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Conference Paper

Natural rate chimera and bond pricing reality

Beiträge zur Jahrestagung des Vereins für Socialpolitik 2020: Gender Economics

Provided in Cooperation with:

Verein für Socialpolitik / German Economic Association

Suggested Citation: Brand, Claus; Goy, Gavin W; Lemke, Wolfgang (2020) : Natural rate chimera and bond pricing reality, Beiträge zur Jahrestagung des Vereins für Socialpolitik 2020: Gender Economics, ZBW - Leibniz Information Centre for Economics, Kiel, Hamburg

This Version is available at:

<https://hdl.handle.net/10419/224546>

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DNB Working Paper

No. 666 / January 2020

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DeNederlandscheBank

EUROSYSTEEM

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* Views expressed are those of the authors and do not necessarily reflect official positions of De Nederlandsche Bank.

Working Paper No. 666

January 2020

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Natural Rate Chimera and Bond Pricing Reality*

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January 2020

Abstract

Incorporating arbitrage-free term-structure dynamics into a semi-structural macro-model, we jointly estimate the real equilibrium interest rate (r^*), trend inflation, and term premia for the United States and the euro area, using a Bayesian approach. The natural real rate and trend inflation are cornerstones determining equilibrium yields across maturities and macroeconomic trends. Taking into account the secular decline in equilibrium rates, term premia exhibit cyclical behavior over the business cycle, rather than the commonly reported trend. Our estimates suggest a fall in r^* from a pre-crisis level of about 3% to around zero, but estimates are subject to sizeable uncertainty. Including survey expectations can lift r^* estimates for recent quarters by a margin.

Keywords: Natural rate of interest, r^* , equilibrium real rate, arbitrage-free Nelson-Siegel term structure model, term premia, unobserved components, Bayesian estimation

JEL Classification: C11, C32, E43, G12, E44, E52.

*We are grateful for suggestions by and discussions with Daniel Buncic, Fabio Franco, Elmar Mertens, Marco del Negro, Michael Kiley, Philip Lane, Anh Nguyen, Andrea Papetti, Fulvio Pegoraro, Paolo Pesenti, and Glenn Rudebusch. Also, we thank seminar participants at De Nederlandsche Bank and the European Central Bank, as well as participants of the joint DNB/ECB workshop on the natural rate of interest in Amsterdam, 2019, the Computational Economics and Finance conference in Ottawa, 2019 and the Computational Finance and Econometrics conference in London, 2019. The views expressed in the paper are those of the authors and do not necessarily reflect those of the ECB, De Nederlandsche Bank or the Eurosystem.

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Non-technical summary

Since the 1980s short- and long-term bond yields in advanced economies have exhibited a protracted downward trend and slumped further in the wake of the global financial crisis. These developments have been widely documented as reflecting a decline in inflation, in the natural or equilibrium rate of interest (r^*), and in term premia.

The widely documented fall in r^* indicates exceptional macroeconomic trends and has caused central bank interest rates to be constrained by their effective lower bound (ELB) – a situation preceded only during the Great Depression. The decline in r^* points to lower productivity and potential output growth, a rise in risk aversion, demographic developments, safe-asset scarcity, and possibly increasing inequality as underlying causes.

Commonly defined as the real rate of interest consistent with the economy operating at its potential or natural level and without inflationary or disinflationary pressures, r^* should play a central role in the conduct of monetary policy. However, r^* is unobservable and estimating it is fraught with a host of measurement problems and model-specification choices. As a result, some prominent economists and policymakers have characterized r^* as a poor guide for policy, as exemplified in the introductory quote by former FOMC member Kevin Warsh asserting “*r-star is not a beacon in the sky but a chimera in the eye*”.

But assumptions about equilibrium interest rates are ubiquitous in economics and finance. Beyond macroeconomics, they are of fundamental importance for modeling the term structure of interest rates. Today’s macro-finance models are typically predicated on constant long-run means, ignoring that persistent changes in real interest rate and long-run inflation expectations affect the level towards which nominal interest rates converge over the long run. Assuming convergence to constant means requires low-frequency trends in observed bond yields – like the hump-shaped evolution of US rates since the 1960s – to be explained, by-and-large, by a trend in term premia.

[Bauer and Rudebusch \(2019\)](#) have recently addressed this shortcoming. They show that, when incorporating a time-varying natural real rate and trend inflation into a no-arbitrage term-structure model, changes in short-term interest rate expectations – rather than the term premium – largely reflect the trend decline in yields.

We follow a similar route of incorporating the natural real rate as shifting endpoint into a term structure model. Different to [Bauer and Rudebusch \(2019\)](#), however, we estimate r^* jointly from both yields and macro-economic trends for the United States and the euro area. To this end, we join a semi-structural macro model similar to [Laubach and Williams \(2003\)](#) with an arbitrage-free term structure model. The term structure module is an arbitrage-free dynamic

Nelson-Siegel (AFNS) model with the level factor incorporating a trend that is given by the natural nominal short-term rate, i.e. the sum of natural real rate and trend inflation. The slope and curvature factor, by contrast, are mean-reverting.

We exploit the cross-sectional information in bond yields together with observed inflation and output to jointly estimate potential output growth, output gaps, trend inflation, real equilibrium interest rates and term premia across maturities – both for the United States and the euro area. An alternative version of our econometric model adds survey forecasts of interest rates to further inform estimates of interest rate expectations and term premia.

We use a Bayesian estimation approach. It allows joint estimation of natural rate and yield curve dynamics and facilitates drawing joint statistical inference on natural rate uncertainty and term premia estimates.

We report four main findings. First, taking into account the secular fall in equilibrium rates, term premia exhibit countercyclical behavior over the business cycle, rather than a distinct trend decline as implied by term structure models with fixed long-run means. With the onset of down-turns or recessions, term premia are elevated and during recovery periods they are found to be low.

Second, by virtue of closing the semi-structural approach with an interest rate equation linked to the yield curve module, our estimates of the natural real rate of interest co-move more closely with the model-specific real interest rate than the ones reported by [Holston et al. \(2017\)](#).

Third, we validate that estimates of r^* based on semi-structural models (as in [Laubach and Williams, 2003, 2016](#); [Holston et al., 2017](#)) are subject to very large statistical uncertainty (as documented in [Fiorentini et al., 2018](#)). Compared to this high degree of imprecision, we find uncertainty surrounding term premia estimates to be more modest.

Fourth, we show that in line with earlier studies in the term structure literature, the inclusion of rate forecast surveys affects model estimates of interest rate expectations and thereby the inference about the latent variables of interest. At sample end, including surveys can contribute to lifting equilibrium interest rate estimates and lowering term premia estimates by sizeable margins. We show that our results are also robust to taking the effective lower bound on interest rates into account by reconstructing the short-end of the yield curve using a model-specific shadow-rate estimate.

Our contribution bridges two strands of literature that have largely developed independently: on the one hand, arbitrage-free yield curve models incorporating a time-varying attractor (“shifting end point”); on the other hand semi-structural macro-models to estimate r^* .

“Central bankers should give up the search for the neutral real interest rate [...] In my view, r -star is not a beacon in the sky but a chimera in the eye. The idea of a “neutral” rate is a useful fiction. It makes for an interesting academic thought experiment. In practice, though, it’s unobservable, unpredictable, imprecise and highly variable. That makes it a poor guide for policymakers.” (Warsh, 2018)

“In thinking about how we should set the federal funds rate, many policymakers and economists find the concept of the neutral rate of interest to be a useful figure of reference. So, what does the neutral rate mean? Intuitively, I think of the nominal neutral interest rate as the level of the federal funds rate that keeps output growing around its potential rate in an environment of full employment and stable inflation.” (Brainard, 2018)

1 Introduction

Since the 1980s short- and long-term bond yields in advanced economies have exhibited a protracted downward trend and slumped further in the wake of the global financial crisis. These developments have been widely documented as reflecting a decline in inflation, in the natural or equilibrium rate of interest (r^*), and in term premia.

The natural rate of interest r^* is commonly defined as the rate of interest consistent with the economy operating at its potential level (in the absence of transitory shocks) or its natural level (in the absence of nominal frictions). Inspired by [Wicksell \(1898\)](#), [Woodford \(2003\)](#) established its central role in today's widely used New-Keynesian modeling framework. The prominence of r^* in modern monetary macroeconomics and the present exceptional macroeconomic situation, with policy and short-term money-market interest rates at historically low levels, have spawned a burgeoning literature trying to quantify r^* , estimate its dynamics and understand its driving forces.¹ The literature broadly agrees on a general downward trend in r^* and its fall to levels around zero in the wake of the financial crisis (as far as advanced economies are concerned). It is generally seen as caused by factors including lower productivity and potential output growth, a rise in risk aversion, declining growth rates in the working-age population, rising savings in anticipation of longer retirement periods (at global level), safe-asset scarcity, and possibly increasing inequality and firm profits.²

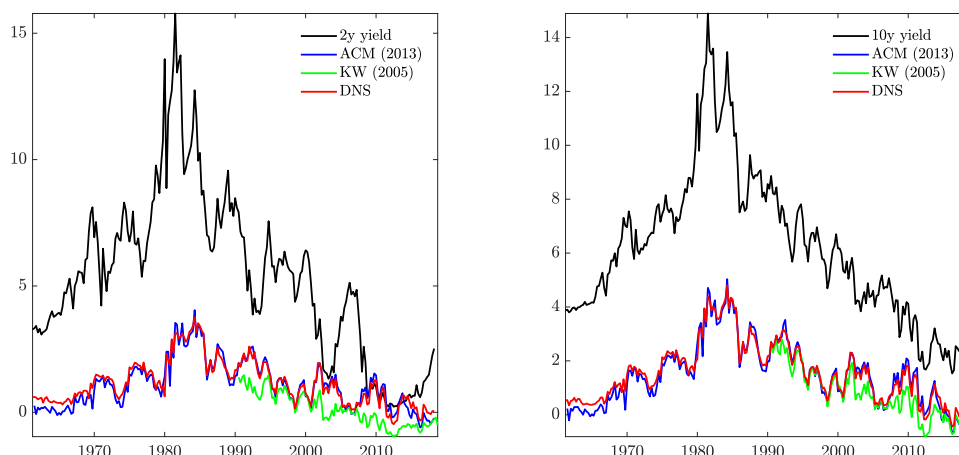
Developments in r^* affect the monetary policy stance:³ when the actual real rate exceeds its natural counterpart, the resulting positive real rate gap has a contractionary effect on the business cycle and in turn dampens inflation – and vice versa for a negative rate gap. Accordingly, central banks should have a keen interest in measuring and understanding the driving forces of the natural real rate. Yet, a host of measurement problems and model specification choices needs to be addressed when estimating r^* for quantitative policy analysis, implying considerable differences across r^* estimates, in their stabilizing properties, and their underlying drivers. As a result, academic economists and policymakers have occasionally characterized r^* as a poor guide for policy, as exemplified in the introductory quote by former FOMC

¹Econometric approaches typically focus on backing out low-frequency components in yields from macroeconomic times series, as e.g. in [Laubach and Williams \(2003, 2016\)](#); [Del Negro et al. \(2017, 2019\)](#); [Fiorentini et al. \(2018\)](#). Structural estimates yielding a contemporaneous stabilization of output gaps from DSGE models have been provided by [Edge et al. \(2008\)](#); [Barsky et al. \(2014\)](#); [Cúrdia et al. \(2015\)](#), and [Gerali and Neri \(2019\)](#), just to name a few. For a review of estimates, drivers and stabilizing properties (for the euro area and the United States), see [Brand et al. \(2018\)](#).

²See e.g. [Rachel and Smith \(2015\)](#); [Rachel and Summers \(2019\)](#); [Gourinchas and Rey \(2019\)](#); [Bielecki et al. \(2018\)](#); [Papetti \(2019\)](#); [Rannenberg \(2018\)](#); [Mian et al. \(2019\)](#); [Caballero et al. \(2017\)](#); [Marx et al. \(2017\)](#); [Gomme et al. \(2011\)](#) amongst a wide range of studies.

³See, e.g., [Weber et al. \(2008\)](#) for a conceptual discussion regarding the usefulness of r^* for monetary policy, and [Neiss and Nelson \(2003\)](#) for a model-based evaluation of the natural rate gap as policy stance indicator.

Figure 1: 2 and 10-year bond yield with common term premia estimates for the United States



Note: The figure shows the 2-year (left) and 10-year (right) yields in black, together with term premium estimates derived from the Adrian et al. (2013) model and a Dynamic Nelson-Siegel model (both own calculations) as well as from Kim and Wright (2005), which are taken from FRED.

member Kevin Warsh asserting “*r-star is not a beacon in the sky but a chimera in the eye*”.

But assumptions about equilibrium interest rates are ubiquitous in economics and finance. Besides being a conceptually important – yet difficult-to-measure – guide post for monetary policy, the natural real rate is also an elementary driving force of the term structure of interest rates. Slow-moving changes in the natural real rate together with potential shifts in the long-run inflation outlook are likely to determine the level towards which nominal short-term interest rates are expected to converge in the long run.

Yet the macro-finance literature has by and large ignored the notion of changing equilibrium rates and low-frequency macroeconomic trends (like potential output growth or central bank credibility) and their important implications for asset pricing. Rather, commonly used term structure approaches specify short-rate dynamics as being stationary around a constant mean.⁴ As a consequence, the low-frequency trends in observed bond yields – like the hump-shaped evolution of US rates since the 1960s – is by-and-large explained by a trend in term premia (see Figure 1), see also Cochrane (2007).

