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$$\frac{n!}{(n-1)!} p^{m-1} (1-p)^{n-m} = p \sum_{\ell=0}^{n-1} \frac{\ell+1}{n} \frac{(n-1)!}{(n-1-\ell)! \ell!} p^{\ell} (1-p)^{n-1-\ell} = p \frac{n-1}{n} \sum_{\ell=0}^{n-1} \left[\frac{\ell}{n-1} + \frac{1}{n-1} \right] \frac{(n-1)!}{(n-1-\ell)! \ell!} p^{\ell} (1-p)^{n-1-\ell} = p^2 \frac{n-1}{n} +$$

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Quantifying Endogeneity of Cryptocurrency Markets

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Abstract:

In this paper we construct a "reflexivity" index for Bitcoin crypto currency that measures the amount of activity generated endogenously within the market. For this purpose we fit a univariate self-exciting Hawkes process with two-classes of parametric kernels to high-frequency trade data that allows for a parsimonious representation of endogenous-exogenous dynamics.

JEL: G140, G150, C580

Keywords: Hawkes process, endogeneity, branching ratio, maximum-likelihood estimation, cryptocurrencies, bitcoin

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1. Introduction

Bitcoin, introduced by Nakamoto (2008), is arguably the most interesting financial innovations of this century. Without a central authority the peer-to-peer network issues an asset that is a potential alternative to fiat currencies with all the necessary features such as a store of value or a medium of exchange. Moreover, it offers various advantages such as rather fast and low-cost transactions, particularly when compared to traditional financial institutions. The blockchain technology also makes the recorded transactions public and transparent, as well as its source code. Recently, the cryptocurrency space has spawned thousands of other similar assets, creating a financial market similar to foreign exchanges of fiat currencies. Nevertheless, its size measured in market capitalization of 0.2 trillion USD is still insignificant compared to 20 trillion USD of the S&P 500 stock index.

At the moment, the public discussion is centred mostly around technical or legal points. Indeed, the market capitalisation has grown significantly, yet the market itself is almost entirely unregulated. Lack of centralised regulatory body along with extreme market swings are typical grounds for criticism. On the other hand, it offers a window into possibly the most laissez-faire market there has ever been, and opens the doors to financial research on the market dynamics. On top of that, the exchanges offer unprecedented public access to market data allowing an in-depth analysis and comparisons to the theories describing its traditional counterparts.

One of the most well-known and controversial is the Efficient Market Hypothesis (EMH) (Malkiel and Fama 1970). It states that prices absorb fully and instantaneously all the available information, thus the sole price driver is the exogenous flow of the pertinent news. Recently, a class of self-exciting point processes was recognized as a suitable tool for disentangling and quantifying the underlying dynamics of the price process (Filimonov and Sornette 2012), as one of its inherent features is a neat separation of the endogenous and exogenous component. This started a new stream of literature on the endo-exo problem for financial markets on which this paper is based.

1.1. *Cryptocurrencies in academia*

The Bitcoin and related assets offer several interesting points to inspect them from. Firstly, the generation of new coins is popularly called “mining”. It is the process of verification of transactions in the network. This is similar to when a bank internally verifies that a sender owns enough liquidity, after which it actually transfers the funds. In cryptocurrency networks, the transactions are verified with algorithms that run on specialised hardware, which is purchased by the miner. For that investment along with the consumed electricity, miners are rewarded with the newly minted coins in new transaction blocks. Thus inflation of the currency is controlled and hardcoded in the currency’s protocol.

The place of Bitcoin in the current financial system is analysed by Baur, Hong, and Lee (2018), who conclude that Bitcoin is a speculative asset and does not serve as an alternative currency or medium of exchange. Kristoufek (2015) inspects Bitcoin through wavelet analysis and finds it to be a hybrid asset where money supply and usage in trade influence its price, which is in accordance with standard economic theory. Kristoufek (2013) studies the influence of Bitcoin price in relation to Google searches, where he finds a significant connection. As of this moment, there is no definite answer on the nature of cryptocurrencies and its place within the traditional system.

This paper shall add to the discussion from the perspective of endogeneity, i.e. the portion of dynamics determined within the market.

1.2. *The endo-exo problem in finance*

The discussion on endogeneity and its evolution in the markets (“endo-exo” problem) was opened by Filimonov and Sornette (2012), wherein the authors fit a univariate exponential Hawkes process to E-mini S&P 500 futures traded from 1998 to 2010. They discuss the reflexivity¹ on a micro scale (i.e., ≤ 1 hour) and report a significant increase in the level of endogeneity over the observed period (from 0.3 in 1998 to 0.8 in 2012) that is attributed to the rise of the algorithmic trading. Hardiman, Bercot, and Bouchaud (2013) revisit this problem with the same dataset, however instead of the fast decaying exponential, they opt for a heavy-tail long memory power-law kernel. While agreeing on the rise in the short term reflexivity, overall the paper concludes that in fact the markets were always operating around the criticality level. The claim made, is that the price dynamics are best described by two separate kernels for long and short term memory, thus taking into account the meso (~ 1 day) market structure.

Since then, this discussion was further elaborated upon several times. First, Filimonov et al. (2014) pointed out numerous estimation related issues in the methodology of Hardiman, Bercot, and Bouchaud (2013), such as an upward bias in the presence of outliers for the power-law kernel norm, hence putting HB’s results in question. Finally, to settle this discussion Hardiman and Bouchaud (2014) developed an empirical approximate estimator of the branching ratio that further supported their claim of the market criticality and its long-memory properties.

