero.

Then, at least for small values of b , from the previous conditions,

$$\begin{aligned} \frac{-v_b^M(b, s)}{u_c(c, n, g)} = \lambda(b, s) &= \frac{-v_b^M(b'(s'), s')}{\theta_b(b'(s'), s')} \text{ all } s' \\ &= \frac{-v_b^M(b'(s), s)}{\theta_b(b'(s), s)} \\ &= \frac{-v_b^M(b'(s), s) u_c(b'(s), s)}{u_c(b'(s), s) \theta_b(b'(s), s)} < \frac{-v_b^M(b'(s), s)}{u_c(b'(s), s)} = \lambda(b'(s), s) \end{aligned}$$

which shows that $\lambda(b, s)$, a measure of tax distortion, increases over time for any fixed s . Since $\lambda(b, s)$ is strictly increasing in b on the good part of the Laffer curve, government liabilities increase over time.

Recall that government liabilities b are stationary in the Ramsey equilibrium. Relative to the Ramsey policy, Markov perfect policies take b as given and decrease u_c in order to decrease the intertemporal value of b . This is implemented through a lower tax rate leading to higher labor, production and consumption and a lower marginal utility of consumption u_c relative to the Ramsey policy. As a result of the lower tax rate, future government liabilities b are higher relative to the Ramsey policy, and therefore government liabilities increase over time. Notice that the difference between the Ramsey policy and Markov perfect policies is that the Ramsey policy commits to a fixed current u_c , while Markov perfect policies manipulate it. Indeed, if u_c were held fixed in the expression $\theta = u_c b$, then $\theta_b = u_c$, so $\lambda(b, s) = \lambda(b'(s), s)$, and the Markov perfect policy would be the same as the Ramsey policy.

A Markov perfect equilibrium can be computed with the following two methods. Given an initial function $\theta(b, s)$, one can determine the other functions by the standard value function iterations method. Then, one can use the just obtained functions to update the function $\theta(b, s)$, and iterate until convergence. Alternatively, given initial functions $\theta(b, s)$ and $v^M(b, s)$, obtain the other functions solving the simple static optimization problem in the definition of Markov perfect equilibrium. Then, using the just obtained functions, update both functions $\theta(b, s)$ and $v^M(b, s)$ simultaneously, and iterate until convergence. This second method corresponds to computing the Markov perfect equilibrium of the infinite horizon economy as the limit as the time horizon goes to infinity of the Markov perfect equilibrium of the finite horizon economy.

We have used the second method to compute a Markov perfect equilibrium for the same numerical example considered at the end of section 4. However, we have restricted the choice of consumption c to be strictly below $2/3\xi(s)$. This amounts to constraining the government not to depart too far from the continuation of the Ramsey equilibrium where consumption is below $0.75\xi(s)$. In particular, the government never chooses a negative tax rate. We hope that the constraint on consumption gives discipline to the government, and allows it to reach a strictly higher value $v^M(b, s)$. Without the constraint, we obtain convergence but we find that $v^M(b, s) = -\infty$ for all positive values of b , and we are not confident in the correctness of the result. Figures 9 and 10 plot the relevant variables as functions of debt d for each

level of s . It is disappointing to notice that the constraint that we have added is severely binding. Also, contrary to our previous argument, consumption and the marginal utility of consumption do not change with b in this numerical example.

6 Ramsey as sustainable equilibrium

In this section, we show how to determine numerically whether the Ramsey equilibrium can be sustained by a sustainable equilibrium, as defined by Chary and Kehoe (1990).

As in the previous section, in each period t , first the state variable s_t , the technology $\xi(s_t)$ and the government real liabilities $b_t(s^t)$ are realized, then the government chooses the tax rate τ_t and public consumption g_t , and finally the households choose private consumption c_t , labor n_t and next-period contingent assets $b_{t+1}(s^{t+1})$. Histories are re-defined to include the government's choices of tax rates and public consumption. Loosely speaking, a sustainable equilibrium is a sequence of functions of the histories such that, at any period and history, the continuation outcome is a competitive equilibrium of the continuation economy, and the government maximizes the continuation value. More precisely, after defining $z_t \equiv \{s_t, \tau_t, g_t\}$, let $z^t \equiv \{z_j\}_{j=0}^t$ be the history of the state up to period t , with $z^{-1} \equiv \{\emptyset\}$. Let $\{g_t(z^{t-1}, s_t) \geq 0, \tau_t(z^{t-1}, s_t) < 1\}_{t=0}^\infty$ be a government strategy, and let $\{c_t(z^t) \geq 0, n_t(z^t) \in [0, \bar{n}], b_{t+1}(z^t, s_{t+1}), q_t(z^t, s_{t+1}) > 0\}_{t=0}^\infty$ be an economy strategy. These strategies generate recursively an outcome, namely an allocation and a tax and price system, and an associated value. Also, at each period t and history $\{z^{t-1}, s_t\}$ the strategies induce continuation strategies and continuation outcomes and values. A sustainable equilibrium is a pair of government and economy strategies such that, at each period t and history $\{z^{t-1}, s_t\}$, the continuation outcome is a competitive equilibrium of the continuation economy, and the government action $\{g_t(z^{t-1}, s_t), \tau_t(z^{t-1}, s_t)\}$ maximizes the continuation value.

Chang (1998) and Phelan and Stacchetti (2001) show that, after adding the continuation value of sustainable equilibria as a state variable, the set of values of all sustainable equilibria can be characterized recursively. Also, the value of any sustainable equilibrium can be obtained with a recursive sustainable equilibrium. Here, we adapt their arguments to our model, modifying them. A recursive sustainable equilibrium is a value correspondence $V(b, s)$, policy functions $c(v, b, s)$, $n(v, b, s)$, $g(v, b, s)$, $b'(v, b, s, s')$, a law of motion $v'(v, b, s, c, n, g, b'(s'), s') : (b', s') \rightarrow V(b', s')$ for the continuation value, and a *future* policy function $\theta(v, b, s)$, satisfying:

- Given $v'(v, b, s, c, n, g, b'(s'), s')$ and $\theta(v, b, s)$, the value correspondence $V(b, s)$ and the

policy functions solve the Bellman equation:

For all b, s , all $v \in V(b, s)$:

$$v = \max_{\{c \geq 0, n \in [0, \bar{n}], g \geq 0, b'(s')\}} \left\{ u(c, n, g) + \sum_{s' \in S} \beta v'(c, n, g, b'(s'), s') \pi(s'|s) \right\}$$

subject to:

$$u_c(c, n, g)b = u_c(c, n, g)c + u_n(c, n, g)n + \sum_{s' \in S} \beta \theta(v'(c, n, g, b'(s'), s'), b'(s'), s') \pi(s'|s)$$

$$c + g \leq \xi(s)n$$

- The future and the current policy functions are the same:

$$\text{For all } b, s, \text{ for all } v \in V(b, s): \theta(v, b, s) = u_c(c(v, b, s), n(v, b, s), g(v, b, s))b$$

First, notice that, if the value correspondence $V(b, s)$ is a value function, so the law of motion for the continuation value can depend only on $(b'(s'), s')$, then the recursive sustainable equilibrium can be sustained without any threat to revert to other sustainable equilibria, and the definition is the same as for Markov perfect equilibria.

