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## ORIGINAL ARTICLE

## LONG RANGE DEPENDENCE FOR STABLE RANDOM PROCESSES

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We investigate long and short memory in  $\alpha$ -stable moving averages and max-stable processes with  $\alpha$ -Fréchet marginal distributions. As these processes are heavy-tailed, we rely on the notion of long range dependence based on the covariance of indicators of excursion sets. Sufficient conditions for the long and short range dependence of  $\alpha$ -stable moving averages are proven in terms of integrability of the corresponding kernel functions. For max-stable processes, the extremal coefficient function is used to state a necessary and sufficient condition for long range dependence.

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## 1. INTRODUCTION

The occurrence of long memory in time series has been known for a long time starting from the work of Hurst (1951). Since then, this phenomenon has been observed and studied in applications in various fields including biophysical data (Burnecki, 2012), network traffics (Pilipauskaitė and Surgailis, 2016), neuroscience (Botcharova *et al.*, 2014), geosciences (Montillet and Yu, 2015), and so on. A typical example in financial applications (see e.g. Cheung and Lai, 1995; Panas, 2001) is a stationary solution of a autoregressive moving average FARIMA( $p, d, q$ ) process with  $\alpha$ -stable innovations. In light of the variety of applications, a wide range of statistical models and methods for long range dependent processes has been developed, see, for instance, Avram and Taqqu ( ), Kasahara *et al.* (1988), Kokoszka and Taqqu (1996) for classical ones, and Magdziarz and Weron (2007) Beran *et al.* (2012), Jach *et al.* (2012), Koul and Surgailis (2018) for more recent developments. For a broader overview, we recommend the books of Doukhan *et al.* (2003), Beran *et al.* (2013), and Samorodnitsky (2016). These instruments rely on the explicit definition of long range dependence (LRD, for short) of a stationary time series or, more generally, a stationary stochastic process  $X = \{X(t), t \in T\}$ . Here and throughout this article, stationarity is understood in the sense that all finite-dimensional distributions of  $X$  are invariant under translations. There are many definitions of LRD in the literature depending on the class of processes to which  $X$  belongs. For instance, if  $X$  has a finite variance the following definition is classical cf. Samorodnitsky, 2016, pp. 194–5:

**Definition 1.1.** A stationary stochastic process  $X = \{X(t), t \in T\}$  on some domain  $T \subset \mathbb{R}$  with  $\mathbb{E}[|X(0)|^2] < \infty$  is called *long range dependent* if

$$\int_T |C(t)| dt = \infty,$$

where  $C(t) = \text{Cov}(X(0), X(t))$ ,  $t \in T$ , is its covariance function. For processes in discrete time, the integral above should be changed to a sum.

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Also,  $X$  is *antipersistent* if  $\int_T |C(t)| dt < \infty$ ,  $\int_T C(t) dt = 0$ , and *short range dependent*, otherwise.

Alternative definitions of long memory rely, for example, on the unboundedness of the spectral density of  $X$  at zero, growth comparisons of partial sums, phase transition in limit theorems for sums or maxima and so on (cf. Heyde and Yang, 1997; Dehling and Philipp, 2002; Samorodnitsky, 2004; Lavancier, 2006; Giraitis *et al.*, 2012; Beran *et al.*, 2013; Paulauskas, 2016; Samorodnitsky, 2016; Jach *et al.*, 2012).

Many of these approaches fail for heavy-tailed stochastic processes whose variance does not exist. Such processes occur, for instance, in modelling of network data, in finance and in insurance (see e.g. Kokoszka and Mikosch, 1997 who call the FARIMA( $p, d, q$ ) process with  $\alpha$ -stable innovations long range dependent if  $d \in (0, 1 - 1/\alpha)$  or Embrechts *et al.*, 1997; Resnick, 2007). To allow for the analysis of long memory behaviour in a broader setting, Kulik and Spodarev (2020) propose to consider the covariance of indicator functions of excursions and introduce

**Definition 1.2.** A real-valued stationary stochastic process  $X = \{X(t), t \in T\}$  where  $T$  is an unbounded subset of  $\mathbb{R}$  is short range dependent (SRD) if

$$\int_T \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \right| \mu(du) \mu(dv) dt < \infty \tag{1}$$

for any finite measure  $\mu$  on  $\mathbb{R}$ . Otherwise, that is, if there exists a finite measure  $\mu$  such that the integral in inequality (1) is infinite,  $X$  is long range dependent. For stochastic processes in discrete time, the integral  $\int_T dt$  should be replaced by the summation  $\sum_{t \in T: t \neq 0}$ .

One major advantage of this definition is that the above covariance exists in any case due to the boundedness of the indicators. Furthermore, the definition turns out to be useful as it offers the applicability of limit theorems for certain functionals of the process of interest.

In practice, however, the computation of the multiple integral in (1) might prove to be tricky. Therefore, we restrict ourselves here to the wide class of positively associated stochastic processes, including the class of infinitely divisible moving average processes with nonnegative kernels (Bulinski and Shashkin, 2007, Chapter 1, Theorem 3.27). This will allow us to eliminate the absolute value in (1).

To introduce the notion of positive association, we need the class  $\mathcal{M}(n)$  of real-valued bounded coordinate-wise nondecreasing Borel functions on  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ . For a real-valued stochastic process  $X = \{X(t), t \in T\}$  and a set  $I \subset T$ , we denote  $X_I = \{X(t), t \in I\}$ .

**Definition 1.3.** A real-valued stochastic process  $X = \{X(t), t \in T\}$  is *positively associated* if  $\text{Cov}(f(X_I), g(X_J)) \geq 0$  for any disjoint finite subsets  $I, J \subset T$  and all functions  $f \in \mathcal{M}(|I|)$  and  $g \in \mathcal{M}(|J|)$ .

By setting  $I = \{0\}$  and  $J = \{t\}$  for  $t \neq 0$ ,  $f(x) = \mathbb{1}\{x > u\}$  and  $g(x) = \mathbb{1}\{x > v\}$  for  $u, v \in \mathbb{R}$ , we have  $f \in \mathcal{M}(|I|)$  and  $g \in \mathcal{M}(|J|)$ . Consequently, for a positively associated stochastic process  $X$ , it holds  $\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) = \text{Cov}(f(X_I), g(X_J)) \geq 0$ , i.e. the absolute value in (1) can be omitted.

In this article, we consider two important subclasses of positively associated stationary processes that satisfy certain stability properties. More precisely, we study  $\alpha$ -stable moving averages and max-stable processes with  $\alpha$ -Fréchet marginals. As these processes are heavy-tailed, the classical definition of LRD (Definition 1.1) does not apply. Instead, we check Definition 1.2.

With regard to this endeavor, we first establish a general framework to compute the double integral  $\int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \mu(du) \mu(dv)$  by inverting the univariate characteristic function  $\varphi(s)$  of  $X(0)$  and the bivariate characteristic function  $\varphi_t(s_1, s_2)$  of  $(X(0), X(t))$ . Thus, our Theorem 2.4 yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \mu(du) \mu(dv)$$

$$\begin{aligned}
 &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{s_1 s_2} \operatorname{Re} \left\{ (\varphi_t(s_1, -s_2) - \varphi(s_1)\varphi(-s_2)) \overline{\psi(s_1)\psi(s_2)} \right\} \\
 &\quad - \frac{1}{s_1 s_2} \operatorname{Re} \left\{ (\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)) \overline{\psi(s_1)\psi(s_2)} \right\} ds_1 ds_2
 \end{aligned}$$

where  $\psi(s) = \int_{\mathbb{R}} \exp\{isx\} \mu(dx)$  is the Fourier transform of measure  $\mu$ .

Integrating this relation with respect to  $t$  will establish short or long range dependence according to Definition 1.2. Subsequently, we will apply this result to get the LRD of *symmetric  $\alpha$ -stable (SaS) moving averages* which are defined as follows.

**Definition 1.4** (Samorodnitsky and Taquq (1994)). Let  $m$  be a measurable function with  $m \in \mathcal{L}^\alpha(\mathbb{R})$ ,  $\alpha \in (0, 2)$ . Then, a SaS moving average process with parameter  $\alpha \in (0, 2)$  and kernel function  $m$  is a stochastic process  $X = \{X(t), t \in \mathbb{R}\}$  defined by

$$X(t) = \int_{\mathbb{R}} m(t-x) \Lambda(dx), \quad t \in \mathbb{R}, \tag{2}$$

where  $\Lambda$  is a SaS random measure with Lebesgue control measure.

Here and throughout the article, we use the notation  $m \in \mathcal{L}^p(A)$ ,  $p > 0$ , to imply that  $\int_A |m(x)|^p dx < \infty$ .

Regarding the SRD/LRD of the process  $X$  given in (2), our main result relies on the notion of  $\alpha$ -spectral covariance  $\rho_t = \int_{\mathbb{R}} (m(-x)m(t-x))^{\alpha/2} dx$ ,  $t \in \mathbb{R}$ , where  $m(x) \geq 0$ ,  $x \in \mathbb{R}$ . The  $\alpha$ -spectral covariance was first introduced by Paulauskas (1976) and its properties were studied in Damarackas and Paulauskas (2014) and Damarackas and Paulauskas (2017). In Paulauskas (2016), it was discussed how the integrability of  $\rho_t$  can be used for the definition of the memory property. Here, we establish by Theorem 3.4 that  $X$  is short range dependent if  $\rho_t \in \mathcal{L}^1(\mathbb{R})$  or, equivalently,  $m \in \mathcal{L}^{\alpha/2}(\mathbb{R})$ . Also, Theorem 3.5 establishes long range dependence if  $\int_{\mathbb{R}} \int_{\mathbb{R}} (m^a(x) \wedge m^a(t)) dx dt = \infty$  where  $a \wedge b$  is the minimum of  $a$  and  $b$ . These results hold also for  $\alpha$ -stable linear time series if integrals are replaced by sums.

To put our results into context, one may refer to other research and discussion on memory properties of  $\alpha$ -stable processes such as Rachev and Samorodnitsky (2002), Maejima and Yamamoto (2003), Samorodnitsky (2004). Also, we demonstrate how our findings are meaningful in practice by detecting LRD in a real world data set consisting of daily log-returns based on the opening price of the Intel corporation share.

Analogously to  $\alpha$ -stable processes, which have become popular as limits of rescaled sums of stochastic processes, *max-stable processes* have become a widely used concept in extreme value analysis occurring as limiting models for maxima. Thus, they have found applications in various areas such as meteorology (see e.g., Coles, 1993; Buishand *et al.*, 2008; Davison and Gholamrezaee, 2012; Oesting *et al.*, 2017), hydrology (Asadi *et al.*, 2015) and finance (Zhang and Smith, 2010). Max-stable processes are defined as follows.

**Definition 1.5.** A real-valued stochastic process  $X = \{X(t), t \in T\}$  is called a *max-stable process* if, for all  $n \in \mathbb{N}$ , there exist functions  $a_n : T \rightarrow (0, \infty)$  and  $b_n : T \rightarrow \mathbb{R}$  such that

$$\left\{ \max_{i=1}^n \frac{X_i(t) - b_n(t)}{a_n(t)}, t \in T \right\} \stackrel{d}{=} \{X(t), t \in T\},$$

where the processes  $X_i$ ,  $i \in \mathbb{N}$ , are independent copies of  $X$ , and  $\stackrel{d}{=}$  means equality in distribution. If the index set  $T$  is finite,  $X$  is also called a *max-stable vector*.