In our research, we fit a various Hawkes processes to Bitcoin mid-price changes to quantify the degree of endogeneity of price dynamics. Our findings allows us to compare the inner workings of Bitcoin microstructure to more traditional assets. One of our contributions is to further the discussion of Filimonov et al. (2014) with evidence about the estimation biases and the effect of the window length by recognizing different market regimes.

1.3. *Outline*

The paper is structured as follows. Section 2 introduces the Hawkes model along with its branching representation that allows us to separate the endo-exo dynamics. Section 3 follows with the description of the dataset. In Section 4 we presents the key finding and Section 5 concludes.

2. Model

A univariate Hawkes process is a linear self-exciting point process with a conditional intensity function, defined as

$$\lambda(t|\mathcal{H}_t) = \mu(t) + \int_0^t g(t - \tau) dN_\tau = \mu(t) + \sum_{i:\tau_i < t} g(t - \tau_i), \quad (1)$$

¹Market reflexivity is a term coined by Soros (1994) emphasizing the positive feedback mechanism for investors’ anticipations leading to self-fulfilling prophecies.

where $\tau_i \geq 0$ denotes the i -th arrival time. The baseline intensity $\mu(t)$ is a deterministic function of time, while $g(\cdot)$ is a (nonnegatively valued) self-excitation function completely determining the covariance properties of the process, often called the memory kernel.

In the framework of Hawkes processes, endogeneity refers to the ability to generate new arrivals from past events. This notion of endogeneity is formalized using an alternative but equivalent view of the process as a branching structure (Hawkes and Oakes 1974). It allows for a linear mapping between arrivals and clusters, where each cluster starts with an immigrant generated from an inhomogenous Poisson process with baseline intensity $\mu(t)$. As per Eq. (1), every arrival triggers a spike in the intensity through the memory kernel, hence generating its own offsprings following inhomogenous Poisson process with intensity g . This cascades through all the offsprings, effectively creating a branch-resembling structure. Eventually, every event can be labeled either as an immigrant, generated exogenously through the deterministic drift without an existing parent, or an endogenously created offspring. The central parameter that controls the size of endogenously generated offspring families, the “branching ratio” n , is defined as the average number of offspring per event, i.e.,

$$n = \int_0^\infty g(t) dt. \quad (2)$$

Conceptually, a branching ratio determines the degree of self-excitation of the process. The latter presents the percentage of the arrivals generated endogenously within the process as a consequence of previous arrivals. Based on the value of n , we can distinguish four different regimes:

- (1) $n = 0$, corresponding to a memoryless inhomogenous Poisson point process,
- (2) $n < 1$, corresponding to a stationary (sub-critical) process,²
- (3) $n = 1$, corresponding to a nearly unstable (critical) process where a generation of offspring lives indefinitely (the existence of such processes was proven by Brémaud and Massoulié (2001) conditioned on $\mu(t) = 0$),
- (4) $n > 1$, (super-critical) corresponding to a nonstationary-explosive process with finite intensity but infinite/non-integrable covariance. (In other words, a single event starts an infinite family and the process explodes.)

This separation provides very succinct overview of the intertwined market dynamics, similar in interpretation to well-established autoregressive processes. It ultimately allows us to describe the endogeneity with a single value comparable across asset classes or instruments.

When inferring the branching ratio from data one has two main options. Stochastic declustering (Zhuang, Ogata, and Vere-Jones 2002), which essentially entails reverse-engineering the clusters or maximum-likelihood estimation (MLE) and then using the definition from Eq. (2). In the remainder of the paper we apply the latter approach, as in our case it is the more straightforward one.

2.1. Parametric Kernels

The two most prominent classes of (parametrized) self-excitation functions are exponential and power-law.

²As long as the immigrant generation process $\mu(t)$ is bounded.

(1) Exponential kernel is parametrized as

$$g(t) = n \frac{1}{\beta} e^{-\frac{t}{\beta}}, \quad t \geq 0, \quad (3)$$

where n represents the branching ratio and β is the decay parameter. It satisfies the Markov property which makes it particularly popular. Furthermore, as shown by Ogata (1981) it allows for a recursive maximum likelihood function simplifications that reduces the computational complexity from $\mathcal{O}(N_T^2)$ to $\mathcal{O}(N_T)$.

(2) Power-law kernel is given by

$$g(t) = n \frac{\varepsilon \tau_0^\varepsilon}{t^{1+\varepsilon}}, \quad t \geq 0, \quad (4)$$

where n represents the branching ratio and ε sets the decay speed. Compared to the exponential kernel it features a long memory that may be better suited for financial markets.

Remark 1. The power-law kernel used is not in fact a true power-law, rather an approximation made out of a power-law weighted sum of exponential kernels

$$g(t) = \frac{n}{Z} \sum_{i=1}^{M-1} a_i^{(1+\varepsilon)} e^{-\frac{t}{\xi_i}},$$

where $a_i = \tau_0 m^i$. The parameters are the branching ratio n , the tail exponent ε and the timescale τ_0 . This approximation, first used by Hardiman, Bercot, and Bouchaud (2013), captures the long-term dependencies while leveraging recursive formulation of the maximum-likelihood function resulting in faster fitting times. In fact, the formulation allows for an arbitrarily close approximation, as the power-law approximation is valid up to durations of the order of m^{M-1} , above which the kernel drops off exponentially. The parameter Z is chosen such that n equals to the true branching ratio of the kernel, $\int_0^\infty g(t) dt = n$, i.e., $Z = \sum_{i=1}^{M-1} (\tau_0 m^i)^{-\varepsilon}$.