In a recent paper, Bauer and Rudebusch (2019) address this shortcoming of standard mean-reverting term structure models by constructing a no-arbitrage term-structure model on yields demeaned by extraneous estimates of time-varying natural real rates and long-run inflation

⁴Finance models, including those that rely on yield curve information (Dai and Singleton, 2000; Cochrane and Piazzesi, 2005; Diebold and Li, 2006; Adrian et al., 2013) and those incorporating macroeconomic variables (Ang and Piazzesi, 2003; Gürkaynak and Wright, 2012; Wright, 2011; Crump et al., 2018), but also structural macro models, such as Kliem and Meyer-Gohde (2017) and references therein, typically do not take trends in equilibrium rates into account.

expectations (dubbed “observed shifting endpoints”). Alternatively, they also estimate an arbitrage-free term structure model with shifting endpoints, but without these endpoints being informed by macroeconomic trends (dubbed “estimated shifting endpoints”). Their work importantly expands a sparse earlier literature taking initial steps towards incorporating ‘shifting end points’ for short-term rate trajectories including [Kozicki and Tinsley \(2001\)](#); [Dewachter et al. \(2014\)](#); [Ajevskis \(2018\)](#); [Christensen and Rudebusch \(2019\)](#). As [Bauer and Rudebusch \(2019\)](#) incorporate the time-varying natural nominal rate (natural real rate plus trend inflation) by either using an average of respective estimates from other studies or by extracting it as a latent variable from bond yield variation alone, part of the trend in bond yields is attributed to short rate expectations rather than having the term premium explaining the bulk of the trend decline in yields.

Our paper follows a similar route and also incorporates the natural real rate as shifting endpoint into a term structure model. But, in contrast to [Bauer and Rudebusch \(2019\)](#), our estimate of r^* is also informed by macroeconomic trends. To this end, we integrate the term structure of interest rates into a semi-structural macro model similar to [Laubach and Williams \(2003\)](#). Thereby, our term structure module is an affine Nelson-Siegel (AFNS) model with the level factor incorporating a stochastic trend determined by the natural nominal short-term rate. The slope and curvature factors, by contrast, are mean-reverting. The natural nominal short-term rate, i.e. the sum of the natural real rate and trend inflation joins the term structure and the macro module. While trend inflation is specified as a simple random walk, the natural real rate is linked to the expected growth rate of potential output as well as a non-growth component capturing other determinants of r^* . Both components of the natural real rate have a stochastic trend. The gap between the actual (model-consistent) real rate and the natural real rate drives the output gap in the IS equation, and the Phillips curve equation prescribes the link between the output gap and inflation.

We estimate the model using a Bayesian approach using data from 1961 Q2–2019 Q2 for the United States and from 1995 Q1–2019 Q2 for the euro area. We exploit the cross-sectional information in bond yields together with observed inflation and output to jointly estimate potential output growth, output gaps, trend inflation, the natural real rate and term premia across maturities. A variant of our econometric model also includes long-run expectations of long-run yields to further inform estimates of the expectations component in yields and the natural rate. Using Bayesian estimation facilitates drawing joint statistical inference on the natural rate and term premia.

We report four main findings. First, taking into account the secular fall in equilibrium rates,

term premia exhibit more cyclical behavior over the business cycle, rather than a distinct trend decline as implied by term structure models with fixed long-run means. In fact, the mean reversion property of term premia is hard-wired in our model in the sense that the level factor and the expectations component of long-term rates depend on the trending natural rate, while term premia and the slope are mean-reverting. This stands in contrast to [Bauer and Rudebusch \(2019\)](#), where the yield curve level, the slope, rate expectations and term premia are *all* driven by the stochastic trend.

Second, by virtue of closing the semi-structural approach with an interest rate equation linked to the yield curve module, our estimates of the natural real rate of interest co-move more closely with the model-specific real interest rate than the ones reported by [Holston et al. \(2017\)](#). As a result, at specific points in time our r^* estimates differ significantly from those reported in [Holston et al. \(2017\)](#): in the euro area for the first half of 2019 we obtain a natural rate of around -2% (using the non-survey version of our model), compared to their estimate of 0% . For the United States at the onset of the Great Inflation in the 1970s these discrepancies are even of an order of magnitude of $3 - 4$ percentage points.⁵ At the same time, our low estimated level of r^* at sample end squares well with recent estimates of advanced-economies r^* having fallen to zero or into negative territory: Estimates from [Gourinchas and Rey \(2019\)](#); [Fiorentini et al. \(2018\)](#); [Kiley \(2019\)](#); [Jorda and Taylor \(2019\)](#) are in a range of 0 to -3 percentage points in recent quarters.

The difference between our estimates and those obtained by [Holston et al. \(2017\)](#) probably originates from “closing” the model by endogenizing the dynamics of the nominal (and via the inflation process also the real) short-term rate using our term structure model which in turn ensures mean reversion of the real rate gap around zero. By contrast, [Holston et al. \(2017\)](#) treat the short-term nominal rate as exogenous, so there is no mechanism to ensure stationarity of the real rate gap. In fact their resulting real rate gap metric can deviate from zero in a rather persistent manner. Other approaches endogenizing the short-term rate by closing the model with a policy rule ([Jorda and Taylor, 2019](#); [Brand and Mazelis, 2019](#)) likewise imply a more stationary real-rate gap and thus different natural rate measures than [Holston et al. \(2017\)](#).

Third, qualitatively similar to other approaches in the literature, estimation uncertainty around natural rate estimates is high. Confirming results by [Fiorentini et al. \(2018\)](#), we observe that uncertainty is especially high for those model specifications that imply small estimated

⁵These numbers refer to the [Holston et al. \(2017\)](#) 2019 Q2 real time estimates obtained from <https://www.newyorkfed.org/research/policy/rstar>. When comparing point estimates across models one has to recognize the substantial estimation uncertainty underlying both estimates as discussed in more detail below. An additional source of uncertainty surrounding these r^* estimates originates from retro-active changes in past natural-rate estimates as new data become available.

coefficients for the slope of the Phillips curve and the real-rate sensitivity in the IS curve. Our joint approach – allowing for the construction of confidence bands around r^* (and trend inflation) and term premia estimates – reveals that uncertainty about estimated term premia is smaller than about the natural rate of interest. Conversely, [Bauer and Rudebusch \(2019\)](#) do not take natural rate uncertainty into account when estimating term premia on the basis of yields adjusted for off-the shelf r^* estimates and long-run inflation expectations. Accordingly, they cannot show statistical inference on what they label the ‘observed shifting endpoint’ version of their study.

Fourth, in line with earlier studies in the term structure literature, the inclusion of interest rate forecast surveys discernibly affects model-implied rate expectations and thereby the inference about the latent variables of interest. Specifically, taking survey information into account lifts the expectations component embodied in long-term bond rates and compresses the term premium, in particular, at sample end, e.g. in 10-year yields, by around one percentage point in the United States and the euro area. Bringing surveys on board also lifts our point estimate of the natural rate by around one percentage point for the United States, and by around two percentage points for the euro area.

Our paper contributes to the macro-finance literature taking into account the “... *need for further integration of financial and macroeconomic approaches to understanding trends in interest rates*”, as recently called for by [Kiley \(2019\)](#). On the one hand, we contribute to the strand of term structure literature that incorporates shifting end points in bond yield dynamics. But compared to this literature, our approach is unique in giving a more direct macroeconomic interpretation to the drivers of the stochastic end points as we embed them in a – still stylized yet complete – system of key macroeconomic aggregates. On the other hand, we contribute to the fast-growing literature of deploying small- or medium-scale econometric macro models to infer the level and dynamics of the natural real rate of interest.

More specifically, our paper adds to those contributions that not only infer the natural short rate but the whole natural yield curve. Within those papers, we are the first – to the best of our knowledge – to estimate jointly a semi-structural macroeconomic system and an arbitrage-free affine term structure model. By contrast, [Brzoza-Brzezina and Kotłowski \(2014\)](#); [Imakubo et al. \(2018\)](#); [Kopp and Williams \(2018\)](#); [Dufrénot et al. \(2019\)](#) all follow a multi-step approach in which yield curve factors are treated as observables. Moreover, [Brzoza-Brzezina and Kotłowski \(2014\)](#); [Imakubo et al. \(2018\)](#) and [Dufrénot et al. \(2019\)](#) do not provide term-premia estimates.

The paper most closely related to our work is [Kopp and Williams \(2018\)](#), yet their approach differs in several aspects. Firstly, the authors choose a model specification in which they re-

place output and its gap measure with unemployment. Consequently, the real rate trend is not linked to potential output growth, as in [Laubach and Williams \(2003\)](#), but instead follows a simple random walk. Secondly, crucial macroeconomic trends, such as the natural rate of unemployment and core inflation, are treated as observables instead of extracting them from the data. Thirdly, our term structure rules out riskless arbitrage across bond prices. Finally, we present estimation results for both the United States and the euro area.

The rest of the paper is organized as follows. Section 2 describes the macro-finance term structure model and compares it to the [Bauer and Rudebusch \(2019\)](#) setup; the Bayesian estimation approach is explained in Section 3; Section 4 presents the empirical results for the United States and the euro area.

2 The Model

We present the main building blocks of our semi-structural macro-finance model. Its macroeconomic components reflect the constraints adopted by [Laubach and Williams \(2003\)](#). While they specify the short-term rate to be exogenous, we close the macro model using an arbitrage-free term structure model: the short-term rate, together with other yields across maturities, depend on yield curve factors that, in turn, are co-driven by macroeconomic fundamentals. We modify our basic model setup by adding a measurement equation incorporating survey information. Another modification excludes yields of short maturities from the measurement equation during periods when the effective lower bound can be considered as having been binding.

2.1 A semi-structural macro model with a term structure

The approach by [Laubach and Williams \(2003\)](#) extends the unobserved components model by [Clark \(1987\)](#), decomposing macro-economic variables into random-walk trends and stationary cycles. Their approach stipulates that the real rate cycle, the output gap and inflation interact through backward-looking IS and Phillips curves. The IS curve is given by

$$\tilde{x}_t = a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + \frac{a_3}{2} (\tilde{r}_{t-1} + \tilde{r}_{t-2}) + \varepsilon_t^{\tilde{x}}, \quad (1)$$

where \tilde{x}_t denotes the output gap, defined as $\tilde{x}_t = x_t - x_t^*$, with x_t and x_t^* denoting log actual and log potential output, respectively, and $\tilde{r}_t = r_t - r_t^*$ denoting the real rate gap, i.e. the difference

between the actual real rate r_t and its natural counterpart r_t^* . Potential output x_t^* satisfies

$$x_t^* = x_{t-1}^* + g_{t-1} + \varepsilon_t^{x^*}, \quad (2)$$

with g_t reflecting the expected growth rate of potential output. The real natural rate r_t^* is the sum of the expected growth rate of potential output and a “catch-all” non-growth component, denoted z_t , i.e.

$$r_t^* = 4g_t + z_t, \quad (3)$$

Here, the coefficient on g_t maps quarterly growth rates into annualized rates of return. Both g_t and z_t follow a random walk

$$g_t = g_{t-1} + \varepsilon_t^g, \quad \text{and} \quad z_t = z_{t-1} + \varepsilon_t^z. \quad (4)$$

The z_t component captures effects such as saving-investment imbalances arising from longer retirement periods, as well as increased demand for safe assets, usually measured by an increase in the convenience yield (Del Negro et al., 2017, 2019), or other financial frictions. While Laubach and Williams (2003) use a trailing average of inflation to approximate inflation expectations and construct ex ante real rates, we define the ex ante real rate r_t as

$$r_t = i_t - E_t \pi_{t+1}, \quad (5)$$

where i_t denotes the nominal short-term interest rate. Inflation expectations $E_t \pi_{t+1}$ are formed in a model-consistent manner.⁶

Our second main equation, the Phillips curve, is given by

$$\tilde{\pi}_t = b_1 \tilde{\pi}_{t-1} + b_2 \tilde{x}_{t-1} + \varepsilon_t^\pi, \quad (6)$$

where $\tilde{\pi}_t = \pi_t - \pi_t^*$, represents the inflation gap, i.e. the difference of inflation from its trend. The latter is also assumed to follow a random walk

$$\pi_t^* = \pi_{t-1}^* + \varepsilon_t^{\pi^*}. \quad (7)$$

As a result, the real rate gap \tilde{r}_t affects – via the output gap – the high-frequency component of inflation $\tilde{\pi}_t$ around its low-frequency trend π^* (henceforth referred to as trend or core inflation). This specification differs from Laubach and Williams (2003) who also impose a unit root

⁶For more details, see Annex section B.

on inflation, but eschew an explicit expression for its stochastic trend.

We close the model by prescribing the law of motion of the nominal short-term interest rate that is, in turn, implied by a complete characterisation of the dynamics of the risk-free yield curve. At each point in time, the cross section of yields of all maturities is assumed to be explained by three factors ('level', L_t , 'slope', S_t , and 'curvature', C_t) with factor loadings across maturities following the functional form of Nelson and Siegel (1987):

$$y_t(\tau) = \mathcal{A}(\tau) + L_t + \theta_s(\tau)S_t + \theta_c(\tau)C_t \quad (8)$$

where $y_t(\tau)$ denotes the τ -period bond yield, and where factor loadings are given by $\theta_s(\tau) = \frac{1 - \exp(-\lambda\tau)}{\lambda\tau}$ and $\theta_c(\tau) = \frac{1 - \exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau)$.

