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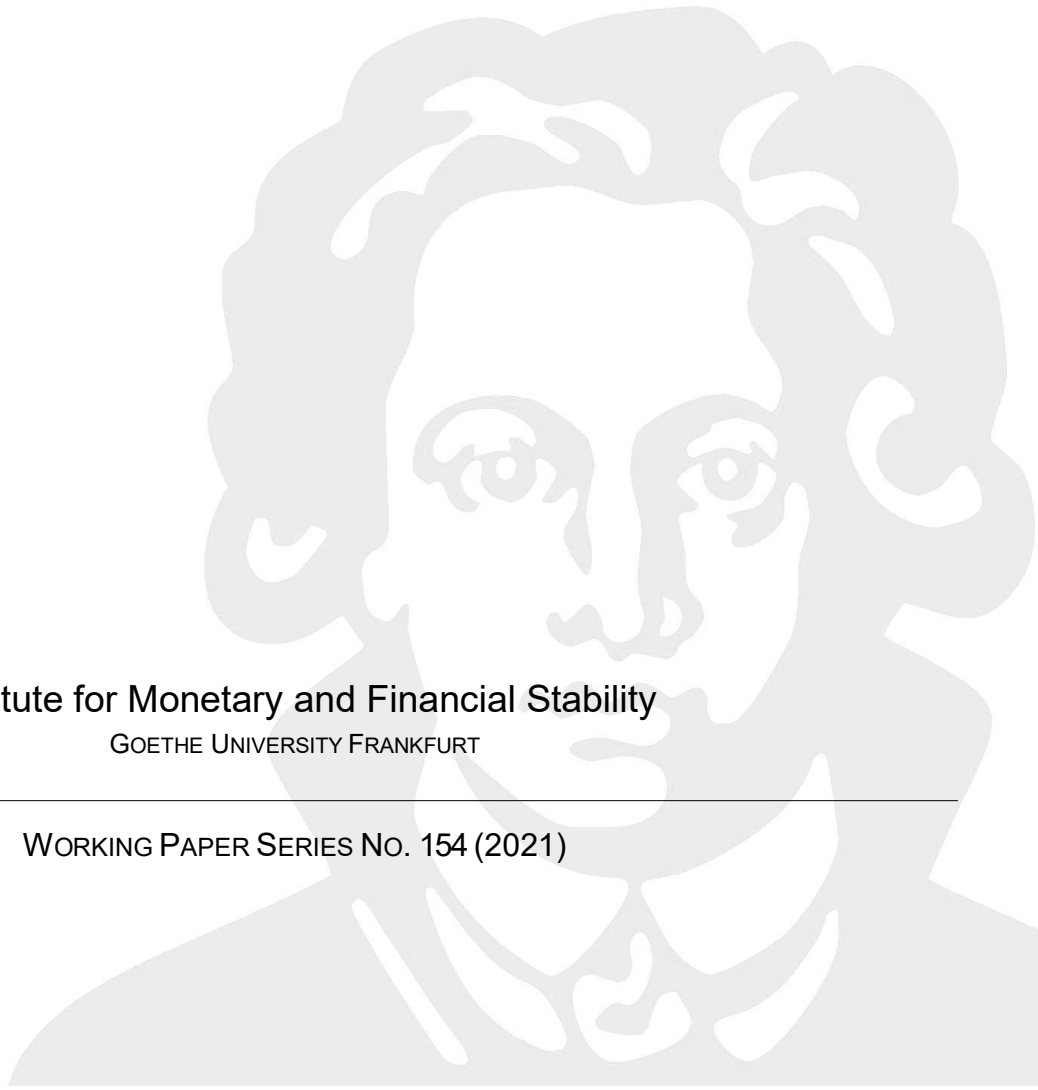
ALEXANDER MEYER-GOHDE

On the Accuracy of Linear DSGE Solution Methods and the Consequences for Log-Normal Asset Pricing



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ON THE ACCURACY OF LINEAR DSGE SOLUTION METHODS AND THE CONSEQUENCES FOR LOG-NORMAL ASSET PRICING

ALEXANDER MEYER-GOHDE

ABSTRACT. This paper demonstrates a failure of standard, generalized Schur (or QZ) decomposition based solutions methods for linear dynamic stochastic general equilibrium (DSGE) models when there is insufficient eigenvalue separation about the unit circle. The significance of this is demonstrated in a simple production-based asset pricing model with external habit formation. While the exact solution afforded by the simplicity of the model matches post-war US consumption growth and the equity premium, QZ-based numerical solutions miss the later by many annualized percentage points.

JEL classification codes: C61, C63, E17

Keywords: Numerical accuracy; Production-based asset pricing; DSGE; Solution methods

1. INTRODUCTION

The asset pricing literature abounds with puzzles and perhaps the most prominent is the equity premium puzzle (Mehra and Prescott, 1985; Mehra, 2003) that seeks to understand how risky assets can command such a high excess return in the face of

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Any and all errors are entirely my own.

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moderate underlying volatility. While many convincing consumption based explanations that modify assumptions on, say, the stochastic properties of the pricing kernel have been offered, production based asset pricing facing the additional challenge of needing to provide a structural cause of these stochastic properties. Providing a structural explanation invariably requires solving a structural model, the most common being dynamic stochastic general equilibrium (DSGE) models, which generally need to be solved numerically. Cochrane (2008, p. 300) expressed concern regarding the accuracy of solution approximations in general equilibrium and this paper points out a surprising degradation of the accuracy of solution approximations in the simplest approximation, linear approximations, and their consequences for the equity premium reported by these methods.

Providing a solution to a DSGE model involves solving a functional equation to determine an unknown function that maps the sequences of variables in the information set into the endogenous variables of the model, resolving expectations of these same endogenous variables (Judd, 1998; Fernández-Villaverde, Rubio-Ramírez, and Schorfheide, 2016). Linear DSGE models, whose solutions are linear functions of these sequences of variables have long been studied, e.g., Blanchard (1979) and Blanchard and Kahn (1980), and modern numerical packages such as Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot, 2011), Gensys (Sims, 2001), (Perturbation) AIM (Anderson and Moore, 1985; Anderson, Levin, and Swanson, 2006), Uhlig's Toolkit (Uhlig, 1999) and Solab (Klein, 2000) not only provide tools for solving a wide range of linear models, but also provide a first step in the solution procedure for many nonlinear methods as well.