For further discussion on kernel properties and their differences see the comprehensive review by Bacry, Mastromatteo, and Muzy (2015).

2.2. Goodness-of-fit Tests

A standard method for assessing the quality of point process fits is the residual analysis. It involves computing the time-deformed series of durations $\{\xi_i\}_{i \in \mathbb{N}}$, defined as

$$\xi_i = \int_{\tau_{i-1}}^{\tau_i} \hat{\lambda}(s) ds,$$

and statistically testing it for theoretical properties. In case that a Hawkes process is an accurate description of the empirical data, residuals of the inter-arrival times are independently and identically distributed (i.i.d.) draws from the exponential distribu-

tion with parameter $\lambda = 1$. We assess these properties using the following standard set of statistical tests:

- (1) Ljung and Box (1978) test (LB) for the absence of autocorrelations to ensure independence of residuals using up to $h = 20$ lags;
- (2) Kolgomorov-Smirnov test (KS) for the distance between the empirical and the theoretical distribution of the residual process;
- (3) Engle and Russell (1998) test (ED) of excess dispersion in the residuals.

In our setting, a fit that passes all three tests simultaneously is considered successful and to explain the data well.

3. Data

Our data set includes all executed transactions on the BitMEX cryptocurrency exchange during the period March - May 2019. This exchange was selected, as it constitutes a largest crypto exchange in terms of the volume traded, particularly in the Bitcoin (BTC) contracts settled in USD. The trading is open 24 hours a day, so that it closely ressembles traditional foreign exchanges. Each trade is recorded with its appropriate time stamp, volume, price and whether or not the transaction changed the last transaction price (uptick or downtick). The available milliseconds resolution of the data is of the highest available granularity on this market.

Even though our dataset tracks all BitMEX traded instruments including exotics such as Cardano(ADA) or Tron(TRX), we restrict attention solely to Bitcoin contracts (ticker XBTUSD), as it accounts for a vast majority of the traded volume (Fig. 1).³

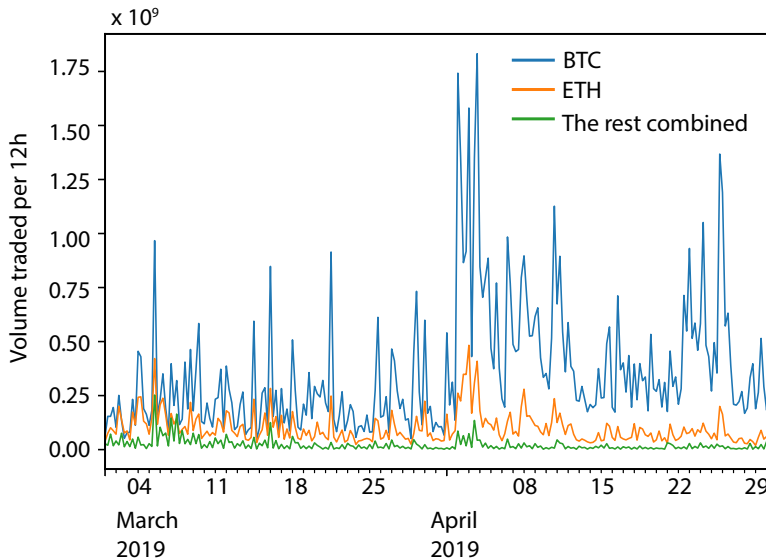


Figure 1. The number of transactions, or rather arrivals in terms of Hawkes process, differ greatly on any given day between selected currencies, where Bitcoin contracts account for almost two thirds of the daily trading activity. Together with the Ethereum market, they account for practically all the trades. The average number of arrivals on the XBTUSD market is almost an order of magnitude higher than for other currencies, with a peak daily activity of around a million recorded trades.

³Full description of all available contracts can be found on website <https://www.bitmex.com/>.

timestamp	ordertype	volume	price	ticktype	arrival
2019-02-02 22:45:58.560	Buy	20	3433.5	PlusTick	0.000
2019-02-02 22:46:03.493	Sell	10	3433.0	MinusTick	4.933
2019-02-02 22:46:06.754	Sell	50	3433.0	ZeroMinusTick	8.194
2019-02-02 22:46:09.639	Sell	4	3433.0	ZeroMinusTick	11.079
2019-02-02 22:46:10.679	Buy	21	3433.5	PlusTick	12.119

Table 1. Description of the market order data. The data come from BitMEX exchange that account approximately for 10% of all volume traded in BTC.

3.1. Measure of the market activity

The quality and informational value of process fits is directly dependent on a definition of the event arrival. We look for a reliable measure of market activity that is the least affected by the microstructure noise omnipresent in high-frequency data. In the finance literature, Hawkes processes were mainly fitted to either the most granular trade data or various price actions near the best price. Although the arrival rate of trades may seem a justifiable metric for market activity, it comes with a drawback as not all trades are equal in their impact due to their volume. Therefore, one would have to consider a marked version of a Hawkes process that is significantly more intricate to fit.⁴

As for the price action, practitioners commonly track four different price quotations each serving a different purpose:

- best bid $b(t)$,
- best ask $a(t)$,
- last transaction price $p_{tr}(t)$,
- midprice $p_m = (a(t) + b(t))/2$.

