

Glawion, Rene; Puche, Marc; Haller, Frédéric

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A GENERAL EQUILIBRIUM MODEL OF EARNINGS, INCOME AND WEALTH HETEROGENEITY

R. M. GLAWION[†], F. E. HALLER[‡], AND M. PUCHE[‡]

ABSTRACT. We develop a general equilibrium model of earnings, income and wealth heterogeneity in continuous time. We extend existing analytical and numerical methods to solve the model. We calibrate the model to U.S. data and find that stochastic interest rates provide a mechanism to link earnings, income and wealth distributions. We use this connection to demonstrate that an increase in unemployment benefits leads to a rise in steady state wealth inequality measured by the Gini coefficient.

JEL Classification: C65, C68, D31, E21

Keywords: Incomplete Markets, Fokker-Planck Equations, Wealth Distributions, Computable General Equilibrium Models

We extend the standard Bewley-Hugget-Aiyagari framework to describe the steady state distributions of earnings, income and wealth simultaneously. We cast our model in continuous time to exploit more powerful mathematical techniques as well as newly developed numerical tools. We provide detailed proofs to explain our numerical methods. Afterwards we calibrate the model to U.S. microdata and show that our model is able to explain the different distributional outcomes of earnings, income, and wealth by assuming a continuous state space of income and heterogeneous interest rates. Finally, we conduct a policy experiment and demonstrate that *ceteris paribus* an increase in unemployment benefits leads to a rise in steady state wealth inequality.

This work combines several results from the theory of heterogeneous agent models which have been established in recent years. First, the core of our model as well as the core of our numerical results build on the work of Achdou et al. (2017). These authors develop a continuous time version of the Aiyagari (1994) model and describe in great detail how to solve it numerically using finite difference methods, namely the upwind scheme. We extend their numerical algorithm to handle multiple continuous state variables. More precisely, we model labor income and interest rates continuously.

The idea to combine these two variables is not new. In a seminal work Benhabib et al. (2011) show that modeling a risk-free interest rates together with idiosyncratic stochastic interest rates lead to a Pareto-tail of the wealth distribution. Khieu and Wälde (2018) adapt this idea and develop a model with both stochastic labor and stochastic interest rates. They achieve a good fit of the wealth distribution. However, the authors assume that interest rates and labor income are two state processes which

[†]UNIVERSITY OF HAMBURG, VON-MELLE-PARK 5, 20146 HAMBURG, GERMANY

[‡]UNIVERSITY OF HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY

E-mail addresses: rene.glawion@uni-hamburg.de, frederic.haller@uni-hamburg.de, marc.puche@uni-hamburg.de.

switch via Poisson-jumps. Further, they only work in partial equilibrium, whereas we work in general equilibrium throughout the whole analysis.

Besides developing new methodology we see our study as an empirical contribution. During the last 25 years a lot of authors have successfully produced models which deliver a good fit of the observed U.S. wealth distribution. For example, Krusell and Smith (1998) use stochastic discount factors and obtain a good fit for the entire wealth distribution as well as for the Gini coefficient. However, they solely model the wealth distribution.

In a comprehensive study Castaneda et al. (2003) also model the earnings process and are able to match both the income and the wealth distribution almost perfectly. Yet, to generate sufficiently thick right tails in the wealth distribution they have to introduce a labor income state where the income level is an order of magnitude larger than it is empirically plausible. This is what Benhabib and Bisin (2018) refer to as “awesome state” or “superstar state”¹.

It therefore seems reasonable to search for other determinants of the wealth distribution, i.e. how it is made up. Quadrini and Rios-Rull (1997) analyze empirical facts of inequality based on the 1992 Survey of Consumer Finances and 1984-85 and 1989-90 Panel Study of Income Dynamics. They find three dimensions of inequality. First, inequality in *earnings* which are related to labor input. Second, *income* which is defined as revenue from all sources before taxes but after transfers. In our model income is composed of earnings plus income generated by wealth, i.e. interest rate payments. Third, *wealth* which is defined as the net worth of households. The authors find that wealth is by far the most concentrated of the three variables, earnings rank second, and income is the most dispersed of the three. The authors contend that “a good theory of inequality should be able to account for the differences among the distributions of earnings, income and wealth [...]”. Yet, economist did not come up with a joint theory of these three measures of inequality. Our study aims to close this gap.

Within our framework we conduct a policy analysis. We find that an increase in unemployment benefits, i.e. the lower bound of earnings, leads to a rise in wealth inequality. At first sight this result seems odd. However, it is inherited from the Bewley precautionary-savings core of our heterogeneous agent model. The unemployment benefits serve as an insurance for agents. Hence, if policy makers rise the lower bound of income agents have less incentive to save and thus the left-tail of the wealth distribution becomes more prominent.

Lastly, we analyze which channels in our model drive the results. We find that we need stochastic labor earnings to generate a left tail and to achieve convergence of the numerical algorithm. Some fraction of agents have to hit the lower bound of earnings to generate sufficient skewness in the wealth distribution. Additionally, for the right tail we need stochastic interest rates. This mechanism puts a small fraction of agents into a “Goldilocks zone”, where they already own a large amount of wealth but also generate

¹For models with superstars see for example Rosen (1981), Garicano and Rossi-Hansberg (2006), Gabaix and Landier (2008), Tervio (2008), and Geerolf (2016).

high returns on investments. Thus, these agents still have a high propensity to save. This result is in line with the findings of Benhabib et al. (2011) and Krusell and Smith (1998).

The rest of the paper proceeds as follows. Section 1 describes our model and the stationary equilibrium. Section 2 provides the numerical algorithms to solve the model as well as the calibration and resulting distributions generated by the model. In section 3 we use the model to analyze how a rise of unemployment benefits deforms the earnings, income, and wealth distribution. Afterwards, in section 4 we show which mechanisms of the model drive the different results. Lastly, section 5 concludes.

1. THE MODEL

For our study we consider a heterogeneous agent model à la Bewley-Hugget-Aiyagari set up in continuous-time following Achdou et al. (2017). The standard Aiyagari-model features stochastic labor endowments which generate some skewness in the distribution of wealth, especially if the labor endowment process is itself skewed and persistent. A large literature indeed studies models in which households face uninsurable idiosyncratic labor income risk (typically referred to as Bewley models). Yet, the standard Bewley models of Aiyagari (1994) and Huggett (1993) produce low Gini coefficients and cannot generate heavy tails, since at high wealth levels, the incentives for precautionary savings taper off and the right tails of the wealth distribution remains this². To model the thick right tail we introduce stochastic returns to the model. This is motivated by the seminal paper of Benhabib et al. (2015) which shows that in Bewley economies with idiosyncratic capital income risk the wealth distribution is a unique ergodic distribution with a fat (Pareto-)tail. Hence, in our model capital income risk induces the necessary mobility across wealth levels to generate a fat-tailed wealth distribution. Another difference to the standard model is our continuous earnings process. This overcomes the aforementioned “superstar” problem.

1.1. Framework. There is a continuum of unit mass of infinite lived households that are heterogeneous in their labor earnings, z , realized interest rate, r , and wealth, a . Each household consists of one individual, and we will speak of households and individuals interchangeably. The state of the economy is the joint distribution of interest rates, wages and wealth. Individuals have standard preferences over utility flows from future consumption c_t discounted at rate $\rho \geq 0$:

$$(1) \quad U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

We assume $u'_c > 0$ and $u''_c < 0$, i.e. the utility function is concave. Throughout the whole analysis the instantaneous utility function is given by:

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \log c_t & \text{for } \gamma = 1, \end{cases}$$

²See Carroll (1997) and Quadrini (2000) for a discussion of these issues.

where $\gamma > 0$ is the coefficient of relative risk aversion.

Each individual earns a wage z_t which is governed by a mean-reverting process

$$(2) \quad dz_t = \varphi(w_t - z_t) dt + \nu z_t dW_t.$$

The parameter φ governs the speed of mean reversion, and w_t corresponds to the mean level of z_t . We let vary the parameter w_t , so that it can be matched with market clearing conditions in equilibrium. For the variance we use the specification of Courtadon (1982). Fischer (2018) argues that this process is well suited to model labor income dynamics³. Hence, ν is a constant which is scaled by the current level of wage. Since w_t has continuous trajectories it cannot take negative values. The wage of the individual is paid as an endowment of the economy's final good.

The standard Bewley-Aiyagari model generates too few agents that hold low levels of wealth, and the concentration of wealth among the richest agents is far too small. In the US-Data 12 percent of the population have below zero wealth, whereas the richest 5 percent of the population hold roughly 53 percent of all the wealth. One of the main purposes of our line of research is to extend the standard macroeconomic framework to allow heterogeneity among agents. Thus, it is important that the heterogeneity in the new framework is quantitatively adequate to match the observed data and wealth distribution. To generate a fatter right tail it is necessary to either make rich agents have a higher propensity to save or to give them higher returns on savings (or both). Krusell and Smith (1998) give them a higher propensity to save by assuming a stochastic discount factor. In contrast, Benhabib et al. (2011) introduces a Merton's portfolio selection problem to enable agents to achieve a higher return on capital, and thus give them a higher savings rate. We follow their approach and introduce two investment possibilities. First, a risk-free bond b_t that is in zero net supply and which return r^f is constant over time and agents. Second, we allow agents to invest their wealth in productive capital k_t . We assume individuals differ in their ability to invest and the realized return follows a mean-reverting process

$$(3) \quad dr_t = \kappa(\theta_t - r_t) dt + \eta\sqrt{r_t} dW_t,$$

where κ governs the speed of mean reversion, and θ_t corresponds to the mean level of r_t . We also let the parameter θ_t vary, so that we can match it with market clearing conditions in equilibrium later. For the variance we use the specification of Cox et al. (1985)⁴.

³The Courtadon process has an inverse Gamma asymptotic distribution. Fischer (2018) shows that its Gini coefficient can be computed directly via

$$Gini(z) = \frac{\Gamma(\alpha - 0.5)}{\Gamma(\alpha)\sqrt{\pi}},$$

where $\alpha = 2\zeta/\nu^2 + 1$. Thus, for parameter combinations which result into an $\alpha < 2$ we obtain Gini coefficients greater than 0.4.

⁴Other specifications for the short-rate process are possible as well. One could also use the Vasicek (1977) specification. In this case η is the constant variance. This model has the advantage that the process r_t can take negative values. Hence, agents are able to realize negative returns on capital. We checked this specification and all algebra is carried out exactly the same as well as our numerical implementation. For other short-rate specifications see Ait-Sahalia (1996) for an excellent overview.

Putting everything together the wealth of an individual follows⁵

$$(4) \quad dk_t + db_t = (b_t r^f + k_t r_t + z_t - c_t) dt + \sigma k_t dW_t.$$

The agents now face an optimal portfolio allocation problem as in Merton (1969). We simplify the problem by writing the budget constraint in terms of wealth or net worth $a_t = b_t + k_t$:

$$(5) \quad da_t = (a_t r^f + k_t (r_t - r^f) + z_t - c_t) dt + \sigma k_t dW_t.$$

Finally, agents also face a borrowing limit

$$(6) \quad a_t \geq -a_{min},$$

where $a_{min} \in \mathbb{R}_+ := [0, \infty)$.