In the general case that $V(b, s)$ is not a function, the law of motion for the continuation value plays a crucial role. The future continuation value $v'(c, n, g, b'(s'), s')$ as a function of the last two arguments is constrained to be in the value correspondence $V(b'(s'), s')$ but can vary with the current choices $c, n, g, b'(s')$. The current government takes as given the law of motion for the continuation value, and correctly believes that the future continuation value depends on its current choices. This is what may give an additional incentive to the current government to make a choice closer to the Ramsey choice.

To determine whether the value $v^R(b, s)$ of the Ramsey equilibrium can be sustained by a sustainable equilibrium, we first determine whether the continuation value $v^C(b, s)$ of the Ramsey equilibrium can be sustained by the following recursive sustainable equilibrium. Let $V(b, s) = \{v^M(b, s), v^C(b, s)\}$, so the value correspondence is made of only two value functions, the one of a Markov perfect equilibrium and the one of the continuation value of the Ramsey equilibrium. If $v = v^M(b, s)$, the law of motion for the continuation value $v'(v, b, s, c, n, g, b'(s'), s')$ is equal to $v^M(b'(s'), s')$ regardless of the current government's choice. If $v = v^C(b, s)$, however, the law of motion for the continuation value $v'(v, b, s, c, n, g, b'(s'), s')$ is equal to $v^C(b'(s'), s')$ if the current government's choice is the Ramsey choice, and it is equal to $v^M(b'(s'), s')$ otherwise. Let the policy functions be the ones associated with the continuation of the Ramsey equilibrium and the Markov perfect equilibrium. The previous is a recursive sustainable equilibrium in the case that $v^C(b, s) \geq v^M(b, s)$ for all (b, s) .

Then, in the case that $v^C(b, s) \geq v^M(b, s)$ for all (b, s) , we only need to establish that, in the first period, $v^R(b, s) \geq v^M(b, s)$ for all (b, s) , which is immediately true since the Ramsey equilibrium is the optimal competitive equilibrium. It follows that the value of the Ramsey equilibrium is higher than the value of the Markov perfect equilibrium at each period and history, and the value $v^R(b, s)$ of the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium.

Figure 11 plots the value functions $v^R(b, s)$, $v^C(b, s)$ and $v^M(b, s)$ for the numerical example described at the end of section 4. All the value functions are decreasing in debt b and increasing in the state s . For this numerical example, $v^R(b, s) > v^C(b, s) > v^M(b, s)$ all (b, s) , although it is not easy to see it in the figure. Since the condition $v^C(b, s) \geq v^M(b, s)$ holds for all (b, s) , the continuation of the Ramsey equilibrium can be sustained as a recursive sustainable equilibrium. Then, the Ramsey equilibrium is a sustainable equilibrium.

We emphasize that the condition $v^C(b, s) \geq v^M(b, s)$ for all (b, s) needs not be true in general, and it ceases to hold when the preferences discount factor β decreases below a threshold value. For the numerical example considered, the threshold value is below $\beta = 0.8$.

Also, notice that the value of the Ramsey equilibrium cannot be sustained by a recursive sustainable equilibrium as defined in this paper. To sustain it as a recursive sustainable equilibrium as in Chang (1998) and Phelan and Stacchetti (2001), one should add to the list of state variables the promised value of government liabilities.

7 Conclusion

Our main result under commitment is that, in a benchmark case, taxes, government expenditures and the primary surplus should all be constant positive fractions of production, and both government liabilities and the fiscal surplus should be positively correlated with production. In addition, we have shown that, for realistic values of the preferences discount factor, the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium.

As explained in the introduction, in models where government expenditures are exogenous, the overall sign of the correlation between the primary surplus and production depends on the relative importance of technology shocks and government expenditures shocks. In this paper, we have shown that the correlation is positive if government expenditures are chosen endogenously. A main direction for research will be to model both the endogenous and the exogenous part of government expenditures, and to quantitatively characterize the optimal fiscal policy.

References

- Atkeson, A. (1991). International lending with moral hazard and risk of repudiation. *Econometrica* 59, 1069-1090.
- Abreu, D., D. Pearce, and E. Stacchetti (1990). Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica* 58 (5), 1041-1063.
- Benhabib, J. and A. Rustichini (1997). Optimal taxes without commitment. *Journal of Economic Theory* 77 (2), 231-259.
- Chamley, C. (1986). Optimal taxation of capital income in general equilibrium with infinite lives. *Econometrica* 54 (3), 607-622.
- Chang, R. (1998). Credible monetary policy in an infinite horizon model: Recursive approaches. *Journal of Economic Theory* 81 (2), 431-67.
- Chari, V. V., L. J. Christiano, and P. J. Kehoe (1994). Optimal fiscal policy in a business cycle model. *Journal of Political Economy* 102 (4), 617-652.

- Chari, V. V., L. J. Christiano, and P. J. Kehoe (1995). Policy analysis in business cycle models. In T. F. Cooley (Ed.), *Frontiers of Business Cycle Research*, Chapter 12. Princeton, N. J.: Princeton University Press.
- Chari, V. V. and P. J. Kehoe (1990). Sustainable plans. *Journal of Political Economy* 98 (4), 783-802.
- Chari, V. V. and P. J. Kehoe (1993). Sustainable plans and debt. *Journal of Economic Theory* 61 (2), 230-261.
- Fernandez-Villaverde, J., and Aleh Tsyvinski (2002). Optimal fiscal policy in a business cycle without commitment. Mimeo.
- Judd, K. L. (1985). Redistributive taxation in a simple perfect foresight model. *Journal of Public Economics* 28 (1), 59-83.
- Klein, P., P. Krusell, and J.-V. Rios-Rull (2004). Time-consistent public expenditures. Mimeo.
- Klein, P. and J.-V. Rios-Rull (2003). Time-consistent optimal fiscal policy. *International Economic Review* 44 (4), 1217-1246.
- Krusell, P., F. M. Martin, and J.-V. Rios-Rull (2004). Time-consistent debt. Mimeo.
- Kydland, F. E. and E. C. Prescott (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy* 85 (3), 473-92.
- Kydland, F. E. and E. C. Prescott (1980). Dynamic optimal taxation, rational expectations, and optimal control. *Journal of Economic Dynamics and Control* 2, 78-91.
- Ljungqvist, L. and T. J. Sargent (2004). *Recursive Macroeconomic Theory*, 2nd edition. The MIT Press.
- Lucas, R. E. J. and N. L. Stokey (1983). Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics* 12 (1), 55-93.
- Phelan, C. and E. Stacchetti (2001). Sequential equilibria in a Ramsey tax model. *Econometrica* 69 (6), 1191-1518.
- Stokey, N. L. (1989). Reputation and time consistency. *American Economic Review, Papers and Proceedings*, 79 (2), 134-139.
- Stokey, N. L. (1991). Credible public policy. *Journal of Economic Dynamics and Control* 15 (4), 627-656.
- Stockman, D. (2001). Balance-budget rules: Welfare loss and optimal policies. *Review of Economic Dynamics* 4 (2), 438-459.
- Zhu (1992). Optimal fiscal policy in a stochastic growth model. *Journal of Economic Theory* 58 (2), 250-289.