The substantial hurdle in these linear methods is finding a solution to a (matrix) quadratic equation, frequently required to be the unique stable solution. For multivariate models with potentially singular coefficient matrices, the standard method is to double the dimension of the problem and employ a generalized Schur or QZ

decomposition of Moler and Stewart (1973). While this algorithm is backward stable for the generalized eigenvalue decompositions for which it was designed, it is not always backward stable for quadratic eigenvalue problems (Tisseur, 2000) and may yield ill-conditioned eigenvalues for quadratic matrix polynomials (Higham, Mackey, and Tisseur, 2006; Higham, Mackey, Tisseur, and Garvey, 2008).

I demonstrate this danger in a simple, two equation canonical real business cycle model with habit formation, chosen as its small dimension enables its symbolic solution. Exploring calibrations that match two calibration targets, post-war US consumption growth volatility and the average equity premium, I find that QZ based solution methods deliver an equity premium often off by several annual percentage points when the calibration pushes the stable and unstable eigenvalues close to the unit circle. While only illustrative and chosen to enable simple analysis, it is remarkable that none of the algorithms from the literature I explore produced any warning that their solutions might suffer from such an economically significant loss of accuracy. More general production based explanations, such as Jermann (1998), Tallarini (2000), and Croce (2014) combine more involved specifications of preferences and mappings from exogenous variables to underlying macroeconomic variables, and must also invariably solve their models numerically. Inasmuch as the solutions thereto are potentially subject to this inaccuracy in the solution of the underlying DSGE models, so too are their asset pricing predictions.

The remainder of the paper is structured as follows. Section 2 introduces the real business cycle model and the log-linear asset pricing approach. In section 3 I turn to the solution of linear DSGE models and the links to the numerical mathematics literature on solving matrix quadratic and generalized eigenvalue problems. Section 4 presents the results from various methods in the DSGE linear solution literature for several calibrations. Finally, I conclude in section 5.

2. A Simple Log Normal DSGE Asset Pricing Model

Here I layout a simple production-based asset pricing model, based on a standard real business cycle model (Kydland and Prescott, 1982; King and Rebelo, 1999) with external habit formation and a power utility kernel.(Constantinides, 1990; Campbell and Cochrane, 1999; Campbell, 2003) The representative household seeks to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, X_t), \ 0 < \beta < 1 \tag{1}$$

where c_t is consumption and X_t the habit stock, subject to

$$c_t + k_t = e^{z_t} k_{t-1}^{\alpha} + (1-\delta) k_{t-1}, \ 0 < \alpha, \delta < 1 \tag{2}$$

where k_t is the capital stock accumulated at time t and z_t is total factor productivity that follows the AR(1) process

$$z_t = \rho z_{t-1} + \omega \varepsilon_t, \ \varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0,1), \ |\rho| < 1, \ 0 < \omega \tag{3}$$

The first order condition of the maximization problem is

$$1 = E_t \left[\underbrace{\beta \frac{u_c(c_{t+1}, X_{t+1})}{u_c(c_t, X_t)}}_{m_{t+1}} \underbrace{\left(\alpha e^{z_{t+1}} k_t^{\alpha} + 1 - \delta \right)}_{R_{t+1}} \right] \tag{4}$$

where m_{t+1} is the stochastic discount factor or pricing kernel and R_{+1} is the (risky) return on capital. Assuming an external habit ($X_t = c_{t-1}$ in equilibrium with h the degree of habit formation) and power or CRRA utility (risk coefficient σ), marginal utility is $u_c(c_t, X_t) = (c_t - h c_{t-1})^{-\sigma}$.

Equations (2)-(4) characterize a equilibrium for the stochastic sequences $\{c_t, k_t, z_t\}_{t=0}^{\infty}$ given a sequence of shocks $\{\varepsilon_t\}_{t=0}^{\infty}$ and initial conditions c_{-1}, k_{-1}, z_{-1}. Defining the steady state, values $\overline{c}, \overline{k}, \overline{z}$ that solve (2)-(4) with $\varepsilon_t = 0 \, \forall t$, equations (2)

and (4) can be log-linearized around these values to yield

$$0 = AE_t[y_{t+1}] + By_t + Cy_{t-1} + Dz_t,\ y_t = \begin{bmatrix} \hat{c}_t & \hat{k}_t \end{bmatrix}' \tag{5}$$

$$z_t = \rho z_{t-1} + \omega\varepsilon_t,\ \varepsilon_t \overset{\text{i.i.d.}}{\sim} N(0,1) \tag{6}$$

a a 2 by 2 system of equations linear in the log-deviations of the endogenous variables, c_t and k_t, from their steady states, $\hat{w}_t \equiv \log w_t - \log \overline{w}$, for $w \in c,k$.

Following Hansen and Singleton (1983); Campbell and Shiller (1988); Campbell (2003), risky (say, R_t from above) and risk-free (via no arbitrage, $1 = E_t[m_{t+1}]R_t^f$) assets can be priced under the implied joint log-normality of the approximation above via

$$1 = E_t\left[e^{\widehat{m_{t+1}}+\widehat{R_{t+1}}}\right],\ \text{and} 1 = \overline{R^f}e^{R_t^f}E_t\left[e^{\widehat{m_{t+1}}}\right] \tag{7}$$

which gives the risk premium as $-cov_t(\widehat{m_{t+1}},\widehat{R_{t+1}})$, following, e.g., Lettau (2003), and can be expressed in terms of the variance of z_t (ω^2) as $\frac{\sigma}{1-h}\alpha Q_{cz}\left(1+\beta(1-\delta)\right)\omega^2$. Importantly, the coefficient Q_{cz}, the impact of technology on (log) consumption must be solved for numerically even in this (log) linear case. To this solution I turn to in the following section.