Best-bid and best-ask constitute a price at which a trader can immediately engage in selling or buying respectively, up to a cumulative volume of standing orders on the given price level; see Fig. 2a. They can be regarded as a proxy for available supply and demand of market makers (liquidity providers). When a buy (resp. sell) market order arrives on an exchange at time t , it is paired with the best ask (resp. bid) price available, completing a trade that forms a last transaction price $p_{tr}(t)$. As trades arrive in a random order with the direction of the trade being a random variable as well, the last transaction price jumps sporadically even without changes in supply and demand. This behavior is called bid-ask bounce and is established as a noise source (Ait-Sahalia and Yu 2008; Black 1986). Therefore, midprice is regarded as a better proxy for asset value, particularly because it does not suffer from the bid-ask bounce (Fig. 2 b)).

A change in midprice can arise due to one of the following:

- (1) Cancellation of an existing limit order at the best bid/ask price;
- (2) Submission of a new limit order at a new best bid/ask price;
- (3) Depleting available volume at the best/ask through market orders.

Even though cases (1) and (2) result from limit orders submitted by liquidity providers (who want to trade), the publicly visible order book does not reflect true

⁴In fact, inclusion of i.i.d. marks independent of the the arrival distribution is a straightforward extension to the process (Chehrizi and Weber 2015). However, these assumptions are most certainly violated for the sequence of trade volumes.

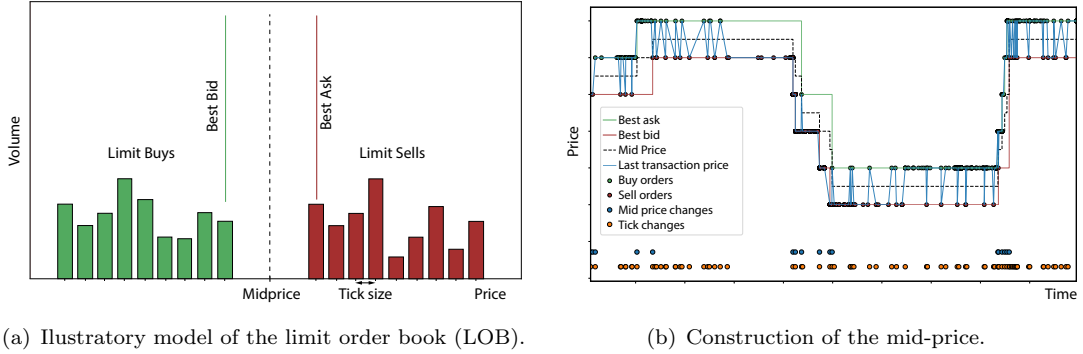


Figure 2. Orderbook and construction of the mid-price from the transactional level data.

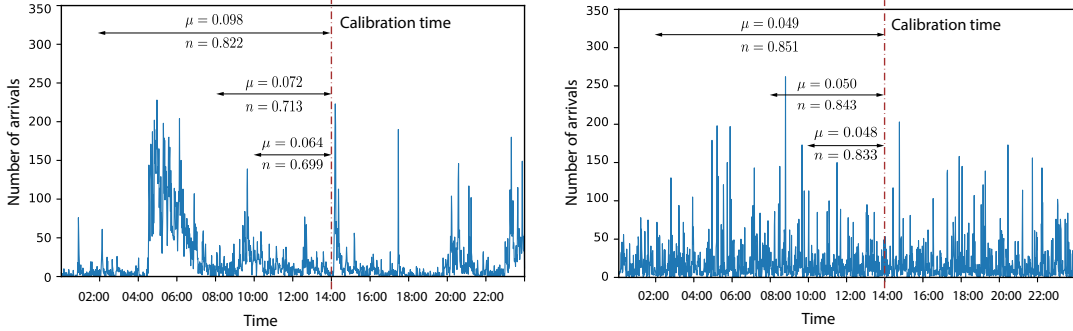
supply and demand of the market. This comes as a consequence of the fact that market participants (particularly large liquidity providers) often do not want to disclose their private information and display their intentions and intended positions. Consequently, in fast markets such as the one for Bitcoin, the majority of cancelled and limit orders represent so-called *ghost* orders (Lewis and Baker 2014) that are used merely to pry out private information (in our dataset cancellations and limit orders account for $\approx 15\%$ of all midprice changes). We argue that only point (3) refers to the actual interactions between supply and demand and as such it is the most reliable source of the signal.

4. Results

In this section we fit a univariate Hawkes process with exponential and power-law kernels to Bitcoin mid-price changes caused by filled orders between 1st March 2019 and 1st May 2019 using maximum-likelihood estimation. MLE is a conventional technique for Hawkes process calibration that attempts to directly solve an optimization program

$$\begin{aligned} \max_{\theta \in \Theta} \quad & \ln \mathcal{L}(\theta | \mathcal{F}_T), \\ \text{subject to} \quad & \theta \geq 0, \end{aligned} \tag{M}$$

where θ represents a vector of kernel and base rate parameters. The likelihood derived by Rubin (1972) is asymptotically normal, efficient and consistent (Ogata 1978). As such, it constitutes a straightforward statistical inference technique for the family of self-exciting point processes. On the downside, its non-convexity in the decay parameter (β , ϵ for exponential, power-law kernels respectively) coupled with extreme flatness of the log-likelihood surface near the optimum (Veen and Schoenberg 2008) makes reliable calibration a challenging task (Mark and Weber 2019). In order to circumvent these problems we solve the optimization program (M) in parallel for a batch of 500 starting guesses and then single out the solution that attains the highest log-likelihood.