It follows from the univariate extreme value theory that the marginal distributions of a max-stable process are either degenerate or follow a Fréchet, Gumbel or Weibull law. While covariances always exist in the Gumbel and Weibull case and, thus, the classical notion of long-range dependence applies, we will consider the case when  $X$  is a stationary max-stable process with  $\alpha$ -Fréchet marginal distributions, that is,  $\mathbb{P}(X(t) \leq x) = \exp(-x^{-\alpha})$  for all  $x > 0$  and some  $\alpha > 0$  and all  $t \in T$ . Here, covariances do not exist if  $\alpha \leq 2$ .

In combination with Definition 1.2, a well-established dependence measure for max-stable stochastic processes allows for an easily tractable condition for short and long memory respectively. More specifically, we use the pairwise extremal coefficient  $\{\theta_t, t \in \mathbb{R}\}$  defined via the relation  $\mathbb{P}(X(0) \leq x, X(t) \leq x) = \mathbb{P}(X(0) \leq x)^{\theta_t}$ , which holds for all  $x > 0$ , to show that a stationary max-stable process with  $\alpha$ -Fréchet marginal distributions is long range dependent if and only if  $\int_{\mathbb{R}} (2 - \theta_t) dt = \infty$  (cf. Theorem 4.3).

To summarize, our article is structured as follows: Section 2 establishes the framework to invert the bivariate characteristic functions. In Section 3, we make use of this framework to find conditions for long range dependence of symmetric  $\alpha$ -stable moving averages and linear time series, while, in Section 4, we investigate long range dependence of a stationary max-stable process with  $\alpha$ -Fréchet marginals. Finally, we model the daily log-returns of an Intel corporation share by a  $S\alpha S$  moving average  $X$  and show that  $X$  is LRD in Section 5. For the sake of legibility, some of the proofs have been left out of the main part of this article. They can be found in the Appendix.

## 2. FROM CHARACTERISTIC FUNCTION TO COVARIANCE OF INDICATORS

We express the covariance of indicators of excursions of random variables above some levels  $u, v$  through their uni- and bivariate characteristic functions. Notice that for random variables  $U$  and  $V$  it holds that

$$\text{Cov}(\mathbb{1}\{U > u\}, \mathbb{1}\{V > v\}) = \mathbb{P}(U \leq u, V \leq v) - \mathbb{P}(U \leq u)\mathbb{P}(V \leq v). \tag{3}$$

**Theorem 2.1.** Suppose  $U$  and  $V$  are identically distributed random variables with marginal characteristic function  $\varphi_U$  and joint characteristic function  $\varphi_{UV}$ . Then, for a finite measure  $\mu$  with its Fourier transform denoted by  $\psi : \mathbb{R} \rightarrow \mathbb{C}, \psi(s) = \int_{\mathbb{R}} \exp\{isx\} \mu(dx)$  it holds that

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{U > u\}, \mathbb{1}\{V > v\}) \mu(du) \mu(dv) \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{s_1 s_2} \left( \varphi_U(s_1) \varphi_U(s_2) - \varphi_{UV}(s_1, s_2) \right) \overline{\psi(s_1) \psi(s_2)} ds_1 ds_2 \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[ \frac{1}{s_1 s_2} \text{Re} \left\{ \left( \varphi_{UV}(s_1, -s_2) - \varphi_U(s_1) \varphi_U(-s_2) \right) \overline{\psi(s_1) \psi(s_2)} \right\} \right. \\ & \quad \left. - \frac{1}{s_1 s_2} \text{Re} \left\{ \left( \varphi_{UV}(s_1, s_2) - \varphi_U(s_1) \varphi_U(s_2) \right) \overline{\psi(s_1) \psi(s_2)} \right\} \right] ds_1 ds_2. \end{aligned} \tag{4}$$

*Proof.* Let  $U'$  and  $V'$  be independent copies of  $U$  and  $V$ . Then

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{U > u\}, \mathbb{1}\{V > v\}) \mu(du) \mu(dv) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \mathbb{1}\{U > u, V > v\} - \mathbb{1}\{U' > u, V' > v\} \right] \mu(du) \mu(dv) \\ &= \lim_{a \rightarrow \infty} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \mathbb{1}\{U > u > -a, V > v > -a\} - \mathbb{1}\{U' > u > -a, V' > v > -a\} \right] \mu(du) \mu(dv). \end{aligned} \tag{5}$$

If we denote the difference of the two indicators by  $f_a(u, v)$ , then by Schilling (2017, Theorem 19.12)<sup>1</sup> we get that the last equality in (5) simplifies to

$$\lim_{a \rightarrow \infty} \frac{1}{4\pi^2} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}_a(s_1, s_2) \overline{\psi(s_1)\psi(s_2)} ds_1 ds_2, \quad (6)$$

where  $\widehat{f}_a(s_1, s_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(s_1 u + s_2 v)} f_a(u, v) du dv$ . By Lemma A.1 one can interchange the expectation and the integrals in (6) and computes

$$\mathbb{E} \widehat{f}_a(s_1, s_2) = \frac{1}{s_1 s_2} \left( \varphi_U(s_1) \varphi_U(s_2) - \varphi_{UV}(s_1, s_2) \right) \quad (7)$$

which is independent of  $a$ . Thus, (6) simplifies to

$$\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{s_1 s_2} \left( \varphi_U(s_1) \varphi_U(s_2) - \varphi_{UV}(s_1, s_2) \right) \overline{\psi(s_1)\psi(s_2)} ds_1 ds_2.$$

The second identity in (4) follows from splitting the integrals into the positive and negative half-lines and substituting afterwards.  $\square$

**Corollary 2.2.** Under the assumptions of Theorem 2.1, suppose that the random vector  $(U, V)$  is symmetric. Then relation (4) simplifies to

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{U > u\}, \mathbb{1}\{V > v\}) \mu(du) \mu(dv) \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[ \frac{1}{s_1 s_2} \left( \varphi_{UV}(s_1, -s_2) - \varphi_U(s_1) \varphi_U(-s_2) \right) \text{Re}\{\overline{\psi(s_1)\psi(s_2)}\} \right. \\ & \quad \left. - \frac{1}{s_1 s_2} \left( \varphi_{UV}(s_1, s_2) - \varphi_U(s_1) \varphi_U(s_2) \right) \text{Re}\{\psi(s_1)\psi(s_2)\} \right] ds_1 ds_2 \end{aligned} \quad (8)$$

$$\begin{aligned} &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{s_1 s_2} \left( \varphi_{UV}(s_1, -s_2) - \varphi_{UV}(s_1, s_2) \right) \text{Re}\{\psi(s_1)\} \text{Re}\{\psi(s_2)\} \\ & \quad + \frac{1}{s_1 s_2} \left( \varphi_{UV}(s_1, -s_2) + \varphi_{UV}(s_1, s_2) - 2\varphi_U(s_1) \varphi_U(s_2) \right) \text{Im}\{\psi(s_1)\} \text{Im}\{\psi(s_2)\} ds_1 ds_2. \end{aligned} \quad (9)$$

*Proof.* Equality (8) follows immediately from  $\varphi_U$  and  $\varphi_{UV}$  being real-valued as characteristic functions of a symmetric random variable and random vector respectively.

Equality (9) follows from  $\text{Re}\{xy\} = \text{Re}\{x\}\text{Re}\{y\} - \text{Im}\{x\}\text{Im}\{y\}$  for any  $x, y \in \mathbb{C}$ .  $\square$

If the stationary real-valued stochastic process  $X = \{X(t), t \in \mathbb{R}\}$  is positively associated, we can apply Theorem 2.1 and, in the symmetric case, Corollary 2.2 to  $X(0)$  and  $X(t)$  to check the long range dependence of  $X$ .

To do so, let  $T = \mathbb{R}$  in integral (1). However, the resulting expressions in (4), (8) or (9) might prove difficult to integrate w.r.t.  $t$  over the whole real line. Thus, it is worth noting that the following lemma allows us to restrict integration to unbounded subsets over which it might be easier to integrate.

<sup>1</sup>We thank René Schilling for his idea which simplifies our original proof.

**Lemma 2.3.** Let  $|\cdot|$  denote the Lebesgue measure on  $\mathbb{R}$  and let  $A \subset \mathbb{R}$  be an arbitrary subset with  $|A^c| < \infty$ . Then, a process  $X = \{X(t), t \in \mathbb{R}\}$  is SRD or LRD iff  $X_A = \{X(t), t \in A\}$  is SRD or LRD respectively.

*Proof.* We split up the integral in relation (1) into  $A$  and  $A^c$

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \text{Cov} \left( \mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\} \right) \right| \mu(du) \mu(dv) dt \\ &= \int_A \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \text{Cov} \left( \mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\} \right) \right| \mu(du) \mu(dv) dt \\ &+ \underbrace{\int_{A^c} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \text{Cov} \left( \mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\} \right) \right| \mu(du) \mu(dv) dt}_{\leq 1}. \end{aligned}$$

As the integral over  $A^c$  is finite in any case, the integral in relation (1) is finite iff  $X_A$  is SRD. □

Now we give the main result of this section showing the use of characteristic functions to check the short or long range dependence of  $X$ .

**Theorem 2.4.** Suppose we have a stationary real-valued, positively associated stochastic process  $X = \{X(t), t \in \mathbb{R}\}$  with absolutely continuous marginal distributions. Denote the univariate characteristic function of  $X(0)$  by  $\varphi$  and the bivariate characteristic function of  $(X(0), X(t))$  by  $\varphi_t$ . Furthermore, let  $A \subset \mathbb{R}$  be an arbitrary subset with  $|A^c| < \infty$ .

(a) Then,  $X$  is short range dependent if

$$\begin{aligned} & \int_A \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \mu(du) \mu(dv) dt \\ &= \frac{1}{2\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[ \frac{1}{s_1 s_2} \text{Re} \left\{ (\varphi_t(s_1, -s_2) - \varphi(s_1)\varphi(-s_2)) \overline{\psi(s_1)} \psi(s_2) \right\} \right. \\ &\quad \left. - \frac{1}{s_1 s_2} \text{Re} \left\{ (\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)) \overline{\psi(s_1)} \psi(s_2) \right\} \right] ds_1 ds_2 dt < \infty \end{aligned} \tag{10}$$

for any finite measure  $\mu$  with Fourier transform  $\psi(s) = \int_{\mathbb{R}} \exp\{isx\} \mu(dx)$ .

(b) Additionally, if  $(X(0), X(t))$  is symmetric for all  $t \in \mathbb{R}$ , then condition (10) rewrites as

$$\begin{aligned} & \int_A \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \mu(du) \mu(dv) dt \\ &= \frac{1}{2\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[ \frac{\varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2)}{s_1 s_2} \text{Re}\{\psi(s_1)\} \text{Re}\{\psi(s_2)\} \right. \\ &\quad \left. - \frac{\varphi_t(s_1, -s_2) + \varphi_t(s_1, s_2) - 2\varphi(s_1)\varphi(s_2)}{s_1 s_2} \text{Im}\{\psi(s_1)\} \text{Im}\{\psi(s_2)\} \right] ds_1 ds_2 dt < \infty. \end{aligned} \tag{11}$$

Otherwise, that is, if there exists a finite measure  $\mu$  such that the integral in (10) is infinite,  $X$  is long range dependent.