An increase in the level factor induces a parallel up-shift of the whole yield curve, an increase in the slope factor increases the short end by more than the long end (hence, strictly speaking, 'negative slope factor') and an increase in the curvature factor accentuates the curvature at short- to medium-term maturities. The parameter λ governs how strongly a change in the slope factor S_t affects the slope of the yield curve and at which maturity the curvature factor has its maximum impact on the yield curve.

The intercept term $\mathcal{A}(\tau)$ does not appear in the original Nelson-Siegel specification: $\mathcal{A}(\tau)$ rules out risk-free arbitrage, as detailed further in Appendix B.2. Besides depending on maturity, $\mathcal{A}(\tau)$ is a function of the Nelson-Siegel factor loadings as well as of factor innovation variances.

If yield factor dynamics were constrained to be stationary, all yields would converge to some fixed mean. In particular, this convergence would imply that the long-horizon expectation of the nominal one-period rate $i_t \equiv y_t(1)$ is constant, i.e. $i_t^* \equiv \lim_{h \rightarrow \infty} E_t i_{t+h} = i^*$. But as our macro module specifies integrated processes for trend inflation and the natural real rate, the long-run Fisher equation, $i_t^* = \pi_t^* + r_t^*$, implies time-variation in the attractor for the nominal short-term rate. We incorporate this time-variation by allowing the level factor to be non-stationary, while imposing stationarity on the slope and curvature factor. Specifically, we decompose

$$L_t = L_t^* + \tilde{L}_t \quad (9)$$

where L_t^* is a non-stationary trend with $\lim_{h \rightarrow \infty} E_t L_{t+h}^* = L_t^*$ and \tilde{L}_t is a zero-mean stationary cyclical component. From (8), we have for the short-term rate

$$i_t = \mathcal{A}(1) + L_t + \theta_s(1)S_t + \theta_c(1)C_t \quad (10)$$

and hence for the limit

$$\lim_{h \rightarrow \infty} E_t i_{t+h} \equiv i_t^* = \mathcal{A}(1) + L_t^* + \theta_s(1)\bar{S} + \theta_c(1)\bar{C}, \quad (11)$$

where \bar{S} and \bar{C} denote the constant long-run means of the slope and curvature factor, respectively. In combination with equation (11), the long-run Fisher equation $i_t^* = \pi_t^* + r_t^*$ pins down the trend component of the level factor as $L_t^* = \pi_t^* + r_t^* - \theta_s(1)\bar{S} - \theta_c(1)\bar{C} - \mathcal{A}(1)$. As L_t^* is a latent process and $\mathcal{A}(1)$ is a free parameter (see Appendix B.2) we set $\mathcal{A}(1) = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ so that the long-run level factor is equal to the nominal short-term natural rate

$$L_t^* = i_t^* \equiv r_t^* + \pi_t^*. \quad (12)$$

For the stationary zero-mean component of the level factor we specify

$$\tilde{L}_t = a_L \tilde{L}_{t-1} + \varepsilon_t^{\tilde{L}},$$

with $|a_L| < 1$. Finally, slope S_t and curvature C_t are assumed to follow a bivariate, stationary VAR that also includes the inflation and output gap:

$$\begin{aligned} S_t &= a_{10} + a_{11}S_{t-1} + a_{12}C_{t-1} + a_{13}\tilde{\pi}_{t-1} + a_{14}\tilde{x}_{t-1} + \varepsilon_t^S, \\ C_t &= a_{20} + a_{21}S_{t-1} + a_{22}C_{t-1} + a_{23}\tilde{\pi}_{t-1} + a_{24}\tilde{x}_{t-1} + \varepsilon_t^C. \end{aligned}$$

Our specification implies a “natural yield curve” at each point in time, i.e. a set of attractors for all maturities. Taking limits on equation (8),

$$\lim_{h \rightarrow \infty} E_t y_{t+h}(\tau) \equiv y_t(\tau)^* = \mathcal{A}(\tau) + L_t^* + \theta_s(\tau)\bar{S} + \theta_c(\tau)\bar{C} \quad \text{for all } \tau. \quad (13)$$

The location of the natural yield curve varies over time with the stochastic drift in the level factor which in turn, according to equation (12), is pinned down by the natural real short-term rate and trend inflation. In the long-run slope and curvature, however, converge to constant means, so the long-run *shape* of the natural yield curve is constant. In particular, the “natural yield spread”

$$y_t^*(\tau) - y_t^*(1) = \mathcal{A}(\tau) + \theta_s(\tau)\bar{S} + \theta_c(\tau)\bar{C} \quad (14)$$

is time invariant as the short-term natural real rate and trend inflation affect the short and the long end of the natural yield curve to the same extent.

The law of motion of the macroeconomic and term structure factors constitutes the tran-

sition equation for our state variables. The observable variables are output and inflation (no measurement error) and a set of bond yields (with measurement errors except for the short rate), so the set of measurement equations is given by:

$$y_t(\tau_i) = \mathcal{A}(\tau_i)L_t + \theta_s(\tau_i)S_t + \theta_c(\tau_i)C_t + \varepsilon_t^{\tau_i}, \quad \text{for a set of } K \text{ maturities } \tau_i = 1, \tau_2, \dots, \tau_K$$

$$x_t = x_t^* + \tilde{x}_t, \tag{15}$$

$$\pi_t = \pi_t^* + \tilde{\pi}_t, \tag{16}$$

with $\varepsilon_t^1 \equiv 0$. Collecting the state variables in a vector ξ_t and the observed yields, output and inflation in the vector ζ_t , the model can be represented in state space form as follows,

$$\zeta_t = \gamma + C\xi_t + Du_t \quad \text{with } u_t \sim NID(0, 1) \tag{17}$$

$$\xi_t = \mu + F\xi_{t-1} + Ge_t \quad \text{with } e_t \sim NID(0, 1), \tag{18}$$

see Appendix B.1 for details.

Finally, we compute a model-consistent term premium $TP_t(\tau)$ of maturity τ as the difference between the model-implied τ -period bond yield and its expectations component (the average short-term rate expectation over the bond's maturity) $\frac{1}{\tau} \sum_{h=0}^{\tau-1} E_t(i_{t+h})$. From (10), the short-term rate is a linear function of the state vector, i.e. $i_t = \beta_0 + \beta_1' \xi_t$, hence $E_t(i_{t+h}) = \beta_0 + \beta_1' E_t(\xi_{t+h})$, and $E_t(\xi_{t+h})$ is obtained by iterating equation (18) forward. The term premium is then given by

$$TP_t(\tau) = y_t(\tau) - \frac{1}{\tau} \sum_{h=0}^{\tau-1} E_t(i_{t+h}). \tag{19}$$

2.2 Yield-curve dynamics compared to Bauer and Rudebusch (2019)

The dynamics of term structure factors as detailed in section 2.1 differs to some extent from the specification in Bauer and Rudebusch (2019). Both approaches rule out arbitrage and imply a factor representation of the yield curve of the form $y_t(\tau) = \mathcal{A}(\tau) + \mathcal{B}(\tau)'F_t$, in our case with $F \equiv (L, S, C)'$. Abstracting from constant intercept terms, both models imply factor dynamics of the form $F_t = \tilde{F}_t + \alpha i_t^*$ where the 3×1 vector \tilde{F}_t contains the stationary cyclical components and i_t^* is the natural nominal rate of interest which is integrated of order one. While α is unconstrained in Bauer and Rudebusch (2019) so that the stochastic trend potentially affects all factors, we impose $\alpha = (1, 0, 0)'$, i.e. only the level factor incorporates the trend.

Using the definition employed in Bauer and Rudebusch (2019), the natural nominal short rate is “unspanned” in both models in the following sense: while it loads contemporaneously

on all bond yields, it only does so through the factors, so that *conditional* on knowing the factors F , the variable i_t^* is non-informative for the yield curve.

In our case, i_t^* links to yields of all maturities with a loading of unity, i.e. moving up i_t^* by one unit, all yields increase by the same amount. This pattern is an immediate implication of i_t^* being part of the level factor, $L_t = \tilde{L}_t + i_t^*$, and not being linked to the other term-structure factors. In [Bauer and Rudebusch \(2019\)](#) a change in i_t^* also lifts the nominal short rate one for one, but, in contrast to our model, longer-term bond yields rise by more. In other words, a change in i_t^* affects not only the level but also the slope of the yield curve. As argued by [Bauer and Rudebusch \(2019\)](#), this slope effect is qualitatively mirroring a pattern in the data (based on regressions of bond yields on off-the-shelf proxies of i^*). This nexus implies that the slope of the curve also exhibits a stochastic trend. [Bauer and Rudebusch \(2019\)](#) show that the term premium incorporates that trend as well, rendering it non-stationary.

In our model, by contrast, both the slope and the term premium are stationary. For the slope $y_t(\tau) - y_t(1)$ this property is immediate to see from (14) as $\lim_{h \rightarrow \infty} E_t(y_{t+h}(\tau) - y_{t+h}(1)) = y_t^*(\tau) - y_t^*(1)$ converges to a time-invariant constant. Expanding the expression for the term premium defined in (19), we have

$$TP_t(\tau) = \mathcal{A}(\tau) + i_t^* + \tilde{L}_t + \theta_s(\tau)S_t + \theta_c(\tau)C_t \quad (20)$$

$$- \frac{1}{\tau} \sum_{h=0}^{\tau-1} E_t(\mathcal{A}(1) + i_{t+h}^* + \tilde{L}_{t+h} + \theta_s(1)S_{t+h} + \theta_c(1)C_{t+h}). \quad (21)$$

Noting that since $E_t(i_{t+h}^*) = i_t^*$ for all h , the i^* terms cancel in the above expression. Moreover, $E_t(\tilde{L}_{t+h})$, $E_t(S_{t+h})$ and $E_t(C_{t+h})$ are all independent of i_t^* or any trending variable.

Our choice specifying that the time-variant equilibrium interest rate affects only the level factor is motivated by the observation that, over the past decades, the level factor exhibits drifting behavior, while slope and curvature exhibit cyclical behavior – as will also be illustrated further below.

2.3 Model extension using survey data

At each point in time, the model determines expectations of macroeconomic variables and bond yields across time horizons – and thereby term premia. For commonly used stationary term-structure models with time invariant long-run mean the persistence of the driving factors may be underestimated and long-term expectations biased towards converging too quickly to this long-run mean. This feature renders the expectations component overly stable with the term premium component capturing too much of the low-frequency movements in yields.

One remedy to attenuating mean reversion is to include survey information (typically in the form of a mean survey forecast of short-term rates over various horizons). It renders interest-rate expectations more persistent and more variable (see [Kim and Wright, 2005](#)).

Conversely, in our setup interest rate expectations at long horizon are pinned down by i_t^* – which is a random walk. In the standard macro-only approach inference about r^* is exclusively relying on backward-looking output gap dynamics. Our model version including the term structure of interest rates implies that r^* (together with π^*) is also inferred from forward-looking bond yield information. Adding surveys can enhance this information set further.

We add long-run Consensus expectations of the 10-year (=40 quarters) yield, denoted by $\hat{E}_t^{sur}y(40)$ to our measurement equation as follows:

$$\hat{E}_t^{sur}y(40) = \mathcal{A}(40) + L_t^* + \theta_s(40)\bar{S} + \theta_c(40)\bar{C} + u_t^e, \quad (22)$$

with $u_t^e \sim \mathcal{N}(0, \sigma_e^2)$ denoting its measurement error and σ_e^2 the associated variance. Consensus Economics survey participants are asked to report their expectation for the average yield realization over the time window 6 to 10 years ahead. For our match of survey data and model counterpart, we treat this time frame as a ‘very long’ horizon; that is, formally, the model expression on the right-hand side of equation (22) corresponds to $\lim_{h \rightarrow \infty} E_t y_{t+h}(40)$, as implied by equation (8). Adding survey information to our setup supports the estimation of the natural level factor $L_t^* = i_t^*$ and – recalling equation (12) – thereby the estimation of r^* and π^* . Appendix 2.3 shows how to modify the state space representation for the survey-extended model.

2.4 Accounting for the effective lower bound

To address misspecification concerns arising from effective lower bound (ELB) constraints, we exclude observed yields up to a certain maturity from our measurement variables during times at which the ELB was arguably binding, treating them as missing observations. Thereby we assume mainly yields with maturities of up to two years to have been constrained by the ELB, in line with [Swanson and Williams \(2014\)](#). Our smoothing algorithm then generates estimates of those yields dropped from the observables, i.e. they arise as projections on inflation, output and longer-term yields. The short rate projection during the constrained times can then be interpreted as a ‘shadow short rate’ similar to [Lombardi et al. \(2018\)](#).

3 Estimation

We estimate the resulting state space model using Bayesian techniques. We use largely uninformative priors, but their choice rules out the “pile-up” problem (Stock and Watson, 1998). A Bayesian approach allows simultaneous estimation of all model parameters and thereby eschews the usual multi-step Maximum-Likelihood approach by Stock and Watson (1998)—a method that would be excessively unwieldy in our more comprehensive model setup.