(a) Empirical data from 1st April 2019. Total number of arrivals, $N_T = 30,683$.

(b) Data simulated from a Hawkes process with parameters $\mu = 0.05$ and $n = 0.85$. Total number of observations, $N_T = 27,540$.

Figure 3. Impact of a differing estimation horizon $T = (4, 6, 12)$ hours on the estimate’s accuracy. In contrast to simulated data coming from a single point process realization, empirical data contain intra-day seasonalities that significantly affect the maximum likelihood estimates.

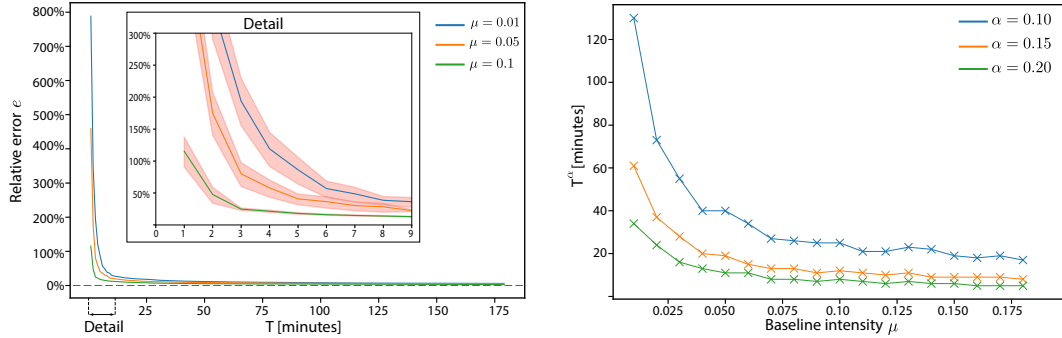
4.1. Optimal estimation horizon

From a practitioner’s standpoint, an important, but in the literature often neglected question is how to determine an optimal estimation horizon T . In the case of a stationary process with constant parameters a greedy approach “*more is better*” would indeed stand true. However, the empirical trade data most likely do not come from a single, long and historically consistent generating process. A more appropriate view is the one of multiple regimes, constantly switching and creating a long history of concatenated processes. This problem, commonly known as the Poisson disorder problem (Peskir and Shiryaev 2002), has been studied in the context of homogenous Poisson process and does not have a straightforward extension to self-exciting processes. Therefore, one has to carefully calibrate the length of the estimation window, such that the history contains a sufficiently large sample for obtaining accurate estimates, while limiting the possibility of calibrating across multiple regimes. Furthermore, an inference from a shorter window justifies the assumption of a constant base rate, even though the empirical flow of mid-price changes clearly contains intra-day seasonalities. On the other hand, short estimation windows limit the memory of the kernel and thus disregard the interdependencies developing over hours, days or longer periods of time. This phenomenon is demonstrated in Fig. 3 where fits of various observation horizon are compared on simulated and empirical data.

In order to resolve this tradeoff, we rely on a robust numerical experiment. As discussed above, we try to identify a minimal window size T such that the amount of observations is sufficient for an “accurate” inference. Suppose a family of exponential Hawkes processes with a moderate branching ratio $n = 0.5$ but variable baseline intensity $\mu \in [0, 0.2]$ representing different market regimes.⁵ We perform an estimation of each process for observation horizon $T \in [60, 10800]$ ⁶ and measure the relative estimation error $e = \frac{\|\hat{\theta} - \theta\|_2}{\|\theta\|_2}$. To ensure robustness of the experiment, we obtain a mean relative error for every individual T using Monte Carlo simulation across 1,000 process realizations. Fig. 4 a) depicts the relationship between calibration window

⁵Markets indeed tend to change in μ rather than in self-excitation as per Wheatley, Wehrli, and Sornette (2019) and evidence from our data.

⁶This range corresponds to a time range 1 minute to 3 hours.



(a) Time horizon - relative error dependence for three representative regimes. Red shaded area represent bootstrapped mean confidence intervals.

(b) The relationship between optimal (minimal) observation horizons and different μ regimes for 3 representative acceptance thresholds.

Figure 4. Monte Carlo analysis of relative error for various baseline regimes. Although the Fig. 3 was constructed using a fixed self-excitation parameter $n = 0.5$, the results indeed serve as a conservative decision tool for the optimal horizon T^α as higher values directly translate into more observations and hence faster reliable calibration. The value of $n = 0.5$ was not chosen arbitrarily, it is the lowest self-excitation measured on non-overlapping 10-min windows using the approximate branching-ratio estimator (Hardiman and Bouchaud 2014).

T and relative error e obtained as a mean across all simulation paths together with bootstrapped 5%, 95% confidence intervals (Efron and Tibshirani 1994).

As expected, higher μ regimes can handle shorter observation horizons without a significant impact on the accuracy. The optimal horizon T^α is selected such that for all $T \geq T^\alpha$, $e_{0.95}(T) < \alpha$ where α is some pre-specified acceptance threshold. In other words, it is the minimal horizon after which the relative error does not surpass α at a 95% confidence level. Finding this threshold for various values of μ gives us Fig. 4b, which might be used as an approximate decision tool for the optimal horizon given some baseline intensity.