3. Solution

Standard numerical solution packages available to economists and policy makers—e.g., Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot, 2011), Gensys (Sims, 2001), (Perturbation) AIM (Anderson and Moore, 1985; Anderson, Levin, and Swanson, 2006), Uhlig's Toolkit (Uhlig, 1999) and Solab (Klein, 2000)—all analyze models that in some way or another can be expressed in the form of the nonlinear functional equation

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)] \tag{8}$$

The model equations (optimality conditions, resource constraints, market clearing conditions, etc.) are represented by the n_y-dimensional vector-valued function $f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_y}$; $y_t \in \mathbb{R}^{n_y}$ is the vector of n_y endogenous variables; and $\varepsilon_t \in \mathbb{R}^{n_e}$ the vector of n_e exogenous shocks with a known distribution, where n_y and n_e are positive integers ($n_y, n_e \in \mathbb{N}$).

The solution to (8) is sought, abstracting from stochastic dependencies that vanish at first order anyway, as the unknown function

$$y_t = y(y_{t-1}, \varepsilon_t), \quad y : \mathbb{R}^{n_y + n_e} \to \mathbb{R}^{n_y} \tag{9}$$

a function in the time domain that maps states, y_{t-1} and ε_t, into endogenous variables, y_t. An analytic form for (9) is rarely available and researchers and practitioners are compelled to find approximative solutions. However, a steady state, $\overline{y} \in \mathbb{R}^{n_y}$ be a vector such $\overline{y} = y(\overline{y}, 0)$ and $0 = f(\overline{y}, \overline{y}, \overline{y}, 0)$ can frequently be recovered, either analytically or numerically, providing a point of expansion around which local solutions may be recovered.

A first-order, or linear approximation, of (8) delivers, analogously to the log linearized model of section 2,

$$0 = AE_t[y_{t+1}] + By_t + Cy_{t-1} + D\varepsilon_t \tag{10}$$

where A, B, C, and D are the derivatives of f in (8) with respect to its arguments and, recycling notation, the y's in (10) refer to (log) deviations of the endogenous variables from their steady states, $\overline{y}$.

In analogy to (9), the standard approach to finding a solution to the linearized model (10) is to find a linear solution in the form

$$y_t = P\, y_{t-1} + Q\, \varepsilon_t \tag{11}$$

a recursive solution in the time domain–solutions that posit y_t as a function of its own past, y_{t-1}, and exogenous innovations, ε_t.

Inserting (11) into (10) and taking expectations ($E_t[\varepsilon_{t+1}] = 0$), yields the restrictions

$$0 = AP^2 + BP + C, \quad 0 = (AP + B)Q + D \tag{12}$$

Generally, a unique P with eigenvalues inside the closed unit circle is sought (I will address this formally below). Lan and Meyer-Gohde (2014) prove the latter can be uniquely solved if such a P can be found. Hence, the hurdle is the former, matrix quadratic equation.

To assist in the analysis, I will formalize the matrix quadratic equation in (12). For A, B, and $C \in \mathbb{R}^{n_y \times n_y}$, a matrix quadratic $M(P) : \mathbb{C}^{n_y \times n_y} \to \mathbb{C}^{n_y \times n_y}$ is defined as

$$M(P) \equiv A P^2 + B P + C \tag{13}$$

with its solutions, called solvents,[1] given by $P \in \mathbb{C}^{n_y \times n_y}$ if and only if $M(P) = 0$. The eigenvalues of the solvent, called latent roots of the associated lambda matrix[2] $M(\lambda) : \mathbb{C} \to \mathbb{C}^{n \times n}$ (here of degree two), are given via

$$M(\lambda) \equiv A \lambda^2 + B \lambda + C \tag{14}$$

The latent roots are (i) values of $\lambda \in \mathbb{C}$ such that $\det M(\lambda) = 0$ and (ii) $n_y - \text{rank}(A)$ infinite roots. An explicit link between the quadratic matrix problem and the quadratic eigenvalue problem is given via

$$\lambda \in \mathbb{C} : \left(A\lambda^2 + B\lambda + C\right) x = 0 \text{ for some } x \neq 0 \tag{15}$$

[1] The analysis proceeds in the complex plane, but the results carry over when solutions are restricted to be real valued due to the eigenvalue separation about the unit circle assumed below, see also Klein (2000).

[2] See, e.g., Dennis, Jr., Traub, and Weber (1976, p. 835) or Gantmacher (1959, vol. I, p. 228).

which has been reviewed extensively by Tisseur and Meerbergen (2001) and for which Hammarling, Munro, and Tisseur (2013) provide a comprehensive method to improve the accuracy of its solutions, is given in Higham and Kim (2000). If a unique stable solution is sought or required, this can be formulated via an adaptation of Blanchard and Kahn's (1980) rank and order conditions to the matrix quadratic formulation above. First assume there exist $2n_y$ latent roots of (14) of which n_y lie inside and n_y outside the unit circle. Second, there exists an $P \in \mathbb{R}^{n_y \times n_y}$ such that $M(P) = 0$ and $|eig(P)| < 1$.

Most methods use a generalized Schur or QZ decomposition (Moler and Stewart, 1973; Golub and van Loan, 2013) of the companion linearization of (10)[3] in some form or another. For the formulation above, the matrix quadratic (12) can be brought into the QZ form as

$$F \begin{bmatrix} I_{n_y} \\ P \end{bmatrix} P = G \begin{bmatrix} I_{n_y} \\ P \end{bmatrix}, \quad F \equiv \begin{bmatrix} I_{n_y} & 0_{n_y \times n_y} \\ 0_{n_y \times n_y} & A \end{bmatrix}, G \equiv \begin{bmatrix} 0_{n_y \times n_y} & I_{n_y} \\ -C & -B \end{bmatrix} \tag{16}$$

where I_{n_y} is an $n_y \times n_y$ identity matrix and $0_{n_y \times n_y}$ is an $n_y \times n_y$ zero matrix.