As common in Bayesian estimation of unobserved components models, we use conjugate priors, the Gibbs sampler and the Durbin and Koopman (2002) simulation smoother to jointly estimate potential output growth, output gaps, core inflation, real equilibrium interest rates, and term premia for the United States and the euro area. In the Gibbs sampler, we use a total of 100,000 draws, of which we use the first 90,000 as burn-in and subsequently retain every fifth draw of the remaining 10,000. The Kalman filter is initialized using the HP-Filter for trends and OLS regressions for the parameters. We calibrate the Nelson-Siegel parameter λ by estimating a yields-only Dynamic Nelson-Siegel (DNS) model in the spirit of Diebold and Li (2006) using maximum likelihood and the Kalman filter.

Including survey data creates missing observations. First, the sample period of US survey data starts only in 1989 Q2 and is thereby shorter than the original data sample. Second, for both the United States and the euro area, survey data are initially only available biannually. For this reason, we adapt the Durbin and Koopman simulation smoother to allow for mixed frequencies and treat missing values as unobserved variables (see Durbin and Koopman (2012), pp. 110-112, for details).

We use conjugate priors for all model parameters and variances, i.e. prior distributions are either normal inverse gamma or normal inverse Wishart. All priors are uninformative with the exception of the variance of shocks to expected potential output growth σ_g^2 . Here, we choose shape and scale parameters of the inverse gamma distribution such that the mean equals 0.0015. Also, for the variance of the innovation to trend inflation $\sigma_{\pi^*}^2$, we choose somewhat conservative shape and scale parameters. Specifically, we assume, a priori, the mean and variance of the disturbance term to equal $\frac{1}{7}$ and $\frac{1}{180}$, respectively, which still allows for an expected annual change in trend inflation of around 0.6%. In order to achieve convergence of the sampler in the model version incorporating surveys, we needed to set fairly tight priors with sizeable means for standard deviations of survey measurement errors, σ_e^{US} and σ_e^{EA} . In other words, we nudge the model to ensure that it does not fit the surveys too tightly. Still, despite allowing for such large survey measurement errors, the inclusion of surveys induces a marked difference to estimated paths of premia and the natural rate as we show below.

We use rejection sampling to ensure correct signs of the loadings of the real rate gap and the output gap in the IS curve and the Phillips curve, respectively, to ensure local stability of individual equations and global stability of the system.

Table 1 summarizes the priors of the main structural parameters, followed by their posterior mean, median as well 5% and 95% credibility bands for each jurisdiction. Finally, we also compare our parameter estimates to those obtained by Holston et al. (2017).

Table 1: Prior and posterior densities of parameter estimates

	Distr.	Prior		US posterior					EA posterior				
		Param. 1	Param. 2	Mean	Median	5%	95%	HLW	Mean	Median	5%	95%	HLW
a_1	\mathcal{N}	1.5	0.5	1.59	1.60	1.45	1.71	1.53	1.64	1.64	1.49	1.78	1.67
a_2	\mathcal{N}	-0.60	0.5	-0.67	-0.68	-0.78	-0.53	-0.59	-0.70	-0.71	-0.84	-0.56	-0.73
$\frac{a_3}{2}$	\mathcal{N}	-0.1	0.05	-0.01	-0.01	-0.03	-0.00	-0.07	-0.02	-0.01	-0.04	-0.00	-0.04
b_1	\mathcal{N}	0.9	1	0.86	0.87	0.80	0.93	0.67	0.60	0.60	0.40	0.79	0.71
b_2	\mathcal{N}	0.15	0.1	0.14	0.13	0.09	0.19	0.08	0.17	0.16	0.07	0.27	0.07
a_L	\mathcal{N}	0.75	1	0.61	0.62	0.28	0.92		0.88	0.90	0.72	0.99	
σ_{Lc}	Γ^{-1}	4	2	0.44	0.44	0.35	0.53		0.31	0.30	0.25	0.37	
σ_S	\mathcal{W}^{-1}	10	$10 \cdot \mathbf{I}_2$	1.06	1.04	0.91	1.24		0.61	0.58	0.48	0.84	
σ_C	\mathcal{W}^{-1}	10	$10 \cdot \mathbf{I}_2$	2.26	2.25	2.06	2.50		1.32	1.26	0.99	1.87	
σ_{π^*}	Γ^{-1}	8	1	0.30	0.29	0.25	0.35		0.24	0.24	0.19	0.29	
σ_{x^*}	Γ^{-1}	4	2	0.59	0.59	0.52	0.67	0.33	0.32	0.32	0.27	0.39	0.4
σ_g	Γ^{-1}	14	0.02	0.04	0.04	0.03	0.05	0.001	0.03	0.03	0.03	0.04	0.05
σ_z	Γ^{-1}	4	2	0.41	0.41	0.32	0.50	0.023	0.32	0.32	0.26	0.38	0.33
$\sigma_{\tilde{\pi}}$	Γ^{-1}	4	2	0.39	0.38	0.34	0.43	0.62	0.32	0.32	0.27	0.38	1
$\sigma_{\tilde{x}}$	Γ^{-1}	4	2	0.40	0.40	0.32	0.49	0.13	0.37	0.37	0.31	0.44	0.29
σ_e^{US}	Γ^{-1}	15	800	1.86	1.86	1.73	2.01						
σ_e^{EA}	Γ^{-1}	15	1800						4.19	4.18	3.73	4.73	

Note: The table shows prior and posterior moments of the structural model parameters, based on 100,000 iterations of the Gibbs sampler of which we discarded the first 90,000 draws and subsequently kept each fifth draw. The first (second) prior parameter equals the mean (variance) of the distribution, in case of the Normal distribution, while it represents the shape (scale) in case of the inverse gamma or inverse Wishart distribution. HLW refers to the published estimates from Holston et al. (2017).

Appendix A describes the data used in the study. We estimate the US version of the model over the sample period 1961 Q2–2019 Q2 and the euro area version over the period 1995 Q1–2019 Q2. Figures reporting comparisons with other established yield curve models show estimates up until 2018 Q2.

4 Results

4.1 Parameter estimates

Table 1 above compares posterior means of parameter estimates from the macro block of the model with those published by Holston et al. (2017). While these estimates are broadly consistent, differences in model specification and estimation methodology impair comparability between our results and those by Holston et al. (2017) (e.g. enforcing stationarity in \tilde{r} by closing the model or differences in the Phillips curve).

The loading coefficients of the real rate gap, a_3 , in the IS equation and of the output gap, b_2 ,

in the Phillips curve are small. In particular, the slope of the IS curve (for both the United States and the euro area) falls below 0.1 (in absolute terms) – the critical threshold beneath which filtering uncertainty rises dramatically, as reported in [Fiorentini et al. \(2018\)](#). The corresponding IS curve estimates by [Holston et al. \(2017\)](#) are a bit higher, yet not exceeding that threshold either. By contrast, our estimates of b_2 are higher than in [Holston et al. \(2017\)](#) and exceed the threshold, likely owing to differences in the Phillips-curve specifications.

The standard deviation of innovations to the Phillips curve, σ_{π} , is estimated to be smaller, possibly reflecting our explicit decomposition of inflation dynamics into a low-frequency stochastic trend and a stationary component. The variance of shocks to the non-growth component z (σ_z) is estimated to be much higher (in case of the United States). We show that closing the model makes r^* track the real rate of interest more closely during the Great Inflation period and accordingly the non-growth component needs to capture a larger share of the persistent wedge between expected potential output growth and the trend in real rates.

4.2 United States

For the United States, [Figure 2](#) and [Figure 3](#) display latent factor estimates of the macroeconomic trends and cycles. Thereby, [Figure 2](#) illustrates the trends in inflation π_t^* , potential output x_t^* as well as the nominal and real natural rate i_t^* and r_t^* , respectively. Specifically, the bottom right-hand chart shows a comparison of our term-structure based r^* estimate with a Bayesian estimate of the specification in [Holston et al. \(2017\)](#).

As argued earlier, closing the standard [Laubach and Williams](#) model with an equation for the short-term rate will cause the natural rate r_t^* estimate to follow the real rate process more closely than in the benchmark [Laubach and Williams](#) model in which there is no mechanism to ensure a stationary real rate gap. The resulting differences in r^* estimates shown in the lower right-hand panel of [Figure 2](#) are particularly pronounced in the 1970s.

As for other latent factors our results suggest that (quarterly) potential output growth fell over the sample period from 1.1% to around 0.4% in 2010 and have stayed low ever since.

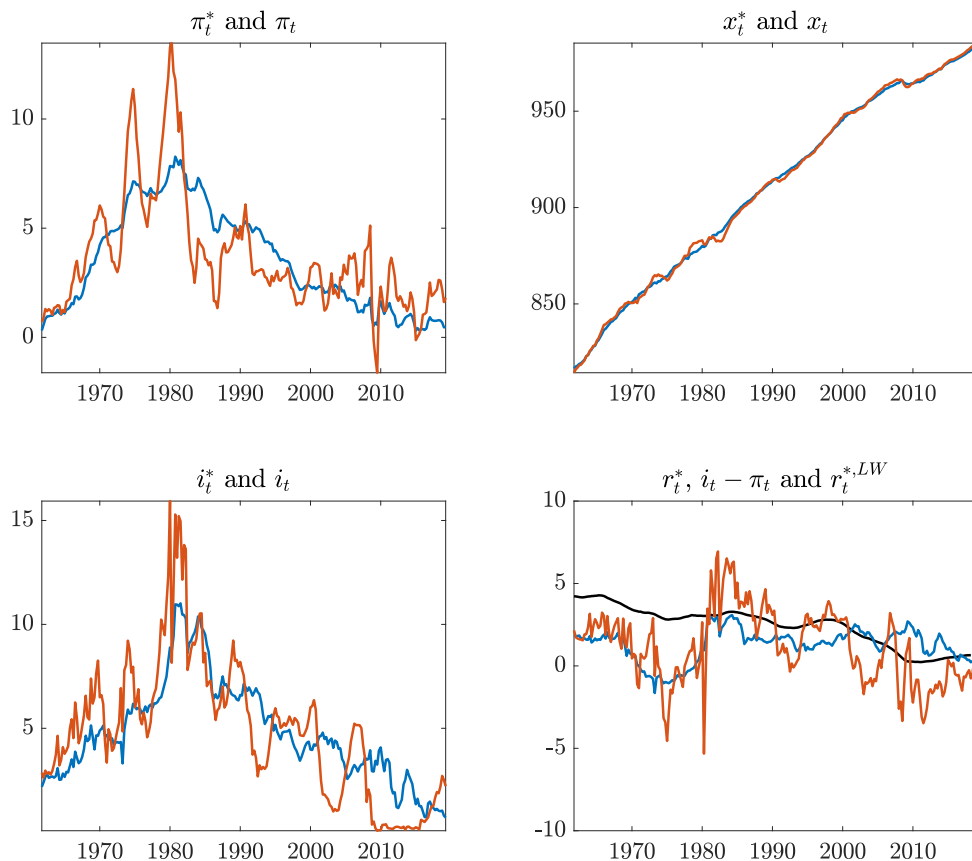
The lower middle and right panels in [Figure 3](#) present inflation and output gap estimates together with NBER recessions. The cycles in these estimates match official recession dates rather well. [Appendix C](#) also illustrates broad consistency in our model-based output gap estimates with institutional ones.

As expected, the z_t component (lower left panel in [Figure 3](#)) of the natural rate is the least precisely estimated – exhibiting confidence bands of several percentage points. This high level of statistical uncertainty about the non-growth component of r^* translates into large error bands

in r^* . As shown in [Fiorentini et al. \(2018\)](#) it is a model-specification feature of [Laubach and Williams \(2003, 2016\)](#), rooted in very weak loading coefficients of gap measures in the IS and the Phillips curves giving rise to a staggering degree of filtering uncertainty. In turn, this high statistical uncertainty in estimating r^* causes i^* or L^* , respectively, to be rather imprecisely estimated too.

Figure 4 plots the AFNS yield curve factors together with their 5-95% credibility bands. Three remarks: First, the path of these yield curve factors corresponds to what would also be obtained from a yields-only AFNS specification. Second, statistical uncertainty in the estimation of the level factor is sizeably larger than of the other yield curve factors. Third, with the yields-only AFNS-factor paths coinciding with those from our specification, the drifting behaviour in the level factor in combination with the cyclical behaviour of the slope and curvature factors validates our model-specification choice that the stochastic term-structure factor loads only into the level component (see section 2.2).