Indeed, let us consider a realizations of a self-exciting process on $[0, T]$, formed as a concatenation of k Hawkes processes with different regime parameters (μ_k, n_k) , each lasting T_k so that $\sum_k T_k = T$. We want to estimate this process on a rolling basis using a smallest possible window T to prevent averaging the fit over multiple regimes. We can approximate the mean intensity of the compound process

$$\Lambda = \sum_{i=1}^k \frac{T_k}{T} \Lambda_k \approx \frac{\text{\#events in } T}{T} \quad (5)$$

and using the identity for the average intensity (Hawkes 1971) and setting n to some high level \bar{n} we recover

$$\underline{\mu} = \Lambda(1 - \bar{n}), \quad (6)$$

a lower bound approximation for the baseline mean μ that can be plugged into Fig. 4b, thus yielding a minimal observation horizon.

Remark 2. Even though the above analysis is purely simulation-based and heuristic, it turns out to be a good approximation and constitutes, to the best of our knowledge, a first proposed workaround for the as of yet unsolved "Hawkes disorder" problem. The importance of this problem can be judged from Fig. 8 of historical price and

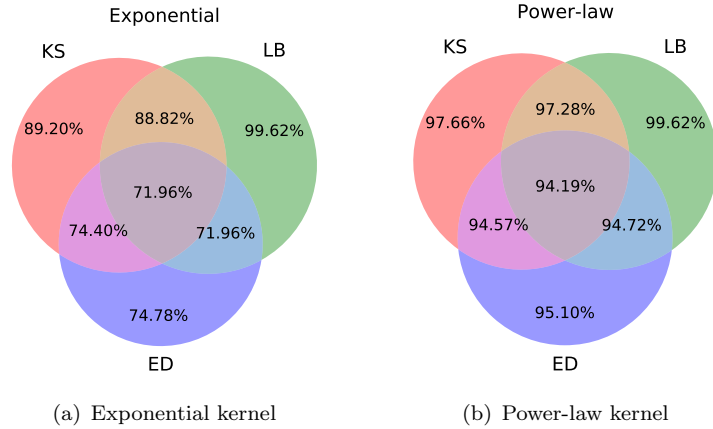


Figure 5. Percentual distribution of significant fits passing the goodness-of-fit criteria.

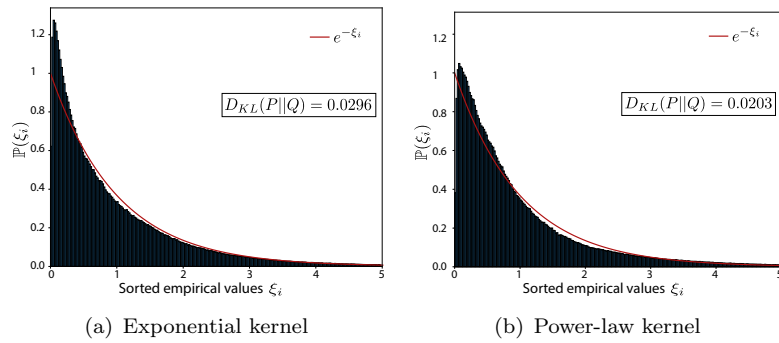


Figure 6. Comparison between theoretical and empirical densities of transformed inter-arrival times ξ_i . The Kullback-Leibler divergence is calculated between the theoretical density $P \sim \text{Exp}(\lambda = 1)$ and empirical density of the residuals Q using the estimator developed in Wang, Kulkarni, and Verdú (2009).

volatility, that clearly features two distinct activity regimes.

4.2. Reflexivity index

Building on the analysis from the previous subsection, we determine the adequate look back period first by measuring the mean intensity, $\Lambda = \frac{416,019 \text{ events}}{5,184,000 \text{ s}} = 0.08$. Next we recover the baseline intensity $\underline{\mu} = \Lambda(1 - \bar{n}) = 0.16$, which roughly corresponds to 60 min look-back period T for 15% total relative error; see Fig. 4b.⁷ For direct comparison with the study conducted by Filimonov and Sornette (2012) on S&P 500 E-mini contracts, we calibrate the process for the two parametric kernels described in Section 2.1 for additional look-back periods of 10 and 30 minutes on a minute-by-minute rolling basis.

To judge the significance of the results, we refer to Tabl. 2 that contains pass rates for the statistical tests together with the Bayes information criterion (BIC) value measured as a mean across all fits.⁸ We observe that with increasing observation horizon the choice of the kernel becomes more consequential, progressively favouring

⁷Value $\bar{n} = 0.8$ was chosen as it is the mean value of branching ratio measured on non-overlapping 10-min windows using the approximate branching ratio estimator (Hardiman and Bouchaud 2014).

⁸Wheatley, Wehrli, and Sornette (2019) confirmed BIC as an effective tool for optimal exogenous-endogenous Hawkes model selection. In their study authors considered estimating Hawkes process with base rate intensity

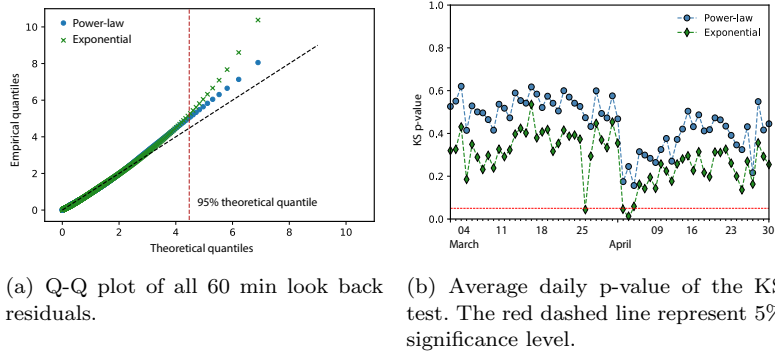


Figure 7. The empirical distribution of transformed arrival times ξ_i .

