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A New Approach to the Irreversible Investment Problem

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A NEW APPROACH TO THE IRREVERSIBLE INVESTMENT PROBLEM

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Abstract

This paper solves the irreversible investment decision problem under uncertainty by a new real options method. It yields a *Shadow Net Present Value rule* such that the investment is triggered only when the *shadow revenue* of the investment reaches the investment cost. This paper hence corrects and extends the conventional Net Present Value (NPV) rule by figuring out the proper NPV of the investment. Meanwhile, the investment policy is shown to be correct and consistent with the real options theory since the *shadow revenue* records the economic value of the investment by deducting the option premium of waiting from the real revenue. This method allows for a generalization of the conventional analysis to all semi-martingale processes with deterministic or stochastic discount rates. Especially, we provide an explicit characterization of the threshold value when the uncertainty is specified as an exponential Lévy process. To highlight the significance of risk preferences on the decision rule, this paper also addresses the utility-based extension of the standard real option model. The investment policy in such a model is identified in the same form as that in the profit-maximization problem but signalled by the *shadow utility process*. The joint effect of risk aversion and jumps on the trigger value is analyzed through examples.

JEL Subject Classification: D81, G3

Keywords: Irreversible investment under uncertainty, Real options, Stochastic representation method, Utility, Risk aversion, Lévy processes.

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1. INTRODUCTION

As an essential activity for firms and economic growth, investment has attracted a great deal of academic attention for decades. Facing with an opportunity to invest, firms have to make the decision whether to invest or not. Traditionally, the investment strategy is determined by the Net Present Value (NPV) method: Investment is made if and only if the discounted revenues are equal to or larger than the costs. As was widely acknowledged (c.f. Dixit and Pindyck (1994) [13] and the literature therein), this conventional decision rule considers the investment only as being *now or never* and neglects the stochastic nature of the project values. Generally, investment is totally or at least partially irreversible. Further due to the uncertainty in the future economic conditions, the investment is often postponed to achieve better information on investment profits. To account for these two effects, contemporary models regard an investment opportunity as a contingent claim on the project value. Such claims are known as real options and are solved by two standard methods, i.e., dynamic programming and contingent claim analysis, as presented in detail by Dixit and Pindyck (1994) [13]. More importantly, the real options theory points out that the NPV of an investment has to cover the option premium of waiting, or the opportunity cost of delaying the investment.

Although the real options approach has been viewed for long as a modern and correct means in academics, it has yet to be popularly adapted by practitioners. According to the 2002 survey of 205 Fortune 1000 CFOs by Patricia Ryan, merely 11.4% used the real options idea, while the NPV method stayed at the top of the list with 96%. One of the most important factors that lead to such a failure in practical application lies in the lack of transparency and simplicity of the real options method (See Teach (2003) [19]). To many managers, the framework is not easy enough to understand. Moreover, the mere concern of shareholders is on the net profit or revenue of the new investment. In this sense, the NPV is surely the most *meaningful* and *straightforward* measure of the investment performance. It is however not really accounted for in the real options method. To promote the wide application of the real options theory, this paper is first going to find an alternative method which provides a correct and economically intuitive decision rule based on the NPV.

As a starting point, the NPV of the investment is formulated as the expected present value (EPV) of all the operating profits after the optimal investment time less the investment cost. In contrast to the EPV that starts accruing at a deterministic time, it is rather difficult to determine the EPV from the investment since the investment time is instead

a stopping time when the project revenue comes to a satisfactory level. Thus, another approach is required here in order to calculate or rewrite the EPV term. In this work, we represent the expected discounted revenue in terms of the EPV of the running supremum of another process. This process is extremely useful, signalling the investment in the decision rule: The optimal investment time is identified as the first moment at which the process reaches the investment cost. In this way, we derive the decision rule by solving a representation problem rather than treating it as an optimal stopping problem. Especially, we define the signalling process as *shadow revenue process*. The key reformulation procedure here is based on the stochastic representation method first proposed in Bank and Föllmer (2003) [3] for various stochastic optimization problems.

By specifying the project's NPV in terms of the EPV of the running supremum of another process, this method gives some economic intuition as follows. First, it exactly coincides with the fact that to maximize the investment profit, the investor is not concerned with the instantaneous revenue of the investment at the moment when the investment is done, but with the future profits it creates after the investment. Surely, it would be optimal to invest at the moment when the project starts to create positive profits (net of all the costs). More precisely, it is a *Shadow NPV rule*: The investment is undertaken if and only if the *shadow revenue* rises up to the investment cost. Thus, we finally achieve a simple optimal investment strategy based on the NPV as desired. Moreover, this method extends and corrects the conventional NPV method by determining the *proper* NPV. Second, the *shadow value* is defined in this work in the sense that it is the true or pure value of the investment that the firm gains after compensating total costs. Moreover, we demonstrate that the *shadow revenue process* is always lower than the expected revenue at any stopping time. The value difference can be interpreted to account for the opportunity cost of delaying the investment. In other words, the *shadow revenue* records the economic value of the investment by deducting the option premium of waiting from the real revenue. In this way, with the trick of reformulating the expected discounted revenue, the new method derives an investment decision rule consistent with the standard real options theory: in addition to the investment cost, the overall revenue has to cover the option premium of waiting.

Another highlight of this method is its applicability to a wide class of general stochastic processes. In this way, we extend the classical real options theory from a Geometric Brownian motion (GBM) to a Lévy process and even all semi-martingale processes that

are economically plausible. The decision problem on a GBM has been fully exploited as in e.g. Dixit and Pindyck (1994) [13]. However, the lognormal distribution is contradictory to the well-known empirical evidence (see, for instance, Yang and Brorsen (1992) [20] as well as Deaton and Laroque (1992) [12]). Indeed, commodity prices and the project value obtained on their basis exhibit significant skewness and kurtosis. More importantly, there is a high probability of large random fluctuations such as crashes or sudden upsurges. As a result, a Lévy process which combines a diffusion process and embedded jumps turns out to be a more correct model description. Moreover, this method also works when the discount rates are modelled as a strictly positive stochastic process.

More powerful than the standard real option pricing methods, this new approach provides explicit characterizations for the threshold value when the uncertainty is specified as an exponential Lévy process. The exercise threshold is obtained as the investment cost multiplied by a correction factor, in the same form as the standard result for the GBM. Hence, this method generalizes the simple decision rule to general exponential Lévy processes, providing a clear qualitative view of the investment strategy. The correction factor is expressed in terms of the supremum process. This result coincides with those of Mordecki (2002) [17] and Boyarchenko and Levendorskii (2002-2004) [7] [8] [9] [10]. Mordecki's work is basically an extension of the discrete-time model on random walks by Darling et al. (1972) [11] and applies well to the general Lévy process. While, Boyarchenko and Levendorskii's method is based on reducing the optimal stopping problem to a free boundary problem for the generalized Black-Scholes (1973) equation [5] in form of pseudo-differential operators.

The management objective of the investment decision problem is the maximization of the expected profit of the project as in the model considered above. This model is standard and can be easily validated in a complete financial market: The cash flow of the investment can be spanned by those products traded in the market such that the value of the project is exactly the EPV of those cash flows under the unique risk neutral measure. Surely, the stochastic representation method is not restricted to those assumptions of market completeness and risk neutrality. However, investors hold in general different attitudes to risk and hence have different preferences for an investment. Consequently, another motivation of the work is to incorporate subjective risk preferences in the irreversible investment valuation. To this end, the standard decision problem is combined

with utility functions which are usually used in economics to define and measure risk preferences. This concept has once been mentioned in Dixit and Pindyck (1994) [13] and developed by Hugonnier and Morellec (2005) [15] for risk aversion and GBM¹. We also apply the stochastic representation method to this utility-based decision problem and derive the optimal investment rule in an analytical form for the case where the decision maker is risk averse with constant relative risk aversion and facing uncertainty modelled by an exponential Lévy process. Similar to the value-maximization decision rule, the *shadow utility process* is obtained such that the investment is initiated whenever the net utility from the investment becomes non-negative. Due to risk aversion, the firm has a relatively high incentive to delay the investment, which in turn leads to a higher threshold compared to risk neutrality. Especially, we study through an example the combined effect of utility consideration and jumps on the trigger value. We demonstrate that under risk aversion and negative jumps, the critical price can be higher or lower than that under the GBM modification. It is argued in the paper that it is not counter-intuitive. Given a certain estimated variance, there are two opposite effects of jumps on the threshold value: on one hand the trigger value rises as a response to possible negative jumps; on the other hand the trigger value declines with the jump coefficients due to the decrease in diffusion uncertainty. The first increasing effect is nevertheless greatly magnified by risk aversion.

The remainder of this paper is organized as follows. Section 2 sets up the model of irreversible investment under uncertainty, analyzes the profit-maximization problem by the stochastic representation method and discusses the inherent economic implications. Particularly, an explicit characterization of the solution is given in Section 3 when the output price is modelled by an exponential Lévy process. Section 4 solves the utility-based problem and derives the optimal decision rule in an analytical form when the firm is subject to a power utility function and an exponential Lévy process modelled uncertainty. Finally, Section 5 concludes with a short summary and remark. Technical details are presented in the Appendix.