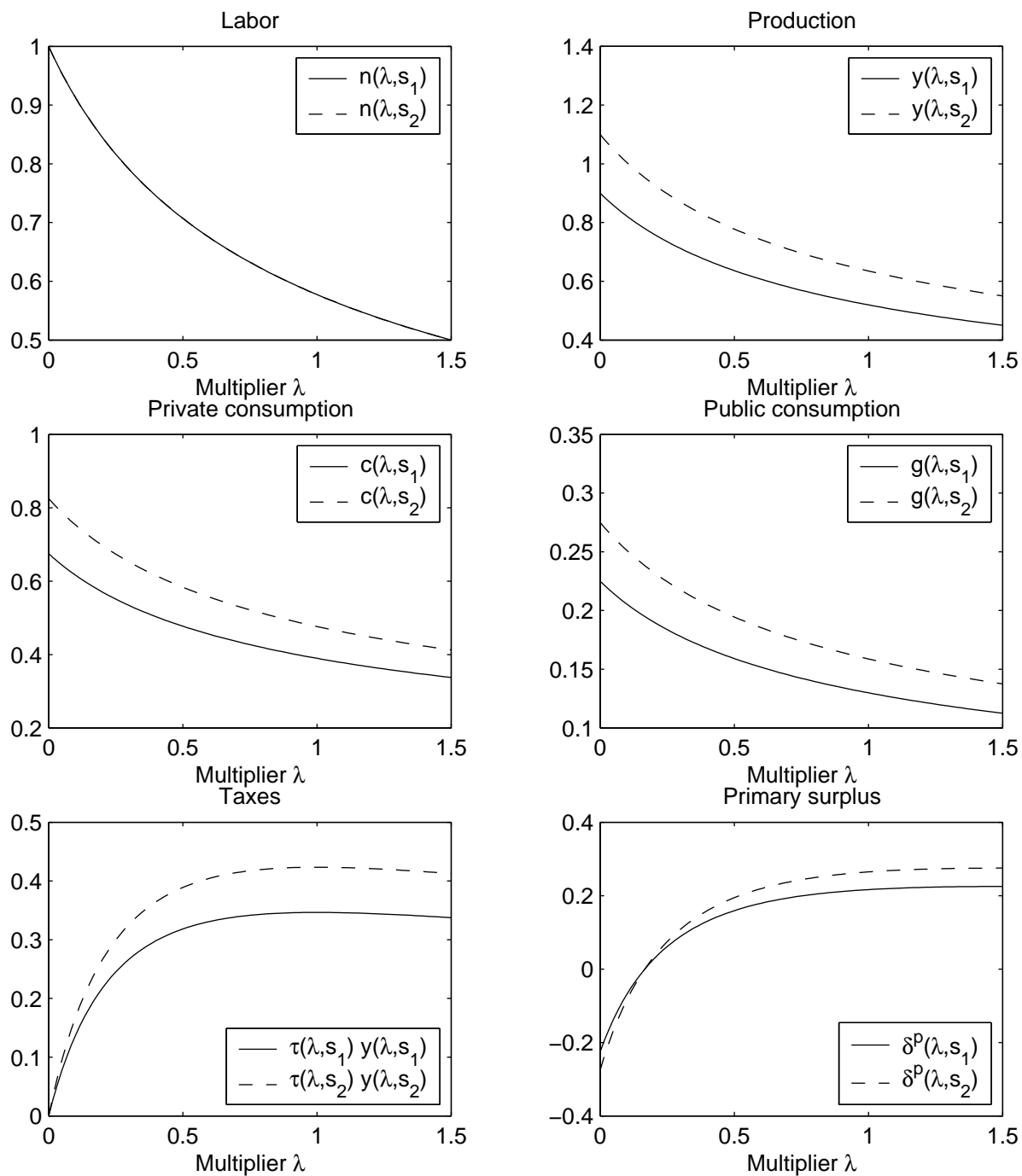


Figure 1: Continuation of the Ramsey equilibrium.

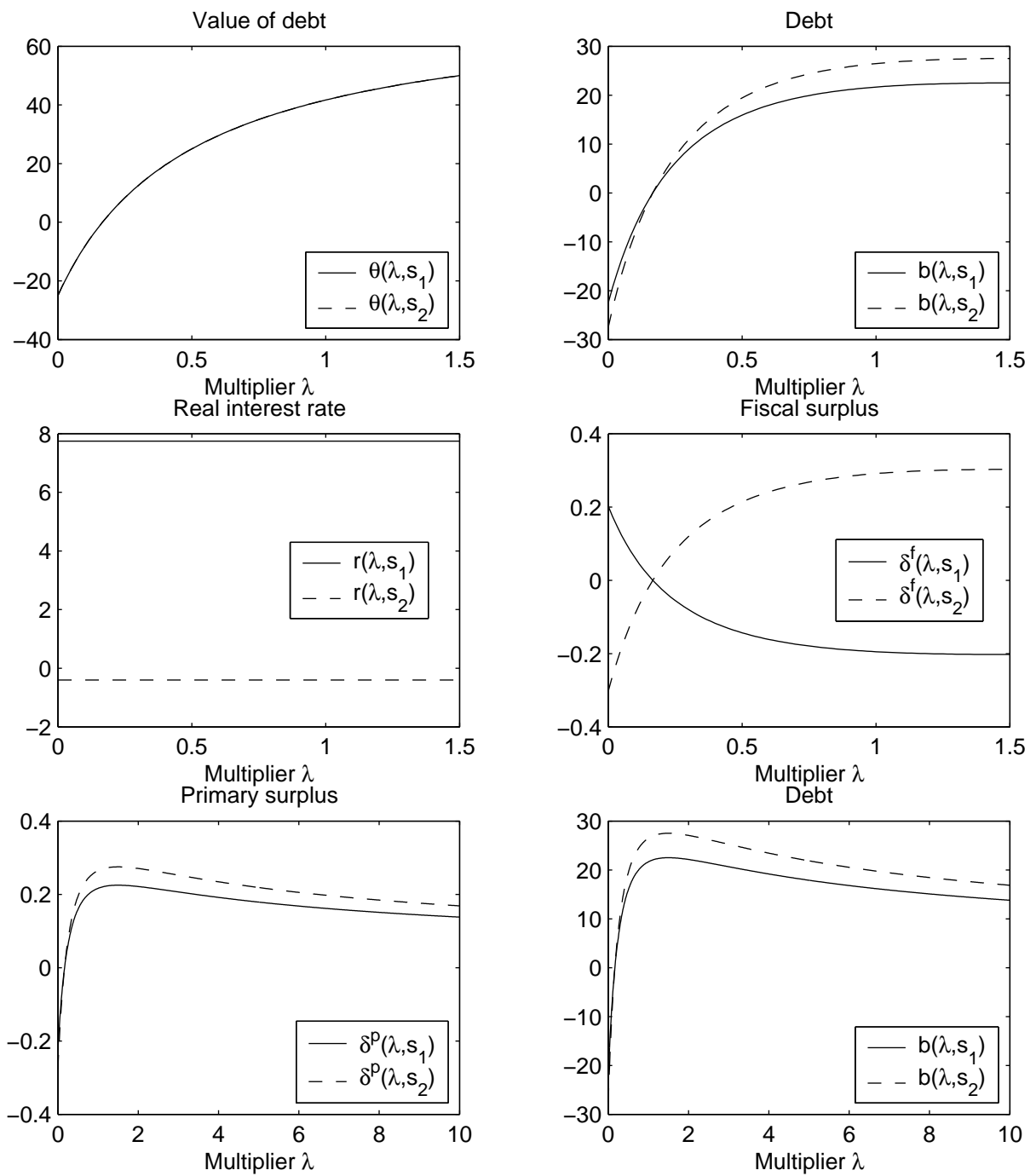


Figure 2: Continuation of the Ramsey equilibrium.