	Exponential kernel					Power-law kernel				
	KS	LB	ED	KS \cap LB \cap ED	Mean BIC	KS	LB	ED	KS \cap LB \cap ED	Mean BIC
10 min	99.32%	96.32%	98.10%	93.97%	212.71	99.79%	96.34%	99.70%	95.90%	211.12
30 min	96.44%	98.72%	90.79%	88.03%	583.35	99.19%	98.72%	98.81%	97.23%	576.03
60 min	89.19%	99.61%	74.78%	71.96%	1150.49	97.65%	99.61%	95.09%	94.19%	1131.62

Table 2. Amount of fits passing each statistical test on confidence level 5%. The joint pass rates of all tests for 60 min time window are provided in Fig. 5. Windows with less than 50 mid-price changes are excluded.

the power-law variant. Indeed on the 10-min time scale the pass rates are almost indistinguishable, with very mild preference for power-law kernel based on the BIC value. This is somewhat expected as the shorter horizon prevents power-law to leverage its long memory property and thus on this time scale one might justify an exponential kernel as a valid kernel well explaining the data.

The situation dramatically changes for 30 minutes and even more so for 60 minutes, where the power-law proves to be a superior choice. This can be deduced from the simultaneous pass rates for all three tests ($KS \cap LB \cap ED$) and is further demonstrated in Fig. 7 that renders a quantile-quantile plot of the process residuals for 60-min time window together with a KS p -values for the two kernels calculated as a daily mean. As a last piece of evidence we include the empirical distribution of the transformed time series (Fig. 6) with computed Kullback-Leibler divergence. With respect to these result we will restrict our attention to the power-law kernel only as it is consistently the best performing one of the two.

Fig. 8 displays the time evolution of the exogenous and endogenous component of the process, and constitutes the Bitcoin reflexivity index for the period. It was obtained from individual fits pooled and averaged in a single point representing a 4-hour period. We observe that the level of endogeneity oscillates around the value $n = 0.8$ and consistently keeps a significant distance from criticality. This makes Bitcoin rather relatable to traditional FX markets, which exhibit very similar values of endogeneity (Lallouache and Challet 2016; Rambaldi, Pennesi, and Lillo 2015). On the other hand, studies on other asset classes report branching ratios strikingly different. For instance Hardiman, Bercot, and Bouchaud (2013) found that futures on equity indices exhibit near criticality levels of the branching ratio, while Filimonov et al. (2014) concluded that within the commodity futures market only around 60% of mid-price changes are of an endogenous origin. This suggests that from the perspective of market microstructure, Bitcoin is closer to being a currency rather than gold, a commodity it is incessantly compared to.

parametrized with log-splines of various degrees from synthetic data. In all cases BIC successfully manages to select the correct generating process.

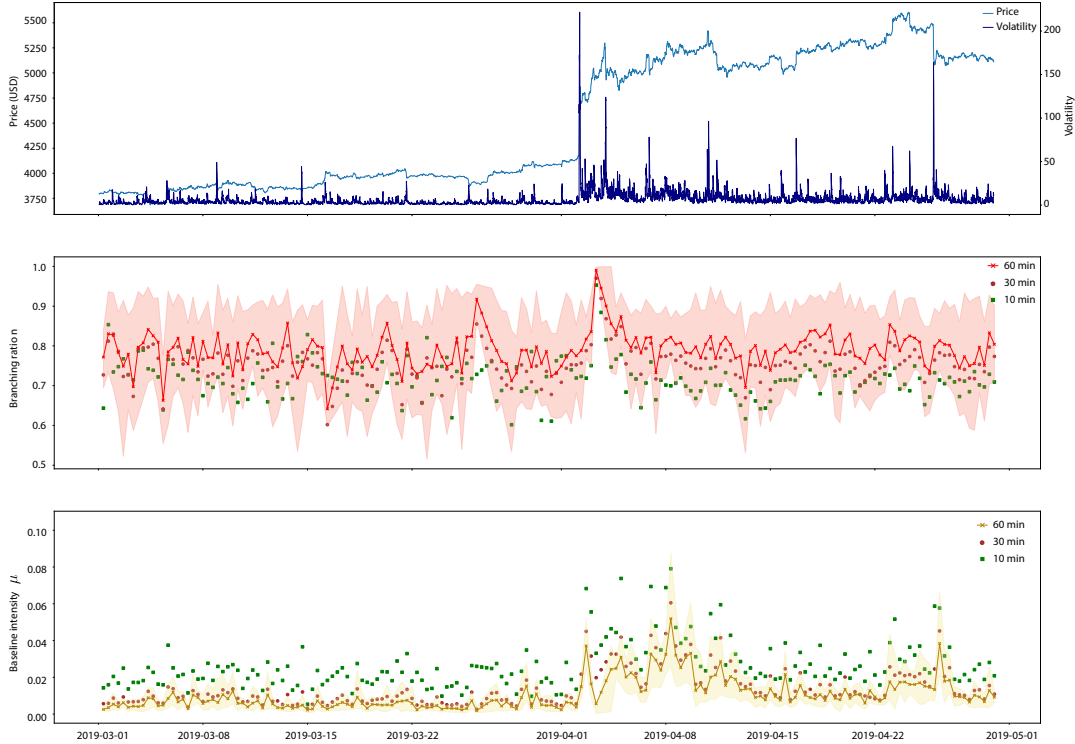


Figure 8. Bitcoin reflexivity index computed as a mean in 4 hour windows. The shaded area represent 5%- and 95%- quantiles.