The capital satisfies $k_t \geq 0$ and hence the borrowing constraint $b_t \geq -a_{min}$ can be written as

$$(7) \quad k_t \leq a_t + a_{min}.$$

Individuals maximize (1) subject to (3), (4) and (6), taking as given the evolution of θ_t and w_t for $t \geq 0$.

There are no private insurance markets for the household specific endowment shocks. However, the wealth accumulation process in Equation (4) creates a mechanism used by agents to self-insure themselves against shocks and allows for consumption smoothing as demonstrated in Bewley (1977).

We close the model by following Aiyagari (1994), hence wealth takes the form of productive capital that is used by a representative firm which also hires labor. The total amount of capital supplied in the economy equals the total amount of capital k . In a stationary equilibrium it is given by

$$K = \int_{\Omega} k(a, r, z) p(a, r, z) d(a, r, z),$$

where $\Omega := [-a_{min}, \infty) \times [0, \infty) \times [0, \infty)$.

We further assume capital depreciates at rate δ and there exists a representative firm with a Cobb-Douglas production function $Y = F(K, L) = K^\alpha L^{1-\alpha}$. Since factor markets are competitive, the wage and the risk-free interest rate are given by

$$(8) \quad \begin{aligned} \theta &= \partial_K F(K, 1) - \delta = \alpha K^{\alpha-1} - \delta, \\ w &= \partial_L F(K, 1) = (1 - \alpha) K^\alpha. \end{aligned}$$

The risky asset is now a real asset in the sense that k_t units produce $r_t k_t$ units of physical output and only positive asset positions are possible. One interpretation of the

⁵Note that we could also account for growth in our setting. Let g be a constant growth rate. By detrending all variables by e^{gt} we obtain our setting.

risky asset is that r_t is the return from owning and running a private firm.⁶ But other interpretations are possible as well.

1.2. Stationary Equilibrium. The individuals consumption-saving decision and the evolution of the joint distribution of their income, interest rates, and wealth can be described by a coupled system of partial differential equations. More precisely, a *Hamilton-Jacobi-Bellman* (HJB) equation and a *Fokker-Planck* (FP) equation also known as *Kolmogorov Forward* equation, i.e. what Lasry and Lions (2007) named a *Mean Field Game*.

Theorem 1.1 (Stationary equilibrium of the stochastic $\theta - w$ -model). *Let the drifts*

$$s := z + ar^f + k(r - r^f) - c,$$

$$\psi := \kappa(\theta - r).$$

$$\mu := \varphi(w - z),$$

be given.

The stationary equilibrium of the stochastic $\theta - w$ -model takes the form:

$$(9a) \quad \rho v = \max_{\substack{0 < c \\ 0 \leq k \leq a + a_{min}}} \left\{ u(c) + \psi v_r + s v_a + \mu v_z + \frac{\eta^2 r}{2} v_{rr} + \frac{\sigma^2 k^2}{2} v_{aa} + \frac{\nu^2 z^2}{2} v_{zz} \right\},$$

$$(9b) \quad 0 = \partial_a \left(\partial_a \frac{\sigma^2 k^2}{2} p - s p \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \psi p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu p \right),$$

with

$$(9c) \quad \int_{\Omega} p(a, r, z) d(a, r, z) = 1,$$

and

$$(9d) \quad \begin{aligned} K &= \int_{\Omega} k(a, r, z) p(a, r, z) d(a, r, z), \\ \theta &= \alpha K^{\alpha-1} - \delta, \\ w &= (1 - \alpha) K^{\alpha}. \end{aligned}$$

We call s the savings policy function.

Proof. See Remark A.3 and Remark B.3. Further, since firms are profit maximizing, define the profit function of the firm $\Pi := F(K, L) - wL - \theta K - \delta K$, and the maximization of Π with respect to K, L yields (9d). \square

From the stationary equilibrium described in (9) we obtain the stationary distributions for earnings, z , net wealth, a , as well as for returns on investment, r . However, as

⁶For example, assume that private firms produce using capital and labor using a constant returns to scale production functions $Z_t f(k_t, l_t)$ as in Angeletos (2007), and define

$$r_t k_t = \max_{l_t} \{ Z_t f(k_t, l_t) - w_t l_t - \delta k_t \}.$$

Then the process for r_t inherits the properties of the process for Z_t . Also see Quadrini (2009) and Cagetti and De Nardi (2006) for related models of private firms.

economists we are also interested in the distribution of the total income. The standard procedure in order to obtain it would be to find the stochastic process of the total income and derive a new stationary equilibrium model (9a)-(9d) with the new variables (a, r, ι) , where ι stands precisely for the total income. This approach is cumbersome and in fact unnecessary, since the information regarding the total income is hidden in the former result. We elucidate it in the following corollary.

Corollary 1.2. *Let $\iota := z + ar^f + k(r - r^f)$ denote the income policy function. Then the stationary income distribution is given by*

$$(10) \quad q(\iota) = \int_{-a_{min}}^{\infty} \int_0^{\infty} \frac{p(a, r, z(a, r, \iota))}{1 + (r - r^f)k_z(a, r, z(a, r, \iota))} dr da,$$

where

$$(11) \quad z(a, r, \iota) = (I + ar^f + (r - r^f)k(a, r, \cdot))^{-1}(\iota)$$

and

$$(I + ar^f + (r - r^f)k(a, r, \cdot))^{-1}$$

denotes the inverse function of $z + ar^f + (r - r^f)k(a, r, z)$ with respect to z .

Proof. See Appendix C. □

Theorem 1.1 and Corollary 1.2 provide the stationary behavior of the economy described by our model. Attempting to find a closed-form solution for (9a)-(9d) can become rather involved if not impossible, so that numerical methods are to be used for solving (9). Nevertheless, this is not an impediment at all, as it will be shown in Section 2. With that, we have acquired the necessary tools—consumption and investment policy function, distributions—in order to calibrate our model to show that it can match empirical observations from microeconomic data.

It is important to note that the optimal problem (9a)-(9d) is nonstandard, in the sense that optimality condition of the capital k alters the “usual structure”, see (Lasry and Lions, 2007, Section 1.2). Moreover, there is a nontrivial fact in Corollary 1.2 which is that k may depend on z , since k_z does not need to vanish identically.

2. NUMERICAL ANALYSIS

Now that we have established the mathematical set-up for our model and provided the necessary results to study the economy described by it in Theorem 1.1 and Corollary 1.2, we move on into presenting numerical results. First, we give a compact version of the algorithms we use to obtain the numerical solution of 9 and 10. We lay out the complete description of the algorithms in Appendix E and provide the codes with this paper. Second, we calibrate the model to match the observed distributions of earning, income and wealth and at the end of the section, we compare the results with the empirical data.

2.1. Algorithm. We base our implementation on the methods described in Achdou et al. (2017), but we differ in some steps. Therefore we describe the whole algorithm

here. For the algorithm we require an auxiliary result that we present here. Even though it seems to be well-known, to the authors' knowledge there is no reference where it can be traced back to and for this reason we give a proof of it in Appendix D⁷. Nevertheless, the proof in the appendix not only proves Theorem 2.1, but pushes the boundary a bit further. It is tailor-made for the class of models we discuss here. First, by defining the linear differential operator associated to (9a) we obtain its formal adjoint to be the linear operator in (9b) by including in $\mathcal{D}(A^*)$ the appropriate boundary conditions. Then we see that p is in fact an element of $\ker A^*$. Third, we argue that we can use these results to obtain a numeric solution of the discretized problem. Thus, other researchers can adopt this method when solving continuous time heterogeneous agent models.

Theorem 2.1. *Let A be the linear operator associated to (9a),*

$$Av = \psi v_r + s v_a + \mu v_z + \frac{\eta^2 r}{2} v_{rr} + \frac{\sigma^2 k^2}{2} v_{aa} + \frac{\nu^2 z^2}{2} v_{zz}, \quad v \in \mathcal{D}(A).$$

Then the solution to the stationary FP Equation (9b) is given by the eigenfunction to the eigenvalue 0 of A^ , the adjoint operator of A , that is,*

$$A^*p = 0.$$

Consequently, if \mathcal{A} is a discretization of A , then the discretization \mathcal{P} of p is the eigenvector corresponding to the 0 eigenvalue of the matrix \mathcal{A}^\top .

Proof. See Appendix D. □

This result gives a neat approach to calculate a stationary distribution from the stationary HJB equation analytically. Additionally, it can be used to efficiently compute the stationary distribution of a sparse matrix numerically. These type of matrices naturally appear in an upwind scheme as stated below and every good software package has pre-implemented algorithms to fastly compute the eigenvalues and eigenvectors of those⁸.

Using the notation of Theorem 2.1, note that the sparsity of the matrix \mathcal{A} corresponding to the discretization of A plays an important role, since the size of the matrix increases as N^3 for an N points mesh in each of the variables (a, r, z) . Were the matrix not sparse, there would be no hope of practically do this process numerically in a feasible amount of time. This sparsity property is one of the main advantages of our continuous time framework. In a discrete framework this property disappears and the amount of computations needed to solve a model in three dimensions skyrockets. To overcome this issue Bayer and Luetticke (2018) recently developed a perturbation method to adapt the sparsity property in a discrete time set-up⁹. However, they need to linearize their

⁷The authors would be thankful for a hint to a reference.

⁸For example, we use the `eigs(AT,1,0)` command in Matlab, where AT denotes the transpose matrix of the stationary, discretized Operator of the HJB equation; the option 1 tells Matlab to return one eigenvalue and 0 denotes the initial guess for the eigenvalue invoking Theorem 2.1.

⁹They also discuss the issue of computational resources needed to solve discrete time heterogeneous agent models with aggregate risk and idiosyncratic states much deeper than we do here.

model around a stationary equilibrium to apply their algorithm. Thus, they loose the non-linear dynamics which can be quite rich as we show in Section 3.

Combining the previous results we now have collected all the ingredients to state our numerical algorithm:

Algorithm 1: Solving Model (9)

- 1 Choose a tolerance $\varepsilon > 0$, a maximum value of iterations it_{max} , and initial guesses v^0, θ^0, w^0, K^0 .
 - 2 **for** $i = 1, \dots, it_{max}$ **do**
 - 3 Given $v^{i-1}, K^{i-1}, \theta^{i-1}, w^{i-1}$ solve the HJB Equation (9a) using an implicit upwind scheme until the error is smaller than ε and set the solution to v^i .
 - 4 Use these values to solve the FP Equation (9b) using Theorem 2.1 and normalize the solution according to (9c). Set p^i as the solution of (9b).
 - 5 With the results update K^i, θ^i, w^i using (9d).
 - 6 **if** *the relative error between $(K^{i-1}, \theta^{i-1}, w^{i-1})$ and (K^i, θ^i, w^i) is smaller than ε* **then**
 - 7 **break**
-

If Algorithm 1 converges before it_{max} is reached, we call the last computed update $(v^i, p^i, K^i, \theta^i, w^i)$ a stationary equilibrium of (9). Importantly, we use the supremum norm for the converge between two consecutive iterations in line 3. Hence, (Barles and Souganidis, 1991, Theorem 2.1) guarantees that by refining the mesh, the approximate solution converges to the viscosity solution¹⁰ of the HJB Equation (9a).