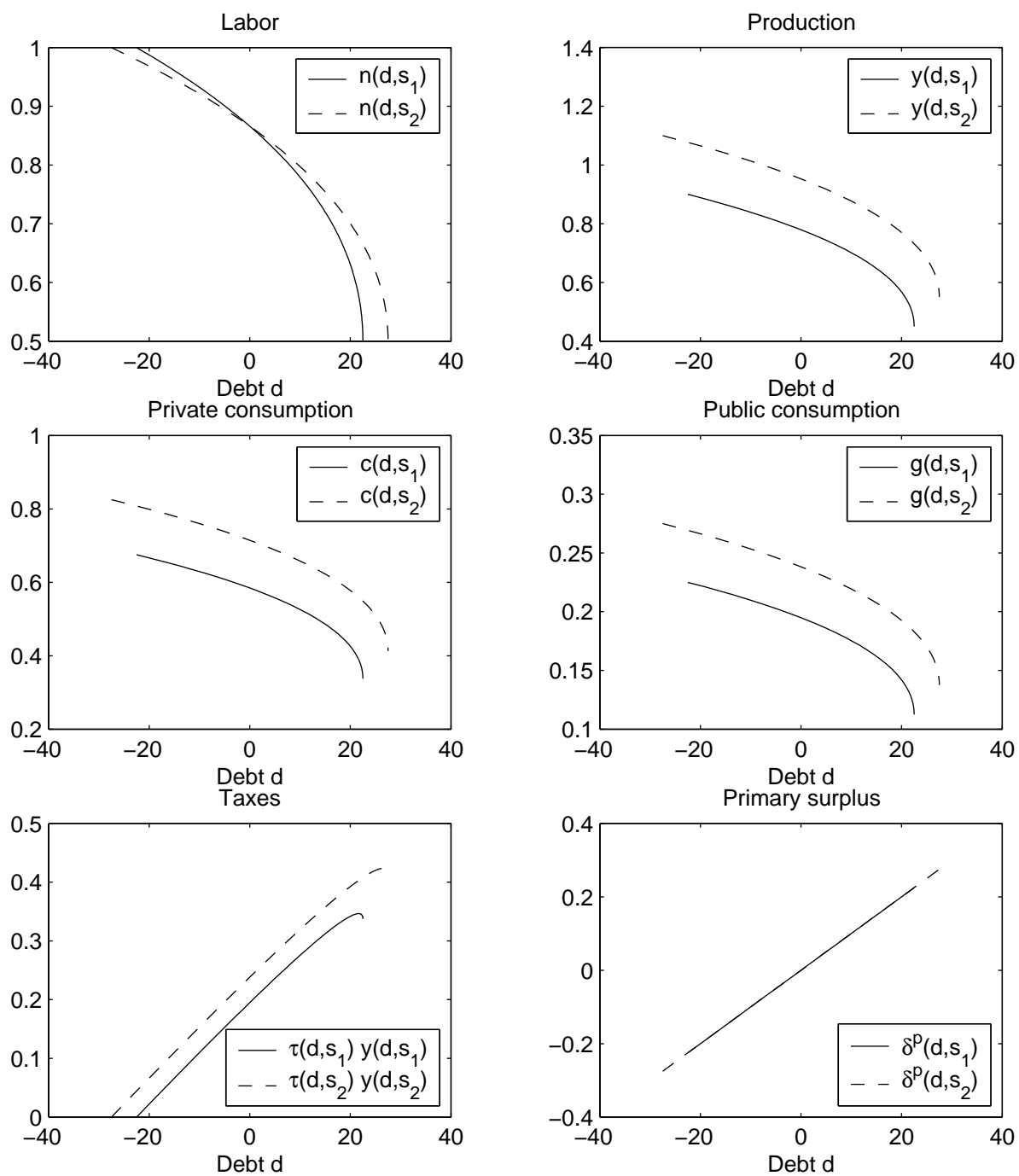


Figure 3: Continuation of the Ramsey equilibrium.