*Proof.*

- (a) Take  $U = X(0)$  and  $V = X(t)$ , where  $t \in \mathbb{R}$  in Theorem 2.1. Then,  $U$  and  $V$  are absolutely continuous and identically distributed random variables. Therefore, the equality in (10) is established by Theorem 2.1. It follows by relation (10) and Lemma 2.3 that  $X$  is SRD. Similarly,  $X$  is LRD if (10) is infinite.
- (b) Follows analogously by using Corollary 2.2. □

### 3. LONG RANGE DEPENDENCE OF $\alpha$ -STABLE MOVING AVERAGES

We investigate the LRD of  $S\alpha S$  moving averages in continuous and discrete time.

By Definition 1.4, a symmetric  $\alpha$ -stable moving average with kernel function  $m \in \mathcal{L}^\alpha(\mathbb{R})$ ,  $\alpha < 2$ , is defined by  $X(t) = \int_{\mathbb{R}} m(t-x) \Lambda(dx)$ ,  $t \in \mathbb{R}$ , where  $\Lambda$  is a symmetric  $\alpha$ -stable random measure.

**Remark 3.1.**

- (a) Note that the  $S\alpha S$  moving average process  $X = \{X(t), t \in \mathbb{R}\}$  is stationary,  $X(0)$  is absolutely continuous and, by Property 3.2.1 from Samorodnitsky and Taqqu (1994), the random vector  $(X(0), X(t))$  is symmetric for every  $t \in \mathbb{R}$ .
- (b) By Bulinski and Shashkin (2007), Theorems 1.3.5 and 1.3.27,  $X$  is positively associated if the kernel function  $m$  is nonnegative.
- (c) To exclude the trivial case  $X(t) = 0$  for all  $t \in \mathbb{R}$  we always assume that the Lebesgue measure of the set  $\{x \in \mathbb{R} \mid m(x) > 0\}$  is positive.

By Samorodnitsky and Taqqu (1994), Proposition 3.4.2, the characteristic function  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$  of  $X(t)$ ,  $t \in \mathbb{R}$ , is given by

$$\varphi(s) = \exp \left\{ - |s|^\alpha \int_{\mathbb{R}} |m(x)|^\alpha dx \right\}, \quad s \in \mathbb{R}. \tag{12}$$

Moreover, the bivariate characteristic function  $\varphi_t : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$  of  $(X(0), X(t))$ ,  $t \in \mathbb{R}$  is given by

$$\varphi_t(s_1, s_2) = \exp \left\{ - \int_{\mathbb{R}} |s_1 m(-x) + s_2 m(t-x)|^\alpha dx \right\}, \quad s_1, s_2 \in \mathbb{R}. \tag{13}$$

Before we get to our main result, we need to introduce the  $\alpha$ -spectral covariance of a stable vector as defined by Damarackas and Paulauskas (2017, equation (11)). Let  $\mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$  be the unit circle. Recall that a random vector  $Z = (X_1, X_2)$  is symmetric  $\alpha$ -stable with parameter  $\alpha$  if there exists a finite measure  $\Gamma$  on  $\mathbb{S}^1$ , the so-called *spectral measure*, such that the characteristic function of  $Z$  is given by

$$\mathbb{E} e^{i\langle s, Z \rangle} = \exp \left\{ - \int_{\mathbb{S}^1} |\langle s, x \rangle|^\alpha \Gamma(dx) \right\}, \quad s \in \mathbb{R}^2,$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner product on  $\mathbb{R}^2$ .

**Definition 3.2.** Suppose  $(X_1, X_2)$  is an  $\alpha$ -stable random vector with spectral measure  $\Gamma$ , then the  $\alpha$ -spectral covariance of  $X_1$  and  $X_2$  is given by

$$\rho = \int_{\mathbb{S}^1} |s_1 s_2|^{\alpha/2} \operatorname{sgn}(s_1 s_2) \Gamma(d(s_1, s_2)). \tag{14}$$

Let us calculate the  $\alpha$ -spectral covariance of  $(X(0), X(t))$ ,  $t \in \mathbb{R}$ , where  $X$  is a  $S\alpha S$  moving average.



**Lemma 3.3.** Suppose  $X = \{X(t), t \in \mathbb{R}\}$  with  $X(t) = \int_{\mathbb{R}} m(t-x) \Lambda(dx)$  is a SaS moving average process. Then, the  $\alpha$ -spectral covariance of  $(X(0), X(t)), t \in \mathbb{R}$ , is given by

$$\rho_t = \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) \operatorname{sgn}(m(-x)m(t-x)) dx. \tag{15}$$

*Proof.* Denote  $m_1(x) = m(-x)$  and  $m_2(x) = m(t-x)$ , Proposition 3.4.3 in Samorodnitsky and Taqqu (1994) and the symmetry of  $\Lambda$  yields that  $(X(0), X(t))$  is SaS with spectral measure  $\Gamma$  defined for all Borel sets  $A \subset \mathbb{S}^1$  by

$$\begin{aligned} \Gamma(A) &= \frac{1}{2} \int_{g^{-1}(A)} \left(m_1^2(x) + m_2^2(x)\right)^{\alpha/2} dx + \frac{1}{2} \int_{g^{-1}(-A)} \left(m_1^2(x) + m_2^2(x)\right)^{\alpha/2} dx \\ &= \gamma(g^{-1}(A)) + \gamma(g^{-1}(-A)) =: (\gamma \circ g^{-1})(A) + (\gamma \circ g^{-1})(-A), \end{aligned}$$

where

$$g(x) = \left( \frac{m_1(x)}{(m_1^2(x) + m_2^2(x))^{1/2}}, \frac{m_2(x)}{(m_1^2(x) + m_2^2(x))^{1/2}} \right), \quad x \in \mathbb{R}.$$

Hereby  $\gamma$  is an absolutely continuous measure w.r.t. the Lebesgue measure with density  $\frac{1}{2}(m_1^2(x) + m_2^2(x))^{\alpha/2}$ . With  $f(s_1, s_2) = |s_1 s_2|^{\alpha/2} \operatorname{sgn}(s_1 s_2)$  we get

$$\begin{aligned} \int_{\mathbb{S}^1} f d(\gamma \circ g^{-1}) &= \int_{g^{-1}(\mathbb{S}^1)} f \circ g d\gamma = \int_{\mathbb{R}} \frac{m_1^{\alpha/2}(x) m_2^{\alpha/2}(x) \operatorname{sgn}(m_1(x)m_2(x))}{(m_1^2(x) + m_2^2(x))^{\alpha/2}} \gamma(dx) \\ &= \frac{1}{2} \int_{\mathbb{R}} m_1^{\alpha/2}(x) m_2^{\alpha/2}(x) \operatorname{sgn}(m_1(x)m_2(x)) dx. \end{aligned}$$

Thus,

$$\rho_t = \int_{\mathbb{S}^1} |s_1 s_2|^{\alpha/2} \operatorname{sgn}(s_1 s_2) \Gamma(d(s_1, s_2)) = \int_{\mathbb{R}} m_1^{\alpha/2}(x) m_2^{\alpha/2}(x) \operatorname{sgn}(m_1(x)m_2(x)) dx.$$

□

Now, a sufficient condition for the short range dependence of  $X$  can be formulated in terms of  $\rho_t$  or, equivalently, in terms of the kernel function  $m$ .

**Theorem 3.4.** Let  $X = \{X(t), t \in \mathbb{R}\}$  be a SaS moving average process with parameter  $\alpha \in (0, 2)$ , nonnegative kernel function  $m$  and  $\alpha$ -spectral covariance  $\rho_t$  given in (15).  $X$  is SRD if

$$\rho_t \in \mathcal{L}^1(\mathbb{R}), \tag{16}$$

or, equivalently,  $m \in \mathcal{L}^{\alpha/2}(\mathbb{R})$ .

*Proof.* Without loss of generality, assume  $\mu$  is a probability measure. Now, apply Theorem 2.4 to  $X$  for some  $\varepsilon \in (0, \|m\|_{\alpha}^{\alpha})$  and choose  $A = \{t \in \mathbb{R} \mid \rho_t \in [0, \varepsilon]\}$ . It follows from the integrability of  $\rho_t$  that  $\rho_t \rightarrow 0$  as  $t \rightarrow \pm\infty$ . Thus, there exists a constant  $\tilde{t} > 0$  such that  $\rho_t < \varepsilon$  for all  $t \in \mathbb{R}$  where  $|t| > \tilde{t}$ . Hence, it holds that  $A^c \subset \{t \in \mathbb{R} \mid |t| \leq \tilde{t}\}$  and  $|A^c| < \infty$ .

Obviously, the right-hand side of the equality in (11) is bounded by

$$\begin{aligned}
 & \frac{1}{2\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2)|}{s_1 s_2} \underbrace{|\operatorname{Re}\{\psi(s_1)\}\operatorname{Re}\{\psi(s_2)\}|}_{\leq 1} ds_1 ds_2 dt \\
 & + \frac{1}{2\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, -s_2) + \varphi_t(s_1, s_2) - 2\varphi(s_1)\varphi(s_2)|}{s_1 s_2} \underbrace{|\operatorname{Im}\{\psi(s_1)\}\operatorname{Im}\{\psi(s_2)\}|}_{\leq 1} ds_1 ds_2 dt \\
 & \leq \frac{1}{\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, -s_2) - \varphi(s_1)\varphi(s_2)|}{s_1 s_2} ds_1 ds_2 dt \\
 & + \frac{1}{\pi^2} \int_A \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)|}{s_1 s_2} ds_1 ds_2 dt =: \frac{1}{\pi^2} (I_1 + I_2). \tag{17}
 \end{aligned}$$

By inequalities (A.6) and (A.7) in Lemma A.3 we get

$$I_1, I_2 \leq \frac{8\pi}{\alpha^2 \|m\|_\alpha^{2\alpha}} \int_A \frac{\rho_t}{\sqrt{\|m\|_\alpha^{2\alpha} - \rho_t^2}} dt \leq \frac{8\pi}{\alpha^2 \|m\|_\alpha^{2\alpha}} \frac{1}{\sqrt{\|m\|_\alpha^{2\alpha} - \varepsilon^2}} \int_A \rho_t dt < \infty.$$

Next, show that condition (16) holds true iff  $m \in \mathcal{L}^{\alpha/2}(\mathbb{R})$ . By Fubini's theorem we get

$$\begin{aligned}
 \int_{\mathbb{R}} \rho_t dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx dt \\
 &= \int_{\mathbb{R}} m^{\alpha/2}(-x) \left( \int_{\mathbb{R}} m^{\alpha/2}(t-x) dt \right) dx \\
 &= \left( \int_{\mathbb{R}} m^{\alpha/2}(-x) dx \right)^2 = \|m\|_{\alpha/2}^\alpha < \infty.
 \end{aligned}$$

□

Naturally, one may also ask for sufficient conditions for the long range dependence of  $X$ . Such a condition is given by

**Theorem 3.5.** Let  $X = \{X(t), t \in \mathbb{R}\}$  be a  $S\alpha S$  moving average process with parameter  $\alpha \in (0, 2)$  and nonnegative kernel function  $m$ . Then,  $X$  is long range dependent if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (m^\alpha(x) \wedge m^\alpha(t)) dx dt = \infty. \tag{18}$$

*Proof.* Given in the appendix. □

Additionally, if the kernel function  $m$  is eventually monotonic, then we can simplify condition (18) as follows.

**Corollary 3.6.** Let  $X = \{X(t), t \in \mathbb{R}\}$  be a  $S\alpha S$  moving average process with parameter  $\alpha \in (0, 2)$  and non-negative kernel function  $m \in \mathcal{L}^\alpha(\mathbb{R})$  which is eventually monotonic, i.e. there is a number  $a > 0$  such that  $m$  is

monotonically decreasing on  $(a, \infty)$  or monotonically increasing on  $(-\infty, -a)$ . Then,  $X$  is long range dependent if

$$\int_a^\infty t m^\alpha(t) dt = \infty \quad \text{or} \quad \int_{-\infty}^{-a} t m^\alpha(t) dt = -\infty. \tag{19}$$

Additionally, if  $m$  is symmetric, the two sufficient conditions (19) are equivalent.