Furthermore, the measured levels of endogeneity seem to decrease with longer look back periods as opposed to the base rate estimates that exhibit an inverse, increasing relationship. A probable explanation is that with shortening look back period one effectively discards events whose impact was not yet fully dissipated. Thus the estimation erroneously attributes an excessive amount of realized intensity to the exogenous component instead of attributing it to the self-excitation itself. This confirms that even events developed on longer time scales (tens of minutes and longer) play an important role in the microstructure and cannot be omitted by restricting the observation horizon.

Lastly, we point out that based on the daily profile of the estimates $\hat{\mu}$ and \hat{n} (Fig. 9), Bitcoin behaves like a true 24/7 market. Again we bring up the comparison with the FX and equity markets where one clearly observes the lunch lull in the form of the typical U-shape activity that has to be accounted for during the estimation.

5. Conclusion

We have constructed a reflexivity index for the Bitcoin market that indicates that the underlying dynamics for the mid-price changes are approximately 80% determined within the market itself. This value of the branching ratio is comparably lower than for equity indices and higher than for commodities Bitcoin is often compared to. It approximately corresponds to the branching ratios of FX markets for currencies, with which Bitcoin shares certain key traits (Barber et al. 2012; Grinberg 2012). Furthermore, our study highlights the importance of an approximate estimation horizon in

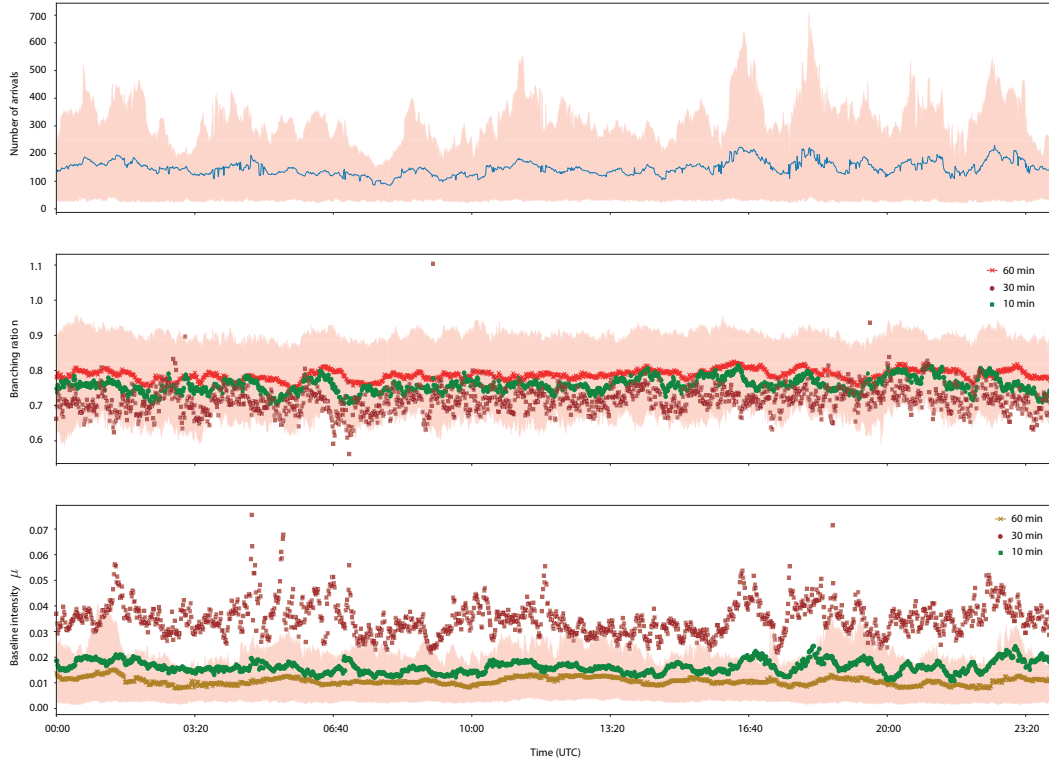


Figure 9. Bitcoin reflexivity; daily profile computed as a mean across all significant fits. The shaded area represent 5%- and 95%-quantiles.

a non-stationary regime and its impact on reliable calibration of Hawkes processes to empirical data. We formulate the Hawkes disorder problem and propose a simple simulation driven workaround technique. Last but not least, our findings suggest that bitcoin mid-price generating process features long-memory properties and therefore inappropriate choice of Exponential kernel produces unreliable results. This can be seen on the time-horizon and exo-endo tradeoff, where shorter windows disregard the developments on larger timescales, thus produce a significant upward bias in the exogenous baseline rate. To conclude, we have demonstrated that power-law Hawkes processes is an exceptionally good model for the mid-price dynamics in Bitcoin markets.

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