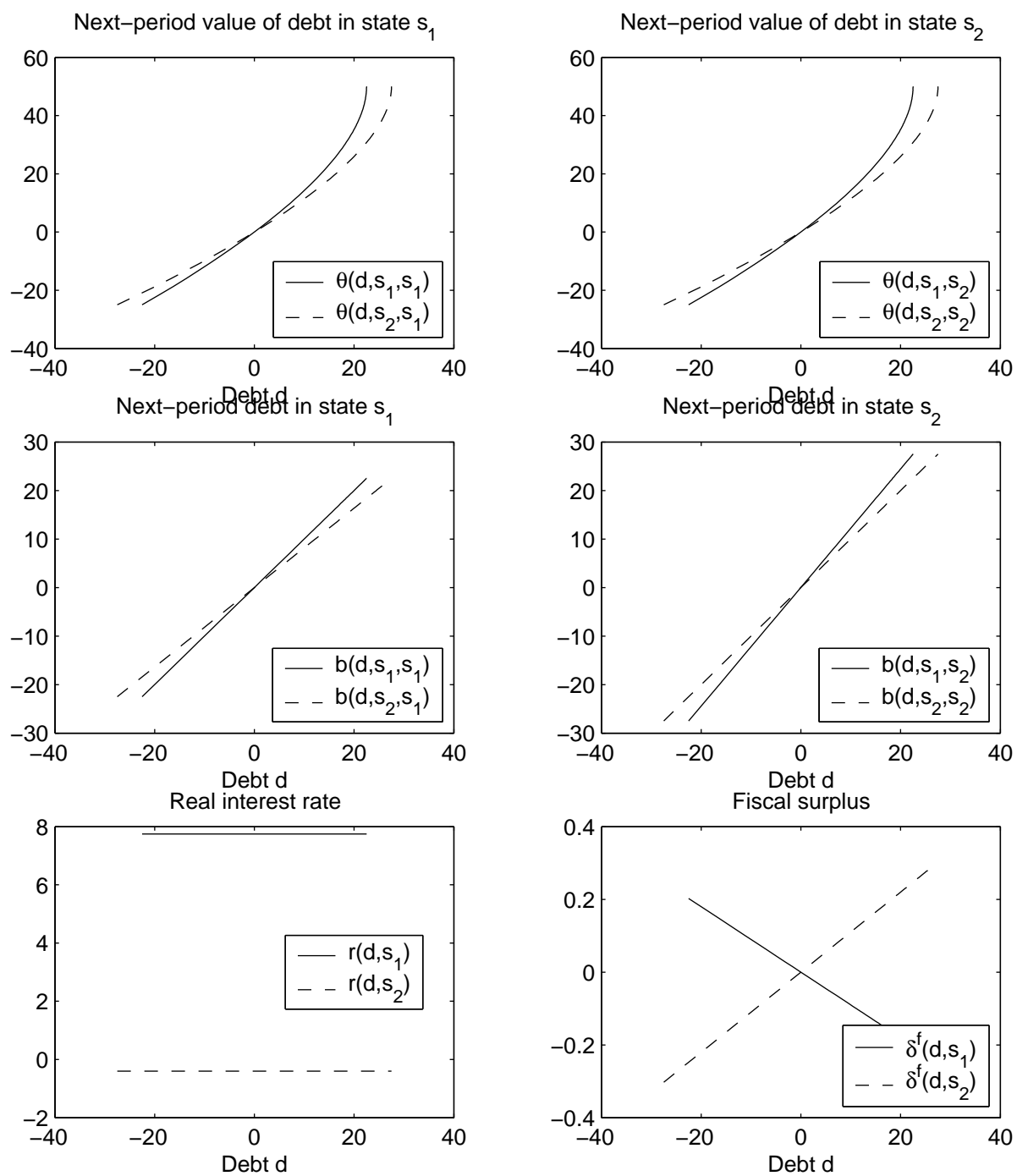


Figure 4: Continuation of the Ramsey equilibrium.

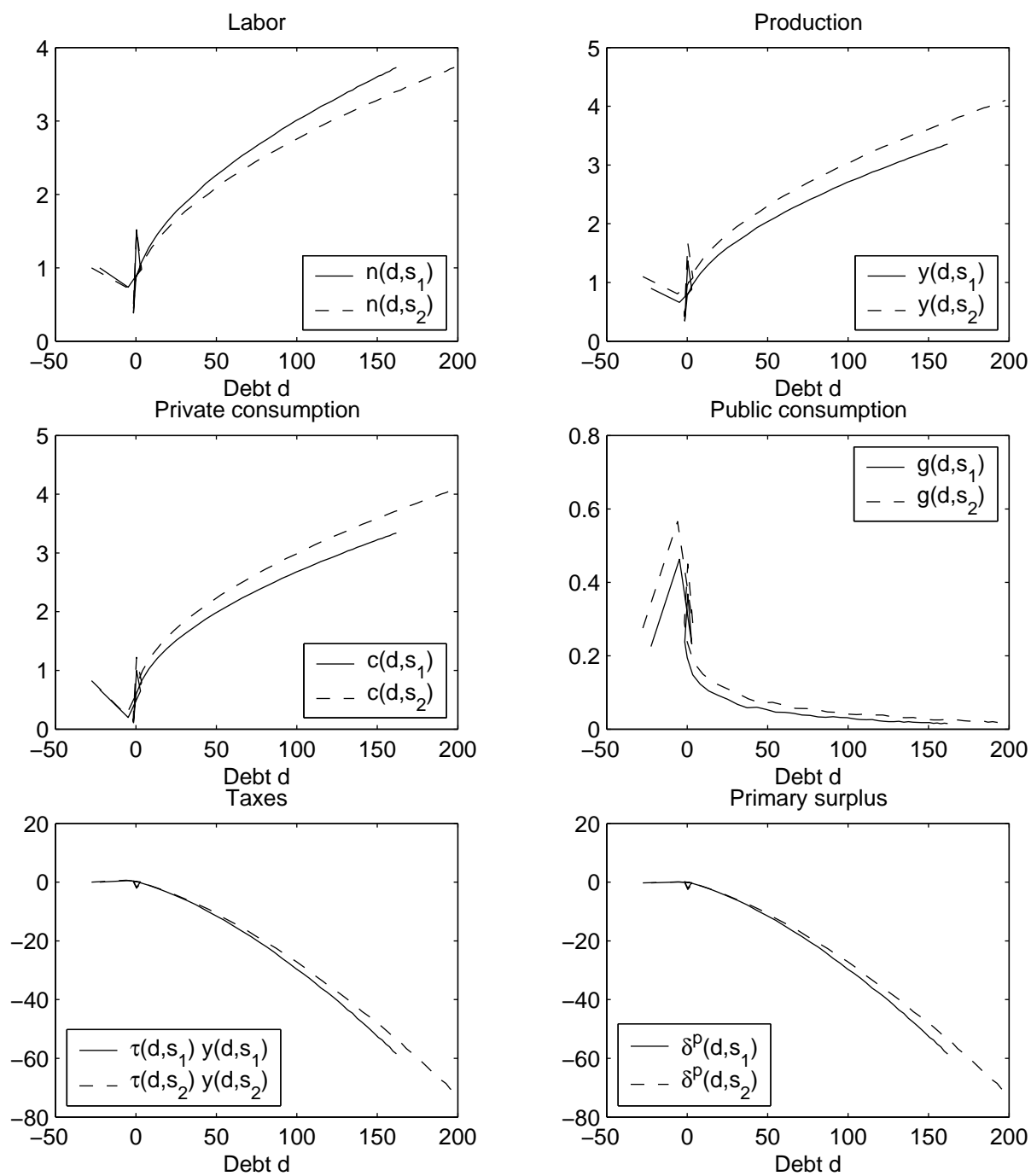


Figure 5: Ramsey equilibrium.