2. REAL OPTIONS AND THE NEW EVALUATION METHOD

This paper solves the irreversible investment decision problem with a new method in the real options literature. In order to facilitate the derivation and interpretation, we consider the well-established irreversible investment model (Pindyck (1988) [18], McDonald and Siegel (1986) [16] and also presented in Dixit and Pindyck (1994) [13]). In this framework,

¹Henderson and Hobson (2002) address also utility maximization model but focus on the pricing and hedging of a non-traded asset.

the resemblance between investment decision problems and perpetual American options can be easily recognized. The new evaluation approach, stochastic representation method, is then introduced and interpreted afterwards in detail.

2.1. Irreversible Investment Decision Problem. Consider a firm who has an opportunity to invest in a project with a fixed scale, infinite life and no marginal cost. This project requires only an initial investment cost I which is supposed to be constant over time. The investment is irreversible in the sense that the investment cost is sunk. The project generates then a continuous stream of cash flows by producing a commodity good of quantity Q ever after the investment date τ . The spot price of the firm's output P evolves stochastically conditional on the economic situation. For convenience, Q is fixed to be 1, i.e., the project yields a unit flow of output forever. Given these assumptions, the firm has to decide whether to take project or not and most importantly it has to decide on the time to invest, i.e., the time which maximizes the expected profit. Formally, the firm faces the problem

$$(1) \quad F = \max_{\tau \in \mathcal{T}([0, \infty))} \mathbb{E} \left[e^{-\rho\tau} \left(\int_{\tau}^{\infty} e^{-\rho(s-\tau)} P_s ds - I \right)^+ \right]$$

$$(2) \quad = \max_{\tau \in \mathcal{T}([0, \infty))} \mathbb{E} [e^{-\rho\tau} (V_{\tau} - I)],$$

where ρ is the constant discount factor measured in the physical measure \mathbb{P} on the probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, the expected value is taken with respect to \mathbb{P} , $V_t = \mathbb{E} \left[\int_t^{\infty} e^{-\rho(s-t)} P_s ds \mid \mathcal{F}_t \right]$ represents the expected overall revenue or operating profit from the investment at time t and the maximization is taken over $\mathcal{T}([0, \infty))$, the class of all stopping times. In addition, we assume that

$$(3) \quad \mathbb{E} \left[\int_0^{\infty} e^{-\rho t} P_t dt \right] < \infty,$$

and the filtration is quasi-left-continuous, i.e.,

$$(4) \quad \mathcal{F}_{\tau} = \mathcal{F}_{\tau-}$$

for any predictable stopping time $\tau \in \mathcal{T}([0, \infty))$.

Obviously, the profit maximization problem is in structure analogous to a perpetual American call option which is written on the future revenue of the investment. That is why an opportunity to invest is usually referred to as a real option, an option contingent on real assets. In this context, the firm has an optimal stopping problem at hand. The standard real options theory offers two solution methods: dynamic programming and contingent claim analysis. Both methods are relevant to solving a stochastic differential

equation. In this paper, the optimal stopping problem is solved by an alternative stochastic representation method as fully explained in the next subsection.

2.2. Stochastic Representation Approach and Shadow NPV Rule. The EPV of a perpetual cash flow accumulated from a deterministic point in time is clear and can be easily obtained. The difficulty of our problem lies in the fact that the EPV begins at a stopping time when the project value reaches a satisfactory level. In order to derive a decision rule based on this essential and straightforward concept of NPV, we rewrite the EPV of the project revenue in terms of the EPV of the running supremum of another process. More explicitly, the EPV of the operating profit from investment at time τ , $e^{-\rho\tau}V_\tau$, is represented in form of

$$(5) \quad e^{-\rho\tau}V_\tau = \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho t} \sup_{\tau \leq v \leq t} \xi_v dt \middle| \mathcal{F}_\tau \right]$$

by some progressively measurable process $(\xi_t)_{t \in [0, \infty)}$ with upper-right continuous paths. This representation then allows for a characterization of the optimal stopping time as the first moment at which the obtained exercise signal process ξ_t hits the investment cost, I .

Theorem 2.1. *Suppose that the decision problem of an irreversible investment specified as (1) admits the stochastic representation (5) in terms of the **shadow revenue process** $(\xi_t)_{t \in [0, \infty)}$ which is progressively measurable with upper-right continuous paths. Then, the level passage time when the process ξ rises up to the investment cost, i.e.,*

$$\tau^* = \inf\{t \geq 0 \mid \xi_t \geq I\}$$

maximizes the investment value over all stopping times $\mathcal{T}([0, \infty))$.

Proof. Bank and El Karoui (2004) [2] give a detailed technical analysis of the representation form (5). In particular, they show that the representation form is valid whenever $e^{-\rho\tau}V_\tau$ is uniformly integrable and upper-semicontinuous in expectation². Hence, we have to show first whether the regularity conditions are satisfied in our construction or not. As assumed in conditions (3) and (4), we have an upper-bounded expected discounted revenue and a quasi-left-continuous filtration. Clearly, the uniform integrability is guaranteed by

²The uniform integrability is a basic condition in the optimal stopping problem to guarantee the existence of a finite solution. Upper-semicontinuity in expectation is precisely stated as follows: $\limsup_n \mathbb{E}[X_{\tau^n}] \leq \mathbb{E}[X_\tau]$ for any monotone sequence of stopping times τ^n ($n = 1, 2, \dots$) converging to some stopping time $\tau \in \mathcal{T}$ almost surely.

condition (3). According to the definition, the discounted revenue is

$$\begin{aligned} e^{-rt}V_t &= \mathbb{E} \left[\int_t^\infty e^{-\rho s} P_s ds \middle| \mathcal{F}_t \right] \\ &= \underbrace{\mathbb{E} \left[\int_0^\infty e^{-\rho s} P_s ds \middle| \mathcal{F}_t \right]}_{:=M_t} - \underbrace{\int_0^t e^{-\rho s} P_s ds}_{:=A_t}, \end{aligned}$$

where A_t is predictable and absolutely continuous. Moreover, we can show that M_t is a martingale as follows: for any $u < t$

$$\begin{aligned} \mathbb{E}[M_t | \mathcal{F}_u] &= \mathbb{E} \left[\mathbb{E} \left[\int_0^\infty e^{-\rho s} P_s ds \middle| \mathcal{F}_t \right] \middle| \mathcal{F}_u \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-\rho s} P_s ds \middle| \mathcal{F}_u \right] = M_u. \end{aligned}$$

This martingale is cadlag, quasi-left-continuous if the filtration is quasi-left-continuous. As the sum of M_t and A_t , $e^{-rt}V_t$ is then always quasi-left-continuous, i.e.,

$$\limsup_n e^{-r\tau^n} V_{\tau^n} = e^{-r\tau} V_\tau \quad \text{a.s.}$$

for any monotone sequence of stopping times τ^n ($n = 1, 2, \dots$) converging to some stopping time $\tau \in \mathcal{T}$, whenever condition (4) is fulfilled. It hence yields the required upper-semicontinuity in expectation of $e^{-rt}V_t$.

In the following, we are going to prove that the project gives a positive net profit at the optimal investment time and some loss would be created for any earlier and later investment. Following the optimal investment policy, the investment is undertaken at the first time when ξ reaches I . It gives then the project in value

$$\begin{aligned} F &= \mathbb{E} [e^{-\rho\tau^*} V_{\tau^*} - e^{-\rho\tau^*} I] \\ &= \mathbb{E} \left[\mathbb{E} \left[\int_{\tau^*}^\infty \rho e^{-\rho t} \sup_{\tau^* \leq v \leq t} \xi_v dt \middle| \mathcal{F}_{\tau^*} \right] - e^{-\rho\tau^*} I \right] \\ &\geq \mathbb{E} \left[\mathbb{E} \left[\int_{\tau^*}^\infty \rho e^{-\rho t} I dt \middle| \mathcal{F}_{\tau^*} \right] - e^{-\rho\tau^*} I \right] \\ &= 0. \end{aligned}$$

This shows that the investment at τ^* always brings about a non-negative profit. Suppose that the investment is initiated earlier at time point $\hat{\tau} < \tau^*$. The net profit at that moment

is obtained as

$$\begin{aligned}\hat{F} &= \mathbb{E} \left[\mathbb{E} \left[\int_{\hat{\tau}}^{\infty} \rho e^{-\rho t} \sup_{\hat{\tau} \leq v \leq t} \xi_v dt \middle| \mathcal{F}_{\hat{\tau}} \right] - e^{-\rho \hat{\tau}} I \right] \\ &= \mathbb{E} \left[\int_{\tau^*}^{\infty} \rho e^{-\rho t} \sup_{\hat{\tau} \leq v \leq t} \xi_v dt - e^{-\rho \tau^*} I \right] + \mathbb{E} \left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t} \sup_{\hat{\tau} \leq v \leq t} \xi_v dt - e^{-\rho \hat{\tau}} I + e^{-\rho \tau^*} I \right],\end{aligned}$$

where the second step is achieved by splitting the integral into two parts. Because $\xi_{\tau^*} \geq I > \xi_t$ whenever $t \in [0, \tau^*)$, we can easily derive

$$(6) \quad \sup_{\hat{\tau} \leq v \leq t} \xi_v < I \quad \text{for } t \in [\hat{\tau}, \tau^*)$$

and

$$\sup_{\hat{\tau} \leq v \leq t} \xi_v = \sup_{\tau^* \leq v \leq t} \xi_v \quad \text{for } t \in [\tau^*, \infty).$$