Applying the QZ or generalized Schur decomposition (unitary Q and Z and upper triangular S and T with $Q^* F Z = S$ and $Q^* G Z = T$), Higham and Kim (Theorem 3 2000) prove that all solvents or solutions of (16) are of the form $P = Z_{21} Z_{11}^{-1} = Q_{11} S_{11}^{-1} T_{11} Q_{11}^{-1}$. The decomposition is intricately related to the quadratic eigenvalue problem (15) via

$$\lambda \in \mathbb{C} : (F\lambda - G)y, \text{ where } y = \begin{bmatrix} x' & x'\lambda \end{bmatrix} \text{ for some } x \neq 0 \tag{17}$$

$$\lambda \in \mathbb{C} : Q(S\lambda - T)\tilde{y}, \text{ where } \tilde{y} = Z^* \begin{bmatrix} x' & x'\lambda \end{bmatrix} \text{ for some } x \neq 0 \tag{18}$$

[3] Instead of the method of undetermined coefficients taken for expediency here, a multivariate pivoted Blanchard (1979) approach that delivers the solution constructively is presented in the appendix.

where the eigenvalues in both lines are identical following from unitary equivalence (Moler and Stewart, 1973) and hence identical to the eigenvalues in (15) and the latent roots of (14). From the upper triangularity of S and T it follows that the spectrum or set of eigenvalues of the pencil $P_{FG}(\lambda) = F\lambda - G$ is determined by the diagonal entries of S and T

$$\rho(P_{FG}) = \left\{ t_{ii}/s_{ii}, \text{ if } s_{ii} \neq 0; \infty, \text{ if } s_{ii} = 0; \emptyset, \text{ if } s_{ii} = t_{ii} = 0; \; i = 1, \ldots, 2n_y \right\} \tag{19}$$

where s_{ii} and t_{ii} denote the i'th row and i'th column of S and T respectively.

Ordering the decomposition so that the eigenvalues outside the unit circle are in the lower right blocks of S and T (hence S_{22} and T_{22}), the necessary and sufficient assumptions for a unique stable solution for y_t of (10) to exist are (1) Regularity: $P_{FG}(z)$ is called regular if there exists a $z \in \mathbb{C}$ such that $det(Fz - G) \neq 0$; (2) Order: Of the $2n_y$ generalized eigenvalues, there are exactly n_y stable roots inside the unit circle, and consequently, exactly n_y unstable roots outside the unit circle; (3) Rank: Z_{11}, the upper right block of Z, is nonsingular If and only if these three assumptions are fulfilled does a unique solution P stable with respect to the closed unit circle exist. Consequentially, the overwhelming majority of the linear solution methods provided to researchers and practitioners in the standard numerical solution packages enumerated at the beginning of the section can be summarized by this single matrix decomposition.

Binder and Pesaran (1997) and Anderson (2010) are two prominent methods that solve for P in a substantially different manner. Binder and Pesaran's (1997) "fully recursive method" works directly with the matrix quadratic (12) and iterates on

$$\tilde{P}_k = I_{n_y} - \tilde{A}\tilde{P}_{k-1}^{-1}\tilde{C}, \text{ where } \tilde{A} \equiv B^{-1}A, \tilde{C} \equiv B^{-1}C, \tilde{P}_0 \equiv I_{n_y} \tag{20}$$

Delivering the solution to the matrix quadratic (12) as $P = -\tilde{P}_N^{-1}\tilde{C}$ for some maximum iteration N. Anderson (2010) applies the bi-orthogonality that arises from the

separation of stable and unstable solutions to solve for the left invariant space associated with unstable solutions via[4]

$$\begin{bmatrix} y_t \\ E_t[y_{t+1}] \end{bmatrix} = \begin{bmatrix} 0_{n_y \times n_y} & I_{n_y} \\ -A^{-1}C & -A^{-1}B \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} \Rightarrow \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 0_{n_y \times n_y} & I_{n_y} \\ -A^{-1}C & -A^{-1}B \end{bmatrix} = \mathscr{M} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \tag{21}$$

where the vectors of V span the invariant space associated by unstable eigenvalues. This gives $y_t = -V_2^{-1}V_1 y_{t-1}$ as the solution to the homogenous problem, i.e., the matrix quadratic (12), $P = -V_2^{-1}V_1$. The key commonality of Binder and Pesaran (1997) and Anderson (2010) in contrast to the methods above is that they avoid the QZ or generalized Schur decompositon.

Apart from Anderson (2008), very little attention has been paid to comparing the accuracy of different algorithms for linear models[5] and to numerically addressing the assumptions necessary for the existence of a unique stable solution.[6] Improvements in the accuracy of the solution to linear DSGE models has implications for many nonlinear solutions as well. Anderson, Levin, and Swanson (2006) demonstrate that even small inaccuracies in lower orders compound to larger errors in the computation of higher, nonlinear solutions such as in Jin and Judd (2002).

Studies concerning the numerical robustness of generalized eigenvalue problems date back at least to Stewart (1972) and Wilkinson (1979), who provided examples of essentially arbitrary results from the QZ algorithm in the presence of nearly singular

[4] This assumes that A is invertible, the general case can be found in Anderson (2010) and is merely slightly more involved, utilizing the shuffle-algorithm of Luenberger (1978) to yield an invertible A.

[5] This is in stark contrast to the many studies that examine the accuracy of different nonlinear methods. See Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) for an overview.