The exact steps are feazed in Appendix E. If Algorithm 1 converged we obtain the optimal controls for the consumption and savings policy functions as well as the steady state values for aggregate capital, K , the mean level of wages, w , and interest rates, θ , and the stationary distributions for earnings, interest rates and wealth. Hence, we still need to compute the stationary distribution of overall income. However, invoking Corollary 1.2 this constitutes no obstacle at all. The following code snipped sketches how to recover the income distribution (10):

2.2. Calibration and Results. We continue calibrating the model and show that it can replicate some key facts of the empirically observed distributions. To this end, we use parameters that can be found in the existing literature. Table 1 presents the exact calibration of our structural parameters.

Using Algorithms 1 and 2 we now solve the model and describe some of the outcoming results.

Table 2 summarizes the top facts. Our model is able to replicate all empirically observed Gini indices. In fact, we match the Gini index of earning exactly, since we calibrate the earnings process directly. Nevertheless, our model also replicates the Gini

¹⁰For an introduction to the topic of viscosity solutions of second order equations see Lions (1983) and for an overview on the topic check Lions and Souganidis (1988).

Algorithm 2: Obtaining the income distribution (10)

- 1 Let $(v^i, p^i, K^i, \theta^i, w^i)$ be the solution obtained in Algorithm 1.
- 2 Compute $J := 1 + (r - r^f)k_z(a, r, z)$ and check that $J > 0$.
- 3 Define $\iota := z + ar^f + k(r - r^f)$ and interpolate $(v^i, p^i, K^i, \theta^i, w^i)$ and (a, r, z, ι) .
- 4 Find $z(a, r, \iota)$ as in (11) numerically using the interpolated variables, that is, the following holds

$$\iota = z(a, r, \iota) + ar^f + k(a, r, z(a, r, \iota))(r - r^f).$$

- 5 Compute $q(a, r, \iota) := p(a, r, z(a, r, \iota))$ and redefine

$$J(a, r, \iota) := 1 + (r - r^f)k_z(a, r, z(a, r, \iota)).$$

- 6 Calculate $q(\iota)$ as in (10) by using

$$q(\iota) = \int_{-a_{min}}^{\infty} \int_0^{\infty} \frac{q(a, r, \iota)}{J(a, r, \iota)} dr da.$$

TABLE 1. Calibration of the θ -model.

Parameter	Explanation	Value	Target/Source
ρ	Discount Factor	0.05	Achdou et al. (2017)
γ	Coefficient of relative risk aversion	2	Achdou et al. (2017)
r^f	risk-free rate	0.01	Safe assets' return
σ	Volatility of wealth	0.2	Vol. of Stock ind.
κ	Speed of Mean Reversion Investments	0.72	
η	Volatility of Investment	0.2	Vol. of Stock ind.
ζ	Speed of Mean Reversion Wages	0.2	To match the data
ν	Volatility of Wages	2	To match the data
α	Capital's Share of Output	1/3	
δ	Depreciation rate	0.025	

TABLE 2. Results of the calibrated model. (Source: SCF 2013 Data. Gini coefficient are taken from (Kuhn and Rios-Rull, 2016, Tab. 3)).

	Model	Data
Wealth Top 1%	15.27%	34%
Wealth Top 5%	39.21%	53%
Wealth Top 10%	52.12 %	67%
Wealth Top 20%	72.13%	85%
Wealth Top 30%	85.26%	88%
Wealth Bottom 50%	0.61%	0.7%
Gini Earnings	0.3931	0.39
Gini Income	0.6348	0.58
Gini Wealth	0.8174	0.85

indices of income and wealth as a result of optimal decisions of agents in a stationary general equilibrium, which are matched indirectly.

The model also captures the bottom tail of the wealth distribution almost exactly. It only fails to fully account for the thick, right tail of the wealth distribution. The top 30% share is matched well, but the very top of the distribution generates too few mass.

One could possibly fix this by including jumps into the earnings process and hence, get some “superstars”. For instance Wang (2007) proposes a jump-diffusion process for income and obtains a fat, right tail of the income distribution which propagates through to the wealth distribution.

3. THE EFFECTS OF UNEMPLOYMENT BENEFITS ON WEALTH INEQUALITY

Having available a model which explains the distributions of earnings, income, and wealth we can now focus on the implications of different policy experiments. One of the most important questions of every policy is “Who gains and who loses?”. Standard representative agent models cannot take such considerations into account, since they only feature one agent by construction. Hence, we need a model like ours to evaluate all the distributional impacts of a policy.

We illustrate the mechanisms in our model by analyzing a rise in unemployment benefits. When we think of inequality we always have to answer the question “inequality in what?” simultaneously. First, it is no surprise that rising the lower bound of earnings reduces the Gini coefficient in overall earnings, and hence reduces inequality in earnings. Countryman (1999) documents this effect empirically for Canada in the time between 1975 to 1996 and Koeniger et al. (2007) for eleven OECD countries between 1973 and 1998. Our model also features this mechanism and Figure 1 shows the quantitative results. However, the more interesting questions are the effects on income and wealth inequality.

To analyze these redistributive effects, we first calibrate our model according to Table 1 again. Afterwards, we vary the risk-free rate and the variable of interest, i.e. unemployment benefits. In our model we interpret unemployment benefits as the minimum grid-point of the z -process. Hence, the lowest possible value of z represents an agent who is not working.

Figure 2 displays the results for the income inequality. Our general equilibrium model features highly non-linear effects. Thus, the feedback effect on income is not clear. However, for US values of unemployment benefits of 50% inequality in the income distribution is rising¹¹. But for slightly higher values income inequality plummets in our calibration.

We present the results on wealth inequality in Figure 3. The model implicates that a rise in unemployment benefits leads to an increase in the Gini index of wealth, and hence to an increase in overall wealth inequality. Again, the equilibrium effects are highly non-linear. As in the income distribution there is a window where inequality is decreasing, but this window vanishes with a rising risk-free rate. Further, by comparing Figures 2 and 3 we see that the sharp fall in income inequality comes along with an exploding inequality in wealth.

¹¹(Nickell, 1997, Tab. 4) summarizes “replacement rates,” which show what share of income is replaced by unemployment benefits, and the duration of these benefits for different OECD countries.

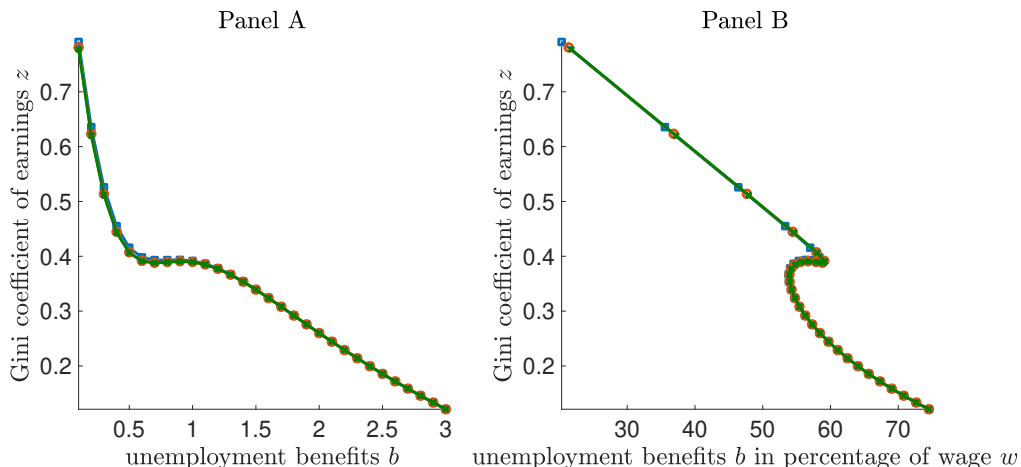


FIGURE 1. **Gini coefficient of earnings for different levels of unemployment benefits**

Panel A shows the Gini coefficient of earnings for different levels of unemployment benefits. The Gini coefficient is decreasing in the level of unemployment benefits in all specifications. Panel B shows the Gini coefficient of earnings for different levels of unemployment benefits in relation to the equilibrium wage for different levels of the risk-free rate r^f . The blue- \square -line refers to a risk-free rate of 1%, the orange- \circ -line to a risk-free rate of 2% and the green- \times -line to a risk-free rate of 3%. Ceteris paribus inequality is falling with an increasing level of unemployment benefits.

The increase in wealth inequality is self-fulfilling prophecy. Within a Bewley-framework one can think of unemployment benefits as an insurance against loosing the own job¹². Hence, if policy makers rise the lower bound of earnings, agents with low incomes will in optimum choose lower savings and thus, in general equilibrium, reduce their overall wealth. With a low level of unemployment benefits in relation to the mean wage, also low incomes have to save more to smooth their consumption intertemporally.

4. CHANNELS OF INEQUALITY

We now shed light on the different drivers of inequality in our model. The model features two main sources of heterogeneity. First, agents receive different labor earnings. Second, agents differ in their ability to invest. In our setting it is straightforward to set each process constant and elaborate the effects on the different distributions.

To begin with, we switch off the optimal portfolio choice problem, by giving each agent the same rate of return to investment, which is equal to the risk-free rate. Hence, the only source of inequality arises from heterogeneity in earnings. This setting is close to the standard Bewley-Aiyagari model. The only difference is that our model still features a continuous state space of possible labor earnings. However, we expect the

¹²In our diffusion process framework one can think of “loosing” as multiple negative realizations of a normal random variable and thus, reaching the lower bound of earnings.

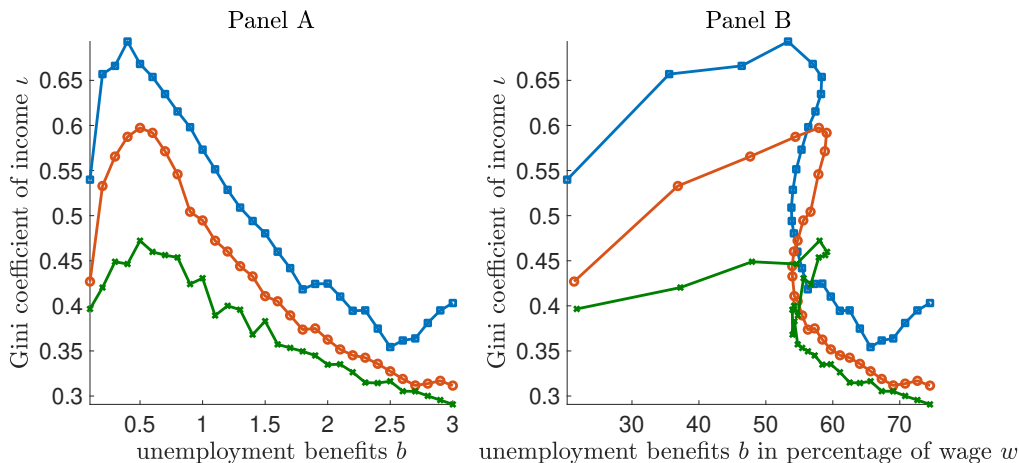


FIGURE 2. **Gini coefficient of income for different levels of unemployment benefits**

Panel A shows the Gini coefficient of income for different levels of unemployment benefits. Panel B shows the same Gini coefficient but displays the unemployment benefits in relation to the equilibrium wage for different levels of the risk-free rate r^f . The blue- \square -line refers to a risk-free rate of 1%, the orange- \circ -line to a risk-free rate of 2% and the green- \times -line to a risk-free rate of 3%. Ceteris paribus for low levels of unemployment benefits inequality in income is rising, but then declines sharply. For unrealistic high values of b inequality in income starts rising again.

standard results of these models, more precisely we loose the flexibility to obtain a thick upper tail of the wealth distribution.