On this basis, \hat{F} can be further reduced as

$$\begin{aligned}\hat{F} &= \mathbb{E} \left[\int_{\tau^*}^{\infty} \rho e^{-\rho t} \sup_{\tau^* \leq v \leq t} \xi_v dt - e^{-\rho \tau^*} I \right] + \mathbb{E} \left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t} (\sup_{\hat{\tau} \leq v \leq t} \xi_v - I) dt \right] \\ &= F + \mathbb{E} \left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t} (\sup_{\hat{\tau} \leq v \leq t} \xi_v - I) dt \right].\end{aligned}$$

That is, \hat{F} is a sum of F , the net profit of the project invested at the optimal time τ^* , and another term which is definitely negative due to (6). Therefore, an earlier investment yields a lower project value.

Consider a later investment at time $\tau' > \tau$. We check in this case the difference in the project value at the optimal time and the moment τ' :

$$\begin{aligned}F - F' &= \mathbb{E} \left[\int_{\tau^*}^{\infty} \rho e^{-\rho t} \sup_{\tau^* \leq v \leq t} \xi_v dt - e^{-\rho \tau^*} I \right] - \mathbb{E} \left[\int_{\tau'}^{\infty} \rho e^{-\rho t} \sup_{\tau' \leq v \leq t} \xi_v dt - e^{-\rho \tau'} I \right] \\ &= \mathbb{E} \left[\int_{\tau^*}^{\tau'} \rho e^{-\rho t} \sup_{\tau^* \leq v \leq t} \xi_v dt + \int_{\tau'}^{\infty} \rho e^{-\rho t} \max \left\{ \sup_{\tau^* \leq v < \tau'} \xi_v, \sup_{\tau' \leq v \leq t} \xi_v \right\} dt - e^{-\rho \tau^*} I \right] \\ &\quad - \mathbb{E} \left[\int_{\tau'}^{\infty} \rho e^{-\rho t} \sup_{\tau' \leq v \leq t} \xi_v dt - e^{-\rho \tau'} I \right] \\ &= \mathbb{E} \left[\int_{\tau'}^{\infty} \rho e^{-\rho t} \left(\max \left\{ \sup_{\tau^* \leq v < \tau'} \xi_v, \sup_{\tau' \leq v \leq t} \xi_v \right\} - \sup_{\tau' \leq v \leq t} \xi_v \right) dt \right] \\ &\quad + \mathbb{E} \left[\int_{\tau^*}^{\tau'} \rho e^{-\rho t} \left(\sup_{\tau^* \leq v \leq t} \xi_v - I \right) dt \right],\end{aligned}$$

where we write the running supremum in F into the maximum of the two running supremum before and after τ' in the second step. Obviously, the first term is always nonnegative no matter a new all time high is achieved before or after time τ' . Furthermore, the second

term is also shown to be non-negative since $\sup_{\tau^* \leq v \leq t} \xi_v \geq \xi_{\tau^*} \geq I$ for any $t \in [\tau^*, \infty)$. In all, we have $F \geq F'$. This completes the proof that τ^* is the optimal investment time for the firm to maximize the project value. \square

In this way, the optimal stopping problem is reduced to a representation problem based on the stochastic representation method first proposed by Bank and Föllmer (2003) [3] for various stochastic optimization problems. The representation form (5) is valid whenever the two regularity conditions (3) and (4) are satisfied. In general, there always exists a unique solution of ξ_t to this problem.

Remark 2.2. *In general, the regularity conditions (3) and (4) are relatively weak and well satisfied. First, the investment decision problem is well-posed or makes economically sense only when (3) is true. Moreover, for a semi-martingale process $(X_t)_{t \in [0, \infty)}$ and its generated filtration \mathcal{F} ,*

$$X_\tau = X_{\tau-} \quad \text{for any predictable stopping time } \tau \in \mathcal{T}([0, \infty))$$

is the only requirement to achieve the quasi-left-continuity of the filtration. Intuitively speaking, one cannot tell in advance when the jumps of X will take place. For instance, the filtration generated by a Brownian motion (BM) or a Lévy process is always quasi-left-continuous as the σ -field of a BM is absolutely continuous and for the latter case the stopping time at which a jump occurs is never predictable (see Protter (1988) [18] Chapter 3 p.p. 105).

The intuition behind this method is as follows. The holder of a real option would like to maximize the EPV of the net profit from the investment. That means, the investor does not care about the instantaneous value of the project at the investment time, but about its future proceeds after the investment. By means of this method, the EPV of all future revenues is then specified by the running supremum of the process ξ_t . Formally, it can be expressed as:

$$\begin{aligned} \mathbb{E} \left[e^{-\rho\tau} (V_\tau - I)^+ \right] &= \mathbb{E} \left[\mathbb{E} \left[\int_\tau^{+\infty} \rho e^{-\rho t} \sup_{\tau \leq v \leq t} \xi_v dt \middle| \mathcal{F}_\tau \right] - e^{-\rho\tau} I \right] \\ &= \mathbb{E} \left[\int_\tau^{+\infty} \rho e^{-\rho t} \sup_{\tau \leq v \leq t} \xi_v dt - e^{-\rho\tau} I \right]. \end{aligned}$$

Obviously, the trick of this method is to represent the EPV of the project revenue in terms of the supremum of another process.

According to the optimal investment policy, the investment is undertaken if and only if ξ_t becomes equal to or greater than the investment cost. Otherwise, some positive revenues are lost. Earlier exercise, i.e. when $\xi_t < I$, is also not optimal since the investment at such a time yields only negative payoff. Although ξ_t is not the real revenue from the investment, it takes the role of initiating the investment. Especially, we define ξ_t as the *shadow revenue process* and the decision rule on this basis as the *Shadow NPV rule*. It states that the investment is taken if and only if the *shadow revenue* rises up to cover the investment cost. In this sense, this method indeed extends and corrects the conventional NPV method by identifying the proper net present value.

Shadow value is defined in this work as the value of the investment that the firm really gains from the project after compensating all the costs incurred. Thus, the final obtained *Shadow NPV* measures exactly the willingness of the decision maker to give up money and also time for the investment opportunity. Therefore, it is not only the market value or the real revenue from the investment less the initial investment cost. Additionally, subjective valuation of the investment should be considered. Under uncertainty, investors are reluctant to invest and may be waiting for better information. During the waiting process, the firm may be losing other opportunities to gain profit, hence increasing the opportunity cost of undertaking the investment. Therefore, we argue that the *shadow value* records the true or pure expected benefit embedded in the real revenue of the investment after deducting the full opportunity cost. In other words, the *shadow NPV* accurately captures the expected economic value of the investment in an uncertain environment. Furthermore, we show (in Appendix 1) that the *shadow revenue* is always smaller than or at least equal to the created operating profit at any stopping time, i.e., $V_\tau \geq \xi_\tau$ for all the stopping times $\tau \in \mathcal{T}([0, \infty))$. The discrepancy between these two values accounts exactly for the opportunity cost of delaying the investment. In this sense, the *shadow revenue* measures the expected economic value of the investment which is the real revenue less the option premium of waiting. Thus, it becomes optimal to invest when the *correct* NPV becomes non-zero. Clearly, the investment rule obtained by this method is fully consistent with the established result in the real options theory.

Remark 2.3.

- (a) *The same argument can be applied to the so-called exit problem. Consider the same firm who has already invested and produced a unit output at price P_t . The firm contemplates scrapping the investment for a value C , once the price declines and results in loss. In this context, the firm would like to maximize the payoff of the*

investment abandonment

$$\max_{\tau \in \mathcal{T}([0, \infty))} \mathbb{E} \left[e^{-\rho\tau} \left(C - \int_{\tau}^{\infty} e^{-\rho(s-\tau)} P_s ds \right)^+ \right].$$

That is, the firm has a put on the investment at hand. By means of this method, the EPV of the future revenues that would be lost after exit is reduced to a representation in terms of the infimum process of the shadow revenue process ξ as

$$e^{-\rho\tau} V_{\tau} = \mathbb{E} \left[\int_{\tau}^{\infty} \rho e^{-\rho t} \inf_{\tau \leq v \leq t} \xi_v dt \middle| \mathcal{F}_{\tau} \right],$$

where $V_t = \mathbb{E} \left[\int_t^{\infty} e^{-\rho(s-t)} P_s ds \middle| \mathcal{F}_t \right]$. The optimal investment time is then characterized as the first time when the **shadow revenue process** becomes equal to or lower than the gain of exit C , namely,

$$\tau^{**} = \inf\{t \geq 0 \mid \xi_t \leq C\}.$$

- (b) This method works also when the discount rate is not constant but stochastic with strictly positive values.