[6] Heilberger, Klarl, and Maußner (2015) provides an exception, showing that, theoretically, if the rank assumption for the QZ decomposition is fulfilled for one ordering of the eigenvalues that conforms to the unit circle separation, it holds for any ordering that conforms to the same.

pencils, i.e. violation of the regularity assumption above. The computation of eigenvalues numerically is likewise subject to finite precision. Hammarling, Munro, and Tisseur (2013) provide a comprehensive study on improving the accuracy of quadratic eigenvalue problems. Anguas, Bueno, and Dopico (2019) provides a comparison of different conditioning numbers for the eigenvalues of matrix polynomials and conditioning numbers of polynomial eiqenvalues can be obtained via eigenvalues for perturbations of the polynomial or pseudospectra (see Tisseur and Higham, 2001; Higham and Tisseur, 2002). Specifically, Tisseur and Higham (2001), Mengi and Overton (2005), and Michiels, Green, Wagenknecht, and Niculescu (2006) apply pseudospectra to stability radii in continuous-time applications. I will turn to the pseudospectrum for insight into the saddle-path stability vis-a-vis the unit circle in the problem laid out above.

Specifically, the pseudospectrum provides a perturbed analog to the spectrum or set of eigenvalues/latent roots of (14) and (15)

$$\rho_\epsilon(M) = \{\lambda \in \mathbb{C} : (M(\lambda) + \Delta M(\lambda))x = 0 \text{ for some } x \neq 0 \text{ and } \Delta M(\lambda) \tag{22}$$

$$\text{with } \|\Delta A\| \leq \epsilon\alpha_A, \|\Delta B\| \leq \epsilon\alpha_B, \|\Delta C\| \leq \epsilon\alpha_C\} \tag{23}$$

where $\Delta M(\lambda)$ represents the perturbation of the quadratic[7]

$$\Delta M(\lambda) \equiv \Delta A\, \lambda^2 + \Delta B\, \lambda + \Delta C \tag{24}$$

and the α_i's control the perturbation, which are set as $\alpha_X = |X|$ using the 2-norm following Tisseur (2000). As shown in Tisseur and Higham (2001), this 2-norm definition of the pseudopectrum corresponds to the backward errors of the eigenvalues.

As proven in Tisseur (2000), while the QZ or generalized Schur algorithm is numerically stable for the generalized eigenvalue problem (Stewart, 1972), this is not

[7]This is perhaps easier to see via the identity $M(\lambda) + \Delta M(\lambda) = (A + \Delta A)\lambda^2 + (B + \Delta B)\lambda + (C + \Delta C)$.

the case for the quadratic eigenvalue problem, as it does not respect the structure of the latter. To see this, first define the pseudospectrum of (17) analogous to above

$$\rho_\epsilon(P_{FG}) = \{\lambda \in \mathbb{C} : (P_{FG}(\lambda) + \Delta P_{FG}(\lambda))x = 0 \text{ for some } x \neq 0 \text{ and } \Delta P_{FG}(\lambda) \tag{25}$$

$$\text{with } \|\Delta F\| \leq \epsilon\alpha_F, \|\Delta G\| \leq \epsilon\alpha_G\} \tag{26}$$

comparing the perturbations involved in (25) with (22)

$$\Delta P_{FG}(\lambda) \equiv \Delta F\lambda - \Delta G = \begin{bmatrix} \Delta F_{11} & \Delta F_{12} \\ \Delta F_{21} & \Delta F_{22} \end{bmatrix} \lambda - \begin{bmatrix} \Delta G_{11} & \Delta G_{12} \\ \Delta G_{21} & \Delta G_{22} \end{bmatrix} \tag{27}$$

$$\neq \left(\begin{bmatrix} I_{n_y} & 0_{n_y \times n_y} \\ 0_{n_y \times n_y} & \Delta A \end{bmatrix} \lambda - \begin{bmatrix} 0_{n_y \times n_y} & I_{n_y} \\ -\Delta C & -\Delta B \end{bmatrix} \right) \begin{bmatrix} I_{n_y} \\ I_{n_y}\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta M(\lambda) \end{bmatrix} \tag{28}$$

inspection underscores that, in general, perturbations of the QZ or generalized Schur of the companion linearization (16) do not respect the specific structure in the underlying matrix quadratic problem (12).

While the backward stability in the calculation of the eigenvalues does not fully characterize the numerical stability of the solution of linear DSGE models, it highlights the potential for QZ or generalized Schur-based algorithms to underperform algorithms that preserve the structure of the matric quadratic problem. I will turn to demonstrating this in the next section using the specific model of section 2.

4. Results

In this section I investigate the accuracy of the different methods, QZ- and non-QZ-based, from section 3 in solving the model of section 2. The model was chosen to be as simple as possible, in order to enable the symbolic solution of the underlying matric quadratic problem; see Higham and Kim (2000) who argue that Matlab can successfully solve two-dimensional matrix quadratic problems reliably. I provide numerical results for two calibrations, see table 1, labeled standard and extreme.

The standard calibration follows the RBC literature (see, e.g., King and Rebelo, 1999) with the degree of habit formation, h and curvature in the utility function, σ, elevated to match an equity premium of 7.8 in annual percentage points following (Mehra, 2003) for the post-war US and ω, the standard deviation of the technology shock, adjusted to deliver a standard deviation of consumption growth, $std(\log c_t)$, in line again with the post-war US experience. The extreme calibration is chosen to bring the eigenvalue separation between the stable and unstable pencils closer together, while maintaining the match of the exact solution to the equity premium and consumption growth volatility.

Besides assessing whether the different solution methods are able to recover the exact solutions for the two calibration targets, I examine the underlying causes of a degeneration in accuracy. Namely the largest absolute deviation in the matrices for the linear solution or policy function (11), P and Q, and the largest absolute difference in the finite eigenvalues of the quadratic eigenvalue problem (15) via

$$\max(|\Delta eig|) \equiv \max(|\lambda_{exact} - \lambda_{method}|) \tag{29}$$

$$\max(|\Delta P|) \equiv \max(|P_{exact} - P_{method}|) \tag{30}$$

$$\max(|\Delta Q|) \equiv \max(|Q_{exact} - Q_{method}|) \tag{31}$$

Additionally, I provide plots of the pseudospectra of the matrix quadratic (22) and of the QZ companion linearization (25). The results that are referred to as exact are solved symbolically and evaluated using Mathlab's VPA (variable precision arithmetic) with 100 digits of accuracy.