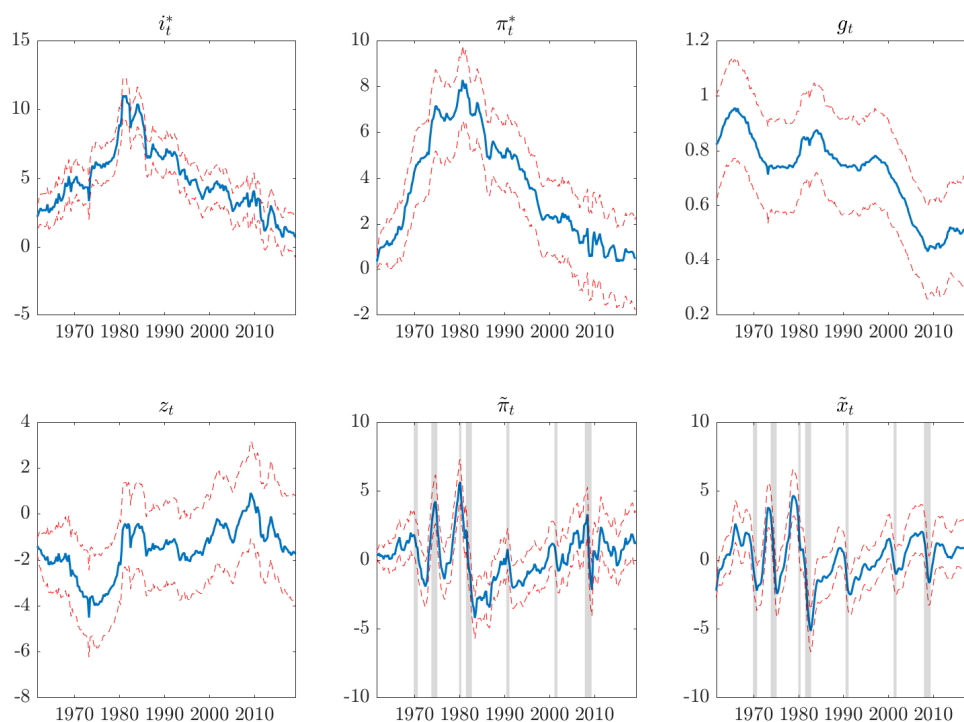
Figure 2: US macroeconomic observables and trends



Note: The figure shows the estimated trends (in blue) and the observed macro-variables (in red). The black solid line reports our own Bayesian estimates of the specification in [Holston et al. \(2017\)](#).

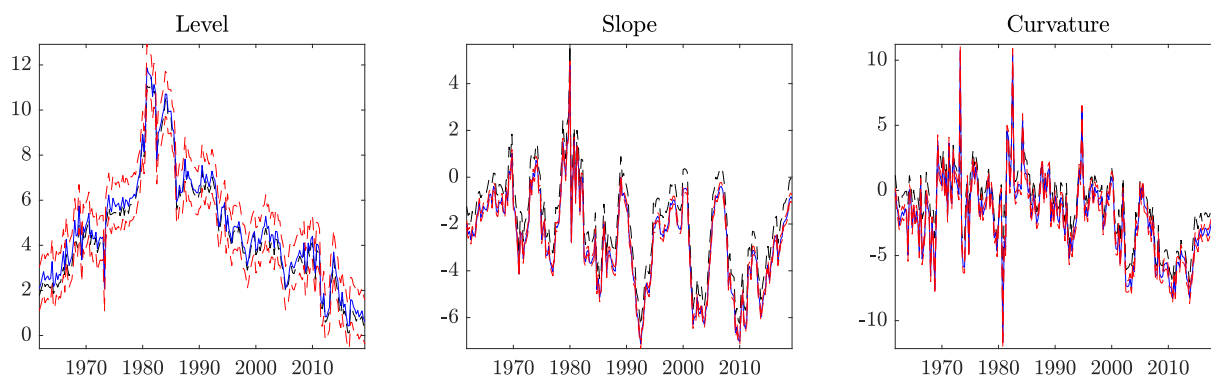
Including surveys on long-run interest rate expectations as specified in section 2.3 (starting in 1989 Q2) can cause rather persistent differences in r^* estimates. At sample end r_t^* is estimated

Figure 3: US latent macro variables



Note: The figure shows the estimated latent states of the model in blue together with their 5% and 95% credibility bands in red-dashed. Shaded areas represent NBER recessions.

Figure 4: US yield curve factors

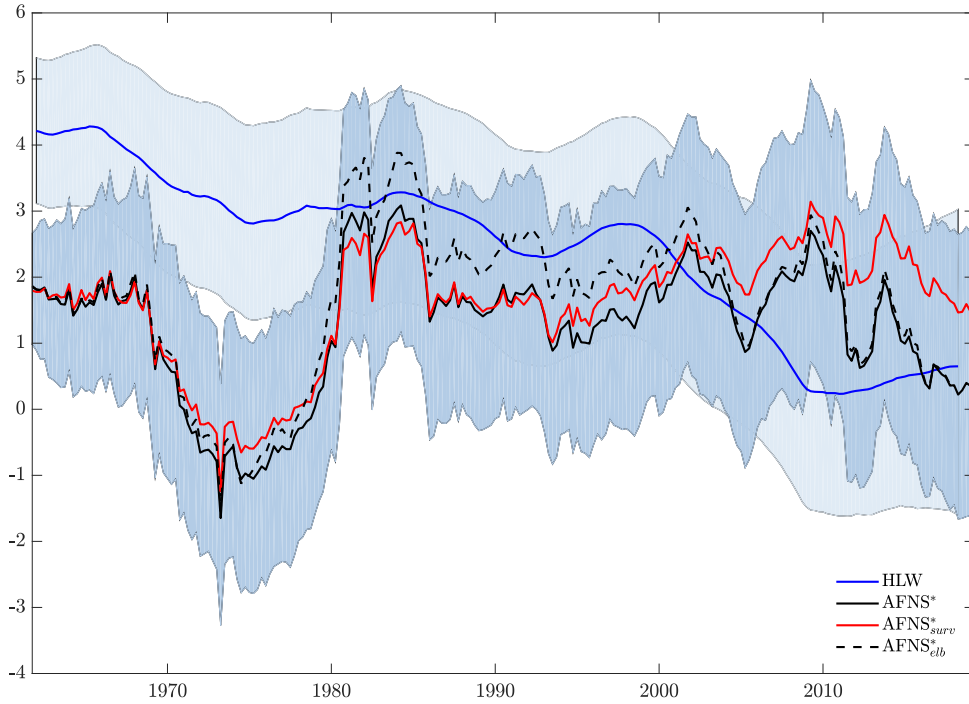


Note: The figure shows the yield curve factors in blue with respective 5% and 95% credibility bands in red. For comparison, the black-dashed lines represent yield curve factors obtained from a yields-only AFNS model.

to be 1.5% in case survey information is employed, while it is as low as 0.3% otherwise.

Figure 6 displays decompositions of the 2 and 10-year yields into the expectations component and the term premium as defined in equation (19). Term premia move in a countercyclical fashion and tend to be higher at the beginning of recessions than otherwise. Error bands around term premia estimates are consistently smaller than those around r^* estimates.

Figure 5: US natural rate estimates



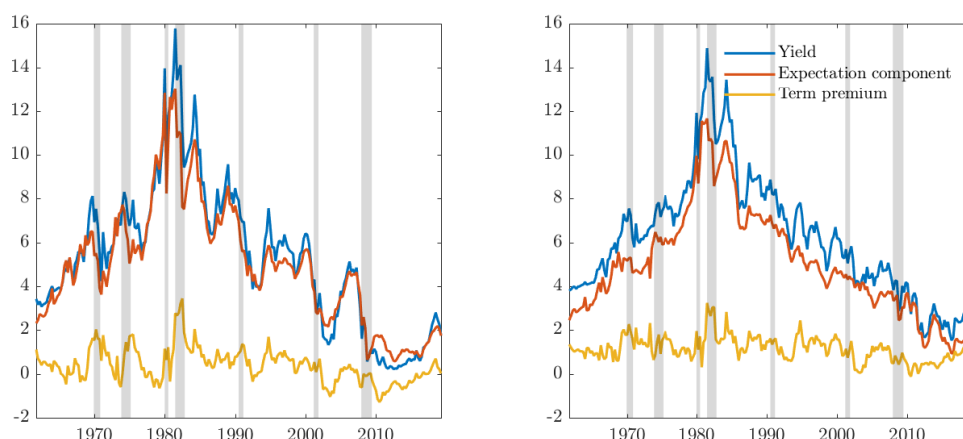
Note: The figure shows the natural rate estimate in black with 5% and 95% confidence bands depicted by the blue-shaded area. The red line depicts our r_t^* estimate when including surveys. For comparison, our Bayesian replication of the Holston et al. (2017) estimate is in blue with 5% and 95% credibility bands given by the light-blue-shaded area. The dashed black line corresponds to the natural rate estimate when accounting for the effective lower bound as explained in section 2.4

In Figure 7, we compare our term-premia estimates with those from commonly-used models featuring constant steady states for risk-free yields (see, among others, Kim and Wright, 2005; Adrian et al., 2013). Across both modeling classes, the timing of peaks and troughs largely coincides. Term premia have slumped precipitously in the wake of the global financial crisis and with the start of the Federal Reserve’s large-scale asset purchases at the end of 2008. They also display the ‘taper-tantrum’ run-up in 2013. However, since then, while the measures from the literature have further extended their consistent decline starting in the 1980s, our term-premia estimates have continued to edge up. Overall, they display a more distinct mean-reverting behavior compared to the secular downward trend typically reported in the literature.

4.3 Euro area

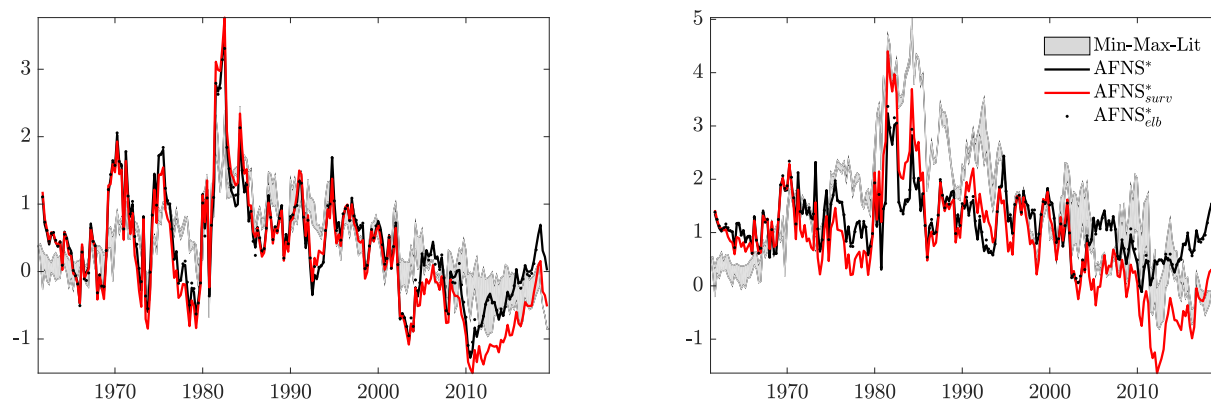
For the euro area, macroeconomic observables and their trends are shown in Figure 8. Similar to the United States, the r_t^* estimate closely tracks a proxy for the real rate, as can be seen in the lower right panel of Figure 8. The macroeconomic states of the estimated model for the

Figure 6: Decomposition of US yields



Note: The figure shows the decomposition of the 2y (left) and 10y (right) yields in blue into expectation component (red) and the term premium (yellow). Own calculations. NBER recessions in gray.

Figure 7: Comparing US term premia

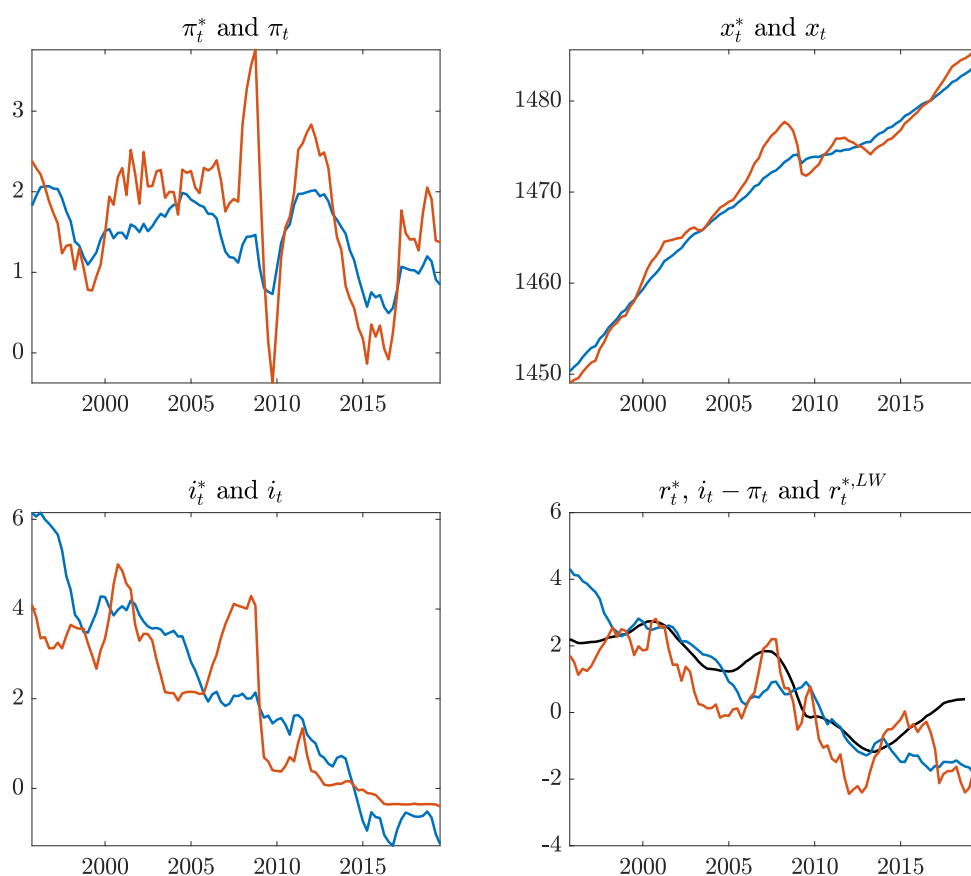


Note: The figure compares our baseline term premia estimates, denoted $AFNS^*$ and those derived including survey information, denoted $AFNS^*_{surv}$, for the 2y (left) and 10y (right) bonds with the min-max-range several estimates in the literature, including Kim and Wright (2005) (taken from FRED), Adrian et al. (2013) and a DNS model following Diebold and Li (2006) (both according to our own estimates).

euro area are illustrated in Figure 9. Uncertainty around these states appears somewhat higher than for the United States, yet given the significantly shorter estimation sample, this comes as no surprise. The results suggest that quarterly potential growth has fallen from around 0.6% in the mid 1990s to about 0.2% in 2010, and recovering only marginally to 0.3% since then. Both the inflation and output gap show a consistent cyclical pattern and align well with the official OECD recession dates (see Annex C). Similar to the United States, the yield curve factors in the euro area line up tightly with their counterparts from a yields-only AFNS model, with the variance for the level factor being larger than for either the slope or curvature factor (see Figure 10).