As a short summary, the irreversible investment decision problem is solved by finding the solution of a stochastic representation problem in terms of the running supremum/infimum process of the *shadow revenue process*. Obviously, the *shadow revenue process*, ξ_t , is the key process in this method, signalling the optimal exercise rule. In particular, this exercise signal process is universal in the sense that it is the single reference process determining optimal investment times for any possible investment costs. This property would be favorable in more complicated investment decision problems, for instance, sequential investments as well as capital expansion programs.

This approach fits all semi-martingale processes which are economically plausible and hence often used in finance, provided that the mild regularity condition is satisfied. Generally, numerical methods have to be used to specify the universal exercise signal process. To some cases, e.g., exponential Lévy processes, explicit formulae are already available. Particularly, the solution is in a simple and intuitive form such that the expected future operating profit from the investment has to cover not only the investment cost but also the opportunity cost of delaying the investment. Thus, in this sense, this method generalizes the solution of real options in the GBM model and provides additional interpretations even within the GBM model framework.

3. EXPLICIT SOLUTION FORMULAE FOR IRREVERSIBLE INVESTMENT PROBLEMS

One outstanding advantage of this method is the capability of providing an explicit formula of the critical investment value for an exponential Lévy process. Lévy processes are a general class of Markov processes with independent identically distributed increments and can be decomposed into a continuous Gaussian process and a pure jump process (see, for example, Bertoin (1996) [4]). Hence, it is frequently used to capture the significant skewness and kurtosis of commodity prices as empirically observed in for instance Yang and Brorsen (1992) [20] as well as Deaton and Laroque (1992) [12]. In this section, a thorough analysis is provided to characterize analytical solutions to the irreversible investment decision problem where the underlying uncertainty is modelled by a general exponential Lévy process.

To put it in a formal way, assume that the exogenous output price is generated by the following stochastic process

$$(7) \quad P_t = P_0 e^{Y_t},$$

where $P_0 > 0$ is the initial price and $Y = (Y_t)_{t \geq 0}$ is a Lévy process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ with initial value $Y_0 = 0$. Before moving on to derive the solution, a short excursion for Lévy processes is provided³. The Lévy-Laplace exponent $\Psi(z)$ of the Lévy process Y is defined via

$$\mathbb{E}[e^{zY_t}] = e^{t\Psi(z)}.$$

$\Psi(z) = \rho$ is the corresponding characteristic equation of Y_t for $\rho > 0$. Define $\bar{Y}_t = \sup_{0 \leq s \leq t} Y_t$ and $\underline{Y}_t = \inf_{0 \leq s \leq t} Y_t$ as the running supremum and infimum of Y_t . The main technique for solving the problem in this paper is the Wiener-Hopf factorization

$$\frac{\rho}{\rho - \Psi(z)} = \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} e^{z\bar{Y}_t} dt \right] \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} e^{z\underline{Y}_t} dt \right] = \Psi_\rho^+(z) \Psi_\rho^-(z).$$

It is possible to obtain analytical forms of the Wiener-Hopf left and right factors $\Psi_\rho^+(z)$ and $\Psi_\rho^-(z)$ respectively as the factorization is unique. For instance, for the case of a GBM, the characteristic equation has one positive and one negative root as β^+ and β^- . Then the two factors are given by

$$\Psi_\rho^+(z) = \frac{\beta^+}{\beta^+ - z} \quad \text{and} \quad \Psi_\rho^-(z) = \frac{\beta^-}{\beta^- - z}.$$

³More details on Lévy processes and their properties are found in Bertoin (1996) [4] and literature cited therein.

Furthermore, Boyarchenko and Levendorskiĭ (2002a) [7] derive a general solution form to regular Lévy processes of exponential type as we assume here.

The Investment Threshold and Project Value. In this context, a closed-form characterization can be found for the critical shadow revenue process identifying the investment initiating time:

Theorem 3.1. *Under assumption of (7), the solution of representation problem (5), namely, the shadow revenue process is obtained as $\xi_v = P_v/\kappa$ with*

$$\kappa = (\rho - \Psi(1)) \mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right],$$

where $\tau(\rho)$ is an independent exponentially distributed time with parameter ρ .

Proof. The proof is given in Appendix 2. □

According to the above theorem, the shadow revenue of the investment is determined to be the revenue of the investment divided by a constant factor. The optimal investment time can be then rewritten as

$$\tau^* = \inf\{t \geq 0 \mid P_t \geq \kappa I\}.$$

It suggests that

$$V_{\tau^*} \geq \mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] I,$$

where the expectation term is always larger than 1 as $e^{\bar{Y}_t} \geq e^{Y_0} = 1$ for all $t \in [0, \infty)$. Thus, it gives the following investment rule: an investor undertakes the investment at the first time when the expected revenue reaches or exceeds the investment cost multiplied by a correction factor⁴. Indeed, it gives the same form of the trigger value as in Dixit and Pindyck (1994) [13]. In this sense, this new method is more favorable because it generalizes explicit formulae to an exponential Lévy process. This will be addressed below further with specific examples.

The remaining problem is how to solve κ and the value of the option to invest

$$F = \mathbb{E} \left[e^{-\rho\tau^*} (V_{\tau^*} - I)^+ \right].$$

⁴Alternatively, we can obtain $P_{\tau^*} \geq \rho I / \mathbb{E} \left[e^{Y_{\tau(\rho)}} \right]$ after applying the Wiener-Hopf formula. It is a modified Jorgensonian trigger value which includes a risk premium for the marginal revenue product above the Jorgensonian user cost of capital, ρI , due to the irreversibility and uncertainty.

Thanks to some mathematical properties of Lévy processes, they can be obtained in analytical form. Moreover, simple explicit formulae are possible for those Lévy processes with only negative jumps, as stated in the following theorem and shown in Appendix 3.

Theorem 3.2. κ in the threshold value $P_{\tau^*} = \kappa I$ is calculated in explicit and simple form for some special cases:

- (a) In general, $\kappa = (\rho - \Psi(1)) \Psi_{\rho}^{+}(1)$.
- (b) For a Lévy process with no positive jumps, $\kappa = (\rho - \Psi(1)) \frac{\beta^{+}}{\beta^{+}-1}$ where β^{+} is the unique positive root of the characteristic equation of Y_t .

With the knowledge of κ , the value of the option to invest is given

- (a) $F = I \left[\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] \mathbb{E} \left[e^{-\rho\tau^{*} + (Y_{\tau^{*}} - y^{*})} \right] - \mathbb{E} \left[e^{-\rho\tau^{*}} \right] \right]$,
where y^{*} is the value of Y at the time point τ^{*} and the Laplace transforms of the two expectations are obtained as follows:

$$\int_0^{\infty} e^{-qy} \mathbb{E} \left[e^{-\rho\tau^{*} + (Y_{\tau^{*}} - y)} \right] dy = \frac{1}{q+1} \left(1 - \frac{\Psi_{\rho}^{+}(-q)}{\Psi_{\rho}^{+}(1)} \right)$$

and

$$\int_0^{\infty} e^{-qy} \mathbb{E} \left[e^{-\rho\tau^{*}} \right] dy = \frac{1 - \Psi_{\rho}^{+}(-q)}{q}.$$

- (b) In particular, $F = \left(\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] - 1 \right) \left(\frac{P_0}{\kappa} \right)^{\beta^{+}} I^{1-\beta^{+}}$ for a Y_t with no positive jumps.

It is worth noting that $\kappa > 0$ should be always true to make economic sense. It is satisfied whenever

$$\mathbb{E} \left[\int_0^{\infty} e^{-\rho t} P_t dt \right] < \infty,$$

which is exactly the condition required for uniform integrability. In particular, it is valid for the GBM case if and only if $\beta^{+} > 1$, i.e., $\mu + \frac{1}{2}\sigma^2 < \rho$ where μ and σ are the drift and volatility of the GBM. Intuitively, the expected growth rate of the revenue is bounded from above by the time cost, namely, the discount factor ρ . Otherwise, the discounted payoff is a submartingale and goes to infinity with increasing time. In this sense, the regularity condition coincides with that in Dixit and Pindyck (1994) [13] and with that in Boyarchenko and Levendorskii (2004a) [9] to guarantee that the EPV of the project is finite as time goes to infinity.

Case Studies. Two specific examples of the irreversible investment model are provided in this subsection in order to well illustrate this method.