Table 2 contains the results for the standard calibration. The first line contains the equity premium predicted by the different methods and all of the methods successfully predict an equity premium of 7.8 annual percentage points, likewise the volatility of consumption growth, the third line, is identical across methods. Upon

	h	β	δ	α	σ	ρ	ω
Standard	0.966	0.99	0.025	0.36	98.1	0.95	0.134
Extreme	1-3.907E-05	1-1.750E-10	0.6715	1-5.751E-05	9.151	1-5.184E-04	3.068E-03

TABLE 1. Calibrations

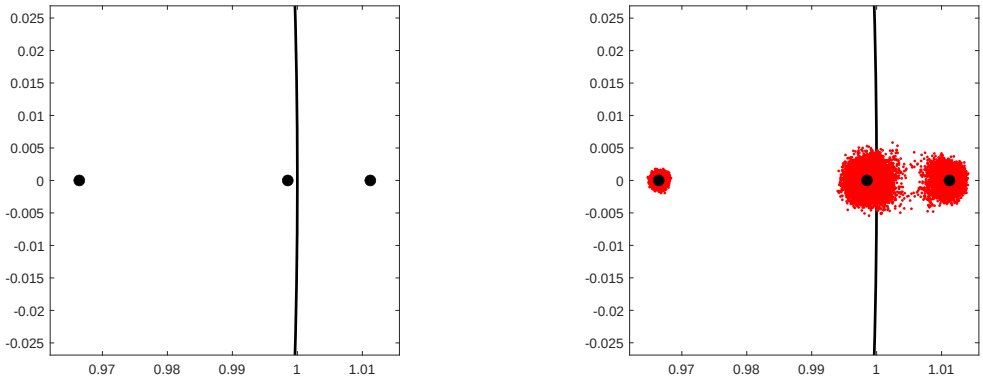
closer examination, the second line, the exact equity premium and that predicted by the varying methods differ to varying degrees. The most accurate method being that of Binder and Pesaran (1997) with all QZ-based methods apart from Dynare displaying degrees of accuracy several orders of magnitude lower. As laid out in Villemot (2011), Dynare reduces the problem solved with the QZ algorithm by, among others, eliminating zero column variables in the A and C matrices of the linear system (10); this is in line with one of the suggestions by Hammarling, Munro, and Tisseur (2013) to improve the accuracy of the quadratic eigenvalue problem. This is reflected in the fifth line of the table, where the largest error in the finite eigenvalues calculated by Dynare are in line with the non-QZ-based methods, those of the remaining QZ-based methods are several orders of magnitude larger, and that of Binder and Pesaran (1997) demonstrating the smallest error. The errors in the resulting matrices for the linear solution or policy function (11), P and Q are of the same order of magnitude. Despite the differences in the accuracy of calculating the eigenvalues, all of the methods yield the same eigenvalue separation between the stable and unstable pencils. Based on this standard calibration, the differences in the solutions generated by the different methods are of no economic consequence.

Figure 1 plots the pseudospectra for the extreme standard of the matrix quadratic (22) – in blue – and of the QZ algorithm (25) – in red – against the exact eigenvalues – in black – for two different sizes of perturbations. In the left panel, the pseudospectra are not visible, as they overlap with the exact results for perturbations of this size. For slightly larger perturbations (right panel), the pseudospectrum of the QZ algorithms

Measure			QZ-Based Methods				Alternatives	
	Data	Exact	Klein (2000)	Sims (2001)	Uhlig (1999)	Dynare	Anderson (2010)	BP (1997)
$E[rp]$	7.8	7.8	7.8	7.8	7.8	7.8	7.8	7.8
$\Delta E[rp]$			2.31E-08	-1.43E-09	2.27E-08	-3.08E-12	1.54E-11	1.71E-12
$std(\log c_t)$	0.566	0.566	0.566	0.566	0.566	0.566	0.566	0.566
Eig. Sep.		1.27E-02	1.27E-02	1.27E-02	1.27E-02	1.27E-02	1.27E-02	1.27E-02
$\max(\lvert\Delta eig\rvert)$			4.12E-11	2.47E-12	4.13E-11	1.32E-14	2.86E-14	3.09E-15
$\max(\lvert\Delta P\rvert)$			3.93E-11	2.36E-12	3.88E-11	7.17E-14	1.43E-14	1.51E-15
$\max(\lvert\Delta Q\rvert)$			3.21E-11	1.99E-12	3.14E-11	4.25E-15	2.13E-14	2.40E-15

TABLE 2. Results: Standard Calibration

- For Dynare, refer to Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011). BP (1997) refers to Binder and Pesaran (1997).



(A) $1e-6$ (B) $2e-5$

FIGURE 1. Pseudospectrum: Standard Calibration x-axis: real component, y-axis: imaginary component, large black dots: eigenvalues, black curve: unit circle, small red dots: pseudospectrum QZ, small blue dots: pseudospectrum matrix quadratic

encompasses the unit circle while that of the matric quadratic remains invisible at this scale. This, following Tisseur and Higham (2001), indicates that the backward error in calculating the eigenvalues is not only larger than under the QZ algorithm than with the matrix quadratic, consistent with Tisseur (2000), but also that the stable and unstable eigenvalues are potentially indistinguishable numerically.

Table 3 contains the results for the extreme calibration and the resulting predictions for the two calibration targets now differ significantly across methods. While the non-QZ-based methods continue to maintain a significant match with the calibration targets, lines 1 and 3, the QZ-based methods including Dynare now miss predict the equity premium by at least 75 annual basis points and as much as 3 annual percentage points, errors of genuine economic significance. The second line, containing the differences of the equity premium predicted by the different methods and the exact solution, now show the algorithm of Anderson (2010) being more accurate than the method of Binder and Pesaran (1997). The differences in the accuracies of the

predicted calibration targets ultimately stem from differences in the accuracies in matrices for the linear solution or policy function (11), P and Q, as can be seen in the last two lines of the table. Importantly, not a single algorithm displayed a warning that their solutions might be inaccurate.