Our term-structure estimates of r^* exhibit a trend that is broadly comparable to those from

Figure 8: Euro area macroeconomic observables and trends

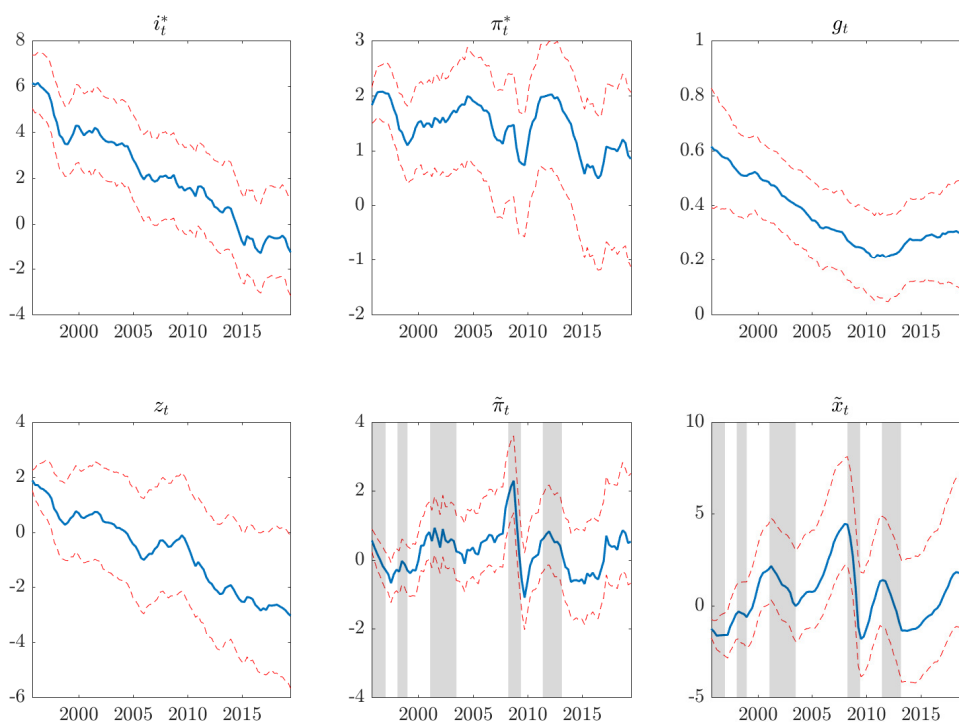


Note: The figure shows the estimated trends (in blue) and the observed macro-variables (in red). The black solid line reports our own Bayesian estimates of the specification in [Holston et al. \(2017\)](#).

the benchmark [Holston et al.](#) model as illustrated in [Figure 11](#). As a result of our estimates following the real rate process more closely, r_t^* is estimated to have fallen from around 4% in the mid 1990s into negative territory recently. Similar to the United States, we again observe a widening gap between our baseline estimates and those that include survey information starting around 2005 (see [Figure 5](#)). Including survey measures, r_t^* increases to around 0% at the sample end in 2019 Q2. While these real equilibrium rate estimates are exceptionally low, they are consistent with a growing literature reporting estimates around zero or below for advanced economies ([Gourinchas and Rey, 2019](#); [Fiorentini et al., 2018](#); [Kiley, 2019](#); [Jorda and Taylor, 2019](#)).

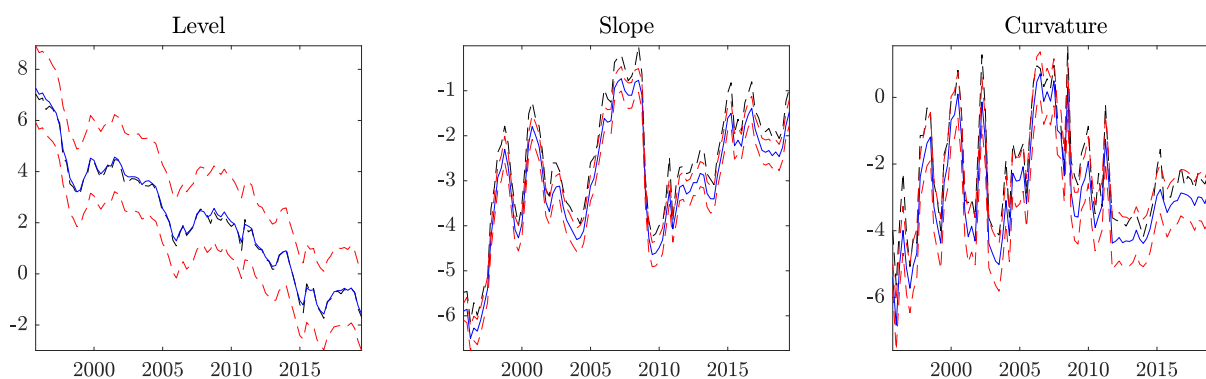
For the euro area, we find that the uncertainty bands around the r^* estimates based on our macro-finance approach are much smaller than corresponding estimates based on the macro-only model by [Holston et al.](#): The width of the uncertainty band resulting from the Bayesian estimation of our macro-finance model is around four percentage points on average, relative to staggering 12.5 percentage points obtained from estimating the macro-only specification. This

Figure 9: euro area latent macro variables



Note: The figure shows the estimated latent states of the model in blue together with their 5% and 95% credibility bands in red. Shaded areas represent CEPR recessions.

Figure 10: Euro area yield curve factors

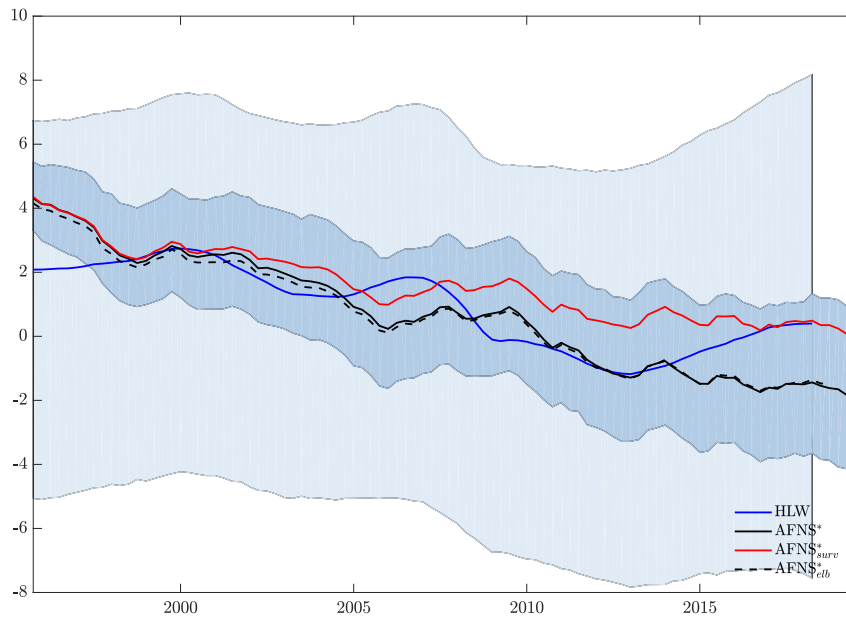


Note: The figure shows the yield curve factors in blue with respective 5% and 95% credibility bands in red. For comparison, the black-dashed lines represent yield curve factors obtained from a yields-only AFNS model.

may be surprising, given that the estimation sample of the our model is substantially smaller, covering only 1995 Q2-2019 Q2 instead of 1972 Q3-2018 Q1 in the original model. To be sure, the differences in r^* uncertainty reflect the impact of changes in parameter estimates, in particular in the cyclical interaction of the macro-economic gap measures. As we happen to estimate the IS and the Phillips curve to be measurably steeper for the euro area than reported in Holston et al. (2017) parameter-specific filtering uncertainty will be smaller, again as expected on the

basis of Fiorentini et al. (2018).

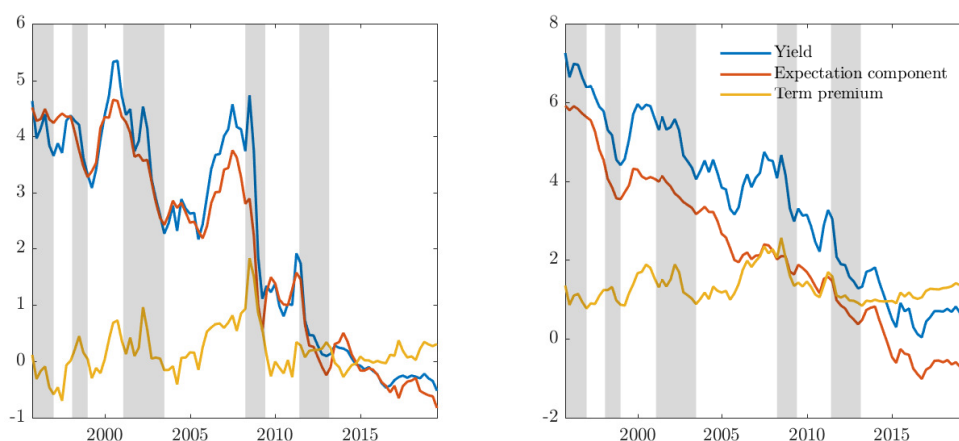
Figure 11: Euro area natural rate estimates



Note: The figure shows the natural rate estimate in black with 5% and 95% credibility bands depicted by the blue-shaded area. The red-dashed line is our r_t^* estimate when including surveys. For comparison, our Bayesian replication of the Holston et al. (2017) estimate is in blue with 5% and 95% credibility bands given by the light-blue-shaded area. Our term-structure based estimates are obtained from the sample period 1995 Q2-2019 Q2 and the replication of Holston et al. (2017) is based on the sample period 1972 Q3-2018 Q1. The dashed black line corresponds to the natural rate estimate when accounting for the effective lower bound as explained in section 2.4

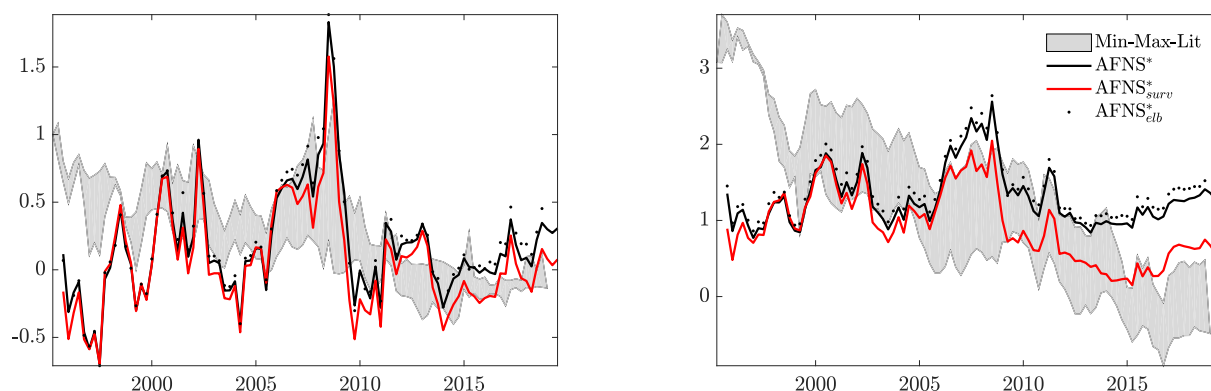
Figure 12 decomposes the 2 and 10-year yields into their expectations and term premium components. Also for the euro area, term premia show a countercyclical pattern. Moreover, the fall in the expectations component explains, by and large, the fall in yields in the period succeeding the the global financial crisis. Figure 13 further illustrates the impact of accounting for secular macroeconomic trends on term premia estimates for the euro area relative to commonly used approaches. As with our results for the United States, across both modeling classes, the timing of peaks and troughs largely coincides, marked specifically by the slump in estimated term premia with the start of the global financial crisis. But there are considerable differences in levels, with our estimates being more volatile (especially at the short-end, illustrated here using the 2 year rate in the left panel of Figure 13). Importantly, they do not show the consistent fall as of 1995 exhibited by term premia estimates from Geiger and Schupp (2018), a Diebold and Li (2006) DNS model, or Adrian et al. (2013), but exhibit cyclical dynamics.