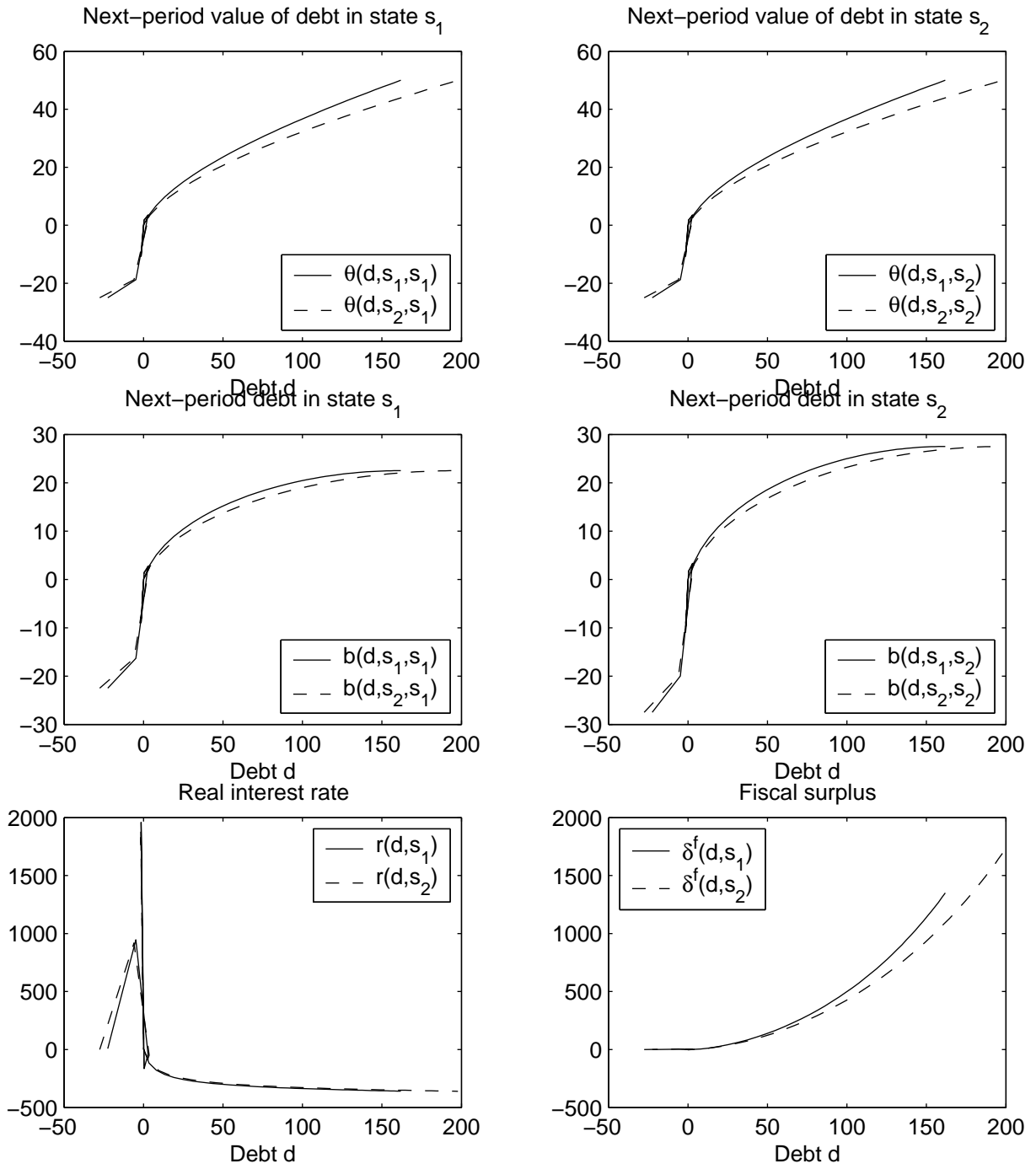


Figure 6: Ramsey equilibrium.

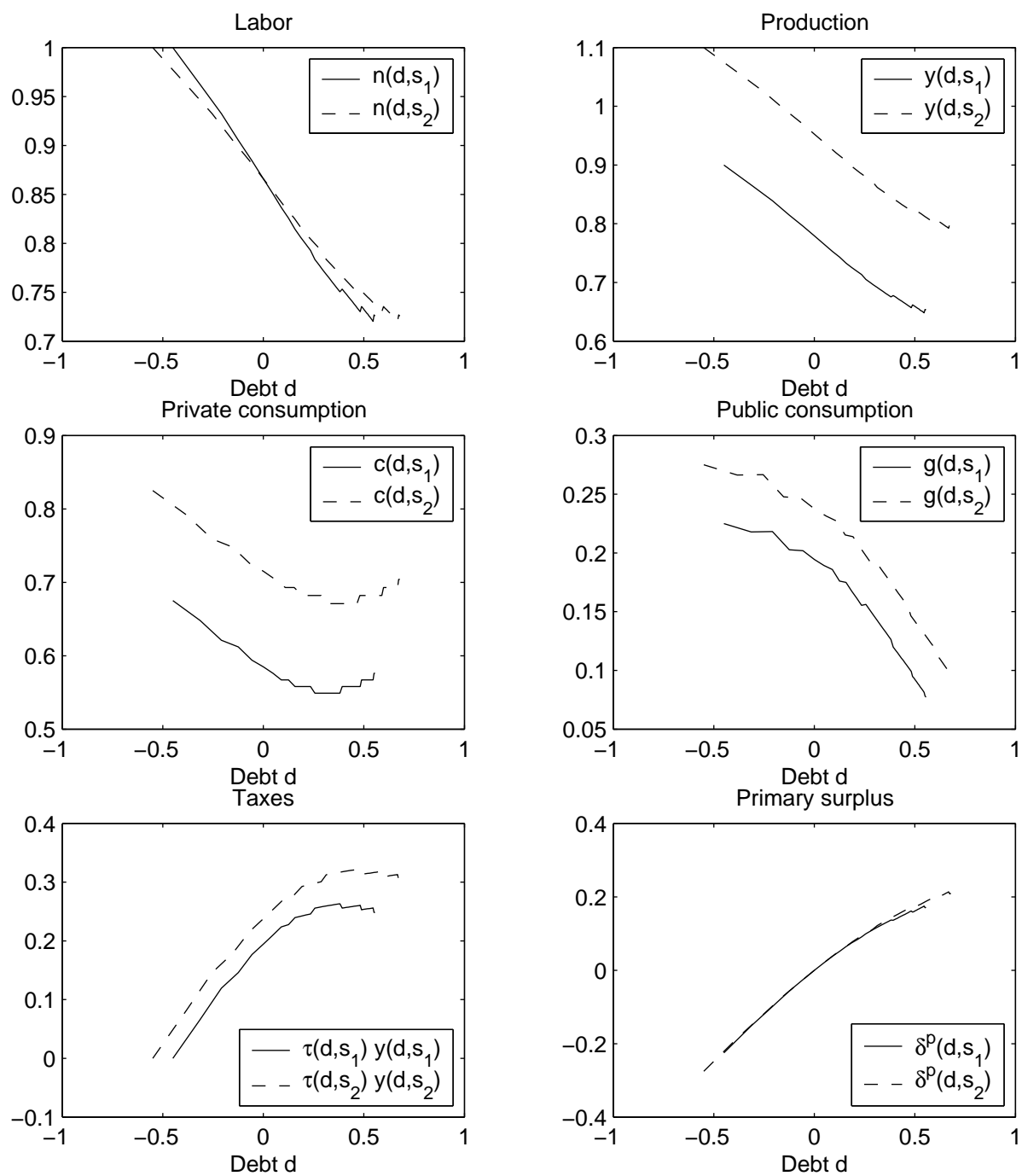


Figure 7: Ramsey equilibrium. $\beta = 0.5$.