*Proof.* Suppose  $m$  is monotonically decreasing on  $(a, \infty)$  and compute the integral (18). Thus, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (m^\alpha(x) \wedge m^\alpha(t)) dx dt \geq \int_a^\infty \left( \int_a^t m^\alpha(t) dx + \int_t^\infty m^\alpha(x) dx \right) dt \geq \int_a^\infty t m^\alpha(t) dt - a \|m\|_\alpha^\alpha.$$

The claim follows from the fact that  $m \in \mathcal{L}^\alpha(\mathbb{R})$ . The case of  $m$  monotonically increasing on some interval  $(-\infty, -a)$  follows analogously. □

Now let us give an example of a kernel function  $m \in \mathcal{L}^\alpha(\mathbb{R})$  whose corresponding SaS moving average is long range dependent if  $m \notin \mathcal{L}^{\alpha/2}(\mathbb{R})$ .

**Example 3.7.** Suppose we have a SaS moving average process  $X = \{X(t), t \in \mathbb{R}\}$  with parameter  $\alpha \in (0, 2)$  and nonnegative kernel function  $m(x) \sim C|x|^{-\delta}$  as  $|x| \rightarrow \infty$  where  $\delta > \frac{1}{\alpha}$  and  $C > 0$ . Obviously,  $m \in \mathcal{L}^\alpha(\mathbb{R})$  and  $m(x) \geq \frac{C}{2}|x|^{-\delta}$  where  $|x| \geq a$  for some  $a > 0$ . Notice that

$$\int_a^\infty t \cdot \left(\frac{C}{2}|t|^{-\delta}\right)^\alpha dt = \left(\frac{C}{2}\right)^\alpha \int_a^\infty t^{1-\delta\alpha} dt = \infty$$

if  $1 - \delta\alpha \geq -1$  or, equivalently,  $\delta \leq \frac{2}{\alpha}$ . Analogously to the proof of Corollary 3.6, this implies that  $m$  fulfills (18) if  $m \notin \mathcal{L}^{\alpha/2}(\mathbb{R})$ . Thus,  $X$  is long range dependent if  $\delta \in \left(\frac{1}{\alpha}, \frac{2}{\alpha}\right]$  by Theorem 3.5 and is short range dependent if  $\delta > \frac{2}{\alpha}$  by Theorem 3.4.

**Remark 3.8.** On one hand, this example was given to illustrate that the conditions (19) in Corollary 3.6 are useful when the kernel function itself is not eventually monotonic but rather asymptotically equivalent to such a function. On the other hand, this example motivates our conjecture that a SaS moving average is long range dependent iff  $m \notin \mathcal{L}^{\alpha/2}(\mathbb{R})$ . However, the conjecture’s proof is still to be found.

Similar results as above can be obtained for symmetric  $\alpha$ -stable linear time series  $Y$ .

**Definition 3.9** (Hosoya (1978)). Let  $\{Z(t), t \in \mathbb{Z}\}$  be a sequence of i.i.d. SaS random variables with characteristic function  $\varphi(s) = \exp\{-\tau^\alpha |s|^\alpha\}$ ,  $\tau > 0$ ,  $0 < \alpha < 2$ ,  $s \in \mathbb{R}$ . Let  $\{a_j, j \in \mathbb{Z}\}$  be a sequence of numbers such that  $\sum_{j=-\infty}^{+\infty} |a_j| < \infty$ . The stochastic process defined by

$$Y(t) = \sum_{j=-\infty}^{+\infty} a_j Z(t-j), \quad t \in \mathbb{Z}, \tag{20}$$

is called a *linear SaS time series*.

Notice that  $Y$  can be written as a continuous time SaS moving average with parameter  $\alpha \in (0, 2)$  and kernel function  $m(x) = \sum_{j=-\infty}^{\infty} a_j \mathbb{1}_{[(j-1)\tau^\alpha, j\tau^\alpha)}(x)$  sampled at time instants  $t \in \mathbb{Z}$ . Thus,  $Y$  is positively associated if the coefficients  $a_j$  are nonnegative for all  $j \in \mathbb{Z}$ . Moreover, the function  $\rho_t$  simplifies to  $\rho_t = \sum_{j=-\infty}^{\infty} a_{-j}^{\alpha/2} a_{t-j}^{\alpha/2}$ ,  $t \in \mathbb{Z}$ .

**Remark 3.10.** Theorems 3.4 and 3.5 as well as Corollary 3.6 apply to linear processes with the obvious substitute of  $\sum_{t=-\infty}^{\infty}$  for  $\int_{\mathbb{R}} dt$ . Indeed, let  $Y = \{Y(t), t \in \mathbb{Z}\}$  be a stationary SaS time series with parameter  $\alpha < 2$  and nonnegative coefficients  $\{a_j, j \in \mathbb{Z}\}$ . If

$$\sum_{t=-\infty}^{\infty} \rho_t < \infty \quad \text{or, equivalently,} \quad \sum_{j=-\infty}^{\infty} a_j^{\alpha/2} < \infty, \tag{21}$$

then  $Y$  is short range dependent. If

$$\sum_{j=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (a_j^\alpha \wedge a_t^\alpha) = \infty, \tag{22}$$

then  $Y$  is long range dependent. Additionally, if the coefficients  $a_j$  are for some  $a > 0$  monotonically increasing for all  $j < -a$  or monotonically decreasing for all  $j > a$ , then  $Y$  is long range dependent if  $\sum_{j=a}^{\infty} j a_j^\alpha = \infty$  or  $\sum_{j=-\infty}^{-a} j a_j^\alpha = -\infty$  respectively.

#### 4. LONG RANGE DEPENDENCE OF MAX-STABLE PROCESSES

We demonstrate that it is possible to use already existing dependence properties to check Definition 1.2 instead of inverting characteristic functions as in the previous sections.

Any max-stable process is positively associated, see for instance Proposition 5.29 in Resnick (2013). Its dependence properties are typically summarized by its pairwise extremal coefficients  $\{\theta_t, t \in T\}$  defined via

$$\mathbb{P}(X(0) \leq x, X(t) \leq x) = \mathbb{P}(X(0) \leq x)^{\theta_t} \quad \text{for all } x > 0,$$

(cf. Schlather and Tawn, 2003). By a series expansion of the logarithm, it can be seen that  $\theta_t = 2 - \lim_{x \rightarrow \infty} \mathbb{P}(X(t) > x \mid X(0) > x)$ . Thus,  $\theta_t \in [1, 2]$ , where  $\theta_t = 2$  corresponds to the case of (asymptotic) independence between  $X(0)$  and  $X(t)$  while  $\theta_t = 1$  means full dependence. Even though the joint distribution of  $(X(0), X(t))$  is not uniquely determined by  $\theta_t$ , this characteristic turns out to be a useful tool for the identification of dependence properties. For instance, Stoev (2008), Kabluchko and Schlather (2010) and Dombry and Kabluchko (2017) provide necessary and sufficient conditions for ergodicity and mixing of a max-stable process in terms of its pairwise extremal coefficients.

Here, we focus on the property of long-range dependence given by Definition 1.2. We obtain bounds for  $\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\})$ ,  $t \in T, u, v > 0$ , by making use of the following lemma.

**Lemma 4.1.** Let  $(X, Y)$  be a bivariate max-stable random vector with  $\alpha$ -Fréchet margins,  $\alpha > 0$ , and extremal coefficient  $\theta \in [1, 2]$ . Then, we have

$$\exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha} + \frac{2-\theta}{(u \vee v)^\alpha}\right) \leq \mathbb{P}(X \leq u, Y \leq v) \leq \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha} + \frac{2-\theta}{(u \wedge v)^\alpha}\right)$$

for all  $u, v > 0$ .

*Proof.* It is well-known that the cumulative distribution function of a bivariate max-stable random vector  $(X, Y)$  with  $\alpha$ -Fréchet margins is of the form

$$\mathbb{P}(X \leq u, Y \leq v) = \exp \left( - \mathbb{E} \left[ \left( \frac{W_X}{u} \vee \frac{W_Y}{v} \right)^\alpha \right] \right), \quad u, v \geq 0,$$

where  $(W_X, W_Y)$  is a bivariate random vector with  $\mathbb{E} W_X^\alpha = \mathbb{E} W_Y^\alpha = 1$  cf. Chapter 5 in Resnick (2013), for instance. This so-called *spectral vector*  $(W_X, W_Y)$  is closely connected to the extremal coefficient via the relation  $\theta = \mathbb{E}(W_X^\alpha \vee W_Y^\alpha)$ . Thus, we obtain

$$\begin{aligned} -\log \mathbb{P}(X \leq u, Y \leq v) &= \mathbb{E} \left[ \left( \frac{W_X}{u} \right)^\alpha \vee \left( \frac{W_Y}{v} \right)^\alpha \right] \\ &= \mathbb{E} \left[ \left( \frac{W_X^\alpha}{(u \vee v)^\alpha} + W_X^\alpha \cdot \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbf{1}\{u \leq v\} \right) \right. \\ &\quad \left. \vee \left( \frac{W_Y^\alpha}{(u \vee v)^\alpha} + W_Y^\alpha \cdot \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbf{1}\{v < u\} \right) \right]. \end{aligned} \tag{23}$$

Distinguishing between the two cases  $u \leq v$  and  $v < u$ , it can be seen that the right-hand side of (23) can be bounded from above by

$$\begin{aligned} &\frac{1}{(u \vee v)^\alpha} \mathbb{E} (W_X^\alpha \vee W_Y^\alpha) + \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbb{E} (W_X^\alpha \cdot \mathbf{1}\{u \leq v\} + W_Y^\alpha \cdot \mathbf{1}\{v < u\}) \\ &= \frac{\theta}{(u \vee v)^\alpha} + \left( \frac{1}{u^\alpha} + \frac{1}{v^\alpha} - \frac{2}{(u \vee v)^\alpha} \right) \cdot 1 = -\frac{2 - \theta}{(u \vee v)^\alpha} + \frac{1}{u^\alpha} + \frac{1}{v^\alpha}, \end{aligned}$$

where we used the fact that  $\mathbb{E} W_X^\alpha = \mathbb{E} W_Y^\alpha = 1$ . This gives the lower bound stated in the lemma. Analogously, we obtain

$$\begin{aligned} -\log \mathbb{P}(X \leq u, Y \leq v) &= \mathbb{E} \left[ \left( \frac{W_X}{u} \right)^\alpha \vee \left( \frac{W_Y}{v} \right)^\alpha \right] \\ &= \mathbb{E} \left[ \left( \frac{W_X^\alpha}{(u \wedge v)^\alpha} - W_X^\alpha \cdot \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbf{1}\{v < u\} \right) \right. \\ &\quad \left. \vee \left( \frac{W_Y^\alpha}{(u \wedge v)^\alpha} - W_Y^\alpha \cdot \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbf{1}\{u \leq v\} \right) \right] \\ &\geq \frac{1}{(u \wedge v)^\alpha} \mathbb{E} (W_X^\alpha \vee W_Y^\alpha) - \left( \frac{1}{(u \wedge v)^\alpha} - \frac{1}{(u \vee v)^\alpha} \right) \cdot \mathbb{E} (W_X^\alpha \cdot \mathbf{1}\{v < u\} + W_Y^\alpha \cdot \mathbf{1}\{u \leq v\}) \\ &= \frac{\theta}{(u \wedge v)^\alpha} - \left( \frac{2}{(u \wedge v)^\alpha} - \frac{1}{u^\alpha} - \frac{1}{v^\alpha} \right) \cdot 1 = -\frac{2 - \theta}{(u \wedge v)^\alpha} + \frac{1}{u^\alpha} + \frac{1}{v^\alpha}, \end{aligned}$$

which gives the corresponding upper bound. □

**Remark 4.2.** Note that the lower bound in Lemma 4.1 corresponds to the bound given in Strokorb and Schlather (2015). For the so-called Molchanov–Tawn model, we even have

$$\mathbb{P}(X \leq u, Y \leq v) = \exp \left( -\frac{1}{u^\alpha} - \frac{1}{v^\alpha} + \frac{2 - \theta}{(u \vee v)^\alpha} \right), \quad u, v > 0,$$

that is, the bound is sharp in this case.