Case I. Geometric Brownian Motion: A GBM is most often used in the irreversible investment model to characterize the uncertainty. Assume that the output price follows a GBM with constant drift μ and volatility σ :

$$(8) \quad P_t = P_0 e^{Y_t} \quad \text{and} \quad Y_t = \mu t + \sigma B_t,$$

where $(B_t)_{t \geq 0}$ is the standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. As is well known, a Lévy process pins down to a GBM when the jump component is absent. In this case, a simple and well-known analytical solution for the investment threshold can be easily achieved to be

$$P_{\tau^*} = \kappa I = (\rho - \mu - \frac{1}{2}\sigma^2) \frac{\beta^+}{\beta^+ - 1} I,$$

where β^+ is the positive root of the characteristic equation $\frac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0$.

Referring back to the basic model in Dixit and Pindyck (1994) [13], the trigger value of the investment is the investment cost multiplied by a correction factor $\frac{\beta}{\beta-1}$, where $\beta > 1$ is the positive root of the fundamental quadratic equation which coincides with the characteristic equation. Thus, this new method recovers the standard result for the simplest case of GBM.

Case II. GBM Combined with a Compound Poisson Process: The dynamics of the price is in this case modelled by a combination of a GBM and a jump component characterized by a compound Poisson process with random jump sizes. The randomness from the jump component causes complicated computations. As pointed out by Dixit and Pindyck (1994) [13], numerical methods to such cases have to be used. However, explicit formulae can even be found by means of this new method.

Consider the model

$$(9) \quad P_t = P_0 e^{Y_t} \quad \text{and} \quad Y_t = \mu t + \sigma B_t + \sum_{k=1}^{N_t} J_k,$$

where $(N_t)_{t \geq 0}$ is a Poisson process of intensity λ and $J = (J_k)_{k \in \mathbb{N}}$ is a sequence of independent identically distributed random variables with density

$$f(j) = \begin{cases} p c^+ e^{-c^+ j} & j \geq 0, \\ (1-p) c^- e^{c^- j} & j < 0. \end{cases}$$

where the parameters $c^\pm > 0$ and $0 \leq p \leq 1$. Under this assumption, the project value at time t has in all N_t possible upward and downward jumps which occur with probability p

and $1 - p$, respectively. Each positive/negative jump is exponentially distributed with the parameter c^+/c^- . This specific model has the Lévy-Laplace exponent

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \lambda p \frac{z}{c^+ - z} - \lambda(1 - p) \frac{z}{c^- + z}.$$

Accordingly, the optimal investment threshold is determined by solving κ

$$\kappa = \left(\rho - \mu - \frac{1}{2}\sigma^2 - \frac{\lambda p}{c^+ - 1} + \frac{\lambda(1 - p)}{c^- + 1} \right) \Psi_\rho^+(1),$$

where the left Wiener-Hopf factor is found to be

$$\Psi_\rho^+(1) = \frac{\beta_1^+}{\beta_1^+ - 1} \frac{\beta_2^+}{\beta_2^+ - 1} \frac{c^+ - 1}{c^+}$$

given the two positive roots $\beta_{1/2}^+$ of the characteristic equation of $\Psi(z) = \rho$.

4. REAL OPTIONS WITH SUBJECTIVE RISK PREFERENCES

Profit-maximization is the standard model in real options literature based on the assumption that the financial market is complete and the cash flows of the project can be spanned by those existing products in the market. In this setup, risk neutrality is presumed such that the risk-free interest rate is the *correct* discount factor. However, these assumptions are not relevant to the real world with different risk preferences and typically an incomplete market. Undoubtedly, the irreversible investment problem can be though properly dealt with by the dynamic programming method and also by our new method, both of which do not require such strict conditions. Nevertheless, decision related parameters and hence the investment policy are greatly affected by the risk attitude of investors. For instance, discounted rates are subjective assessments of investors based on trading prices and their own outlook for future prospects. As a result, it is necessary to incorporate subjective risk preferences when evaluating investment value.

Typically, risk preferences are fully characterized by the utility function which in turn determines the decision maker's subjective valuation for the project. Following this idea, we consider a utility-based real options problem and deal with it also by applying the stochastic representation method. Our new method works well for a general class of increasing and concave utility function which accounts for risk aversion, provided that the objective maximization function is bounded to be finite. An analytically tractable and intuitive solution is obtained when the decision maker has a constant relative risk aversion (CRRA) and the uncertainty is modelled by an exponential Lévy process.

4.1. Utility-Based Irreversible Investment Decision Problem. Consider again the model in Section 2. In addition, we define an expected utility function to represent the risk aversion of the decision maker

$$p \longmapsto \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(p_t) dt \right],$$

where the utility function $U(\cdot)$ is supposed to be increasing, concave and continuously differentiable. In this case, the firm maximizes the expected utility of the investment

$$(10) \quad F = \max_{\tau \in \mathcal{T}(0, \infty)} \mathbb{E} \left[e^{-\rho \tau} \left(\int_\tau^\infty e^{-\rho(t-\tau)} (U(P_t) - U(rI)) dt \right)^+ \right],$$

where $U(rI)$ is the utility lost which could be otherwise gained by investing risk-free cash flow stream rI .

To guarantee the well-posedness of the problem, the expected utility of the future revenue of the investment has to be finite, namely, $\mathbb{E} \left[\int_0^\infty e^{-\rho t} U(P_t) dt \right] < \infty$. This condition is also required to maintain the regularity condition. The optimal strategy of the utility-based irreversible investment is then figured out by the following theorem.

Theorem 4.1. *A risk averse investor who has to decide on the investment timing in the problem (10) will undertake the investment at time*

$$\tau^* = \inf \left\{ t \geq 0 \mid \xi_t \geq \frac{U(rI)}{\rho} \right\},$$

where ξ_t is the solution to the representation of the form

$$(11) \quad \mathbb{E} \left[\int_\tau^\infty e^{-\rho t} U(P_t) dt \mid \mathcal{F}_\tau \right] = \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho t} \sup_{\tau \leq v \leq t} \xi_v dt \mid \mathcal{F}_\tau \right].$$

This result can be easily obtained by the same argument as for Theorem (2.1). In this model, ξ_t can be interpreted as the *shadow utility* that the firm gains from the investment. That is, by investing the firm gives up some utility while postponing the project as the investment is irreversible. The obtained *shadow utility* measures then the direct utility of the project deducted by the utility which is lost while waiting for better information. In this way, investment occurs only when the subjective valuation of the project amounts high enough to cover the full investment cost which takes the subjective valuation of the real option into account.

The theorem provides the utility-maximizing investment rule for any increasing and concave utility function. To derive a specific investment decision and to examine the impact of risk aversion on the decision, the model has to be further specified. In the

following, we consider the case of CRRA with uncertainty modelled by an exponential Lévy process.

4.2. Explicit Solutions for a CRRA Utility Function and an Exponential Lévy Process. A power utility function

$$(12) \quad U(x) = \frac{1}{1-\alpha} x^{1-\alpha}$$

is the typical modification to account for CRRA, where the parameter $\alpha > 0$, $\alpha \neq 1$ is the coefficient of relative risk aversion. The higher it is, the more risk averse the firm is. In particular, it is risk neutral when $\alpha = 0$, which is exactly the original profit-maximization problem. The price of the produced good follows the specification of an exponential Lévy process as given in (7). Under this construction, a simple application of the result above yields the following Theorem.

Theorem 4.2. *Suppose that the decision maker takes the power utility (12) and faces uncertainty modelled by an exponential Lévy process. Whenever the exponential growth rate of the utility is bounded from above by the discount factor, i.e., $\rho > \Psi(1-\alpha)$, the utility-based irreversible investment model is well defined and the shadow utility process ξ_t is obtained as*

$$\xi_t = \frac{\theta}{1-\alpha} P_t^{1-\alpha},$$

where $\theta = \frac{1}{\rho} \mathbb{E} \left[e^{(1-\alpha)Y_{\tau(\rho)}} \right]$. The expected utility under the optimal investment rule is given by

$$F = \frac{\mathbb{E} \left[e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}} \right]}{\rho} \mathbb{E} \left[U(P_{\tau(\rho)}) \right] - \mathbb{E} \left[e^{-\rho\tau^*} \right] \frac{U(rI)}{\rho}.$$

Proof. Following the line of the idea of Theorem (2.1), the solution to the representation form (11) is easily obtained by constructing $\xi_t = \frac{\theta}{1-\alpha} P_t^{1-\alpha}$ and by applying the strong Markov property of Lévy processes. With the value of θ , the expected utility of the investment becomes trivial after some computation. Hence, it is not addressed any more in the paper.