The accuracies in the calculation of the finite eigenvalues roughly reflect the differences in the accuracy of the predicted calibration targets, yet the calculated eigenvalue separations are roughly the same, being on the order of $2E-05$. This is significant for two reasons. First, the separation is several orders of magnitude lower than under the standard calibration, pointing to a near overlap in the stable and unstable pencils. Second, their rough agreement may provide the basis for a diagnistic

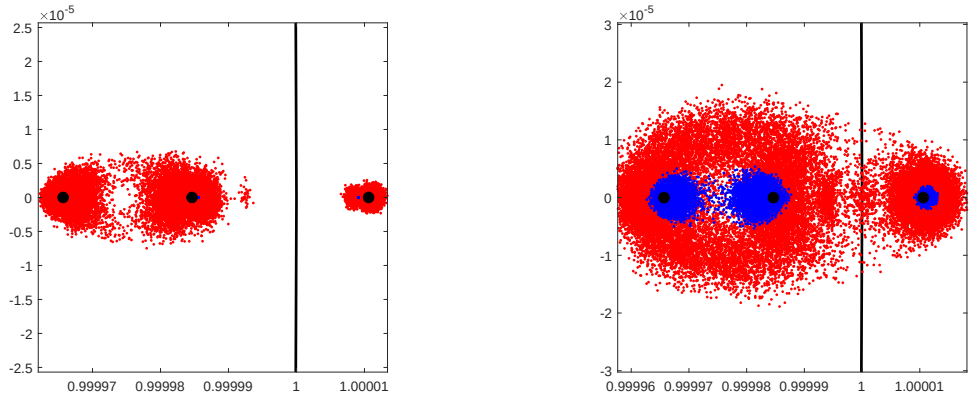
Figure 2 plots the pseudospectra for the extreme calibration of the matrix quadratic (22) – in blue – and of the QZ algorithm (25) – in red – against the exact eigenvalues – in black – for two different sizes of perturbations. In contrast to the results for the standard calibration in figure 2, the finite eigenvalues are all much closer to the unit circle (see the scale on the x-axis) and dispersion away from the exact eigenvalues is visible with perturbations several orders of magnitude smaller. Again, the pseudospectrum of the QZ algorithm bleeds across the unit circle for smaller perturbations than does the matrix quadratic (right panel).

Table 4 contains a summary of results from additional alternate calibrations (see the appendix, Table 5), in all calibrations, the parameters are chosen to match the equity premium of 7.8 and standard deviation of consumption growth of 0.566%. Calibrations I and II are alternative standard calibrations, holding all parameters apart from h, σ and ω constant. Calibration I has a higher curvature in the utility function, σ, and a lower degree of habit formation, h, and calibration II vice versa than in the standard calibration above. As in the standard calibration, the eigenvalue separation is on the order of $1E-02$ and all methods successfully recover the equity

Measure			QZ-Based Methods				Alternatives	
	Data	Exact	Klein (2000)	Sims (2001)	Uhlig (1999)	Dynare	Anderson (2010)	BP (1997)
$E[rp]$	7.8	7.8	7.05	4.75	7.05	6.47	7.8	7.8
$\Delta E[rp]$			0.754	3.05	0.754	1.33	-8.31E-07	-3.06E-06
$std(\log c_t)$	0.566	0.566	0.529	0.41	0.529	0.491	0.567	0.567
Eig. Sep.		2.82E-05	2.60E-05	2.09E-05	2.60E-05	2.55E-05	2.82E-05	2.82E-05
$\max(\lvert\Delta eig\rvert)$			1.84E-06	6.51E-06	1.84E-06	5.73E-06	1.55E-12	1.57E-11
$\max(\lvert\Delta P\rvert)$			1.01E-06	4.07E-06	1.01E-06	1.74E-06	1.13E-12	4.01E-12
$\max(\lvert\Delta Q\rvert)$			0.00127	0.00515	0.00127	0.00224	1.40E-09	5.16E-09

TABLE 3. Results: Extreme Calibration

- For Dynare, refer to Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011). BP (1997) refers to Binder and Pesaran (1997).



(A) $5e-11$ (B) $2e-10$

FIGURE 2. Pseudospectrum: Extreme Calibration x-axis: real component, y-axis: imaginary component, large black dots: eigenvalues, black curve: unit circle, small red dots: pseudospectrum QZ, small blue dots: pseudospectrum exact

		$E[rp]$					
	Eig. Sep.	Klein (2000)	Sims (2001)	Uhlig (1999)	Dynare	Anderson (2010)	BP (1997)
I	1.1E-02	7.8	7.8	7.8	7.8	7.8	7.8
II	1.05E-02	7.8	7.8	7.8	7.8	7.8	7.8
III	1.17E-04	8.31	8.17	8.31	6.5	7.8	7.8
IV	1.97E-05	12.7	7.98	12.7	7.8	7.8	7.8
V	3.2E-05	7.74	11.7	7.74	7.8	7.8	7.8
VI	2.9E-05	6.99	9.64	6.99	7.8	7.8	7.8

TABLE 4. Results for additional calibrations I-VI, see the appendix
For Dynare, refer to Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot (2011). BP (1997) refers to Binder and Pesaran (1997).

premium. Calibration III is similar to the extreme calibration above, but with a slightly reduced degree of habit formation, h, and discount factor, β, compensated by an increased curvature in the utility function, σ. With an eigenvalue separation

on the order of $1E-04$, the QZ methods demonstrate significant deviations in their predicted equity premia as above, though now some methods over and some methods under predict the premium. Both the alternative methods successfully match the premium; under this calibration, the accuracy of the Binder and Pesaran (1997) method exceeds that of Anderson (2010) (not shown in the table, full results are available online). Calibrations IV-VI provide further examples of the arbitrary results from QZ methods. With the eigenvalue separation on the order of $1E-05$, some methods do very well for some calibrations yet worse for others, with all the calibrated parameters arguably very similar (see the appendix, Table 5). Interestingly, Dynare in all of these calibrations successfully predicts the equity premium.