Using Equation 3, Lemma 4.1 yields the following bounds for  $\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\})$ :

$$\begin{aligned} \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \cdot \left[\exp\left(\frac{2-\theta_t}{(u \vee v)^\alpha}\right) - 1\right] &\leq \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \\ &\leq \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \cdot \left[\exp\left(\frac{2-\theta_t}{(u \wedge v)^\alpha}\right) - 1\right] \end{aligned}$$

for  $u, v > 0$  and  $\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) = 0$  if  $u \wedge v \leq 0$  due to the  $\alpha$ -Fréchet margins. Consequently, one obtains for the integral in (1) that

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \int_T \left[\exp\left(\frac{2-\theta_t}{(u \vee v)^\alpha}\right) - 1\right] dt \mu(du) \mu(dv) \\ &\leq \int_T \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\})| \mu(du) \mu(dv) dt \\ &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \int_T \left[\exp\left(\frac{2-\theta_t}{(u \wedge v)^\alpha}\right) - 1\right] dt \mu(du) \mu(dv). \end{aligned} \quad (24)$$

From the lower and the upper bound in (24), we directly obtain a necessary and sufficient condition for long range dependence. Interestingly, unlike in case of  $\alpha$ -stable processes, the criterion does not depend on  $\alpha > 0$ .

**Theorem 4.3.** Let  $X = \{X(t), t \in T\}$  be a stationary max-stable process with  $\alpha$ -Fréchet marginal distributions,  $\alpha > 0$ , and pairwise extremal coefficients  $\{\theta_t, t \in T\}$ . Then,  $X$  is long range dependent if and only if

$$\int_T (2 - \theta_t) dt = \infty. \quad (25)$$

*Proof.* First, assume that (25) holds. Choosing the finite measure  $\mu = \delta_{\{1\}}$  as the Dirac measure, we obtain from the lower bound in (24) and the inequality  $\exp(x) \geq 1 + x$  for all  $x \geq 0$  that

$$\begin{aligned} &\int_T \int_{\mathbb{R}} \int_{\mathbb{R}} |\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\})| \delta_{\{1\}}(du) \delta_{\{1\}}(dv) dt \\ &\geq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \int_T \left[\exp\left(\frac{2-\theta_t}{(u \vee v)^\alpha}\right) - 1\right] dt \delta_{\{1\}}(du) \delta_{\{1\}}(dv) \\ &= \exp(-2) \cdot \int_T [\exp(2 - \theta_t) - 1] dt \geq \exp(-2) \cdot \int_T (2 - \theta_t) dt = \infty. \end{aligned}$$

Conversely, assume that (25) does not hold, that is,

$$C = \int_T (2 - \theta_t) dt < \infty.$$

As  $0 \leq 2 - \theta_t \leq 1$  for all  $t \in T$ , we obtain that

$$\int_T (2 - \theta_t)^k dt \leq \int_T (2 - \theta_t) dt = C$$

for all  $k \in \mathbb{N}$  and therefore

$$\begin{aligned} \exp(-u^{-\alpha}) \int_T [\exp((2 - \theta_t)u^{-\alpha}) - 1] dt &= \exp(-u^{-\alpha}) \int_T \sum_{k=1}^{\infty} \frac{u^{-\alpha k}}{k!} (2 - \theta_t)^k dt \\ &= \exp(-u^{-\alpha}) \sum_{k=1}^{\infty} \frac{u^{-\alpha k}}{k!} \int_T (2 - \theta_t)^k dt \leq \exp(-u^{-\alpha}) \sum_{k=1}^{\infty} \frac{u^{-\alpha k}}{k!} C \leq C \end{aligned}$$

for all  $u \geq 0$ . Combining this inequality with the upper bound in (24), we have

$$\begin{aligned} &\int_T \int_{\mathbb{R}} \int_{\mathbb{R}} |\text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\})| \mu(du) \mu(dv) dt \\ &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_T \exp\left(-\frac{1}{u^\alpha} - \frac{1}{v^\alpha}\right) \int_T \left[\exp\left(\frac{2 - \theta_t}{(u \wedge v)^\alpha}\right) - 1\right] dt \mu(du) \mu(dv) \\ &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_T \exp\left(-\frac{1}{(u \wedge v)^\alpha}\right) \int_T \left[\exp\left(\frac{2 - \theta_t}{(u \wedge v)^\alpha}\right) - 1\right] dt \mu(du) \mu(dv) \\ &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} C \mu(du) \mu(dv) \leq C \mu^2(\mathbb{R}_+) \end{aligned}$$

for any finite measure  $\mu$  on  $\mathbb{R}$ . Thus,  $X$  is short range dependent. □

**Example 4.4.** Here, we consider two popular examples of max-stable processes, namely the extremal Gaussian process and the Brown–Resnick process.

1. The extremal Gaussian process (Schlather, 2002) is a max-stable process with 1-Fréchet marginal distributions and finite-dimensional distributions of the form

$$\mathbb{P}(X(t_1) \leq x_1, \dots, X(t_d) \leq x_d) = \exp\left(-\sqrt{2\pi} \max_{i=1, \dots, d} \frac{\max\{W(t_i), 0\}}{x_i}\right), \quad t_i \in T, x_i > 0,$$

where  $\{W(t), t \in T\}$  is a centered stationary Gaussian process on  $T = \mathbb{R}$ . The extremal coefficients of the extremal Gaussian process are given by

$$\theta_t = 1 + \sqrt{1 - \frac{1 + \rho_t}{2}}, \quad t \in T,$$

where  $\rho_t = \text{Corr}(W(t), W(0))$  denotes the correlation function of the underlying Gaussian process  $W$ . Provided that  $\rho_t \geq 0$  for all  $t \in T$ , we have that  $\theta_t \leq 1 + \sqrt{1/2}$ , and, consequently,

$$\int_T (2 - \theta_t) dt \geq \int_{\mathbb{R}} (1 - \sqrt{1/2}) dt = \infty,$$

that is, the process is long range dependent by Theorem 4.3.

2. The Brown–Resnick process (Kablichko *et al.*, 2009) is a max-stable process with 1-Fréchet marginal distributions and finite-dimensional distributions of the form

$$\mathbb{P}(X(t_1) \leq x_1, \dots, X(t_d) \leq x_d) = \exp\left(-\max_{i=1, \dots, d} \frac{\exp(W(t_i)) - \frac{1}{2} \text{Var}[W(t_i)]}{x_i}\right),$$



$t_i \in T$ ,  $x_i > 0$ , where  $W$  is a centered Gaussian process with stationary increments on  $T = \mathbb{R}$ . The extremal coefficients of the Brown–Resnick process can be expressed in terms of the variogram  $\gamma(t) = \mathbb{E}[(W(t) - W(0))^2]$ ,  $t \in \mathbb{R}$ , of the underlying Gaussian process  $W$  via the relation

$$\theta_t = 2\Phi\left(\frac{\sqrt{\gamma(t)}}{2}\right), \quad t \in \mathbb{R},$$

where  $\Phi$  denotes the standard normal distribution function.

Now assume that there exists some constant  $C > 8$  such that  $\gamma(t) \geq C \log |t|$  for  $|t|$  being sufficiently large. From Mill's ratio  $1 - \Phi(x) \sim x^{-1}\varphi(x)$  as  $x \rightarrow \infty$  with  $\varphi$  being the standard normal density function, it follows that

$$\begin{aligned} 2 - \theta_t &= 2[1 - \Phi(\sqrt{\gamma(t)}/2)] \leq 2[1 - \Phi(\sqrt{C \log |t|}/2)] \\ &\sim \frac{4}{\sqrt{C \log |t|}} \varphi(\sqrt{C \log |t|}/2) = \frac{2\sqrt{2}}{\sqrt{\pi C \log |t|}} |t|^{-C/8}, \quad |t| \rightarrow \infty, \end{aligned}$$

is integrable. Thus, by Theorem 4.3, the Brown–Resnick process is SRD if

$$\liminf_{|t| \rightarrow \infty} \gamma(t) / \log |t| > 8,$$

which is true, for instance, for any fractional Brownian motion  $W$ .

If, in contrast, the variogram  $\gamma$  of the underlying Gaussian process  $W$  is bounded as in the case of a stationary process, we obtain that  $\sup_{t \in T} \theta_t < 2$ . Thus, analogously to the case of the extremal Gaussian process, the Brown–Resnick process can be shown to be LRD.

Note that these conditions also appear in the literature when analyzing the existence of a mixed moving maxima (M3) representation of a Brown–Resnick process: In Kabluchko et al. (2009), it is shown that a M3 representation exists if  $\liminf_{|t| \rightarrow \infty} \gamma(t) / \log |t| > 8$ . In case of a bounded variogram, however, the resulting Brown–Resnick is not even mixing. As sufficient and necessary conditions for the existence of a M3 representation are stated in terms of the asymptotic behavior of the sample paths of the underlying Gaussian process rather than in terms of its variogram (cf. Wang and Stoev, 2010; Dombry and Kabluchko, 2017, for instance), to the best of our knowledge, there is no general treatment of the gap between these two cases. Similarly, for SRD/LRD, a detailed analysis of further cases is beyond the scope of this article.

**Remark 4.5.** Using known dependency properties allows to avoid complex calculation such that no restrictions on the index set  $T$  are required. In particular, all the results are also valid for max-stable random fields, that is, the case that  $T \subset \mathbb{R}^d$ .

## 5. APPLICATION TO DATA

We want to motivate our theoretical results by showing their applicability to real world data. To do so, let us consider the daily log-returns of the Intel corporation share from March 3, 2013 to August 21, 2017 depicted in Figure 1. Preliminary analysis has shown that the marginal distribution of these log-returns fits reasonably well to that of a symmetric  $\alpha$ -stable distribution with estimated index of stability  $\hat{\alpha} = 1.56$  and scale parameter  $\hat{\sigma} = 0.0072$  as depicted in Figure 2. For simplicity, here we use the simple and consistent estimation procedure proposed by McCulloch (1986).