The utility maximization problem is well-posed if and only if

$$\mathbb{E} \left[\int_0^\infty e^{-\rho s} U(P_s) ds \right] < \infty,$$

or equivalently

$$\mathbb{E} \left[\int_0^\infty e^{-\rho s + (1-\alpha)Y_s} ds \right] < \infty.$$

Using Fubini's theorem and Lévy-Laplace exponent $\Psi(z)$ yields

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho s+(1-\alpha)Y_s} ds\right] &= \int_0^\infty e^{-\rho s}\mathbb{E}\left[e^{(1-\alpha)Y_s}\right] ds \\ &= \int_0^\infty e^{-\rho s}e^{s\Psi(1-\alpha)} ds.\end{aligned}$$

Clearly, $\rho > \Psi(1-\alpha)$ is the necessary condition for the well-posedness of the problem. \square

4.3. Effect of Risk Aversion on the Decision. The critical expected utility from the project for issuing the investment is identified as

$$\mathbb{E}\left[\int_{\tau^*}^\infty e^{-\rho t}U(P_t) dt\middle|\mathcal{F}_{\tau^*}\right] \geq \gamma \cdot \frac{U(rI)}{\rho},$$

where $\gamma = \mathbb{E}\left[e^{(1-\alpha)\bar{Y}_{\tau(\rho)}}\right]$. At first glance, this result gives an image that the investment rule is almost the same as that of the profit-maximization problem but in terms of utility.

Despite in the similar form, utility-maximization highlights the risk attitude of investors and its effect on the decision, compared to the profit-maximization problem. The first influence of maximizing the utility is the occurrence of investment even when the project revenue (before netting of the cost) creates a negative utility. In case of slight risk aversion ($0 \leq \alpha < 1$), the threshold utility is always positive and the multiplicative factor γ always larger than 1. While, the expected utility turns out to be negative, when investors are more risk averse with $\alpha > 1$. The net utility is nevertheless positive as $\gamma < 1$. Intuitively, investment gives higher utility than merely holding the money I , although the cash flow from the project is too risky according to their subjective judgement. As a result, the firm in both cases would like to invest if and only if the utility from the project covers both the utility of the investment cost and the lost utility due to the delay in investment.

Furthermore as argued by Hugonnier and Morellec (2005) [15], in the GBM model, the critical investment level is quite high relative to the profit-maximization threshold and it increases monotonically with the relative risk aversion coefficient α . The intuition behind is clear: the decision maker is risk averse and would prefer the project with less risk. Hence, when facing uncertainty in the future revenue, he has a strong incentive to delay the investment. This is also true when extending the model to more general processes, e.g., a Lévy process with possible unexpected shocks.

Theorem 4.3 (Comparative statics analysis of the risk aversion). *The threshold value is obtained as $P^* = \eta^{-\frac{1}{1-\alpha}}rI$ with $\eta = \mathbb{E}\left[e^{(1-\alpha)\bar{Y}_{\tau(\rho)}}\right]$ for any exponential Lévy process utility maximization model. It is monotonically decreasing in the risk aversion coefficient α .*

Proof. The proof is provided in Appendix 4. □

Nevertheless, how do jumps affect the threshold value combined with risk aversion? In order to answer this question, we provide a specific example where a firm has to make an investment decision for a project. Assume that the output log-price has the first two central moments $m_1 = -0.03$ and $m_2 = 0.04$. In addition, the discount rate is supposed to be $\rho = 15\%$ ⁵. Clearly, the decision rule is dependent on the model he chooses: the estimated moments have to be fitted to the model and hence may result in different trigger values. Suppose that the firm mainly focuses on 4 investment scenarios:

- I.* The project value is Gaussian distributed; and profit is the correct measure.
- II.* The project value is Gaussian distributed; managers are nevertheless risk averse and choose to check the utility that the project creates.
- III.* The project value is specified by a jump-diffusion process with negative jumps since e.g., more competitors may come into the market in the future and the price is greatly influenced by another product's price etc.; and profit is the correct measure.
- IV.* Project value is specified by a jump-diffusion process; and managers are also risk averse.

If the project value is normally distributed, the stochastic process of P_t is simply a GBM as described in Equation (8) and the drift and volatility are completely determined by $\mu = m_1$ and $\sigma^2 = m_2$. The jump-diffusion process with negative jumps is then characterized as a special case of (9) with $p = 0$. In this case, the first two moments are obtained as

$$m_1 = \Psi'(0) \quad \text{and} \quad m_2 = \Psi''(0).$$

It helps to uniquely determine the drift and volatility term by the values of m_1 , m_2 , c and λ .

Figure 1 gives the critical price value for the four different scenarios. Profit maximization is one special case of utility-based model. Thus, the investment thresholds for Scenario *I* and *III* are the two red circles corresponding to $\alpha = 0$. The trigger values of Scenario *II* and *IV* are as expected heavily dependent on the relative risk aversion coefficient α . It is shown in the figure that P_{τ^*} in both cases increases monotonically with α , confirming and generalizing Hugonnier and Morellec's result (2005) [15]. In other words, risk aversion increases firms' initiative to postpone the investment. Consequently, the profit-maximization model in general gives a *wrong* investment decision when the investor is indeed risk averse.

⁵A relatively low expectation of the log-price and a high discount factor are assumed in order to draw a picture for a certain interval of α which is large enough for illustration. Such a seemingly unreal assumption is required to make all the chosen parameters economically sensible.

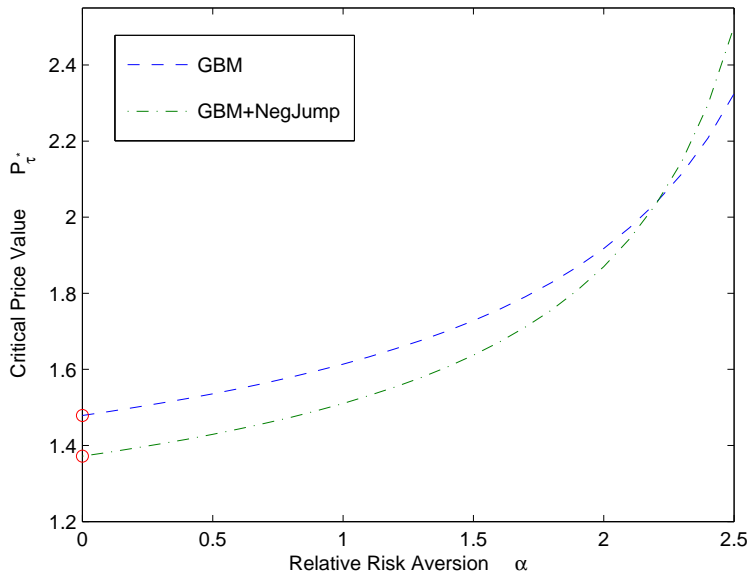


FIGURE 1. Investment Thresholds of Four Models with Parameters $c = 7.5$ and $\lambda = 1.0$

Moreover, with the same first two moments, it is the Non-Gaussian model whose investment trigger value is more affected by α .

It is well acknowledged that in the profit maximizing model a GBM gives a higher threshold value than a jump-diffusion process with negative jumps. It is in fact one essential argument in the literature, recommending the introduction of jumps. However, this property is not maintained when we include additionally utility in the model. A jump-diffusion process gives a lower critical value only when α is small. With a large α , utility consideration may give a higher value than that for a GBM. We found this fact in many numerical analysis even for a jump-diffusion process with positive jumps.

To get more insights on the impact of jumps, we plot in Figure 2 investment trigger values for a family of jump-diffusion processes with various jump coefficients and two different relative risk aversion coefficients of $\alpha = 0$ and $\alpha = 1.75$. In the profit-maximization model ($\alpha = 0$), P_{τ^*} always decreases with λ and c . That is, given certain fixed estimates on the log-price's mean and variance, the risk neutral decision maker is less hesitant to invest when he expects either higher negative jumps or negative jumps with a larger probability. It is because the addition of jump terms decreases the diffusion uncertainty which is in effect the biggest “lost of information” and hence involves the highest uncertainty in comparison to other processes with the same instantaneous volatility. To clarify it, the

Brownian motion (BM) is standard normally distributed and could be obtained as the limit of the average sum of a large number of i.i.d. random variables with finite variance. The larger the number is, the closer their distributions are and thus the more the information is missing! In this sense, the BM modification follows the principle of maximum entropy or minimum information while remaining consistent with the given knowledge—the estimated variance (see Boyarchenko and Levendorskii (2004) [9] and also the detailed mathematical argument in Bouchaud and Potters (2000) [6]). Consequently, the decision maker is better informed of the future profits of the investment or equivalently faces less uncertainty by increasing the relevant parameters of negative jumps. It in turn decreases the threshold value.

However, an increase in jumps has an ambiguous effect on the trigger value when considering utility maximization or risk aversion. As observed in the right plot of Figure 2 for $\alpha = 1.75$, P_{τ^*} rises with λ when $c = 5$ but decreases when $c = 7.5$. To our knowledge, there are two opposite effects of jumps on the trigger value for the case of a fixed estimate on variance: It decreases on one hand the diffusion volatility of the project which in turn lowers the threshold value; on the other hand (negative) jumps increase the trigger value as a response to the expectation of gloomy economic conditions. In case of risk aversion, the second effect of jumps is greatly exaggerated, as risk averse investors would prefer to a larger extent waiting in order to avoid a great loss. Therefore, the critical price value turns out to be much higher than that in the profit-maximization model although with the same jump coefficients. Moreover, P_{τ^*} even increases with λ when $c = 5$. The declining relationship is recovered only when the jump term is high enough (as in the case of $c = 7.5$) to overcome the second effect additionally driven by the risk aversion.