We display the results in column three of Table 3. We are still able to match the Gini coefficient of earnings exactly. The model also matches the lower half of the wealth distribution. But we loose the capability of fitting the very top of the wealth distribution. To this end this is no surprise. As aforementioned in the Bewley-Aiyagari model wealthy agents have a lower propensity to save, since their earnings from income are already huge in relation to their labor earnings. Thus, they have no incentive to save more as insurance against labor market fluctuations.

Next, we switch off the earnings process. This causes some inconveniences. First, setting z constant we do not obtain an earnings distribution by construction. Second, neither $r^f + k_a(r - r^f)$ nor $k_r(r - r^f) + k(r > r^f)$ is strictly positive/negative definite. Thus, we cannot use the techniques of Corollary 1.2 to infer an income distribution. Third, the resulting wealth distribution is degenerated. Nevertheless, we include the results in Table 3 for completeness.

5. CONCLUSION

We propose a new heterogeneous agent model which explains the different shapes of earnings, income and wealth distributions in general equilibrium in interaction with each

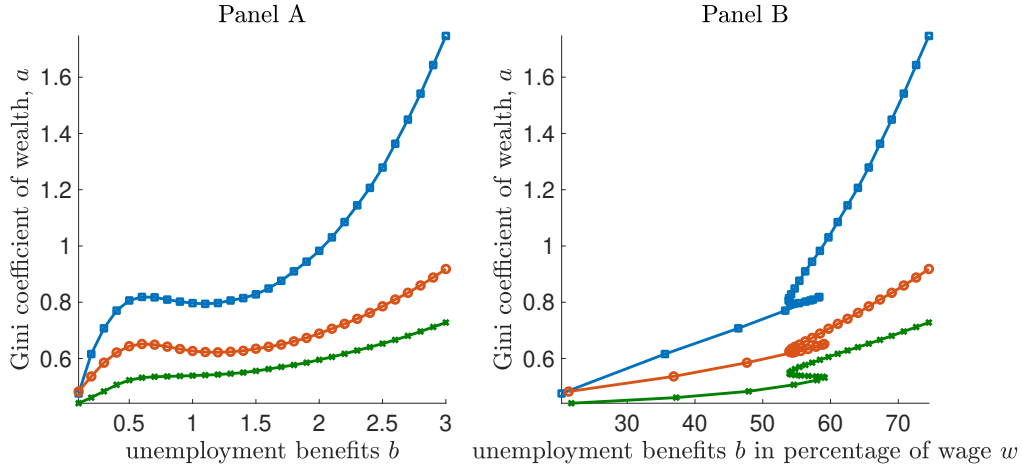


FIGURE 3. **Gini coefficient of wealth for different levels of unemployment benefits**

Panel A shows the Gini coefficient for different levels of unemployment benefits. Panel B shows b in relation to the wage for different levels of the risk-free rate r^f again. The blue- \square -line refers to a risk-free rate of 1%, the orange- \circ -line to a risk-free rate of 2% and the green- \times -line to a risk-free rate of 3%. Ceteris paribus inequality is rising with an increasing level of unemployment benefits. However, there is a small window of unemployment benefits where inequality is decreasing but we observe this effects only for unemployment benefits higher than 55% of the mean wage. This is much higher than US unemployment benefits.

TABLE 3. **Channels of Inequality**

The Table summarizes the channels of inequality. The full model refers to our benchmark model (9). First, we compare the benchmark model to a the model with a constant labor process. Hence, in this specification every agents receives the same labor income. Second, we compare it with a model without the portfolio problem. In this specification all agents generate the same returns to investment. We calibrate all parameters according to Table 1 again.

	Full Model	r constant	z constant	Data
Wealth Top 1%	15.27%	4.8%	385%	34%
Wealth Top 5%	39.21%	16.9%	457%	53%
Wealth Top 10%	52.12 %	29.1%	446%	67%
Wealth Top 20%	72.13%	46.6%	400%	85%
Wealth Top 30%	85.26%	68.44%	400%	88%
Wealth Bottom 50%	0.61%	0.96%	-300%	0.7%
Gini Earnings	0.39	0.39	—	0.39
Gini Income	0.64	0.43	—	0.58
Gini Wealth	0.82	0.63	5.14	0.85

other. We develop the model in continuous time and provide new methods to solve the model. Further, we analyze a policy change in our benchmark model. We demonstrate that an increase of the lower bound of wages reduces the inequality in labor earnings. But the trade-off is higher inequality in wealth. Afterwards we shed light on the different drivers of our model. We argue that we need both inequality in labor earning and returns on investment to give an accurate description of the empirically observed distributions.

Our model can now be used in several ways. On the one hand we now have a model that describes the most important distributions of inequality in an economy. Hence, researchers can use our core to analyze richer models which also include a government sector. This is especially useful to analyze the distributional effects of policies. As shown in Section 3 it is often not clear which distributions are effected by a policy change and in which form. For example, a government can use different tools in form of taxes to redistribute wealth in an economy. But what are the effects of such policies for the earnings and income distributions? And should it use taxes on labor earnings, capital earnings or wealth directly to achieve its goals?

On the other hand our framework can be used for estimation. There is still no established technique for estimating a continuous time heterogeneous agent model. Parra-Alvarez et al. (2017) recently made some progress by using maximum-likelihood estimation. However, they use the standard Aiyagari model of Achdou et al. (2017) as an example for their estimation procedure. Thus, they are not able to recover realistic parameters, because there simply is no parameter combination of the Aiyagari model which explains the observed wealth distribution¹³. Our model can overcome this issue since we showed there is a calibration which fits the observed distributions.

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¹³The authors would like to thank Olaf Posch for pointing this out.

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APPENDIX A. DERIVATION OF THE HJB EQUATIONS

We derive the *Hamilton-Jacobi-Bellman equations* corresponding to the $\theta - w$ -model. To this end, we recapitulate it in concise form as an Itô diffusion. The model is given by the stochastic processes

$$(12) \quad \begin{aligned} da_t &= \left(r_t^f a_t + k_t(a_t, r_t, z_t)(r_t - r_t^f) + z_t - c_t(a_t, r_t, z_t) \right) dt + \sigma k_t dW_t^1, \\ dr_t &= \psi(t, r_t) dt + \eta \sqrt{r_t} dW_t^2, \\ dz_t &= \mu(t, z_t) dt + \nu z_t dW_t^3, \end{aligned}$$

where $\sigma, \eta, \nu > 0$, $r^f \in C(\mathbb{R}_+, \mathbb{R})$, $\psi, \mu \in C(\mathbb{R}_+^2, \mathbb{R})$ and (W_t^1, W_t^2, W_t^3) denotes a 3-dimensional Brownian motion. Here, $(c, k) : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}_+^2$ denotes our *Markov-control*. Further, we wish to extend this system by the additional constraints

$$(13) \quad 0 \leq k_t(a_t, r_t, z_t) \leq a_t + a_{min}, \quad 0 < c_t(a_t, r_t, z_t)$$

for some fixed $a_{min} \in \mathbb{R}_+$. Accordingly, we define

$$D_a := \{ (c, k) \in \mathbb{R}_+^2 \mid 0 \leq k \leq a + a_{min}, 0 < c \}.$$

The set of such solutions is denoted by

$$\mathcal{S} := \{(a, r, z, c, k) \text{ as described above satisfies (12) and (13)}\}.$$

and

$$\mathcal{S}_{t_0, a_0, r_0, z_0} := \{(a, r, z, c, k) \in \mathcal{S} \mid (a_{t_0}, r_{t_0}, z_{t_0}) = (a_0, r_0, z_0)\}.$$

For the formulation of the optimal control problem, assume that

$$(H1) \quad \forall (a, r, z, c, k) \in \mathcal{S}, \tau \in \mathbb{R}_+ : \quad E_\tau \int_\tau^\infty e^{-\int_\tau^t \rho(\xi) d\xi} |u(t, c_t(a_t, r_t, z_t))| dt < \infty,$$

for some *instantaneous utility function* $u \in C^1(\mathbb{R}_+ \times (0, \infty), \mathbb{R})$ and *time preference rate* $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$. Our optimal control problem is now described by the *optimal cost function* which is defined by

$$(14) \quad v(t_0, a_0, r_0, z_0) := \sup_{(a, r, z, c, k) \in \mathcal{S}_{t_0, a_0, r_0, z_0}} E_{t_0} \int_{t_0}^\infty e^{-\int_{t_0}^t \rho(\xi) d\xi} u(t, c_t(a_t, r_t, z_t)) dt,$$

for $(t_0, a_0, r_0, z_0) \in \mathbb{R}_+ \times \Omega$.

For the treatment of this classic and well-known optimal control problem, we follow the rigorous approach of Øksendal (2014) in order to derive the Hamilton-Jacobi-Bellman equation.

Theorem A.1 (The Hamilton-Jacobi-Bellman equation).

Suppose that $V \in C^2(\mathbb{R}_+ \times \Omega, \mathbb{R})$ ¹⁴. Then

$$(15) \quad \begin{aligned} \rho(\tau)v(\tau, a, r, z) = & \sup_{(c,k) \in D_a} \left\{ u(\tau, c) + v_\tau(\tau, a, r, z) \right. \\ & + (ar_\tau^f + k(r - r_\tau^f) + z - c)v_a(\tau, a, r, z) + \psi(\tau, r)v_r(\tau, a, r, z) + \mu(\tau, z)v_z(\tau, a, r, z) \\ & \left. + \frac{\sigma^2 k^2}{2} v_{aa}(\tau, a, r, z) + \frac{\eta^2 r}{2} v_{rr}(\tau, a, r, z) + \frac{\nu^2 z^2}{2} v_{zz}(\tau, a, r, z) \right\}, \end{aligned}$$

for all $(\tau, a, r, z) \in \mathbb{R}_+ \times \Omega$.

Proof. In order to derive the Hamilton-Jacobi-Bellman equation (15), we aim to use (Øksendal, 2014, Theorem 11.2.1). To this end, we first present an optimization problem fitting this framework. This will yield a Hamilton-Jacobi-Bellman equation from which we will derive the statement of this theorem.