Furthermore, we model this time series using a linear S $\alpha$ S process  $Y(t) = \sum_{j=-\infty}^{+\infty} a_j Z(t-j)$ ,  $t \in \mathbb{Z}$  with  $\alpha \in (0, 2)$ , as described in Definition 3.9. By Remark 3.10, we can apply our previous continuous-time results from Section 3

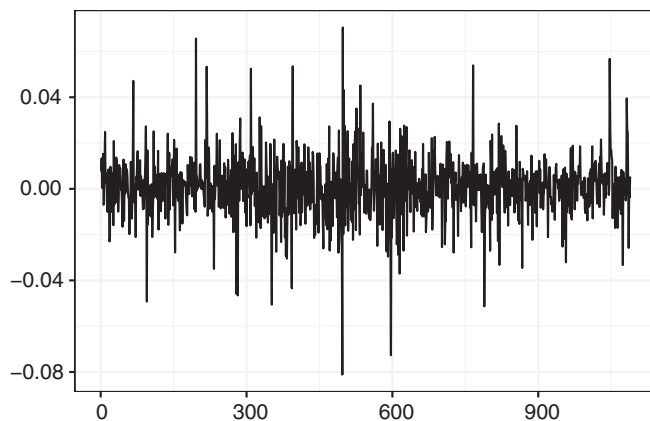


Figure 1. Daily log-returns based on the opening price of the Intel corporation share from March 3, 2013 to August 21, 2017

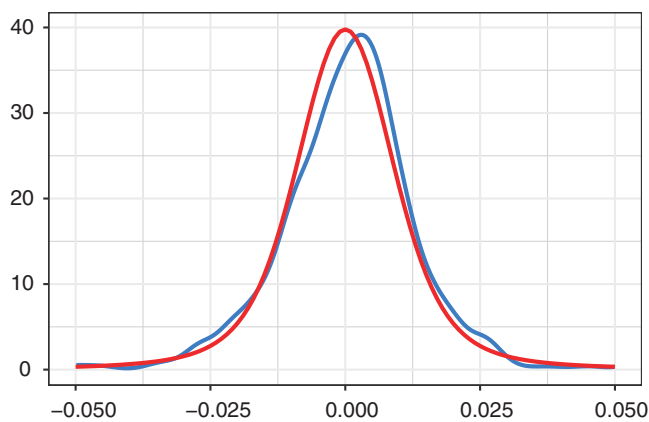


Figure 2. Estimated density of the log-returns (in blue) compared to the theoretical density of a symmetric  $\alpha$ -stable distribution with index of stability  $\hat{\alpha} = 1.56$  and scale parameter  $\hat{\sigma} = 0.0072$  (in red)

by considering a continuous-time S $\alpha$ S moving average  $X$  with a piecewise constant kernel function and interpreting the time series  $Y$  as  $X$  sampled at time instances  $t \in \mathbb{Z}$ .

To do so, we estimate a continuous-time kernel function by a non-parametric approach and check the conditions in Theorems 3.4 and 3.5. By Example 3.7, it suffices to check the condition  $m \in \mathcal{L}^{\alpha/2}(\mathbb{R})$ , if the estimated kernel function exhibits power decaying tails.

However, to the best of our knowledge, there is no universally applicable non-parametric approach for kernel estimation in our setting. For instance, the procedure proposed by Kampf *et al.* (2020) estimates the kernel of a S $\alpha$ S moving average under certain conditions posed on the underlying kernel function  $m$ . However, the authors of this particular article conclude that under their assumptions,  $m$  must be bounded and  $m \in \mathcal{L}^p(\mathbb{R})$  for all  $p \in (1/a, \infty]$  where  $a > \max\{2, 1/\alpha\}$  which, in particular, implies that  $m \in \mathcal{L}^{\alpha/2}(\mathbb{R})$ . Consequently, Theorem 3.4 implies that this kernel estimation procedure is applicable in our setting only if  $X$  is SRD.

Therefore, let us choose a simple parametric minimal contrast method based on the codifference

$$\tau(t) = \|X(0)\|_{c,\alpha}^\alpha + \|X(t)\|_{c,\alpha}^\alpha - \|X(0) - X(t)\|_{c,\alpha}^\alpha$$

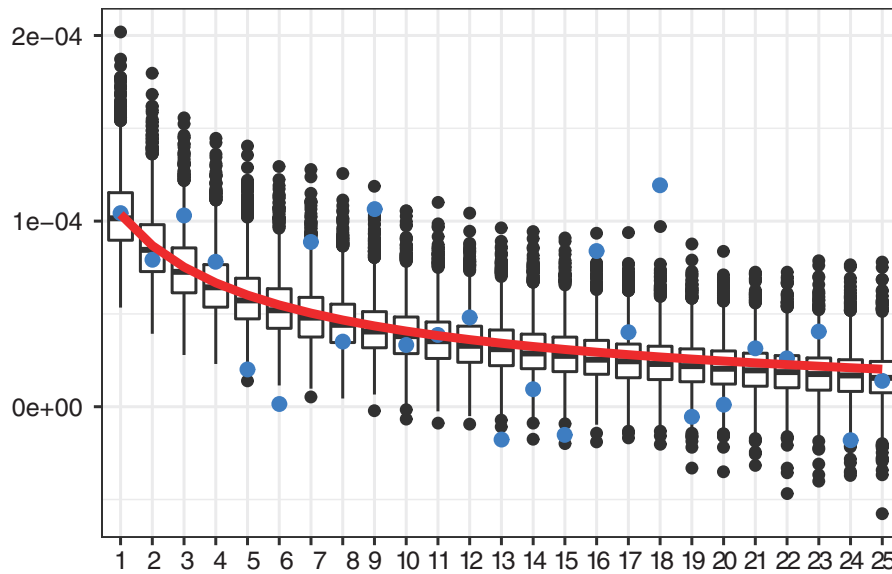


Figure 3. Parametric bootstrap of  $\tau(t)$ . Black: boxplots of  $\hat{\tau}(t)$  based on 2000 S&S time series simulated with kernel parameters  $\hat{\delta}$  and  $\hat{c}$ ; blue:  $\hat{\tau}(t)$  based on data; red: kernel function with parameters  $\hat{\delta}$  and  $\hat{c}$  (plotted as a line for better visibility)

of  $X(0)$  and  $X(t)$  as defined in Samorodnitsky and Taqu (1994, Definition 2.10.1). Here  $\|\cdot\|_{c,\alpha}$  describes the covariation norm of a S&S random variable given by Samorodnitsky and Taqu (1994, Definition 2.8.1).

We assume that the kernel function of  $X$  is causal, that is, supported on the positive half-line, and parametrized like

$$m(t) = \sum_{k=1}^{\infty} \frac{c}{1+k^\delta} \mathbb{1}\{t \in [k-1, k), t \geq 0\},$$

where  $c, \delta > 0$ . By Example 3.7, the process  $X$  is well-defined iff  $\delta > 1/\alpha$  and long range dependent iff  $\delta \leq 2/\alpha$ .

By Samorodnitsky and Taqu (1994, Example 3.6.2) we have that  $\|X(0)\|_{c,\alpha}^\alpha = \|X(t)\|_{c,\alpha}^\alpha = \|m\|_\alpha^\alpha$  and  $\|X(0) - X(t)\|_{c,\alpha}^\alpha = \|m(\cdot) - m(t - \cdot)\|_\alpha^\alpha$ . By simple calculations, we get that for all  $t \in \mathbb{N}$

$$\begin{aligned} \|m\|_\alpha^\alpha &= c^\alpha \sum_{k=1}^{\infty} (1+k^\delta)^{-\alpha}, \\ \|m(\cdot) - m(t - \cdot)\|_\alpha^\alpha &= c^\alpha \left( \sum_{k=1}^t (1+k^\delta)^{-\alpha} + \sum_{k=1}^{\infty} \left| (1+k^\delta)^{-1} - (1+(k+t)^\delta)^{-1} \right|^\alpha \right). \end{aligned}$$

Note here that despite the process  $X$  being defined on the whole real line, we are only interested in sample time points  $t \in \mathbb{N}$  for our data analysis. Consequently, the codifference  $\tau(t)$  of  $X(0)$  and  $X(t)$  writes

$$\tau(t) = c^\alpha \left( \sum_{k=1}^{\infty} (1+k^\delta)^{-\alpha} + \sum_{k=t+1}^{\infty} (1+k^\delta)^{-\alpha} - \sum_{k=1}^{\infty} \left| (1+k^\delta)^{-1} - (1+(k+t)^\delta)^{-1} \right|^\alpha \right). \tag{26}$$

By Samorodnitsky and Taqu (1994, Prop. 2.8.2) it holds that  $\|X(t)\|_{c,\alpha}^\alpha = \sigma$  and  $\|X(0) - X(t)\|_{c,\alpha}^\alpha = \sigma_t$  for all  $t \in \mathbb{N}$  where  $\sigma$  and  $\sigma_t$  are the scale parameters of  $X(0)$  and  $X(0) - X(t)$  respectively. We compare the theoretical quantity (26) to

$$\hat{\tau}(t) = 2\hat{\sigma} - \hat{\sigma}_t$$

where  $\hat{\sigma}$  and  $\hat{\sigma}_t$  are estimators of  $\sigma$  and  $\sigma_t$ , respectively. Again, we use the approach proposed by McCulloch (1986) to estimate  $\sigma$ . When estimating  $\hat{\sigma}_t$ , we do the same based on computed observations  $X(i) - X(i+t)$ ,  $i = 1, \dots, n-t$ , where  $n$  denotes the length of the original sample.

Now, let us estimate the parameters  $\delta$  and  $c$  by minimizing the  $\mathcal{L}^2$ -distance of  $\tau$  and  $\hat{\tau}$  based on the first 25 time instances. We choose this value based on the computation costs of a higher threshold and on the fact that this value has proven useful in simulation studies as part of the preliminary analysis. Using our procedure, we estimate  $\hat{\delta} = 1.0474$  and  $\hat{c} = 0.0019$ . To validate our estimation, we ran a parametric bootstrap and simulated 1000 S&S time series with kernel parameters  $\hat{\delta}$  and  $\hat{c}$  to get a grasp on the variance of  $\tau$  in our chosen model. The results of our parametric bootstrap are depicted in Figure 3. It shows that despite some discrepancies with regard to the empirically computed  $\hat{\tau}$ , which is to be expected in real data, our model exhibits a reasonable fit for the data set.

Notice that, due to the empirical estimation of the scale parameters  $\sigma$  and  $\sigma_t$ , it happens that  $\hat{\tau}(t) < 0$  for some  $t > 0$ , which is impossible for the theoretical codifference  $\tau$ . This, however, does not substantially affect the quality of the fit of  $\hat{\tau}$  to  $\tau$ . By the same parametric bootstrap we find the standard deviations of 0.1802 for  $\delta$  and 0.0005 for  $c$ . It holds that  $\hat{\delta} = 1.0474 < 2/\alpha = 1.2821$  which implies by Example 3.7 that the log-returns of Intel Corporation are long range dependent.

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#### DATA AVAILABILITY STATEMENT

The data used in this article was provided by a free data sample from <https://www.quandl.com/data/EOD/INTC>.

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APPENDIX A.