5. CONCLUSION

The literature treats the irreversible investment decision problem under uncertainty as an option on real assets and solves the optimal stopping problem by means of contingent claim analysis or dynamic programming method. In this paper, we analyze the same real options model but with an alternative approach – the stochastic representation method. This method starts with the EPV of the project, the natural and meaningful definition in economics and represents it in a form of the EPV of the running supremum of another process. By solving the representation problem, the investment decision rule is identified in terms of the *shadow revenue process* such that the investment is initiated at the first moment at which the *shadow net present value* becomes non-negative. The obtained rule

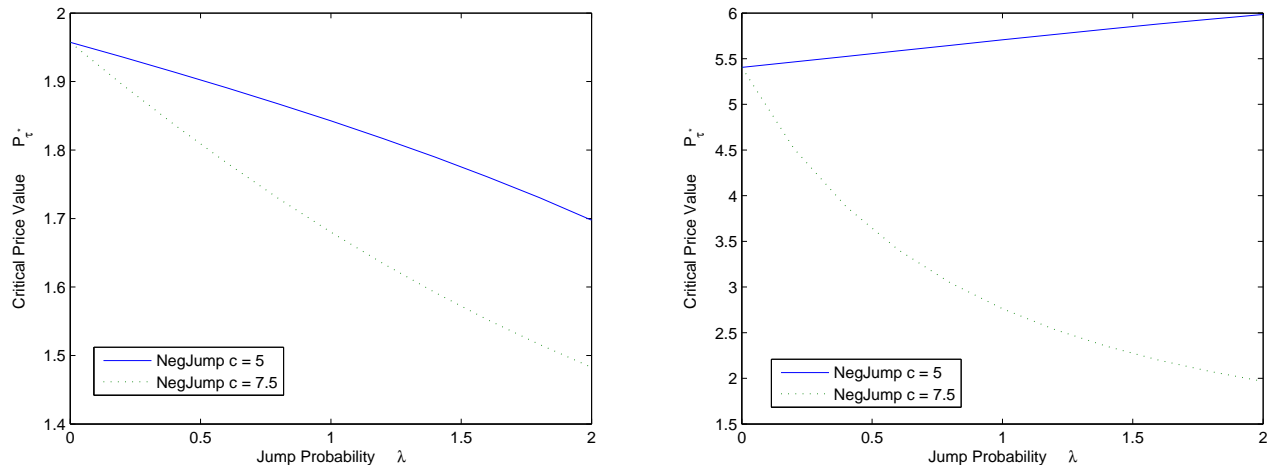


FIGURE 2. Investment Trigger Value vs. Jump Coefficients
for $\alpha = 0$ (left) and $\alpha = 1.75$ (right) with Estimates $m_1 = -0.06$ and $m_2 = 0.16$

is demonstrated to be consistent to that given by the standard real options theory: The critical investment revenue has to cover not only the investment cost but also the opportunity cost of delaying the investment. More importantly, our new method extends and corrects the conventional NPV method by figuring out the *proper* net present value. This formulation in terms of EPV gives a clear and intuitive understanding of the investment strategy and then enables a wide application of real options theory in reality.

Compared to the existing standard approaches, this method is advantageous for the applicability to a large class of stochastic processes (all semi-martingale processes) as well as the feasibility of giving an explicit characterization of the solution for an exponential Lévy process. It is demonstrated in the paper that the closed-form characterization for exponential Lévy processes is obtained almost as easily as in the Gaussian case by solving the fundamental characteristic equations. Moreover, the result defines the optimal investment timing as the first moment when the underlying project value rises to or exceeds κ times the investment cost, which confirms and generalizes the well-known result in the literature for the case of a GBM.

The technique is also applied to the utility-maximizing real options problem. In this model, the utility function is incorporated to measure the risk preferences of the decision maker such that the strict assumptions of the standard real options theory, market completeness and risk neutrality, are not necessarily required. The utility maximization problem is similarly reduced into a representation problem but in terms of the running

supremum of the *shadow utility process*. In particular, when the firm is risk averse with constant relative aversion and faces an exponential Lévy process modelled uncertainty, the investment rule is obtained in an analytically tractable form. Due to risk aversion, the firm has more incentive to delay the investment, which results in a higher threshold. In this sense, the standard real options decision rule may be wrong by assuming a risk preference free framework. With fixed estimates of the first two central moments of the underlying log-price processes, the threshold value under the Gaussian modification can be lower or higher than that under the jump-diffusion process modification with negative jumps, heavily depending on the constant risk aversion coefficient and jump component parameters. We argue that this is not a counter-intuitive result: Given the estimated variance, the increasing effect of negative jumps on the trigger value is greatly magnified by the risk aversion even if the jump term simultaneously lowers the trigger value by means of decreasing diffusion uncertainty.

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Appendix 1: Proof of $e^{-\rho\tau}V_\tau \geq e^{-\rho\tau}\xi_\tau$ at Any Stopping Time $\tau \in \mathcal{T}([0, \infty))$.

$$\begin{aligned} e^{-\rho\tau}V_\tau &= \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho t} \sup_{\tau \leq v \leq t} \xi_v dt \mid \mathcal{F}_\tau \right] \\ &\geq \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho t} \xi_\tau dt \mid \mathcal{F}_\tau \right] \\ &= e^{-\rho\tau} \xi_\tau. \end{aligned}$$

Appendix 2: Proof of Theorem (3.1).

Proof. Based on the specification of P_t , the left-hand side of Equation (5) is first calculated as

$$e^{-\rho\tau}V_\tau = e^{-\rho\tau} \frac{P_0 e^{Y_\tau}}{\rho - \log \mathbb{E}[e^{Y_1}]},$$

where $\log \mathbb{E}[e^{Y_1}]$ records the time increasing rate of the price process and is as defined equal to $\Psi(1)$.

Construct the *shadow revenue process* in form of $\xi_v = P_v/\kappa$. Then by substituting the general exercise signal, the representation equation is reduced into

$$\begin{aligned} e^{-\rho\tau} \frac{P_0 e^{Y_\tau}}{\rho - \Psi(1)} &= \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho t} \sup_{\tau \leq v \leq t} P_0 \frac{\exp(Y_v)}{\kappa} dt \mid \mathcal{F}_\tau \right] \\ &= e^{-\rho\tau} P_0 e^{Y_\tau} \mathbb{E} \left[\int_\tau^\infty \rho e^{-\rho(t-\tau)} \sup_{\tau \leq v \leq t} e^{Y_v - Y_\tau} dt \mid \mathcal{F}_\tau \right] / \kappa. \end{aligned}$$

This can be further simplified by using the property of Lévy processes that $Y_v - Y_\tau$ has the same distribution as $Y_{v-\tau}$ and is independent of the σ -field \mathcal{F}_τ

$$\begin{aligned} e^{-\rho\tau} \frac{P_0 e^{Y_\tau}}{\rho - \Psi(1)} &= e^{-\rho\tau} P_0 e^{Y_\tau} \mathbb{E} \left[\int_0^\infty \rho e^{-\rho t} \sup_{0 \leq v \leq t} e^{Y_v} dt \right] / \kappa \\ &= e^{-\rho\tau} P_0 e^{Y_\tau} \mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] / \kappa, \end{aligned}$$

where $\bar{Y}_t = \sup_{s \leq t} Y_s$ and $\tau(\rho)$ is an independent exponentially distributed time with parameter ρ . Clearly, $\xi_v = P_v/\kappa$ provides the solution to the representation problem (5) if and only if $\kappa = (\rho - \Psi(1)) \mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right]$. \square

Appendix 3: Proof of Theorem (3.2).