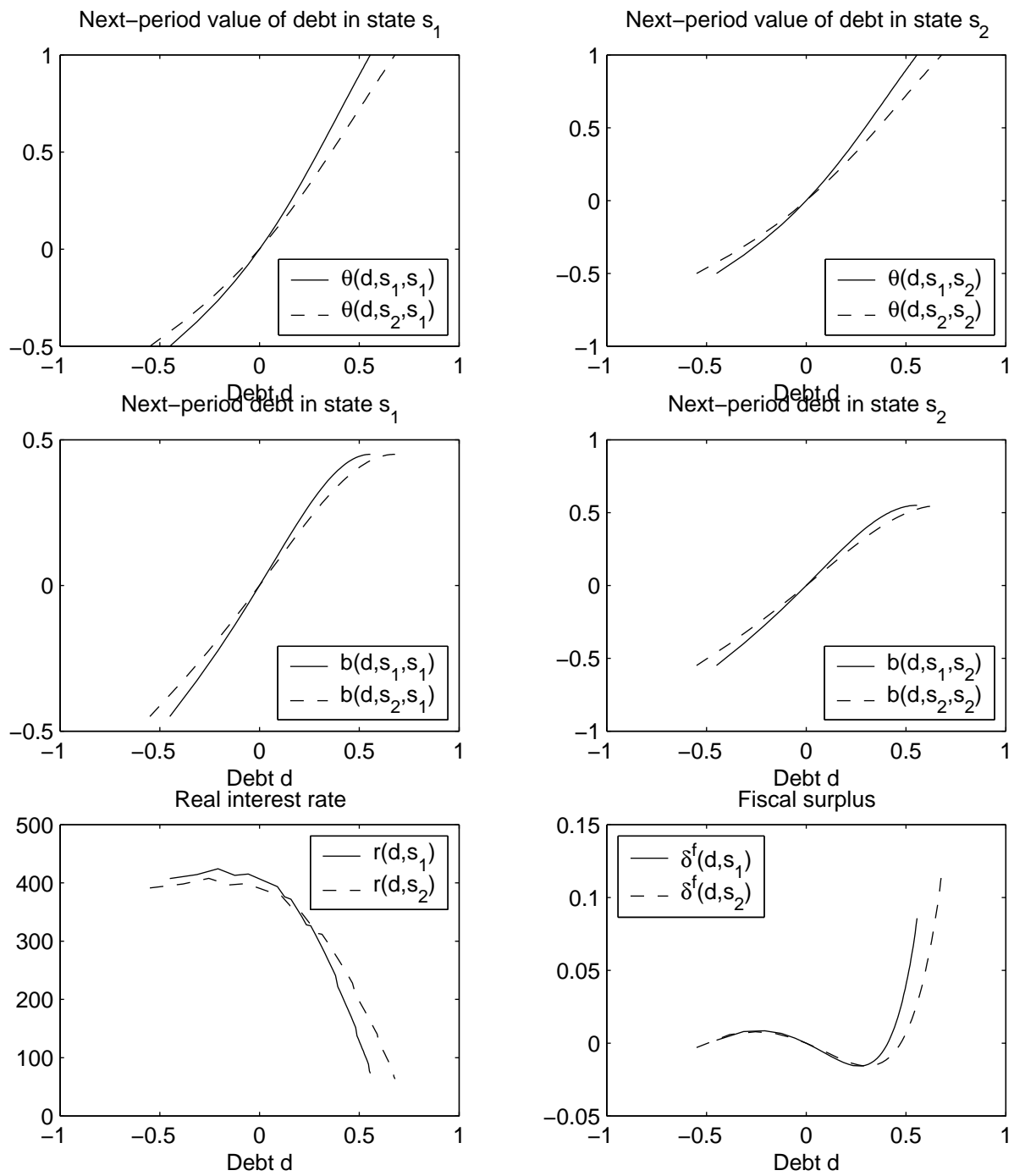


Figure 8: Ramsey equilibrium. $\beta = 0.5$.

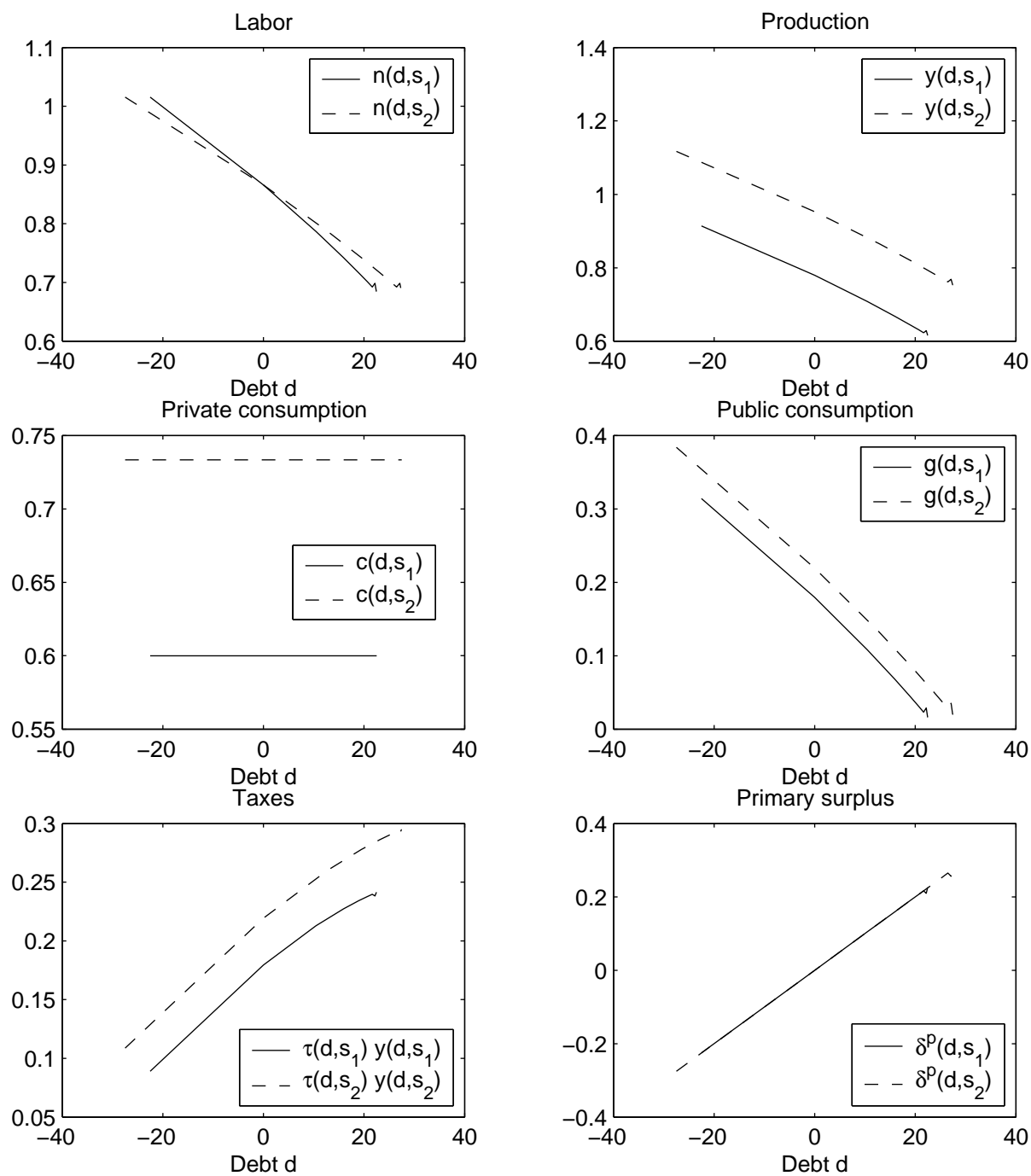


Figure 9: Markov perfect equilibrium.