**Lemma A.1.** Suppose  $U$  and  $V$  are identically distributed random variables with marginal characteristic function  $\varphi_U$  and joint characteristic function  $\varphi_{UV}$ . Also, let  $U'$  and  $V'$  be independent copies of  $U$  and  $V$  respectively. Then, for a Fourier transform of a finite measure  $\mu$  denoted by  $\psi : \mathbb{R} \rightarrow \mathbb{C}$ ,  $\psi(s) = \int_{\mathbb{R}} \exp\{isx\} \mu(dx)$  it holds that

$$\mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}_a(s_1, s_2) \overline{\psi(s_1)\psi(s_2)} ds_1 ds_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \widehat{f}_a(s_1, s_2) \overline{\psi(s_1)\psi(s_2)} ds_1 ds_2,$$

where  $\widehat{f}_a(s_1, s_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(s_1u+s_2v)} f_a(u, v) du dv$ ,  $a > 0$  and

$$f_a(u, v) = \mathbb{1}\{U > u > -a, V > v > -a\} - \mathbb{1}\{U' > u > -a, V' > v > -a\}.$$

*Proof.* Let us define  $g_a(s_1, s_2) = \widehat{f}_a(s_1, s_2) \mathbb{1}\{s_1, s_2 \in (-r, r)\}$  for  $r > 0$  and denote

$$a_r = \int_{\mathbb{R}} \int_{\mathbb{R}} g_a(s_1, s_2) \overline{\psi(s_1)\psi(s_2)} ds_1 ds_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{g}_a(s_1, s_2) \mu(s_1) \mu(s_2),$$

where we have used (Schilling, 2017, Theorem 19.12) in the last equality and  $\widehat{g}_a$  is the inverse Fourier transform of  $g_a$ . It can be computed by tedious, yet simple calculations as

$$\begin{aligned} \widehat{g}_a(s_1, s_2) &= \left( \int_{-r}^r \frac{e^{-is_1x}}{ix} (e^{ixU} - e^{-ixa}) dx \right) \left( \int_{-r}^r \frac{e^{-is_2y}}{iy} (e^{iyV} - e^{-iya}) dy \right) \\ &\quad - \left( \int_{-r}^r \frac{e^{-is_1x}}{ix} (e^{ixU'} - e^{-ixa}) dx \right) \left( \int_{-r}^r \frac{e^{-is_2y}}{iy} (e^{iyV'} - e^{-iya}) dy \right). \end{aligned} \tag{A.1}$$

Computing one of these factors gives us

$$\begin{aligned} \int_{-r}^r \frac{e^{-is_1x}}{ix} (e^{ixU} - e^{-ixa}) dx &= \int_{-r}^r \int_{-a}^U e^{-is_1x} e^{ixt} dt dx = \int_{-a}^U \int_{-r}^r e^{ix(t-s_1)} dx dt \\ &= 2 \int_{r(U-s_1)}^{r(a-s_1)} \frac{1}{t} \frac{e^{it} - e^{-it}}{2i} dt = 2 \int_{r(U-s_1)}^{r(a-s_1)} \frac{\sin t}{t} dt < C \end{aligned}$$

for some finite constant  $C > 0$  which is independent of  $U, V$  and  $r$ . This constant exists because the function  $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$  is bounded. Thus,  $|\widehat{g}_a(s_1, s_2)| \leq 2C^2$  for all  $s_1, s_2 \in \mathbb{R}$  by equality (A.1) and  $|a_r| \leq 2C^2 \mu^2(\mathbb{R})$ . Therefore, we can apply the dominated convergence theorem and  $\mathbb{E}[\lim_{r \rightarrow \infty} a_r] = \lim_{r \rightarrow \infty} \mathbb{E}[a_r]$ .  $\square$

**Lemma A.2.** (a) Suppose  $a, b \in \mathbb{R}$  and  $\alpha \in (0, 2)$ . Then it holds that

$$\left| |a|^\alpha + |b|^\alpha - |a - b|^\alpha \right| \leq 2|a|^{\alpha/2}|b|^{\alpha/2}.$$

(b) Suppose  $a, b \geq 0$  and  $\alpha \geq 0$ , then  $|a - b|^\alpha \leq a^\alpha + b^\alpha$ .

(c) Let  $\alpha > 0, a \geq 0, b \geq 0$ , then

$$\int_0^1 \int_0^1 \frac{(as_1 + bs_2)^\alpha - |as_1 - bs_2|^\alpha}{s_1 s_2} ds_1 ds_2 \geq C_\alpha (a^\alpha \wedge b^\alpha), \tag{A.2}$$

where  $C_\alpha = \frac{2}{\alpha} \int_0^1 \frac{(1+u)^\alpha - (1-u)^\alpha}{u} du \geq \frac{4(2^\alpha - 1)}{\alpha(\alpha + 1)}$ .

*Proof.* (a) Recall that the covariance function of a fractional Brownian motion  $B^H = \{B_t^H, t \in \mathbb{R}\}$  with Hurst index  $H \in (0, 1)$  equals

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \tag{A.3}$$

where  $s, t \in \mathbb{R}$ . Using the Cauchy-Schwarz inequality we have that

$$|\mathbb{E}[B_t^H B_s^H]|^2 \leq \mathbb{E}[(B_t^H)^2] \mathbb{E}[(B_s^H)^2] = |t|^{2H} |s|^{2H}. \tag{A.4}$$

Denoting  $\alpha = 2H$  and combining (A.3) and (A.4) we get  $\left| \frac{1}{2}(|t|^\alpha + |s|^\alpha - |t - s|^\alpha) \right|^2 \leq |t|^\alpha |s|^\alpha$ , or, equivalently,

$$\left| |t|^\alpha + |s|^\alpha - |t - s|^\alpha \right| \leq 2|t|^{\alpha/2}|s|^{\alpha/2}.$$

(b) Suppose  $a < b$ , then

$$a^\alpha + b^\alpha - |a - b|^\alpha = a^\alpha + b^\alpha - (b - a)^\alpha \geq a^\alpha + b^\alpha - b^\alpha = a^\alpha \geq 0.$$

The claim follows analogously if  $a \geq b$ .

(c) For simplicity, consider the case  $0 < a < b$ . After the change of variable, the integral in (A.2) rewrites

$$\begin{aligned} \int_0^a \int_0^b \frac{(z_2 + z_1)^\alpha - |z_1 - z_2|^\alpha}{z_1 z_2} dz_1 dz_2 &= 2 \iint_{0 \leq z_1 \leq z_2 \leq a} \frac{(z_2 + z_1)^\alpha - (z_2 - z_1)^\alpha}{z_1 z_2} dz_1 dz_2 \\ &+ \int_0^a \int_a^b \frac{(z_2 + z_1)^\alpha - (z_2 - z_1)^\alpha}{z_1 z_2} dz_2 dz_1 \geq 2 \int_0^a \left[ \int_0^{z_2} \frac{(z_2 + z_1)^\alpha - (z_2 - z_1)^\alpha}{z_1} dz_1 \right] \frac{dz_2}{z_2} \\ &= 2 \int_0^a z_2^{\alpha-1} dz_2 \left[ \int_0^1 \frac{(1+u)^\alpha - (1-u)^\alpha}{u} du \right] = C_\alpha a^\alpha. \end{aligned} \tag{A.5}$$

□

**Lemma A.3.** Let  $X = \{X(t), t \in \mathbb{R}\}$  be a  $S\alpha S$  moving average process with parameter  $\alpha \in (0, 2)$ , nonnegative kernel function  $m \in \mathcal{L}^\alpha(\mathbb{R})$ ,  $m(x) > 0$  on a set of positive Lebesgue measure and  $\alpha$ -spectral covariance  $\rho_t$ . Let  $\varphi$  and  $\varphi_t, t \in \mathbb{R}$ , be the characteristic functions given in (12) and (13). Then,

$$I_1 = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, -s_2) - \varphi(s_1)\varphi(s_2)|}{s_1 s_2} ds_1 ds_2 \leq \frac{8\pi}{\alpha^2 \|m\|_\alpha^{2\alpha}} \frac{\rho_t}{\sqrt{\|m\|_\alpha^{2\alpha} - \rho_t^2}}, \tag{A.6}$$



$$I_2 = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{|\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)|}{s_1 s_2} ds_1 ds_2 \leq \frac{8\pi}{\alpha^2 \|m\|_\alpha^{2\alpha}} \frac{\rho_t}{\sqrt{\|m\|_\alpha^{2\alpha} - \rho_t^2}}. \tag{A.7}$$

*Proof.* First, compute for  $\alpha \in (0, 2)$  the absolute value of the difference of characteristic functions in  $I_1$  for any  $s_1, s_2 > 0$ :

$$\begin{aligned} |\varphi_t(s_1, -s_2) - \varphi(s_1)\varphi(s_2)| &= \varphi_t(s_1, -s_2) \cdot \left| 1 - \frac{\varphi(s_1)\varphi(s_2)}{\varphi_t(s_1, -s_2)} \right| \\ &= \varphi_t(s_1, -s_2) \cdot \left| 1 - \exp \left\{ - \int_{\mathbb{R}} \underbrace{\left( |s_1 m(-x)|^\alpha + |s_2 m(t-x)|^\alpha - |s_1 m(-x) - s_2 m(t-x)|^\alpha \right)}_{\geq 0 \text{ by Lemma A.2(b)}} dx \right\} \right| \\ &\leq \varphi_t(s_1, -s_2) \cdot \int_{\mathbb{R}} \left( |s_1 m(-x)|^\alpha + |s_2 m(t-x)|^\alpha - |s_1 m(-x) - s_2 m(t-x)|^\alpha \right) dx \\ &\hspace{15em} \left( \text{by using } |e^{-x} - 1| \leq x \text{ for all } x \geq 0 \right) \\ &\leq \varphi_t(s_1, -s_2) \cdot 2s_1^{\alpha/2} s_2^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x) m^{\alpha/2}(t-x) dx, \end{aligned} \tag{A.8}$$

where we have used Lemma A.2(a) in the last inequality. Similarly, we compute  $|\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)|$  for the case  $\alpha \in (1, 2)$  as

$$\begin{aligned} |\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)| &= \varphi(s_1)\varphi(s_2) \cdot \left| \frac{\varphi_t(s_1, s_2)}{\varphi(s_1)\varphi(s_2)} - 1 \right| \\ &= \varphi(s_1)\varphi(s_2) \cdot \left| \exp \left\{ - \int_{\mathbb{R}} \underbrace{\left( |s_1 m(-x) + s_2 m(t-x)|^\alpha - |s_1 m(-x)|^\alpha - |s_2 m(t-x)|^\alpha \right)}_{\geq 0 \text{ since } \alpha > 1} dx \right\} - 1 \right| \\ &\leq \varphi(s_1)\varphi(s_2) \cdot \int_{\mathbb{R}} \left( |s_1 m(-x) + s_2 m(t-x)|^\alpha - |s_1 m(-x)|^\alpha - |s_2 m(t-x)|^\alpha \right) dx \\ &\leq \varphi(s_1)\varphi(s_2) \cdot 2s_1^{\alpha/2} s_2^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x) m^{\alpha/2}(t-x) dx \\ &\leq \varphi_t(s_1, -s_2) \cdot 2s_1^{\alpha/2} s_2^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x) m^{\alpha/2}(t-x) dx, \end{aligned} \tag{A.9}$$

where, again, we have used Lemma A.2(a) in the last inequality. Using the same arguments, we get for the case  $\alpha \in (0, 1)$  that

$$\begin{aligned} |\varphi_t(s_1, s_2) - \varphi(s_1)\varphi(s_2)| &= \varphi_t(s_1, s_2) \cdot \left| 1 - \frac{\varphi(s_1)\varphi(s_2)}{\varphi_t(s_1, s_2)} \right| \\ &= \varphi_t(s_1, s_2) \cdot \left| 1 - \exp \left\{ - \int_{\mathbb{R}} \underbrace{\left( |s_1 m(-x)|^\alpha + |s_2 m(t-x)|^\alpha - |s_1 m(-x) + s_2 m(t-x)|^\alpha \right)}_{\geq 0 \text{ since } \alpha < 1} dx \right\} \right| \end{aligned}$$

$$\begin{aligned} &\leq \varphi_t(s_1, s_2) \cdot 2s_1^{\alpha/2} s_2^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx \\ &\leq \varphi_t(s_1, -s_2) \cdot 2s_1^{\alpha/2} s_2^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx, \end{aligned} \tag{A.10}$$