5. Conclusion

This paper has provided a concrete example, calibrated to macroeconomic and financial data, of almost arbitrary numerical results from QZ-based methods, when the underlying DSGE macroeconomic model has insufficient separation between the backward looking, or stable, and the forward looking, or unstable, components of the solution. This example was chosen to be as small as possible, two endogenous variables, to enable an exact, symbolic solution. While certainly much more work remains to deliver conditioning-like numbers when solving even linear DSGE models, the results here provide a first step in that direction and, perhaps more importantly, serve as a cautionary tale. Not a single one of the methods gave the user any indication that the numerical solution it provided might imply an equity premium off by several percentage points.

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APPENDIX

5.1. **Multivariate pivot derivation of the linear solution.** While this derivation contains nothing substantially new compared with, say Klein (2000), its formulation commensurate with (10) enables a straightforward application of Blanchard's (1979) forward method, making the derivations potentially more transparent and accessible than existing expositions in the literature.

Rearranging the model (10) into the companion linearization yields

$$F\begin{bmatrix} y_t \\ E_t[y_{t+1}] \end{bmatrix} = G\begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} + \begin{bmatrix} 0_{n_y \times n_\varepsilon} \\ D \end{bmatrix} \varepsilon_t, \quad F \equiv \begin{bmatrix} I_{n_y} & 0_{n_y \times n_y} \\ 0_{n_y \times n_y} & A \end{bmatrix}, G \equiv \begin{bmatrix} 0_{n_y \times n_y} & I_{n_y} \\ -C & -B \end{bmatrix} \tag{A1}$$

where I_{n_y} is an $n_y \times n_y$ identity matrix and $0_{n_y \times n_y}$ is an $n_y \times n_y$ zero matrix.

The generalized Schur decomposition (unitary Q and Z and upper triangular S and T with $Q^*FZ = S$ and $Q^*GZ = T$) of the matrix pencil $P_{FG}(z) = Fz - G$, can be ordered arbitrarily to form

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} E_t[w^s_{t+1}] \\ E_t[w^u_{t+1}] \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} w^s_t \\ w^u_t \end{bmatrix} + Q^* \begin{bmatrix} 0_{n_y \times n_\varepsilon} \\ D \end{bmatrix} \epsilon_t \tag{A2}$$

With any generalized Schur decomposition of $P_{DE}(z)$, the spectrum or set of eigenvalues of the pencil $P_{DE}(z)$ is determined by the diagonal entries of S and T

$$\rho(P_{DE}) = \{t_{ii}/s_{ii},\ \text{if } s_{ii} \neq 0;\ \infty,\ \text{if } s_{ii} = 0;\ \emptyset,\ \text{if } s_{ii} = t_{ii} = 0;\ i = 1,\ldots,2n_y\} \tag{A3}$$

where s_{ii} and t_{ii} denote the i'th row and i'th column of S and T respectively. Ordering the decomposition so that the unstable eigenvalues are in the lower right blocks of S and T (hence S_{22} and T_{22}), this lower block can be solved forward following Blanchard (1979) to yield

$$w_t^u = \lim_{j\to\infty} \left(T_{22}^{-1}S_{22}\right)^j E_t\left[w_{t+j}^u\right] - T_{22}^{-1}\underbrace{\left[\{Q^*\}_{21} \quad \{Q^*\}_{22}\right]}_{\equiv\{Q^{-1}\}_{2\bullet}}\underbrace{\left[0'_{n_y\times n_\varepsilon} \quad D'\right]'}_{\equiv\hat{D}}\varepsilon_t = -T_{22}^{-1}\{Q^*\}_{2\bullet}\hat{D}\varepsilon_t \tag{A4}$$

where the invertibility of T_{22} and convergence of $\lim_{j\to\infty}\left(T_{22}^{-1}S_{22}\right)^j$ follow directly from the ordering above. Using the definition $Z\left[w_t^{s\prime} \quad w_t^{u\prime}\right]' = \left[y_{t-1}' \quad y_t'\right]'$ delivers

$$w_t^u = \left[Z_{21}^* \quad Z_{22}^*\right]\left[y_{t-1}' \quad y_t'\right]' = -T_{22}^{-1}\left[\{Q^*\}_{21} \quad \{Q^*\}_{22}\right]\left[0'_{n_y\times n_\varepsilon} \quad D'\right]'\varepsilon_t \tag{A5}$$

where * indicates the complex conjugation of Z that delivers its inverse by virtue of it being a unitary matrix. If the necessary and sufficient assumptions for a unique stable solution for y_t of (10) from the main text hold, the unique stable solution for y_t is given by

$$y_t = Z_{21}Z_{11}^{-1}y_{t-1} - \left(Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}\right)T_{22}^{-1}\left[\{Q^*\}_{21} \quad \{Q^*\}_{22}\right]\left[0'_{n_y\times n_\varepsilon} \quad D'\right]'\varepsilon_t \tag{A6}$$

where ${Z_{22}^*}^{-1} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ and ${Z_{22}^*}^{-1}Z_{21}^* = -Z_{21}Z_{11}^{-1}$ follow from the properties of unitary matrices.

	h	β	δ	α	σ	ρ	ω
I	0.8617	0.99	0.025	0.36	324.3	0.95	8.355E-02
II	1-9.857E-05	0.99	0.025	0.36	6.109	0.95	6.175E-02
III	1-1.008E-04	1-8.991E-06	0.6402	1-5.680E-04	51.53	1-6.066E-05	7.742E-04
IV	1-6.829E-06	1-5.863E-08	0.6562	1-2.652E-05	1+2.591E-08	1-3.437E-03	1.594E-02
V	1-4.294E-06	1-1.012E-12	0.4727	1-9.990E-05	1+7.590E-08	1-9.628E-04	7.898E-03
VI	1-5.070E-06	1-4.259E-08	0.6539	1-5.715E-05	1+4.755E-05	1-1.221E-03	7.102E-03

TABLE 5. Additional Calibrations I-VI

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