Consider the following system of equations

$$(16) \quad \begin{aligned} da_t &= \left((a_t + a_{min})k'_t(a_t, r_t, z_t)(r_t - r_t^f) - c_t(a_t, r_t, z_t) + r_t^f a_t + z_t \right) dt \\ &\quad + \sigma k'_t(a_t, r_t, z_t) \cdot (a_t + a_{min}) dW_t^1, \\ dr_t &= \psi(t, r_t) dt + \eta \sqrt{r_t} dW_t^2, \\ dz_t &= \mu(t, z_t) dt + \nu z_t dW_t^3, \end{aligned}$$

with $(c, k') : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}_+ \times [0, 1]$ subject to the constraint

$$(17) \quad 0 \leq k'_t(a_t, r_t, z_t) \leq 1, \quad 0 \leq a_t + a_{min},$$

for all $t \in \mathbb{R}_+$ for which the set of solutions $S'_{\tau, a, r, z}$ is defined as for the original system of equations (12). Here we define the optimal value function v' at (s, a_0, r_0, z_0) as

$$\sup_{\substack{(a, r, z, c, k) \in \\ S'_{\tau, a_0, r_0, z_0}}} E_\tau \int_\tau^\infty e^{-\int_0^t \rho(\xi) d\xi} u(t, c_t(a_t, r_t, z_t)) dt.$$

I.e., we overall conducted the change of variables

$$\begin{aligned} k_t(a, r, z) &= k'_t(a, r, z) \cdot (a + a_{min}), \\ v(\tau, a, r, z) &= e^{\int_0^\tau \rho(\xi) d\xi} \cdot v'(\tau, a, r, z). \end{aligned}$$

Note that if $a + a_{min} = 0$, then (13) and (17) guarantee that this assignment is indeed one-to-one.

¹⁴Indices on v represent partial derivatives and not time-dependency as with other parameters.

Applying (Øksendal, 2014, Theorem 11.2.1)¹⁵ we obtain

0 =

$$\sup_{(c,k') \in \mathbb{R}_+ \times [0,1]} \left\{ \frac{1}{2} \left[\eta^2 r v'_{rr}(\tau, a, r, z) + \nu^2 z^2 v'_{zz}(\tau, a, r, z) + \sigma^2 k'^2 (a + a_{min})^2 v'_{aa}(\tau, a, r, z) \right] \right. \\ \left. + e^{-\int_0^\tau \rho(\xi) d\xi} u(\tau, c) + \left[(a + a_{min}) k'(r - r_\tau^f) - c + r_\tau^f a + z \right] v'_a(\tau, a, r, z) \right. \\ \left. + v'_r(\tau, a, r, z) + \psi(\tau, r) v'_r(\tau, a, r, z) + \mu(\tau, z) v'_z(\tau, a, r, z) \right\}$$

Together with $v'_\tau(\tau, a, r, z) = e^{-\int_0^\tau \rho(\xi) d\xi} v_\tau(\tau, a, r, z) - \rho(\tau) e^{-\int_0^\tau \rho(\xi) d\xi} v(\tau, a, r, z)$ and undoing the change of variables yields the first statement of the theorem. \square

Proposition A.2. *In the context of Theorem A.1, assume that the supremum in (15) is attained for some $(\tau, a, r, z) \in \mathbb{R}_+ \times \Omega$ at $(c^*, k^*) \in D_a$. If (c^*, k^*) lies in the interior of D_a , $\overset{\circ}{D}_a$, then*

$$0 = u_c(\tau, c^*) - v_a(\tau, a, r, z), \\ 0 = \sigma^2 k^* v_{aa}(\tau, a, r, z) + (r - r_\tau^f) v_a(\tau, a, r, z).$$

More generally, if $u_c(\tau, \cdot)$ is invertible, $v_{aa}(\tau, a, r, z) \neq 0$, $v_a(\tau, a, r, z) \lim_{c \rightarrow 0} u(\tau, c) = -\infty$ and $\lim_{c \rightarrow \infty} \frac{u(\tau, c)}{c} = 0$, then

$$c^* = u_c(\tau, \cdot)^{-1}(v_a(\tau, a, r, z)), \\ k^* \in \arg \max_{k \in \Lambda} \left\{ \frac{1}{2} \sigma^2 k^2 v_{aa}(\tau, a, r, z) + k(r - r_\tau^f) v_a(\tau, a, r, z) \right\}, \\ \Lambda := \left\{ 0, -\frac{(r - r_\tau^f) v_a(\tau, a, r, z)}{\sigma^2 v_{aa}(\tau, a, r, z)}, a + a_{min} \right\} \cap [0, a + a_{min}].$$

Proof. For the given $(\tau, a, r, z) \in \mathbb{R}_+ \times \Omega$, define

$$A(c) = u(\tau, c) - c v_a(\tau, a, r, z), \\ B(k) = \frac{1}{2} \sigma^2 k^2 v_{aa}(\tau, a, r, z) + k(r - r_\tau^f) v_a(\tau, a, r, z), \\ R = v_\tau(\tau, a, r, z) + z v_a(\tau, a, r, z) + \psi(\tau, r) v_r(\tau, a, r, z) + \mu(\tau, z) v_z(\tau, a, r, z) \\ + \frac{1}{2} \left[\eta^2 r v'_{rr}(\tau, a, r, z) + \nu^2 z^2 v'_{zz}(\tau, a, r, z) \right],$$

for $(c, k) \in D_a$. Then (15) reads

$$\rho(\tau) v(\tau, a, r, z) = \sup_{(c,k) \in D_a} A(c) + B(k) + R \\ = \max_{(c,k) \in \overset{\circ}{D}_a} A(c) + B(k) + R \\ = \max_{(c,k) \in \overset{\circ}{D}_a} A(c) + \max_{(c,k) \in \overset{\circ}{D}_a} B(k) + R.$$

¹⁵(Øksendal, 2014, Theorem 11.2.1) makes more technical assumptions than displayed in our Theorem. These are of course necessary and implicitly assumed, but omitted for better exposure.

Since c^*, k^* are maxima of respectively A and B , we know by the first order condition that

$$\begin{aligned} 0 &= A_c(c^*) = u_c(s, c^*) - v_a(\tau, a, r, z), \\ 0 &= B_k(k^*) = \sigma^2 k^* v_{aa}(\tau, a, r, z) + (r - r_\tau^f) v_a(\tau, a, r, z). \end{aligned}$$

For the second statement of the theorem, note that these conditions read

$$\begin{aligned} c^* &= u_c(\tau, \cdot)^{-1}(v_a(\tau, a, r, z)), \\ k^* &= -\frac{(r - r_\tau^f) v_a(\tau, a, r, z)}{\sigma^2 v_{aa}(\tau, a, r, z)}. \end{aligned}$$

Since $\lim_{c \rightarrow \infty} \frac{u(\tau, c)}{c} = 0$ and $v_a(\tau, a, r, z) > 0$, we derive $\lim_{c \rightarrow \infty} u(\tau, c) - cv_a(\tau, a, r, z) = -\infty$. Together with $\lim_{c \rightarrow 0} u(\tau, c) = -\infty$, we ensure that the supremum of $A(c)$ is indeed a maximum in c^* . Note that if the supremum of $B(k)$ is not achieved in k^* , it is in either 0 or $a + a_{min}$. \square

Remark A.3. Sometimes stationary systems arise, i.e., $\rho, u, r^f, u, \gamma, \mu$ in (12) are time-independent functions. An example for this situation is (9a). This additional structure can be exploited in order to simplify the characterisation of the value function. In the following we discuss how the lack of time-dependency can be reflected in the solution of the Hamilton-Jacobi-Bellman equation (15).

Since the system described by (12) is autonomous and we consider an infinite time-horizon, there is a trivial one-to-one correspondence between the solution sets $\mathcal{S}_{t_0, a_0, r_0, z_0}$ and $\mathcal{S}_{t_1, a_0, r_0, z_0}$ for every $t_0, t_1 \in \mathbb{R}_+$, namely by means of the time shift $t \mapsto t - t_0 + t_1$. This mapping being a homoeomorphism of derivative equal to 1, the transformation formula directly yields

$$\begin{aligned} v(t_0, a_0, r_0, z_0) &= \sup_{(a, r, z, c, k) \in \mathcal{S}_{t_0, a_0, r_0, z_0}} E_{t_0} \int_{t_0}^{\infty} e^{\rho t} u(c_t(a_t, r_t, z_t)) dt \\ &= \sup_{(a, r, z, c, k) \in \mathcal{S}_{t_1, a_0, r_0, z_0}} E_{t_1} \int_{t_1}^{\infty} e^{\rho t} u(c_t(a_t, r_t, z_t)) dt \\ &= v(t_1, a_0, r_0, z_0), \end{aligned}$$

for all $(t_0, a_0, r_0, z_0), (t_1, a_0, r_0, z_0) \in \mathbb{R}_+ \times \Omega$. In the context of Theorem A.1, this translates to $v_t = 0$ and we may rewrite (15) as

$$\begin{aligned} (18) \quad \rho v(r, a, z) &= \\ &\sup_{(c, k) \in D_a} \left\{ u(c) + \gamma(r) v_r(r, a, z) + \mu(z) v_z(a, r, z) + (ar^f + k(r - r^f) + z - c) v_a(a, r, z) \right. \\ &\quad \left. + \frac{1}{2} \left[\eta(r)^2 v_{rr}(a, r, z) + \nu(z)^2 v_{zz}(a, r, z) + \sigma^2 k^2 v_{aa}(a, r, z) \right] \right\}. \end{aligned}$$

We also indicate that similar simplifications can be performed for the results of Proposition A.2. In this light we also see that the optimal Markov-control may be chosen time-invariant, even if we did not explicitly require it in the formulation of the time-invariant optimal control problem.

APPENDIX B. DERIVATION OF THE FP EQUATION

Since c and k are time-dependent, the whole problem becomes non-autonomous, and the usual derivation of the FP equation by means of the Dynkin's formula is not valid, see Knäble (2012) for a case in which the derivation is done for a particular class of non-autonomous stochastic processes. For the general derivation of the FP equation with non-autonomous coefficients and transition probabilities we refer to (Öttinger, 1996, Section 3), (Arnold, 1974, Section 5), (Kallianpur, 1980, Section 9) and (Stroock and Varadhan, 1997, Section 5). This requires of the notion of evolution operators, in particular two parameter semigroups.

Since we are dealing with $\Omega \neq \mathbb{R}^3$, for $p \in W_{\text{loc}}^{1,1}(0, \infty; W^{2,2}(\Omega))$ ¹⁶ it is very convenient to define the current $j = (j^a, j^r, j^z)$ by

$$\begin{aligned} j^a(t, a, r, z) &:= \partial_a \frac{\sigma^2 k(t, a, r, z)^2}{2} p(t, a, r, z) - s(t, a, r, z) p(t, a, r, z), \\ j^r(t, a, r, z) &:= \partial_r \frac{\eta^2 r}{2} p(t, a, r, z) - \alpha(t, r) p(t, a, r, z), \\ j^z(t, a, r, z) &:= \partial_z \frac{\nu^2 z^2}{2} p(t, a, r, z) - \mu(t, z) p(t, a, r, z), \end{aligned}$$

which will play a role in the proof of the next result.