Now, plugging the estimates (A.8), (A.9) and (A.10) into  $I_1$  and  $I_2$  we get

$$\begin{aligned} I_1, I_2 &\leq 2 \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi_t(s_1, -s_2)}{s_1^{1-\alpha/2} s_2^{1-\alpha/2}} ds_1 ds_2 \right) \left( \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx \right) \\ &= 2 \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi_t(s_1, -s_2)}{s_1^{1-\alpha/2} s_2^{1-\alpha/2}} ds_1 ds_2 \right) \rho_t. \end{aligned} \tag{A.11}$$

We estimate the integral in (A.11) from above via the density of a bivariate normal law. Thus,

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi_t(s_1, -s_2)}{s_1^{1-\alpha/2} s_2^{1-\alpha/2}} ds_1 ds_2 \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left( \frac{1}{s_1 s_2} \right)^{1-\frac{\alpha}{2}} \exp \left\{ - \int_{\mathbb{R}} |s_1 m(-x) + (-s_2)m(t-x)|^\alpha dx \right\} ds_1 ds_2 \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left( \frac{1}{s_1 s_2} \right)^{1-\frac{\alpha}{2}} \exp \left\{ - \int_{\mathbb{R}} |s_1 m(-x)|^\alpha dx - \int_{\mathbb{R}} |(-s_2)m(t-x)|^\alpha dx \right. \\ &\quad \left. + \int_{\mathbb{R}} |s_1 m(-x)|^\alpha + |(-s_2)m(t-x)|^\alpha - |s_1 m(-x) + (-s_2)m(t-x)|^\alpha dx \right\} ds_1 ds_2 \\ &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left( \frac{1}{s_1 s_2} \right)^{1-\frac{\alpha}{2}} \exp \left\{ - s_1^\alpha \int_{\mathbb{R}} m^\alpha(x) dx - s_2^\alpha \int_{\mathbb{R}} m^\alpha(x) dx \right. \\ &\quad \left. + 2(s_1 s_2)^{\alpha/2} \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx \right\} ds_1 ds_2 \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left( \frac{1}{s_1 s_2} \right)^{1-\frac{\alpha}{2}} \exp \left\{ - (s_1^{\alpha/2} \sigma)^2 - (s_2^{\alpha/2} \sigma)^2 + 2\tilde{\rho}_t(\sigma s_1^{\alpha/2})(\sigma s_2^{\alpha/2}) \right\} ds_1 ds_2, \end{aligned} \tag{A.12}$$

where we denoted

$$\sigma^2 = \rho_0 = \int_{\mathbb{R}} m^\alpha(x) dx = \|m\|_\alpha^\alpha \quad \text{and} \quad \tilde{\rho}_t = \frac{\rho_t}{\sigma^2} = \frac{1}{\sigma^2} \int_{\mathbb{R}} |m(-x)m(t-x)|^{\alpha/2} dx \tag{A.13}$$

in the last equality. Now, we have a substitution

$$\frac{y_i}{\sqrt{2(1-\tilde{\rho}_t^2)}} = \sigma s_i^{\alpha/2}, \text{ or } s_i = \left( \frac{y_i}{\sigma \sqrt{2(1-\tilde{\rho}_t^2)}} \right)^{2/\alpha}, \quad i = 1, 2,$$

so that

$$ds_i = \frac{\frac{2}{\alpha} y_i^{\frac{2}{\alpha}-1}}{(\sigma \sqrt{2(1-\tilde{\rho}_t^2)})^{2/\alpha}} dy_i, \quad i = 1, 2.$$

Then, relation (A.12) rewrites

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left( \frac{(2\sigma^2(1-\tilde{\rho}_t^2))^{2/\alpha}}{y_1^{2/\alpha} y_2^{2/\alpha}} \right)^{1-\frac{\alpha}{2}} \exp \left\{ -\frac{y_1^2 + y_2^2 - 2\tilde{\rho}_t y_1 y_2}{2(1-\tilde{\rho}_t^2)} \right\} \frac{\left(\frac{2}{\alpha}\right)^2 y_1^{\frac{2}{\alpha}-1} y_2^{\frac{2}{\alpha}-1}}{(2\sigma^2(1-\tilde{\rho}_t^2))^{2/\alpha}} dy_1 dy_2 \\ &= (2\sigma^2(1-\tilde{\rho}_t^2))^{-1} \left(\frac{2}{\alpha}\right)^2 \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \exp \left\{ -\frac{y_1^2 + y_2^2 - 2\tilde{\rho}_t y_1 y_2}{2(1-\tilde{\rho}_t^2)} \right\} dy_1 dy_2 \\ &= \frac{4\pi}{\alpha^2 \sigma^2} (1-\tilde{\rho}_t^2)^{-1/2} \underbrace{\int_0^\infty \int_0^\infty \frac{1}{2\pi \sqrt{1-\tilde{\rho}_t^2}} \exp \left\{ -\frac{y_1^2 + y_2^2 - 2\tilde{\rho}_t y_1 y_2}{2(1-\tilde{\rho}_t^2)} \right\} ds_1 ds_2}_{\leq 1 \text{ as a density of bivariate normal distribution}} \\ &\leq \frac{4\pi}{\alpha^2 \sigma^2} (1-\tilde{\rho}_t^2)^{-1/2} = \frac{4\pi}{\alpha^2 \|m\|_{2\alpha}^2} (\rho_0^2 - \rho_t^2)^{-1/2} < \infty \text{ for } \rho_t \neq \rho_0. \end{aligned}$$

Now, show that  $(\rho_0^2 - \rho_t^2)^{-1/2}$  is only infinite on a null set. More specifically, we show that  $\rho_t = \rho_0$  if and only if  $t = 0$ . Recall that the Cauchy–Schwarz inequality (cf. Reed and Simon, 1981, Theorem S.3.) states that

$$\rho_t = \int_{\mathbb{R}} m^{\alpha/2}(-x)m^{\alpha/2}(t-x) dx \leq \int_{\mathbb{R}} m^\alpha(-x) dx = \rho_0,$$

where equality holds if and only if there exists  $\lambda_t \in \mathbb{R}$  such that  $m^{\alpha/2}(-x) = \lambda_t m^{\alpha/2}(t-x)$  a.e. In this case, relation  $\rho_t = \rho_0$  yields  $\lambda_t = 1$ . Note that due to  $m$  being nonnegative, we can rewrite the condition as  $m(-x) = m(t-x)$  a.e. or  $m(x) = m(x+t)$  a.e.; hence,  $m$  is a  $t$ -periodic function with  $m(x) > 0$  on a set of positive Lebesgue measure which contradicts  $m \in \mathcal{L}^\alpha(\mathbb{R})$  because in that case  $m(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . Consequently,  $\rho_t = \rho_0$  if and only if  $t = 0$ .  $\square$

*Proof of Theorem 3.5:* Let us choose  $\mu = \delta_0$  where  $\delta_0$  is the Dirac measure concentrated at zero. Obviously this measure is finite and by Corollary 2.2 we get for  $t \in \mathbb{R}$ :

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \mu(du) \mu(dv) \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{s_1 s_2} \left( \varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2) \right) ds_1 ds_2. \end{aligned} \tag{A.14}$$

We denote  $A = |s_1 m(-x) + s_2 m(t-x)|^\alpha$  and  $B = |s_1 m(-x) - s_2 m(t-x)|^\alpha$ . Then, by  $e^x - 1 \geq x$  for all  $x \in \mathbb{R}$  we estimate

$$\varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2) = \exp \left\{ -\int_{\mathbb{R}} B dx \right\} - \exp \left\{ -\int_{\mathbb{R}} A dx \right\}$$

$$\begin{aligned}
&= \exp \left\{ - \int_{\mathbb{R}} A \, dx \right\} \left( \exp \left\{ \int_{\mathbb{R}} [A - B] \, dx \right\} - 1 \right) \\
&\geq \exp \left\{ - \int_{\mathbb{R}} A \, dx \right\} \int_{\mathbb{R}} [A - B] \, dx.
\end{aligned} \tag{A.15}$$

Thus, we obtain a lower bound for the right-hand side of (A.15):

$$\begin{aligned}
&\exp \left\{ - \int_{\mathbb{R}} |s_1 m(-x) + s_2 m(t-x)|^\alpha \, dx \right\} \\
&\quad \times \int_{\mathbb{R}} [(s_1 m(-x) + s_2 m(t-x))^\alpha - |s_1 m(-x) - s_2 m(t-x)|^\alpha] \, dx.
\end{aligned}$$

Note that for any  $a, b > 0$ ,  $(a+b)^\alpha = ((a+b)^2)^{\alpha/2} \leq 2^{\alpha/2} (a^2 + b^2)^{\alpha/2} \leq 2^{\alpha/2} (a^\alpha + b^\alpha)$ . Thus, for  $s_1, s_2 \in [0, 1]$

$$\exp \left\{ - \int_{\mathbb{R}} |s_1 m(-x) + s_2 m(t-x)|^\alpha \, dx \right\} \geq \exp \left\{ - (s_1^\alpha + s_2^\alpha) 2^{\frac{\alpha}{2}} \|m\|_\alpha^\alpha \right\} \geq e^{-4\|m\|_\alpha^\alpha}.$$

Consequently, it holds

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{1}{s_1 s_2} \left( \varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2) \right) \, ds_1 \, ds_2 \geq \int_0^1 \int_0^1 \frac{1}{s_1 s_2} \left( \varphi_t(s_1, -s_2) - \varphi_t(s_1, s_2) \right) \, ds_1 \, ds_2 \\
&\geq \int_0^1 \int_0^1 \frac{1}{s_1 s_2} \left( \exp \left\{ - \int_{\mathbb{R}} (s_1 m(-x) + s_2 m(t-x))^\alpha \, dx \right\} \right. \\
&\quad \left. \times \int_{\mathbb{R}} \left( (s_1 m(-x) + s_2 m(t-x))^\alpha - |s_1 m(-x) - s_2 m(t-x)|^\alpha \right) \, dx \right) \, ds_1 \, ds_2 \\
&\geq e^{-4\|m\|_\alpha^\alpha} \int_0^1 \int_0^1 \int_{\mathbb{R}} \frac{(s_1 m(-x) + s_2 m(t-x))^\alpha - |s_1 m(-x) - s_2 m(t-x)|^\alpha}{s_1 s_2} \, dx \, ds_1 \, ds_2.
\end{aligned}$$

Now, by Fubini's theorem and Lemma A.2(c), this is greater or equal to

$$e^{-4\|m\|_\alpha^\alpha} C_\alpha \int_{\mathbb{R}} m^\alpha(-x) \wedge m^\alpha(t-x) \, dx.$$

Thus, for  $\mu = \delta_0$  we have that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \text{Cov}(\mathbb{1}\{X(0) > u\}, \mathbb{1}\{X(t) > v\}) \, \mu(du) \, \mu(dv) \geq c \int_{\mathbb{R}} (m^\alpha(-x) \wedge m^\alpha(t-x)) \, dx.$$

Consequently, by Definition 1.2 and Fubini's theorem,  $X$  is long range dependent if

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}} (m^\alpha(-x) \wedge m^\alpha(t-x)) \, dx \, dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} (m^\alpha(-x) \wedge m^\alpha(t-x)) \, dt \, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} (m^\alpha(x) \wedge m^\alpha(t)) \, dt \, dx = \infty.
\end{aligned}$$