Proof. (i) The solution for κ : By the definition of the Wiener-Hopf factorization, it is easy to observe that the expectation form in κ is indeed the left Wiener-Hopf factor $\Psi_\rho^+(1)$ of the Lévy process Y_t . Thus, κ is obtained as

$$\kappa = (\rho - \Psi(1))\mathbb{E}\left[e^{\bar{Y}_{\tau(\rho)}}\right] = (\rho - \Psi(1))\Psi_\rho^+(1).$$

The left Wiener-Hopf factor is readily recognized for the defined Lévy process Y_t . For a Lévy process with no positive jumps, it is more simple to solve by using the well-known exponential distribution of the supremum process $\bar{Y}_{\tau(\rho)}$ (Bertoin (1996) [4], Chapter VII). More precisely, the running supremum at an exponentially distributed time with parameter ρ has an exponential distribution with parameter β^+ , the unique positive root of the characteristic equation. In this way,

$$\begin{aligned}\kappa &= (\rho - \Psi(1))\mathbb{E}\left[e^{\bar{Y}_{\tau(\rho)}}\right] \\ &= (\rho - \Psi(1))\int_0^\infty -\beta^+ e^y e^{-\beta^+ y} dy \\ &= (\rho - \Psi(1))\left[-\frac{\beta^+}{\beta^+ - 1} e^{-(\beta^+ - 1)y}\Big|_0^\infty\right] \\ &= (\rho - \Psi(1))\frac{\beta^+}{\beta^+ - 1}.\end{aligned}$$

(ii) The value of the option to invest: The option value is trivial for the case of Lévy processes with no positive jumps. In this case, there is no upward discontinuity due to the lack of upward jumps, hence zero overshoots over the critical level. Therefore, the project expected value at the optimal investment time is exactly $P_{\tau^*} = \kappa I$. Substituting $I = \frac{P_{\tau^*}}{\kappa}$ and the trigger level of the Lévy process $y^* = Y_{\tau^*} = \ln \frac{\kappa I}{P_0}$ yields the real option value

$$\begin{aligned}F &= \mathbb{E}\left[e^{-\rho\tau^*}(V_{\tau^*} - I)^+\right] \\ &= \mathbb{E}\left[e^{-\rho\tau^*}\left(\frac{P_{\tau^*}}{\rho - \Psi(1)} - \frac{P_{\tau^*}}{\kappa}\right)\right] \\ &= \mathbb{E}\left[e^{-\rho\tau^*}\right]\left(\mathbb{E}\left[e^{\bar{Y}_{\tau(\rho)}}\right] - 1\right)\frac{P_{\tau^*}}{\kappa}.\end{aligned}$$

Then according to the Laplace transform of the hitting time $\mathbb{E}[e^{-\rho\tau_y}1_{[\tau_y < \infty]}] = e^{-y\beta^+}$ where $\tau_y = \inf\{t \geq 0 \mid Y_t \geq y\}$ (Bertoin (1996) [4]), this can be further reduced to

$$\begin{aligned} F &= e^{-y^*\beta^+} \left(\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] - 1 \right) \frac{P_{\tau^*}}{\kappa} \\ &= \left(\frac{\kappa I}{P_0} \right)^{-\beta^+} \left(\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] - 1 \right) I \\ &= \left(\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] - 1 \right) \left(\frac{P_0}{\kappa} \right)^{\beta^+} I^{1-\beta^+}. \end{aligned}$$

However, such a nice form is not possible for a general Lévy process since continuity at the level y^* is not guaranteed any more. Thus, a possible overshoot has to be considered in this context. As

$$\begin{aligned} F &= \mathbb{E} \left[e^{-\rho\tau^*} V_{\tau^*} \right] - \mathbb{E} \left[e^{-\rho\tau^*} I \right] \\ &= I \left[\mathbb{E} \left[e^{-\rho\tau^*} \frac{P_{\tau^*}}{I(\rho - \log \mathbb{E} [e^{Y_t}])} \right] - \mathbb{E} [e^{-\rho\tau^*}] \right] \\ &= I \left[\mathbb{E} \left[e^{-\rho\tau^*} \frac{\kappa P_{\tau^*}}{P^*(\rho - \log \mathbb{E} [e^{Y_t}])} \right] - \mathbb{E} [e^{-\rho\tau^*}] \right] \\ &= I \left[\mathbb{E} \left[e^{-\rho\tau^*} \mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} e^{Y_{\tau^*} - y^*} \right] - \mathbb{E} [e^{-\rho\tau^*}] \right] \right] \\ &= I \left[\mathbb{E} \left[e^{\bar{Y}_{\tau(\rho)}} \right] \mathbb{E} \left[e^{-\rho\tau^* + (Y_{\tau^*} - y^*)} \right] - \mathbb{E} [e^{-\rho\tau^*}] \right] \end{aligned}$$

with $P^* = P_0 e^{y^*} = \kappa I$, the option value is known by calculating the two expectations. Based on the Pecherskii-Rogozin identity⁶, their Laplace transforms are obtained in form of

$$\int_0^\infty e^{-qy} \mathbb{E} \left[e^{-\rho\tau^* + (Y_{\tau^*} - y)} \right] dy = \frac{1}{q+1} \left(1 - \frac{\Psi_\rho^+(-q)}{\Psi_\rho^+(1)} \right)$$

and

$$\int_0^\infty e^{-qy} \mathbb{E} \left[e^{-\rho\tau^*} \right] dy = \frac{1 - \Psi_\rho^+(-q)}{q}.$$

Thus, simple analytical formulae are only possible for some specific cases. In general, numerical methods have to be used to get the final solution. \square

⁶The Pecherskii-Rogozin identity expresses the double Laplace transform of the joint distribution of the first passage time of the level y , $\tau_y = \inf\{t > 0 \mid Y_t \geq y\}$, and the overshoot above the level $Y_{\tau_y} - y$:

$$\int_0^\infty e^{-qy} \mathbb{E} \left[e^{-\alpha\tau_y - \beta(Y_{\tau_y} - y)} \right] dy = \frac{1}{q - \beta} \left(1 - \frac{\Psi_\alpha^+(-q)}{\Psi_\alpha^+(-\beta)} \right),$$

where $\alpha > 0$, $\beta \geq 0$ and $q > 0$. For a more technical proof the reader is referred to Alili and Kyprianou (2005) [1]

Appendix 4: Proof of Theorem (4.3).

Proof. Denote the critical price of the output P^* . Based on Theorem 4.1 and 4.2, it should be identified as

$$\frac{\theta}{1-\alpha}(P^*)^{1-\alpha} = \frac{U(rI)}{\rho}.$$

Further simplification yields then

$$P^* = \left(\frac{1}{\rho\theta}\right)^{\frac{1}{1-\alpha}} rI = \eta^{-\frac{1}{1-\alpha}} rI,$$

where $\eta = \rho\theta = \mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right]$.

To get the comparative statics result wrt. the risk aversion coefficient, we need to calculate $\frac{\partial P^*}{\partial \alpha}$:

$$\begin{aligned} \frac{\partial P^*(\alpha)}{\partial \alpha} &= \eta^{-\frac{1}{1-\alpha}} \left[\left(-\frac{1}{1-\alpha} \right)' \ln \eta + \left(-\frac{1}{1-\alpha} \right) \frac{\eta'(\alpha)}{\eta} \right] rI \\ &= \eta^{-\frac{1}{1-\alpha}} \left[-\frac{1}{(1-\alpha)^2} \ln \eta + \left(-\frac{1}{1-\alpha} \right) \frac{\eta'(\alpha)}{\eta} \right] rI \\ (13) \quad &= -\frac{1}{1-\alpha} \eta^{-\frac{1}{1-\alpha}} \underbrace{\left[\frac{1}{1-\alpha} \ln \eta + \frac{\eta'(\alpha)}{\eta} \right]}_{:=C} rI, \end{aligned}$$

where $\eta'(\alpha) = \frac{\partial}{\partial \alpha} \mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right] = \mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} (-\underline{Y}_{\tau(\rho)}) \right]$ by using the Fubini's theorem and we define especially the term in bracket as C for convenience. Note that $\underline{Y}_{\tau(\rho)}$ is always negative as $Y_0 = 0$. Therefore, we have always $\eta > 0$ and $\eta' > 0$. Again by applying the Fubini's theorem and assuming that the distribution of $\underline{Y}_{\tau(\rho)}$ is $F(\underline{Y}_{\tau(\rho)})$ in the interval $(-\infty, 0]$, we find first that

$$\begin{aligned} \frac{1}{1-\alpha} \ln \eta &= \frac{1}{1-\alpha} \ln \mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right] \\ &= \frac{1}{1-\alpha} \ln \int_{-\infty}^0 e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} dF(\underline{Y}_{\tau(\rho)}) \\ &= \int_{-\infty}^0 \frac{1}{1-\alpha} \ln e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} dF(\underline{Y}_{\tau(\rho)}) \\ &= \int_{-\infty}^0 \underline{Y}_{\tau(\rho)} dF(\underline{Y}_{\tau(\rho)}) \\ &= \mathbb{E} \left[\underline{Y}_{\tau(\rho)} \right]. \end{aligned}$$

In this way, we have C further reduced as

$$\begin{aligned}
C &= \mathbb{E} [\underline{Y}_{\tau(\rho)}] + \frac{\mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} (-\underline{Y}_{\tau(\rho)}) \right]}{\mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right]} \\
&= \frac{\mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} (-\underline{Y}_{\tau(\rho)}) \right] - \mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right] \mathbb{E} [-\underline{Y}_{\tau(\rho)}]}{\mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right]} \\
&= \frac{Cov[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}, -\underline{Y}_{\tau(\rho)}]}{\mathbb{E} \left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}} \right]}.
\end{aligned}$$

Clearly, the sign of the covariance and hence C is heavily dependent on α . For $\underline{Y}_{\tau(\rho)} \in (-\infty, 0]$ always, one can easily find that C turns out to be negative for $0 < \alpha < 1$ and positive for $\alpha > 1$. Taking it back to (13) gives then the final result that $\frac{\partial P^*}{\partial \alpha}$ is always positive for any value of $\alpha \in [0, 1)$ and $(1, \infty)$.

□