Theorem B.1. *Assume that the stochastic differential equations (2), (3) and (5) have a unique solution¹⁷. Let $p_0 \in L^1(\Omega) \cap L^2(\Omega)$ be given with $p_0 \geq 0$ and*

$$\int_{\Omega} p_0(a, r, z) \, d(a, r, z) = 1.$$

Then the PDF p associate to (2), (3) and (5) satisfies

$$(19) \quad \begin{aligned} \partial_t p &= \partial_a \left(\partial_a \frac{\sigma^2 k^2}{2} p - sp \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \alpha p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu p \right), \quad t > 0, \\ p(0) &= p_0, \end{aligned}$$

with

$$\int_{\Omega} p \, d(a, r, z) = 1, \quad \forall t \geq 0.$$

Proof. Let us define the family of time-dependent second order differential operators $A^*(t)$ by

$$A^*(t)p := \partial_a \left(\partial_a \frac{\sigma^2 k(t)^2}{2} p - s(tp) \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \alpha(t)p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu(t)p \right),$$

where we have omitted the dependence on (a, r, z) for clarity.

Denote by $p_{t_0}(a, r, z; t, a', r', z')$ the PDF associate to the transition probability function from t_0 to t of (2), (3) and (5). From (Kallianpur, 1980, Theorem 9.4.4) and

¹⁶For the definition of the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ see (Adams, 1975, Chapters II–IV). For the Bochner spaces $W^{k,p}(0, T; B)$, where B is a Banach space, see (Adams, 1975, Chapter VII) and (Evans, 2010, Section 5.9.2).

¹⁷For the process r_t this is well-known, see Cox et al. (1985). Otherwise, it is enough to assume Lipschitz continuity on the drifts and diffusion terms as well as affine-linear growth, both uniformly in time, see (Øksendal, 2014, Theorem 5.2.1).

subsequent remarks we have that $p_{t_0}(a, r, z; t, a', r', z')$ satisfies

$$(20) \quad \partial_t p_{t_0}(t, a, r, z; a', r', z') = A^*(t, a, r, z) p_{t_0}(t, a, r, z; a', r', z')$$

with

$$p_{t_0}(t_0, a, r, z; a', r', z') = \delta(a - a') \delta(r - r') \delta(z - z').$$

Let $p(t, a, r, z) := p_{t_0}(t, a, r, z; \cdot, \cdot, \cdot) * p_0$. Then $p(t_0, a, r, z) = p_0(a, r, z)$. Moreover, by using the properties of the convolution, if we convolute p_0 with (20) we obtain (19).

In order to show that the mass is conserved, that is, for all $t \geq 0$ it holds that

$$\int_{\Omega} p \, d(a, r, z) = 1.$$

Note that the FP equation can be regarded as a continuity equation which expresses the conservation of the probability when we introduce the probability current j , see (Öttinger, 1996, Section 3.3.5). Since the processes a_t, r_t, z_t are confined in the region Ω , there is no flow probability across the boundary of Ω , that is,

$$\begin{aligned} j_a(t, -a_{min}, r, z) &= 0, & \forall (t, r, z) &\in (0, \infty) \times (0, \infty) \times (0, \infty) \\ j_r(t, a, 0, z) &= 0, & \forall (t, a, z) &\in (0, \infty) \times (-a_{min}, \infty) \times (0, \infty) \\ j_z(t, a, r, 0) &= 0, & \forall (t, a, r) &\in (0, \infty) \times (-a_{min}, \infty) \times (0, \infty). \end{aligned}$$

We can integrate the FP equation in Ω to obtain

$$\int_{\Omega} \partial_t p \, d(a, r, z) = \int_{\Omega} A^*(t) p \, d(a, r, z).$$

By using the Stokes formula —integration by parts in higher dimension— we have that for $p \in \mathcal{D}(A^*(t))$ the following holds

$$\begin{aligned} \int_{\Omega} A^*(t) p \, d(a, r, z) &= \int_{\Omega} \nabla \cdot j \, d(a, r, z) \\ &= \int_0^{\infty} \int_0^{\infty} j_a(t, -a_{min}, r, z) \, dr \, dz \\ &\quad + \int_0^{\infty} \int_{-a_{min}}^{\infty} j_r(t, a, 0, z) \, da \, dz \\ &\quad + \int_0^{\infty} \int_{-a_{min}}^{\infty} j_z(t, a, r, 0) \, da \, dr \\ &= 0. \end{aligned}$$

Hence,

$$\int_{\Omega} \partial_t p \, d(a, r, z) = 0.$$

If we have enough regularity, integration and differentiation commute so that

$$\partial_t \int_{\Omega} p \, d(a, r, z) = 0.$$

Integrating from 0 to t leads

$$\int_{\Omega} p \, d(a, r, z) - \int_{\Omega} p_0 \, d(a, r, z) = 0.$$

Since

$$\int_{\Omega} p_0 \, d(a, r, z) = 1,$$

we have that

$$\int_{\Omega} p \, d(a, r, z) = 1, \quad \forall t \geq 0.$$

□

Remark B.2. Note that the FP equation can be written as an abstract, non-autonomous Cauchy problem of the form

$$\begin{aligned} \dot{p}(t) &= A^*(t)p(t), \quad t > 0, \\ p(0) &= p_0, \end{aligned}$$

with $A^*(t) : \mathcal{D}(A^*(t)) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ defined by

$$A^*(t)p := \partial_a \left(\partial_a \frac{\sigma^2 k(t)^2}{2} p - s(t)p \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \alpha(t)p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu(t)p \right),$$

and

$$\mathcal{D}(A^*(t)) = \{p \in W^{2,2}(\Omega) \mid j_a|_{a=-a_{min}} = j_r|_{r=0} = j_z|_{z=0} = 0\}.$$

The notion of vanishing currents at the boundary of Ω is often called *reflecting boundary*.

Remark B.3. The stationary Fokker-Planck equation arises under the assumption that the coefficients become time-independent and there is no time change, so that the time derivative vanishes, that is,

$$\begin{aligned} 0 &= \partial_a \left(\partial_a \frac{\sigma^2 k^2}{2} p - sp \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \alpha p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu p \right), \\ 1 &= \int_{\Omega} p \, d(a, r, z). \end{aligned}$$

APPENDIX C. INCOME DISTRIBUTION

Here we give a proof of Corollary 1.2. First observe that $z + ar^f + k(r - r^f)$ is the the income ι . The ratio of this quantity per earning is then given by

$$J := \partial_z \iota = 1 + (r - r^f)k_z.$$

From an economic viewpoint it is clear that the quantity J is in most of the cases larger than 1, since the more one earns, the more one invests. However, there might be a region for which the capital is so large, that it does not pay off to invest more in the risky asset k and J is then slightly smaller than 1. Hence, we can assume that $J > 0$ and by the Inverse Function Theorem $z + ar^f + k(a, r, z)(r - r^f)$ is globally locally invertible with respect to z and we denote the inverse by $(I + ar^f + k(a, r, \cdot)(r - r^f))^{-1}$. We now define the transformation $\Psi : \Sigma \rightarrow \Omega$ by

$$\Psi(a, r, \iota) := \begin{pmatrix} a \\ r \\ (I + ar^f + k(a, r, \cdot)(r - r^f))^{-1}(\iota) \end{pmatrix}.$$

Note that Ψ is invertible and the inverse is precisely

$$\Psi^{-1}(a, r, z) = \begin{pmatrix} a \\ r \\ z + ar^f + k(r - r^f) \end{pmatrix}.$$

Hence, we can compute the distribution as a function of (a, r, ι) by using the change of variables formula

$$q(a, r, \iota) = p(\Psi(a, r, \iota)) |\det(D\Psi(a, r, \iota))|.$$

Set $z(a, r, \iota) := \Psi_3(a, r, \iota)$. From the Inverse Function Theorem we also have that

$$D\Psi(a, r, \iota) = ([D\Psi^{-1}](\Psi(a, r, \iota)))^{-1} = ([D\Psi^{-1}](a, r, z(a, r, \iota)))^{-1}.$$

Using the property of the determinant $\det(M^{-1}) = (\det(M))^{-1}$ we further have that

$$\det(D\Psi(a, r, \iota)) = \frac{1}{\det([D\Psi^{-1}](a, r, z(a, r, \iota)))}.$$

It is an easy computation to see that

$$D\Psi^{-1}(a, r, z) = J.$$

Thus,

$$q(a, r, \iota) = \frac{p(a, r, z(a, r, \iota))}{1 + (r - r^f)k_z(a, r, z(a, r, \iota))}.$$

APPENDIX D. STATIONARY DISTRIBUTION AS AN EIGENVECTOR

Here we give a proof of Theorem 2.1. We consider the case in which z_t is modeled by a CIR process, since the mathematical description is simpler as we only have one PDE. Nevertheless, the same argumentation can be used when considering the two weakly coupled PDEs that arise from the two-state Poisson process $z_t \in \{z_1, z_2\}$ like in Achdou et al. (2017).

Recall that $\Omega = [-a_{min}, \infty) \times [0, \infty) \times [0, \infty)$. Observe that Ω can be trivially identified with \mathbb{R}_+^3 by doing the change $a \mapsto a + a_{min}$. Moreover, under such an affine transformation, the differential operator ∂_a remains unchanged, so that without loss of generality we could work with $\Omega \equiv \mathbb{R}_+^3$. Points of Ω will be denoted by (a, r, z) .

Consider the stationary HJB equation

$$(21) \quad \rho v = \max_{\substack{0 < c \\ 0 \leq k \leq a + a_{min}}} \left\{ u(c) + \alpha v_r + s v_a + \mu v_z + \frac{\eta^2 r}{2} v_{rr} + \frac{\sigma^2 k^2}{2} v_{aa} + \frac{\nu^2 z^2}{2} v_{zz} \right\}.$$

Assume that (21) is *well-posed*, that is, there exists a function v such that $v_a > 0$, $v_{aa} < 0$ and the optimal problem has a solution c, k , where c is the optimal cost and k the optimal capital. Note that by using the definition of s we have

$$s v_a + \frac{\sigma^2 k^2}{2} v_{aa} = (z + r^f a - c) v_a + (r - r^f) k v_a + \frac{\sigma^2 k^2}{2} v_{aa}.$$

Hence, if $r \leq r^f$, it turns out that the optimal capital is $k \equiv 0$. For $r > r^f$,

$$k = \min \left\{ -\frac{(r - r^f) v_a}{\sigma^2 v_{aa}}, a + a_{min} \right\}.$$

Also, for $a = -a_{min}$ it follows from (21) that $k \equiv 0$. Note that if v is regular enough, k is continuous at $r = r^f$ and hence k^2 is continuously differentiable at $r = r^f$. Thus, we will further assume that $c \in W^{1,\infty}(\Omega)$ and $k^2 \in W^{2,\infty}(\Omega)$, which is plausible if we take into account the former considerations.

Consider the linear operator associated to (21), namely,

$$\mathfrak{A}v = \alpha v_r + s v_a + \mu v_z + \frac{\eta^2 r}{2} v_{rr} + \frac{\sigma^2 k^2}{2} v_{aa} + \frac{\nu^2 z^2}{2} v_{zz}.$$

We take the formal adjoint operator $A^* : \mathcal{D}(A^*) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ to be

$$A^*p = \partial_a \left(\partial_a \frac{\sigma^2 k^2}{2} p - s p \right) + \partial_r \left(\partial_r \frac{\eta^2 r}{2} p - \alpha p \right) + \partial_z \left(\partial_z \frac{\nu^2 z^2}{2} p - \mu p \right),$$

for $p \in \mathcal{D}(A^*)$.