Figure 12: Decomposition of euro area yields



Note: The figure shows the decomposition of the 2-year (left) and 10-year (right) yields in blue into expectation component (red) and the term premium (yellow). Own calculations. OECD recessions in gray.

Figure 13: Comparing euro area term premia



Note: The figure compares our baseline term premia estimates, denoted $AFNS^*$ and those derived including survey information, denoted $AFNS^*_{surv}$, for the 2y (left) and 10y (right) bonds with the min-max-range several estimates in the literature, including estimates from Geiger and Schupp (2018), and estimates from Adrian et al. (2013) and Diebold and Li (2006) (both own estimates).

5 Conclusion

In this paper, we join two strands of literature: on the one hand, arbitrage-free models of bond yield dynamics incorporating a time-varying attractor (“shifting end point”) for short rate expectations over long horizons – most recently exemplified by the frontier contribution of Bauer and Rudebusch (2019); on the other hand, semi-structural macro models inferring the location and dynamics of the natural real rate of interest – the most prominent example being Holston et al. (2017). Our proposed model captures the joint dynamics of key macroeconomic variables following Holston et al. (2017). Different from their paper, we do not treat the short-term nominal interest rate as exogenous but rather endogenize it by modeling its dynamics as part of a complete arbitrage-free specification of the term structure. The nexus between the

macro and the term structure building blocks of our model is the natural real rate. Relative to its position the actual real rate drives the business cycle; at the same time – together with core inflation – it constitutes the underlying trend of the level of the yield curve.

Paired with a Bayesian estimation approach, our framework allows for simultaneous estimation of key unobservable macro objects like the natural real rate of interest, trend inflation and the output gap, as well as unobservable term premia incorporated in long-term bonds. The joint estimation and quantification of uncertainty distinguishes our method from most other studies in the aforementioned literature that tend to rather rely on multi-step approaches or treating estimates of latent factors as observables.

We validate evidence of a recent decline in the natural rate of interest in advanced economies to levels around zero or into negative territory as reported, e.g. in [Gourinchas and Rey \(2019\)](#); [Fiorentini et al. \(2018\)](#); [Kiley \(2019\)](#); [Jorda and Taylor \(2019\)](#). But our estimates of the natural real rate can deviate from those reported in [Holston et al. \(2017\)](#), as our specification enforces the real rate gap (actual minus equilibrium rate) to mean-revert around zero, a restriction that is absent in their specification.

Trends in nominal rates are also affected by low-frequency movements in inflation. In the United States, the five-decade long gyrations in core inflation measures reflect the Great Inflation, its conquest, and the current environment of low and stable inflation. In both currency areas, core inflation is estimated to have been tepid in recent years.

Consistent with [Bauer and Rudebusch \(2019\)](#), we find that taking into account the secular fall in equilibrium rates, term premia exhibit cyclical behavior over the business cycle, rather than the trend decline reported when using term structure models with a constant steady state.

Our model makes strides towards a proper integration of macro and yield curve dynamics, here with a focus on the natural real rate. Yet, in our view, three further aspects of the semi-structural macro model require further, separate in-depth research: overcoming the staggering imprecision of semi-structural r^* estimates; a better, structurally-informed estimate of the components driving r^* ; and capturing monetary policy easing effects from unconventional policies.

Imprecision of semi-structural r^* estimates is rooted in ill-identified Phillips and aggregate demand curves, likely on account of constraining shocks to the state variables to be orthogonal and homoskedastic. For example, the existing model – following [Laubach and Williams \(2003\)](#) – imposes oil price shocks to be fully reflected in inflation, leaving output contemporaneously unaffected. Similarly, such shocks have produced volatility in inflation and output that is larger and more autocorrelated than demand-side shocks. Preventing the model to pick

up that feature by imposing homoskedasticity may well hinder proper identification of supply and demand-side shocks.

Moreover, an approach of modeling r^* on potential output growth rates and an additional random walk component to capture the persistent gap between trends in growth and the return on safe assets is insufficiently informative of drivers of the natural rate. Overcoming this agnostic approach requires a more structural modeling of trends in working-age population growth, labour and total factor productivity, the demand for productive capital, global saving-investment imbalances, and differences in the pricing of assets across risk classes.

Finally, neglecting effective monetary stimulus from quantitative easing can erroneously lead to estimating too negative a real rate gap and thereby induce an upward bias in r^* estimates. Further extensions would be required to capture such easing effects along the yield curve, requiring changes to the aggregate demand equations and considerations to what extent quantitative easing works through a compression of term premia or along the signalling channel, respectively (see [Li and Wei, 2013](#); [Eser et al., 2019](#)).

We hope to have provided a valid base for such future research.

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A Data

The following table provides an overview of the quarterly data used in this study. For the United States, inflation and GDP data are taken from the FRED-database of the Federal Reserve Bank of St. Louis and yields from [Gürkaynak et al. \(2007\)](#). Sources for euro area data are the ECB's Statistical Data Warehouse, Deutsche Bundesbank, Bloomberg, and Consensus Economics. ACRONYMS refer to codes in the respective databases. Synthetic, pre-1999 euro area data are in fixed composition of member countries (except for HICP which is in full composition for completing the series over the sample period starting 1995-97). The overall sample period covers 1961 Q2–2019 Q2 for the United States and 1995 Q1–2019 Q2 for the euro area.

Table 2: Data used in this study

Variable	US	EA
Consensus long-term expectations of long-term interest rates	Consensus Economics (as of 1989 Q2)	Consensus Economics (as of 1995 Q2)
GDP	GDPC1	MNA.Q.Y.I8.W2.S1.S1.B.B1GQ.Z.Z.Z.EUR.LR.N
Quarter-end zeros-coupon yields	Data by Gürkaynak et al. (2007)	Zero-coupon yields on German government bonds up to 2005 Q4, subsequently midquotes from OIS bid and ask: FM.B.U2.EUR.RT.SI.EUREON3M..ask or FM.B.U2.EUR.RT.SI.EUREON3M..bid, etc. (Sources: Deutsche Bundesbank, ECB Statistical Data Warehouse) .

Continued on next page

Table 2 – *Continued from previous page*

Variable	US	EA
Consumer Prices, all items	CPIAUCSL which is seasonally adjusted	HICP – ICP.M.U2.N.000000.4.INX seasonally adjusted using X-12-ARIMA for data 1995 Q1–1997 Q1, subsequently seasonally adjusted series ICP.M.U2.Y.000000.3.INX.

End of Table

B The state space model

B.1 The baseline model without surveys

The observation equations are given by:

$$\begin{aligned}
 y_t(\tau_i) &= \mathcal{A}(\tau_i) + \tilde{L}_t + L_t^* + \theta_s(\tau_i)S_t + \theta_c(\tau_i)C_t + u_t^{\tau_i}, \quad u_t^{\tau_i} \sim NID(0, \sigma_{\tau_i}^2), \quad i = 1, \dots, K \\
 x_t &= x_t^* + \tilde{x}_t, \\
 \pi_t &= \pi_t^* + \tilde{\pi}_t,
 \end{aligned}$$

where $\theta_s(\tau)$ and $\theta_c(\tau)$ are the Nelson-Siegel loadings defined in the main text, $\mathcal{A}(\tau)$ is defined in the next section, and $L_t^* = r_t^* + \pi_t^*$. We allow for different measurement error variances across observed yields, but assume the short-term rate $i_t \equiv y_t(1)$ to be matched without error, i.e. $\sigma_{\tau_1}^2 = 0$.

The state equations are given by:

$$\begin{aligned}
 \tilde{L}_t &= a_L \tilde{L}_{t-1} + \varepsilon_t^{\tilde{L}}, \\
 S_t &= a_{10} + a_{11}S_{t-1} + a_{12}C_{t-1} + a_{13}\tilde{\pi}_{t-1} + a_{14}\tilde{x}_{t-1} + \varepsilon_t^S, \\
 C_t &= a_{20} + a_{21}S_{t-1} + a_{22}C_{t-1} + a_{23}\tilde{\pi}_{t-1} + a_{24}\tilde{x}_{t-1} + \varepsilon_t^C, \\
 \pi_t^* &= \pi_{t-1}^* + \varepsilon_t^{\pi^*}, \\
 x_t^* &= x_{t-1}^* + g_{t-1} + \varepsilon_t^{x^*}, \\
 g_t &= g_{t-1} + \varepsilon_t^g, \\
 z_t &= z_{t-1} + \varepsilon_t^z, \\
 \tilde{\pi}_t &= b_1 \tilde{\pi}_{t-1} + b_2 \tilde{x}_{t-1} + \varepsilon_t^{\tilde{\pi}}, \\
 \tilde{x}_t &= a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + \frac{a_3}{2} (\tilde{r}_{t-1} + \tilde{r}_{t-2}) + \varepsilon_t^{\tilde{x}}.
 \end{aligned}$$

Given that both the inflation $\tilde{\pi}_t$ and output gap \tilde{y}_t are mean-zero by construction, we have

$$\begin{pmatrix} \bar{S} \\ \bar{C} \end{pmatrix} = \left(\mathbf{I}_2 - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right)^{-1} \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}.$$

In order to calculate the real rate, $r_t = i_t - E_t \pi_{t+1}$, we assume expectations to be model-consistent. Taking conditional expectations of (6) gives

$$E_t \pi_{t+1} = E_t [\pi_{t+1}^* + \tilde{\pi}_{t+1}] = \pi_t^* + b_1 \tilde{\pi}_t + b_2 \tilde{x}_t.$$

Substitution yields

$$r_t \equiv y_t(1) - E_t \pi_{t+1} = y_t(1) - \pi_t^* - b_1 \tilde{\pi}_t - b_2 \tilde{x}_t,$$

and, accordingly, the real rate gap is given by

$$\begin{aligned} \tilde{r}_t &= r_t - r_t^* \\ &= y_t(1) - \pi_t^* - b_1 \tilde{\pi}_t - b_2 \tilde{x}_t - r_t^* \\ &= r_t^* + \pi_t^* + \tilde{L}_t + \theta_s(1)[S_t - \bar{S}] + \theta_c(1)[C_t - \bar{C}] - \pi_t^* - b_1 \tilde{\pi}_t - b_2 \tilde{x}_t - r_t^* \\ &= \tilde{L}_t + \theta_s(1)[S_t - \bar{S}] + \theta_c(1)[C_t - \bar{C}] - b_1 \tilde{\pi}_t - b_2 \tilde{x}_t. \end{aligned} \quad (23)$$

Finally, substituting the latter equation into the IS curve, we have

$$\begin{aligned} \tilde{x}_t &= a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + \frac{a_3}{2} (\tilde{r}_{t-1} + \tilde{r}_{t-2}) + \varepsilon_t^{\tilde{x}} \\ &= a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + \frac{a_3}{2} (\tilde{L}_{t-1} + \theta_s(1)[S_{t-1} - \bar{S}] + \theta_c(1)[C_{t-1} - \bar{C}] - b_1 \tilde{\pi}_{t-1} - b_2 \tilde{x}_{t-1}) \\ &\quad + \frac{a_3}{2} (\tilde{L}_{t-2} + \theta_s(1)[S_{t-2} - \bar{S}] + \theta_c(1)[C_{t-2} - \bar{C}] - b_1 \tilde{\pi}_{t-2} - b_2 \tilde{x}_{t-2}) + \varepsilon_t^{\tilde{x}} \\ &= (a_1 - \frac{a_3 b_2}{2}) \tilde{x}_{t-1} + (a_1 - \frac{a_3 b_2}{2}) \tilde{x}_{t-2} \\ &\quad + \frac{a_3}{2} (\tilde{L}_{t-1} + \theta_s(1)[S_{t-1} - \bar{S}] + \theta_c(1)[C_{t-1} - \bar{C}] - b_1 \tilde{\pi}_{t-1}) \\ &\quad + \frac{a_3}{2} (\tilde{L}_{t-2} + \theta_s(1)[S_{t-2} - \bar{S}] + \theta_c(1)[C_{t-2} - \bar{C}] - b_1 \tilde{\pi}_{t-2}) + \varepsilon_t^{\tilde{x}}. \end{aligned}$$

In state-space representation, the model can be written as

$$\zeta_t = \gamma + C\xi_t + Du_t \quad \text{with } u_t \sim NID(0, 1) \quad (24)$$

$$\xi_t = \mu + F\xi_{t-1} + Ge_t \quad \text{with } e_t \sim NID(0, 1), \quad (25)$$

where

$$\zeta_t = \left(y_t(\tau_1) \quad \dots \quad y_t(\tau_K) \quad x_t \quad \pi_t \right)',$$

and

$$\xi_t = \left(\tilde{L}_t \quad S_t \quad C_t \quad \pi_t^* \quad x_t^* \quad g_t \quad z_t \quad \tilde{\pi}_t \quad \tilde{x}_t \quad \tilde{L}_{t-1} \quad S_{t-1} \quad C_{t-1} \quad \tilde{\pi}_{t-1} \quad \tilde{x}_{t-1} \right)'.$$

B.2 Parameter restrictions to rule out arbitrage in the dynamic Nelson-Siegel model

We explain the no-arbitrage adjustment term $\mathcal{A}(\tau)$ in the yield equation (8). As shown by Christensen et al. (2011) and, in a discrete-time setting, Li et al. (2012), pricing bonds under a specific choice of risk-neutral factor dynamics renders the joint dynamics of bond yields arbitrage-free, gives rise to factor loadings having the Nelson-Siegel functional form, but implies an additional intercept term that is not present in the standard - statistically motivated - Nelson-Siegel formulation.