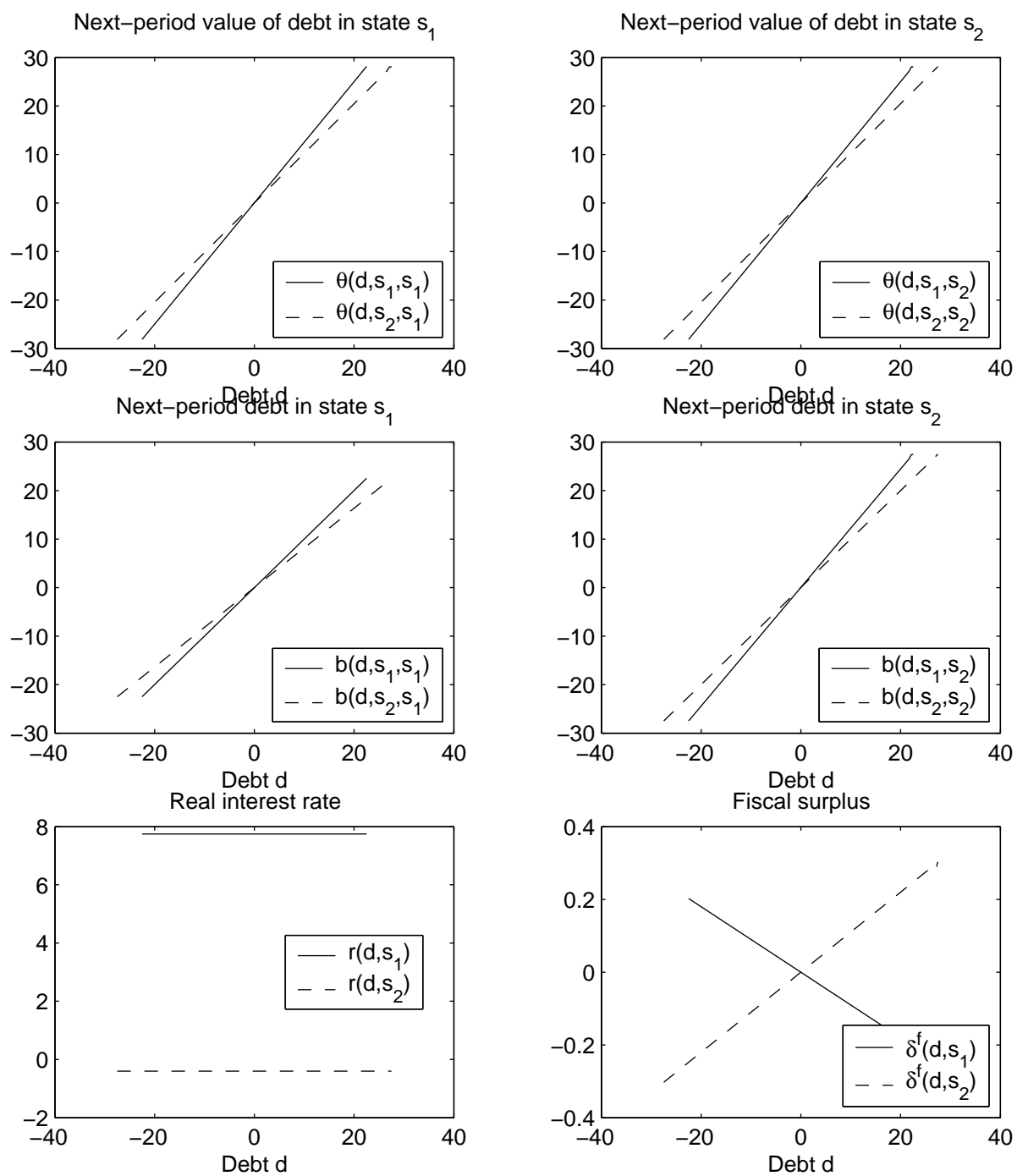


Figure 10: Markov perfect equilibrium.

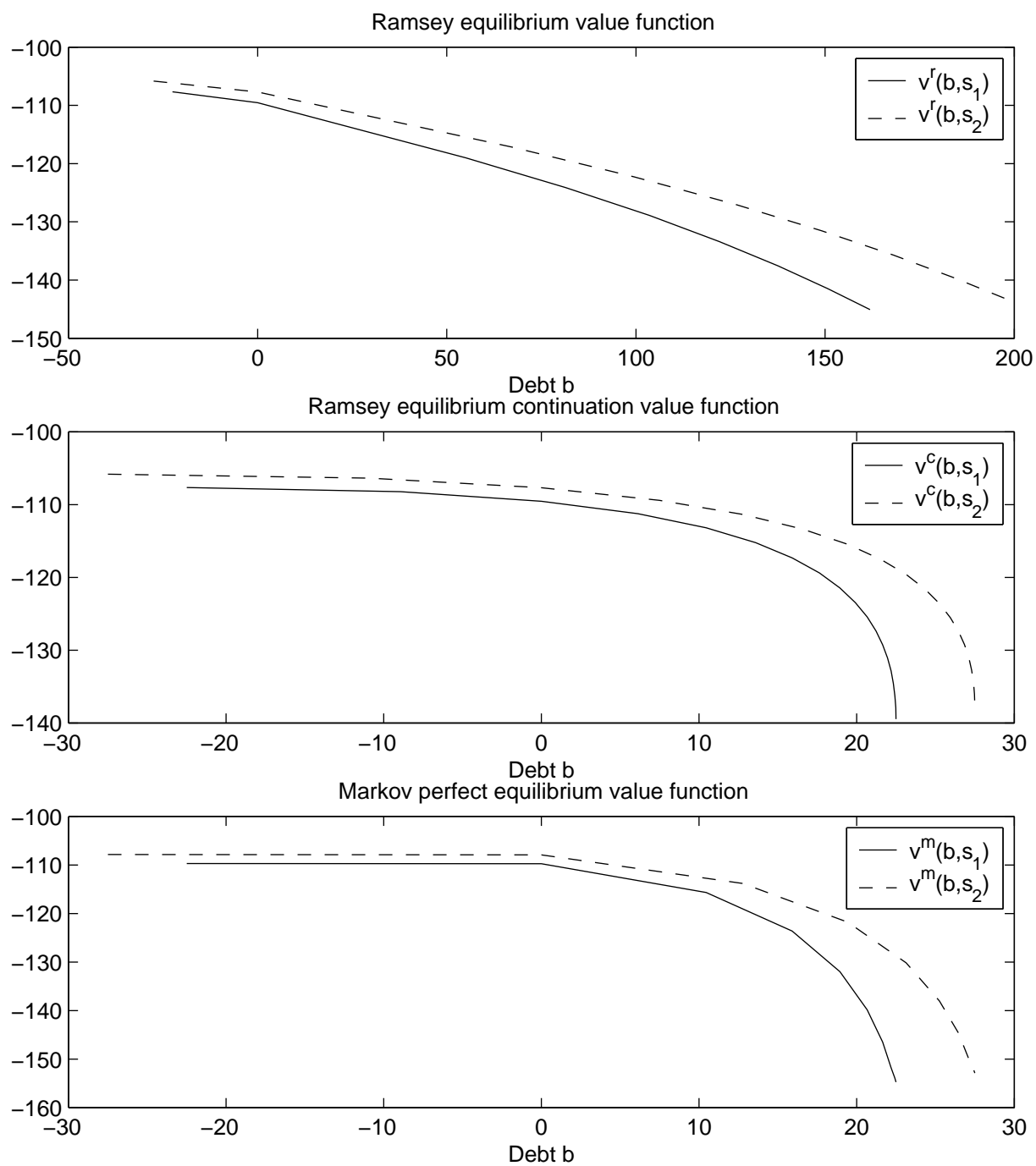


Figure 11: Value functions.