Recall the current $j = (j^a, j^r, j^z)$ which is given by

$$\begin{aligned} j^a &:= \partial_a \frac{\sigma^2 k^2}{2} p - s p, \\ j^r &:= \partial_r \frac{\eta^2 r}{2} p - \alpha p, \\ j^z &:= \partial_z \frac{\nu^2 z^2}{2} p - \mu p, \end{aligned}$$

so that $A^*p = \nabla \cdot j$. The problem $A^*p = 0$ describes the well-known stationary Fokker-Planck equation, and the domain of the operator A^* , $\mathcal{D}(A^*)$, inherits the boundary conditions of the dynamic FP equation described in Appendix B, namely, that the currents vanish at the boundary,

$$\begin{aligned} j^a|_{a=-a_{min}} &= 0, \\ j^r|_{r=0} &= 0, \\ j^z|_{z=0} &= 0, \end{aligned}$$

c.f. (Risken, 1989, Section 4.7.1) and (Öttinger, 1996, Section 3.3.5).

Since Ω is unbounded in a , r and z , the functions need to decay at infinity. Moreover, $p \geq 0$ with

$$\int_{\Omega} p(a, r, z) \, d(a, r, z) = 1.$$

Hence, we are looking for nontrivial solutions of $A^*p = 0$, that is, $p \neq 0$. For that to be possible, the assumption that there is $p \geq 0$ in $\ker A^*$ naturally arises. Existence of solutions and uniqueness —also for the degenerate case, that is, the diffusion coefficients are not strictly positive— of divergence type FP equations can be found in (Bogachev et al., 2015, Chapter 2), in particular (Bogachev et al., 2015, Corollary 2.4.4), for the case $\Omega = \mathbb{R}^3$ and the results can be adapted to our case by considering the vanishing currents at the boundary.

Let

$$(22) \quad \mathcal{D}(A^*) := \{p \in W^{2,2}(\Omega) \mid j^a|_{a=-a_{min}} = j^r|_{r=0} = j^z|_{z=0} = 0\}$$

and note that $\mathcal{D}(A^*)$ is not the maximal domain of A^* . The operator $A^* : \mathcal{D}(A^*) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is *closed* in the sense of (Goldberg, 1966, Chapter II) and *densely defined*, that is, $\mathcal{D}(A^*)$ is dense in $L^2(\Omega)$. Hence, the adjoint operator of A^* is again closed, $(A^*)^* = A$ and $\mathcal{D}(A)$ is dense in $L^2(\Omega)$, see (Tucsnak and Weiss, 2009, Proposition 2.8.1). Moreover, $[\mathcal{R}(A)]^\perp = \ker A^*$ and $[\mathcal{R}(A^*)]^\perp = \ker A$, see (Tucsnak and Weiss, 2009, Remark 2.8.2). A further computation also shows that $Av = \mathfrak{A}|_{\mathcal{D}(A)}v$, where

$$(23) \quad \mathcal{D}(A) = \{v \in W^{2,2}(\Omega) \mid \sigma^2 k^2 V_a|_{a=-a_{min}} = \eta^2 r V_r|_{r=0} = \nu^2 z^2 V_z|_{z=0} = 0\}.$$

Hence, the assumption $p \in \ker A^*$ implies $p \in [\mathcal{R}(A)]^\perp$. Note that the boundary conditions included in $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ impose decays of the functions at the boundaries $a = -a_{min}$, $r = 0$ and $z = 0$.

The nontrivial question of which boundary conditions to impose to v arises when trying to solve (21). It seems reasonable to use the boundary conditions in $\mathcal{D}(A)$ to solve the HJB equation, so that we solve

$$(24) \quad \begin{aligned} \rho v &= \max_{\substack{0 < c \\ 0 \leq k \leq a + a_{min}}} \{u(c) + Av\}, \\ 0 &= A^* p. \end{aligned}$$

We will further assume that 0 is not an accumulation point of the spectrum of A^* , otherwise we could not really guarantee that an approximation of p converges to p and not to another eigenfunction whose eigenvalue is arbitrarily close to 0.

After discretizing the HJB, we end up with an equation of the form

$$\rho \mathcal{V} = u(\mathcal{C}) + \mathcal{A} \mathcal{V},$$

where \mathcal{V}, \mathcal{C} are column vectors with the values of v, c at the nodes, \mathcal{A} is a matrix which corresponds to the discretization of A . Note that a discretization of A^* is in fact \mathcal{A}^\top . Hence, by $[\mathcal{R}(A)]^\perp = \ker A^*$, it follows that there is a row vector $\mathcal{V} \neq 0$ such that $\mathcal{V} \mathcal{A} = 0$. If we transpose the former, we have $\mathcal{A}^\top \mathcal{V}^\top = 0$. Thus, $\mathcal{P} = \mathcal{V}^\top$ is a discretization of p , which is certainly the eigenvector corresponding to the eigenvalue 0 of \mathcal{A}^\top .

APPENDIX E. NUMERIC SCHEME FOR THE EQUILIBRIUM MODEL

Here we elucidate how to solve numerically the equilibrium model (9). The first step is to solve the HJB equation. More precisely, we wish to solve (24). As in Appendix D, we will discuss the CIR case, but the method can be performed analogously with other models.

We begin by deriving a finite difference scheme and use the upwind method. To this end let consider a partition of Ω with M, N, L points in a, r, z respectively. The resulting domain is denoted by $\Omega_{NML} := [a_1, a_M] \times [r_1, r_N] \times [z_1, z_L]$. The points in the nodes will be denoted by a_i, r_j, z_l . Similarly, the evaluation of a function f at a nodal point will be denoted by $f_{ijl} = f(a_i, r_j, z_l)$. Sometimes, in order to avoid any confusions, we will separate the subscripts with a coma, e.g. $f_{1,2,3}$ instead of f_{123} . As in Algorithm 1,

guess v^0, K^0, θ^0, w^0 . Define

$$\begin{aligned}\psi_j^+ &:= \max\{\psi(r_j), 0\}, & \psi_j^- &:= \min\{\psi(r_j), 0\}, \\ s_{ijl}^+ &:= \max\{s(a_i, r_j, z_l; k_{ijl}, c_{ijl}), 0\}, & s_{ijl}^- &:= \min\{s(a_i, r_j, z_l; k_{ijl}, c_{ijl}), 0\}, \\ \mu_l^+ &:= \max\{\mu(z_l), 0\}, & \mu_l^- &:= \min\{\mu(z_l), 0\},\end{aligned}$$

so that

$$\begin{aligned}F_{ijl} &:= u(c_{ijl}) - \rho v_{ijl} + \frac{\psi_j^+}{dr}(v_{ij+1l} - v_{ijl}) + \frac{\psi_j^-}{dr}(v_{ijl} - v_{ij-1l}) \\ &+ \frac{s_{ijl}^+}{da}(v_{i+1jl} - v_{ijl}) + \frac{s_{ijl}^-}{da}(v_{ijl} - v_{i-1jl}) \\ &+ \frac{\mu_l^+}{dz}(v_{ijl+1} - v_{ijl}) + \frac{\mu_l^-}{dz}(v_{ijl} - v_{ijl-1}) \\ &+ \frac{\eta^2 r_j}{2 dr^2}(v_{ij+1l} - 2v_{ijl} + v_{ij-1l}) \\ &+ \frac{\sigma^2 k_{ijl}^2}{2 da^2}(v_{i+1jl} - 2v_{ijl} + v_{i-1jl}) \\ &+ \frac{\nu^2 z_l^2}{2 dz^2}(v_{ijl+1} - 2v_{ijl} + v_{ijl-1}).\end{aligned}$$

A possible way of rewriting the former is

$$\begin{aligned}F_{ijl} &= u(c_{ijl}) + \zeta_j v_{ij+1l} + \chi_j v_{ij-1l} \\ &+ z_{ijl} v_{i+1jl} + x_{ijl} v_{i-1jl} \\ &+ \Theta_l v_{ijl+1} + \Psi_l v_{ijl-1} \\ &+ y_{ijl} v_{ijl} - \rho v_{ijl},\end{aligned}$$

where

$$\begin{aligned}\zeta_j &:= \frac{\psi_j^+}{dr} + \frac{\eta^2 r_j}{2 dr^2}, & \chi_j &:= -\frac{\psi_j^-}{dr} + \frac{\eta^2 r_j}{2 dr^2}, \\ z_{ijl} &:= \frac{s_{ijl}^+}{da} + \frac{\sigma^2 k_{ijl}^2}{2 da^2}, & x_{ijl} &:= -\frac{s_{ijl}^-}{da} + \frac{\sigma^2 k_{ijl}^2}{2 da^2}, \\ \Theta_l &:= \frac{\mu_l^+}{dz} + \frac{\nu^2 z_l^2}{2 dz^2}, & \Psi_l &:= -\frac{\mu_l^-}{dz} + \frac{\nu^2 z_l^2}{2 dz^2},\end{aligned}$$

and

$$y_{ijl} := -\left(\frac{\psi_j^+ - \psi_j^-}{dr} + \frac{s_{ijl}^+ - s_{ijl}^-}{da} + \frac{\mu_l^+ - \mu_l^-}{dz} + \frac{\eta^2 r_j}{dr^2} + \frac{\sigma^2 k_{ijl}^2}{da^2} + \frac{\nu^2 z_l^2}{dz^2}\right).$$

By having used the upwind method, we can guarantee that $y_{ijl} < 0$ independent of the drifts ψ, s, μ .

Since we solve the equation in the bounded domain Ω_{NML} , we need boundary conditions for the extremes of the domain. For $j = 1$ and $j = N$ we have that

$$\frac{v_{i,1,l} - v_{i,0,l}}{dr} = 0, \quad \frac{v_{i,N+1,l} - v_{i,N,l}}{dr} = 0$$

so that

$$\begin{aligned}
F_{i,1,l} &= u(c_{i,1,l}) + \zeta_1 v_{i,2,l} \\
&\quad + x_{i,1,l} v_{i-1,1,l} + z_{i,1,l} v_{i+1,1,l} \\
&\quad + \Theta_l v_{i,1,l+1} + \Psi_l v_{i,1,l-1} \\
&\quad + (y_{i,1,l} + \chi_1) v_{i,1,l} - \rho v_{i,1,l}
\end{aligned}$$

and

$$\begin{aligned}
F_{i,N,l} &= u(c_{i,N,l}) + \chi_N v_{i,N-1,l} \\
&\quad + x_{i,N,l} v_{i-1,N,l} + z_{i,N,l} v_{i+1,N,l} \\
&\quad + \Theta_l v_{i,N,l+1} + \Psi_l v_{i,N,l-1} \\
&\quad + (y_{i,N,l} + \zeta_N) v_{i,N,l} - \rho v_{i,N,l}
\end{aligned}$$

respectively.