Starting from the definition of the state variable ξ_t as in the previous section, we define a factor vector $F_t = [L_t, \bar{\xi}_t]$, where $\bar{\xi}_t$ equals our state vector ξ_t except that the first three elements are re-shuffled so that \tilde{L} appears after the slope and curvature factor S and C . The so-constructed factor vector F_t has the three Nelson-Siegel factors L_t , S_t and C_t lining up upfront. Note further that L results as a linear combination of the states \tilde{L} , g , z and π^* .⁷ We further group $F_t = [F_t^u F_t^o]$ with $F_t^u = [L_t, S_t, C_t]$ and F_t^o capturing the rest of the variables. Based on that partitioning of factors we represent the short-rate equation (10) as

$$i_t = \delta_0 + \delta'_u F_t^u + \delta'_m F_t^m = \delta' F_t$$

with obvious notation. Let $P_t(\tau)$ denote the time- t price of a zero-coupon bond with residual maturity τ . If there are risk-neutral factor dynamics (labelled by \mathbb{Q})

$$F_t = c^{\mathbb{Q}} + \Phi^{\mathbb{Q}} F_{t-1} + v_t^{\mathbb{Q}}, \quad v_t^{\mathbb{Q}} \sim N(0, \Omega) \quad (27)$$

so that bond prices satisfy

$$P_t(\tau) = e^{-i_t} E_t^{\mathbb{Q}} P_{t+1}(\tau - 1), \quad P_t(0) = 1,$$

then the joint evolution of bond prices is arbitrage-free. Moreover, the solution to the pricing equation is exponentially affine in factors

$$P_t(\tau) = \exp(a(\tau) + b(\tau)' F_t)$$

where coefficients $a(\tau)$ and $b(\tau)$ satisfy the well-known difference equations

$$\begin{aligned} a(\tau + 1) &= a(\tau) + b(\tau)' c^{\mathbb{Q}} + \frac{1}{2} b(\tau)' \Omega b(\tau) - \delta_0 \\ b(\tau + 1)' &= b(\tau)' \Phi^{\mathbb{Q}} - \delta', \end{aligned}$$

⁷As $L_t = \tilde{L}_t + i_t^* = \tilde{L}_t + 4g_t + z_t + \pi_t^*$.

with $a(1) = -\delta_0$ and $b(1) = -\delta$. Moreover, as shown by Li et al. (2012), if $\Phi^{\mathbb{Q}}$ is of the form

$$\Phi^{\mathbb{Q}} = \begin{pmatrix} \Phi_{uu}^{\mathbb{Q}} & 0 \\ \Phi_{mu}^{\mathbb{Q}} & \Phi_{mm}^{\mathbb{Q}} \end{pmatrix}, \quad \Phi_{uu}^{\mathbb{Q}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & \lambda e^{-\lambda} \\ 0 & 0 & e^{-\lambda} \end{pmatrix},$$

then $b(\tau)$ exhibits the specific Nelson-Siegel loadings (in price space) for the first three factors L , S and C , and zero on the other factors,

$$b(\tau) = [-n, -\frac{1 - e^{-\lambda n}}{\lambda}, ne^{-\lambda n} - \frac{1 - e^{-\lambda n}}{\lambda}, 0, \dots, 0]'$$

In addition, the zero restrictions on $\Phi^{\mathbb{Q}}$ imply that the expression for $a(\tau)$ simplifies to

$$a(\tau + 1) = a(\tau) + b(\tau)'c^{\mathbb{Q}} + \frac{1}{2}b_u(\tau)'\Omega_{uu}b_u(\tau) - \delta_0, \quad (28)$$

where $b_u(\tau)$ contains the first three elements of $b(\tau)$ and Ω_{uu} is the upper 3-by-3 block of Ω .

Recalling that $F_t = [L_t, \bar{\xi}_t]$ is just an extension of our state vector ξ_t , the transition equation for F_t is readily derived from that of ξ_t described in the previous section. It is affine, as the stipulated (unobserved) risk-neutral dynamics in (27) above, but depends on the physical (no \mathbb{Q} label) parameters:

$$F_t = c + \Phi F_{t-1} + v_t, \quad v_t \sim N(0, \Omega)$$

The variance-covariance matrix Ω of state innovations is the same under both the risk-neutral and the physical measure. For our factor vector $F_t = [L_t, \xi_t]$ it follows from the dynamics of ξ_t and the link of L_t to \tilde{L} , z_t , g_t and π_t^* that Ω_{uu} in (28) is given by

$$\Omega_{uu} = \text{diag}(\sigma_L^2 + \sigma_{\pi^*}^2 + 16\sigma_g^2 + \sigma_z^2, \sigma_s^2, \sigma_c^2),$$

where σ_i^2 denotes the variance of the innovation ε_t^i of variable i in our model. Parameters governing the risk-neutral and physical dynamics are linked as

$$c^{\mathbb{Q}} = c - \Omega^{0.5}\lambda_0, \quad \Phi^{\mathbb{Q}} = \Phi - \Omega^{0.5}\Lambda$$

where λ_0 and Λ ('market prices of risk') are a vector and a matrix, respectively, of appropriate dimension.

Mapping bond prices into yields using $y_t(\tau) = -\frac{1}{\tau} \ln P_t(\tau)$, we have

$$y_t(\tau) = \mathcal{A}(\tau) + \mathcal{B}(\tau)'F_t$$

where $\mathcal{A}(\tau) = -\frac{1}{\tau}a(\tau)$ and $\mathcal{B}(\tau) = -\frac{1}{\tau}b(\tau)$. That is, $\mathcal{B}(\tau)$ has now the Nelson-Siegel loadings for bond yields as the first three entries, and $\mathcal{A}(\tau)$ is the intercept appearing in (8).

The risk-neutral dynamics and cross-sectional pricing equations are parsimoniously parameterized. The Nelson-Siegel tuning parameter λ is calibrated as described in the main text. The relevant variance-covariance matrix Ω_{uu} is implied by the time series estimates under the physical measure as explained above. As we are working with latent factors, the parameter δ_0 in the short-rate equation is not identified and can be arbitrarily calibrated. While it is common to set it to zero, we choose to set $\delta_0 = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ so that (as $a(1) = -\delta_0$) $\mathcal{A}(1) = -a(1) = -\theta_s(1)\bar{S} - \theta_c(1)\bar{C}$ as specified in the main text. Finally, we set the risk-neutral VAR intercept $c^{\mathbb{Q}}$ equal to zero. This is a somewhat ad-hoc choice to prevent additional parameters to enter our setup and is tantamount to imposing a restriction on the market price of risk vector λ_0 . While under that specific choice of $c^{\mathbb{Q}}$ model-implied bond yield dynamics are arbitrage-free, it is eventually an empirical question, whether $c^{\mathbb{Q}} = 0$ is an overly restrictive assumption. Via its impact on $\mathcal{A}(\tau)$, the choice of $c^{\mathbb{Q}}$ affects the (average) slope of the yield curve as argued in the main text. It turns out empirically that the model fits the average slope in the data fairly well so that the parameter restriction appears non-problematic from this perspective.

B.3 Model extension using survey data

The model featuring survey-expectations needs to be expanded. We incorporate long-term Consensus expectations of the 10-year rate 6-10 years in the future, denoted $\hat{E}_t^{sur} y(40)$, to inform estimation of L_t^* . As explained in the main text, treating the 6-10 year horizon as ‘very long’ the model-implied counterpart to the survey data is $\mathcal{A}(40) + \theta_s(40)\bar{S} + \theta_c(40)\bar{C} + L_t^*$.

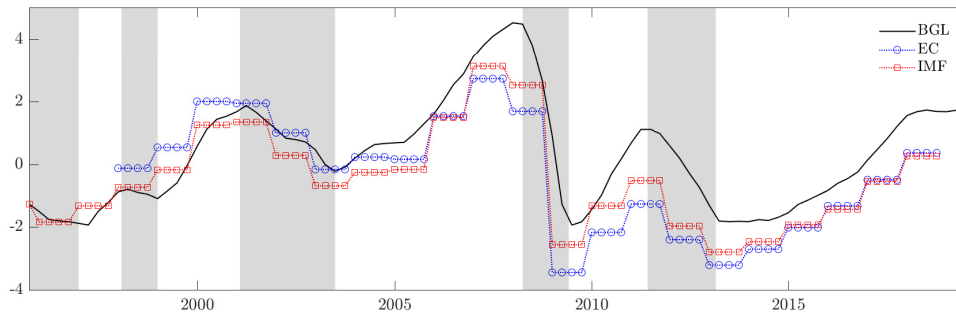
Accordingly, the vector of observables changes to

$$\zeta_t = \left(y_t(\tau_1) \quad \dots \quad y_t(\tau_K) \quad x_t \quad \pi_t \quad \hat{E}_t^{sur} y(40) \right)',$$

with the intercept in the observation equation becoming

$$\gamma = \left(\mathcal{A}(\tau_1) \quad \dots \quad \mathcal{A}(\tau_K) \quad 0 \quad 0 \quad \mathcal{A}(40) + \theta_s(40)\bar{S} + \theta_c(40)\bar{C} \right)',$$

Figure 15: Output gaps in the euro area: Comparison to official estimates

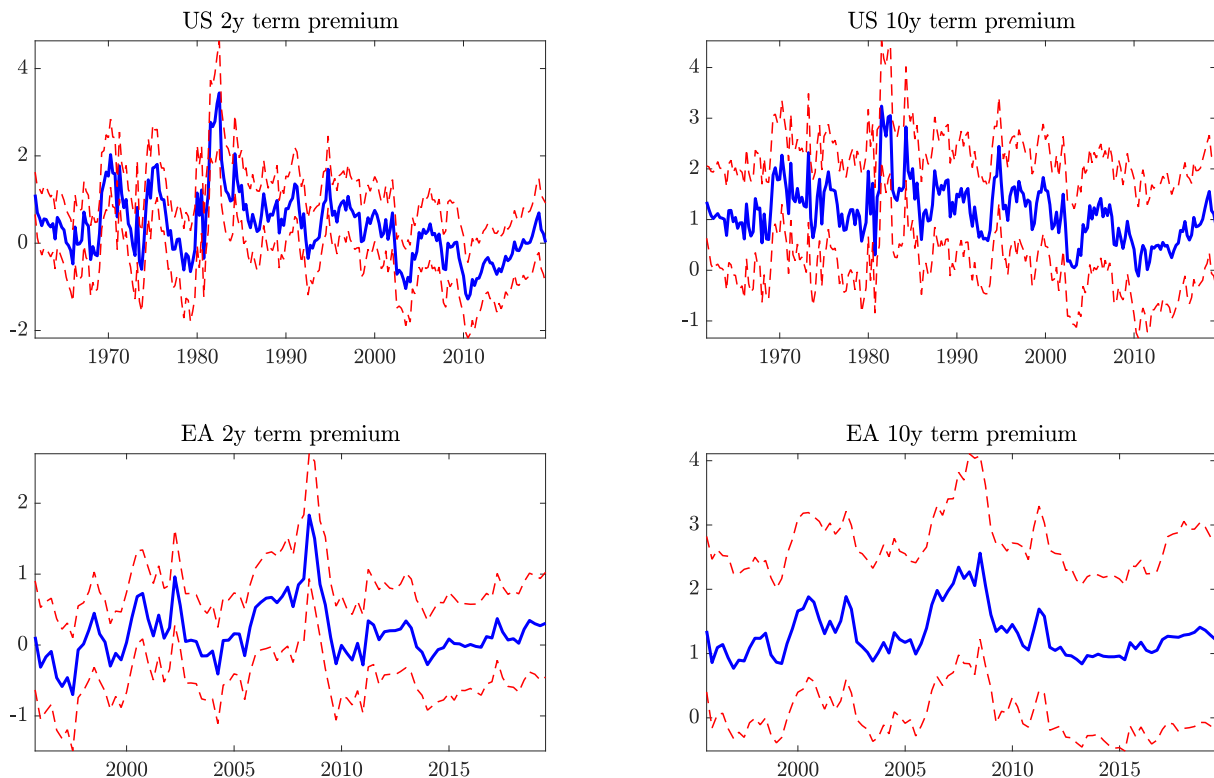


Note: The figure shows institutional output gap measures for the euro area from the European Commission (EC) and the IMF against our model-based estimates. OECD recessions in gray.

C.2 Uncertainty surrounding term premia estimates

Figure 16 shows the 5 and 95% credibility bands of the 2 and 10 year term premia for the United States and the euro area. The width of the credibility bands is slightly larger for longer maturities, and generally smaller for the United States, where the data sample is significantly longer. However, the overall statistical uncertainty about term premia turns out to be sufficiently contained to judge that – on the whole – term premia in long-term yields have not been systematically negative over the last ten years.

Figure 16: Credibility bands around term premia estimates



Note: The figure shows 2 and 10-year term premium estimates for the United States (top) and euro area (bottom) with their 5% and 95% credibility bands.

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