For $l = 1$ and $l = L$ we have that

$$\frac{v_{i,j,1} - v_{i,j,0}}{dz} = 0, \quad \frac{v_{i,j,L+1} - v_{i,j,L}}{dz} = 0,$$

which leads to

$$\begin{aligned}
F_{i,j,1} &= u(c_{i,j,1}) + \zeta_j v_{i,j+1,1} + \chi_j v_{i,j-1,1} \\
&\quad + z_{i,j,1} v_{i+1,j,1} + x_{i,j,1} v_{i-1,j,1} \\
&\quad + \Theta_1 v_{i,j,2} \\
&\quad + (y_{i,j,1} + \Psi_1) v_{i,j,1} - \rho v_{i,j,1},
\end{aligned}$$

and

$$\begin{aligned}
F_{i,j,L} &= u(c_{i,j,L}) + \zeta_j v_{i,j+1,L} + \chi_j v_{i,j-1,L} \\
&\quad + z_{i,j,L} v_{i+1,j,L} + x_{i,j,L} v_{i-1,j,L} \\
&\quad + \Psi_L v_{i,j,L-1} \\
&\quad + (y_{i,j,L} + \Theta_L) v_{i,j,L} - \rho v_{i,j,L}.
\end{aligned}$$

Finally, for $i = 1$ and $i = M$ we use the following conditions. Since we know that at $a = -a_{min}$ the capital satisfies $k(-a_{min}, r, z) = 0$ and we want the consumption to be bounded, for $i = 1$ we set $x_{1,j,l} = 0$, which implies that $v_{0,j,l}$ does not play a role in the system. Given the complexity of the system and the relation between v_{aa} and k , it is rather complicated to find a boundary condition for $i = M$. We note that the optimality condition can be written as

$$k = \frac{r - r^f}{\gamma \sigma^2} \frac{c}{c_a}.$$

We assume that for large values of a , the quotient has the following behavior

$$\frac{c}{c_a} \approx a^{1/\gamma-1},$$

so that

$$k_{M,j,l} = \min \left\{ \frac{r - r^f}{\gamma \sigma^2} a_M^{1/\gamma-1}, a_M + a_{min} \right\}.$$

This implies that

$$\begin{aligned} x_{M,j,l} &= -\frac{s_{M,j,l}^-}{da} + \frac{(r_j - r^f)^2 a_M^{1/\gamma-1}}{da \gamma \sigma^2} (r_j > r^f), \\ y_{M,j,l} &= -\left(\frac{\psi_j^+ - \psi_j^-}{dr} + \frac{s_{ijl}^+ - s_{ijl}^-}{da} + \frac{\mu_l^+ - \mu_l^-}{dz} + \frac{\eta^2 r_j}{dr^2} + \frac{\nu^2 z_l^2}{dz^2} \right) \\ &\quad - \frac{(r_j - r^f)^2 a_M^{1/\gamma-1}}{da \gamma \sigma^2} (r_j > r^f) \end{aligned}$$

Note that the boundary conditions at $i = j = l = 1$ comply with the ones in (23).

We want to solve $F_{ijl} = 0$ using an implicit scheme. We begin with a guess v^0 and compute the optimal capital. We set $k_{1,j,l} = 0$, for $i = 2, \dots, M-1$ we use the optimality condition

$$k_{ijl} = \min \left\{ \frac{-(r_j - r^f) da (v_{i+1jl} - v_{i-1jl})}{2\sigma^2 (v_{i+1jl} - 2v_{ijl} + v_{i-1jl})}, a_i + a_{min} \right\}$$

and the case $i = M$ has been already given.

Next we compute the forward and backward first derivative with respect to a , namely,

$$v_a^f := \frac{v_{i+1jl} - v_{ijl}}{da}, \quad v_a^b := \frac{v_{ijl} - v_{i-1jl}}{da}.$$

At the boundaries we use for $i = 1$

$$v_a^b := u(z_l - a_{min} r^f)$$

and at $i = M$

$$v_a^f := u(z_l + a_M r^f + k_M (r_j - r^f))$$

and with that compute the forward and backward cost from the optimality condition

$$c^f := \dot{u}^{-1}(v_a^f), \quad c^b := \dot{u}^{-1}(v_a^b).$$

With that, we can define the forward and backward savings s^f and s^b by using c^f and c^b respectively. Finally, we set

$$v_a := \phi^f v_a^f + \phi^b v_a^b + \phi^0 \dot{u}(z_l + a_i r^f + k_{ijl} (r_j - r^f)),$$

where ϕ^f is 1 if $s^f > 0$ and 0 otherwise, ϕ^b is 1 if $s^b < 0$ and 0 otherwise and $\phi^0 = 1 - \phi^f - \phi^b$. Now, we can compute the optimal cost c as

$$c := \dot{u}^{-1}(v_a),$$

and the savings using the definition. Note that the step of determining c_{ijl}, k_{ijl} should be one of the first to be done, since it is needed in order to compute s_{ijl} .

Consider a column-vectorization of $F_{ijl}, v_{ijl}, c_{ijl}, k_{ijl}$, so that

$$\mathcal{F} = \mathcal{U} + \mathcal{AV} - \rho I_{MNL} \mathcal{V},$$

where

$$\mathcal{A} = \begin{bmatrix} A_1 + \Psi_1 I_{MN} & \Theta_1 I_{MN} & 0 & \cdots & 0 \\ \Psi_2 I_{MN} & A_2 & \Theta_2 I_{MN} & \ddots & \vdots \\ 0 & \Psi_3 I_{MN} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & A_{L-1} & \Theta_{L-1} I_{MN} \\ 0 & \cdots & 0 & \Psi_L I_{MN} & A_L + \Theta_L I_{MN} \end{bmatrix} \in \mathbb{R}^{MNL \times MNL},$$

$$A_l = \begin{bmatrix} D_{1,l} + \chi_1 I_M & \zeta_1 I_M & 0 & \cdots & 0 \\ \chi_2 I_M & D_{2,l} & \zeta_2 I_M & \ddots & \vdots \\ 0 & \chi_3 I_M & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & D_{N-1,l} & \zeta_{N-1} I_M \\ 0 & \cdots & 0 & \chi_N I_M & D_{N,l} + \zeta_N I_M \end{bmatrix} \in \mathbb{R}^{MN \times MN},$$

$$D_{j,l} = \begin{bmatrix} y_{1,j,l} & z_{1,j,l} & 0 & \cdots & 0 \\ x_{2,j,l} & y_{2,j,l} & z_{2,1,l} & \ddots & \vdots \\ 0 & x_{3,1,l} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & y_{M-1,j,l} & z_{M-1,j,l} \\ 0 & \cdots & 0 & x_{M,j,l} & y_{M,j,l} \end{bmatrix} \in \mathbb{R}^{M \times M}.$$

The column vector corresponding to v is then

$$\mathcal{V} = \begin{bmatrix} v_{1,1,1} \\ \vdots \\ v_{M,1,1} \\ \vdots \\ v_{1,N,1} \\ \vdots \\ v_{M,N,1} \\ \vdots \\ v_{1,1,L} \\ \vdots \\ v_{M,1,L} \\ \vdots \\ v_{1,N,L} \\ \vdots \\ v_{M,N,L} \end{bmatrix}$$

and similarly with $\mathcal{U} = u(\mathcal{C})$. Let $\Delta > 0$ be given. We solve now the implicit problem

$$\frac{\mathcal{V}^{n+1} - \mathcal{V}^n}{\Delta} = \mathcal{U}^n + \mathcal{A}^n \mathcal{V}^{n+1} - \rho I_{MNL} \mathcal{V}^{n+1},$$

that is, beginning with \mathcal{V}^0 we find \mathcal{V}^1 using the former equation, find the new forward and backward variables, update \mathcal{A} and \mathcal{U} and repeat the process until $\|\mathcal{V}^{n+1} - \mathcal{V}^n\|_\infty < \varepsilon$ for some error $\varepsilon > 0$, which implies

$$\|\mathcal{F}^n\|_\infty = \|\mathcal{U}^n + \mathcal{A}^n \mathcal{V}^{n+1} - \rho I_{MNL} \mathcal{V}^{n+1}\|_\infty < \frac{\varepsilon}{\Delta}.$$

For the iteration in which the latter is achieved, we jump into finding the solution of the FP equation. Omitting the n where $\|\mathcal{V}^{n+1} - \mathcal{V}^n\|_\infty < \varepsilon$, we consider c, k, \mathcal{A} at this iteration.

Now, using Appendix D, we have that there is a $\mathcal{P} \in \ker \mathcal{A}^\top$ which is in fact a discretization of the solution to the stationary FP equation. For the discretization of p in the mesh Ω_{NML} , we set p_{ijl} , so that \mathcal{P} is a column vectorization of p_{ijl} . Having now the density function, we can proceed to adjust the parameters. In the end, we wish to find θ and w . To this end, we begin with K^0, θ^0, w^0 and choose a relaxation parameter $\omega \in (0, 1)$. In order to update K , compute with a numerical method the following quantity

$$S = \int_{\Omega_{NML}} p(a_i, r_j, z_l) k(a_i, r_j, z_l) d(a_i, r_j, z_l).$$

Now update K as

$$K = \omega K + (1 - \omega) S$$

and use (9d) to update the new parameters, namely,

$$\begin{aligned} \theta &= \frac{1}{3} K^{-2/3} - \delta, \\ w &= \frac{2}{3} K^{1/3}. \end{aligned}$$

One needs to repeat the process of solving the HJB with these new K^1, θ^1, w^1 , obtain the new stationary density of the FP equation and update again K, θ, w . The stationary equilibrium is reached when the relative error between two consecutive iterations is smaller than a given $\varepsilon > 0$, that is,

$$\left| \frac{\theta^{m+1}}{\theta^m} - 1 \right| + \left| \frac{w^{m+1}}{w^m} - 1 \right| < \varepsilon.$$

Once we have concluded this process, we are able to find the income distribution $q(\iota)$. To this end we compute k_z by using the forward derivative in z and check the condition

$$J_{ijl} := 1 + (r_j - r^f)(k_z)_{ijl} \neq 0.$$

If this holds, we interpolate p, k and k_z . Next we define the income

$$\iota_{ijl} = z_l + a_i r^f + k_{ijl}(r_j - r^f)$$

and set $\iota_{min} := \min\{\iota_{ijl}\}$ and $\iota_{max} := \max\{\iota_{ijl}\}$. Next we interpolate a, r, z, ι and p, k_z . For the sake of simplicity we denote the new number of points by the same letters as before and add assume that the partition in $\iota_n \in [\iota_{min}, \iota_{max}]$ has P points. We denote

by q_{ijn} and J_{ijn} the points of

$$q(a_i, r_j, \iota_n) := p(a_i, r_j, z(a_i, r_j, \iota_n))$$

and

$$J(a_i, r_j, \iota_n) := 1 + (r_j - r^f)k_z(a_i, r_j, z(a_i, r_j, \iota_n))$$

respectively. In order to find the former and the latter we proceed as follows

for $n = 1, \dots, P$:

for $i = 1, \dots, N$:

for $j = 1, \dots, N$:

for $l = 1, \dots, L$:

$$\text{if } |(z_l + a_i r^f + k_{ijl}(r_j - r^f))/\iota_n - 1| < \varepsilon:$$

$$q_{ijn} = p_{ijl}$$

$$J_{ijn} = 1 + (r_j - r^f)(k_z)_{ijl}$$

With that we proceed by computing $q_n := q(\iota_n)$ as in (10), so that

$$q_n = \int_{-a_{min}}^{a_{max}} \int_{r_{min}}^{r_{max}} \frac{q(a_i, r_j, \iota_n)}{J(a_i, r_j, \iota_n)} dr_j da_i.$$