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The n-Player Hirshleifer Contest

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The n -Player Hirshleifer Contest

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Abstract. We characterize the equilibrium set of the n -player Hirshleifer contest with homogeneous valuations. A symmetric equilibrium always exists. It necessarily corresponds to multilateral peace for sufficient noise and uses finite-support randomized strategies otherwise. Asymmetric equilibria are feasible for $n \geq 3$ contestants only, and only for sufficiently small noise. In pure strategies, any asymmetric equilibrium corresponds to one-sided dominance, but there is also a variety of payoff-inequivalent mixed-strategy equilibria for small noise. For arbitrarily small noise, at least two contestants engage in cut-throat competition, while any others become ultimately inactive. Of some conceptual interest is the observation that, for n sufficiently large, the unique equilibrium is multilateral peace.

Keywords. Hirshleifer contest · Nash equilibrium · Rent dissipation · Difference-form contest · All-pay auction

JEL codes. C72, D72, D74

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1. Introduction

Recent years have witnessed a tremendous surge in interest in the game-theoretic analysis of contests.¹ Much of this interest has focused on specific classes of contest technologies that admit a simple axiomatic characterization.² In particular, this includes the Hirshleifer contest (Hirshleifer, 1989), which is one of the canonical models of the difference-form contest. Indeed, within the class of difference-form contests, the Hirshleifer contest is highlighted by the fact that it admits a simple alternative representation in the logit form.

In his seminal contribution, Hirshleifer (1989) identified two main types of pure-strategy Nash equilibria (PSNE) between two contestants, viz. bilateral peace and one-sided dominance. He also offered an insightful discussion of mixed-strategy Nash equilibria (MSNE), yet without explicitly characterizing them. In earlier work (Ewerhart and Sun, 2018), we have shown that the two-player Hirshleifer contest with homogeneous valuations generally admits a unique Nash equilibrium. We also provided an explicit characterization of the Nash equilibrium, which is necessarily symmetric. However, the Hirshleifer contest with more than two contestants has not been studied extensively so far.³

This paper examines the equilibrium set of the n -player Hirshleifer contest with homogeneous valuations. Given that the contest technology exhibits increasing returns to scale, a PSNE is not guaranteed to exist in general. However, a MSNE can be shown to exist for any parameter constellation. Considered are,

¹For an introduction to the theory of contests, see Konrad (2009) or Vojnović (2015). A recent survey is Corchón and Serena (2018).

²See Skaperdas (1996) and Ewerhart (2015b), respectively, for axiom systems applying to a population of varying and constant size. Cubel and Sanchez-Pages (2016) axiomatized difference-form contest success functions more generally.

³We are aware of only three papers that discuss the n -player Hirshleifer contest. To start with, Hirshleifer (1989, pp. 104-105) himself introduced the n -player generalization, and noted the equivalence of representations (1) and (2) below. However, he did not discuss equilibria for $n \geq 3$. Further, the two papers pursuing axiomatic characterizations mentioned before both considered the n -player case, yet likewise did not explore the equilibrium set.

therefore, both PSNE and MSNE. We prepare the equilibrium analysis by making three initial observations. First, as payoffs in the Hirshleifer contest are analytic functions, any randomized best response necessarily has finite support. Second, in any equilibrium, at least $(n - 1)$ of the contestants use the zero bid with positive probability. Third and last, if a contestant i always submits positive bids, then no positive bid is placed by her opponents weakly below contestant i 's lowest bid. Taken together, these results set the stage for our in-depth examination of the equilibrium set.

The main analysis starts by considering symmetric equilibria. Symmetric PSNE necessarily corresponds to *multilateral peace*. Multilateral peace is consistent with any number $n \geq 2$ of contestants, yet only if there is sufficient noise. In fact, multilateral peace is the unique symmetric MSNE whenever multilateral peace is a PSNE. This result is obtained by a new type of argument that exploits, in particular, that bids are strategic substitutes in the relevant domain. For small noise, we show that any symmetric equilibrium is in mixed strategies, where the number of mass points in the distribution of equilibrium bids is bound to grow as the contest technology becomes increasingly deterministic.

Next, we deal with asymmetric equilibria. These are feasible for $n \geq 3$ contestants only, and only for sufficiently small noise. Any asymmetric PSNE reflects *one-sided dominance*, i.e., precisely one contestant is active.⁴ As the identity of the dominant contestant in this type of equilibrium is undetermined, there exist multiple, payoff-inequivalent PSNE. Yet the pure-strategy outcome with just one active bidder is feasible only if the noise parameter is in an intermediate range. For smaller noise, a variety of asymmetric MSNE arise, which may be either semi-mixed or mixed. In a knife-edge case, there is even a continuum of asymmetric

⁴Thus, in this regard, the n -player Hirshleifer contest with more than two contestants has a certain resemblance to a natural monopoly.

MSNE. However, we also construct several robust examples of asymmetric MSNE.

We go on and study the case of arbitrarily small noise. It is shown that, as the technology becomes increasingly deterministic, at least two contestants engage in cut-throat competition, i.e., bid nearly up to the value of the prize. As a consequence, the undissipated rent goes to zero for all contestants. Next, for at least two contestants, which are possibly the same as those that bid up to the value of the prize, the probability weight placed on the zero bid goes to zero. Further, any contestant that does not engage in cut-throat competition becomes ultimately inactive in the limit. Finally, as the noise vanishes, any sequence of MSNE in the Hirshleifer contest that converges in distribution approaches a MSNE of the standard all-pay auction.

The analysis is complemented by a simple but potentially important observation for the case of large populations. Specifically, keeping the decisiveness parameter α fixed, we find that, if the number of contestants n is sufficiently large, then the unique PSNE and even the unique MSNE in the n -player Hirshleifer contest is multilateral peace.⁵ We will conclude that, for more than two contestants, the Hirshleifer technology does not seem to serve well as a model of military conflict.

General classes of difference-form contests have been analyzed for somewhat more than two decades (Baik, 1998). Che and Gale (2000) were the first in comprehensively characterizing equilibria for a class of contests of the difference form with uniform noise. More recently, Cubel and Sanchez-Pages (2020) have generalized that analysis by allowing for more than two contestants and more flexible difference-form contests. However, none of those papers touches upon the questions addressed in the present study.

The remainder of this paper is structured as follows. Section 2 introduces

⁵This observation contrasts, in particular, with the case of the standard n -player Tullock contest where, regardless of n , all contestants are active in the symmetric PSNE (Corcoran, 1988).

the notation and collects some preliminary observations. Symmetric equilibria are characterized in Section 3. Section 4 deals with asymmetric equilibria. The case of arbitrarily small noise is studied in Section 5. Section 6 discusses the equilibrium in large populations. Section 7 concludes. All technical proofs have been relegated to an Appendix.

2. Preliminaries

2.1 Set-up and notation

There are $n \geq 2$ contestants (or players), collected in a set $N = \{1, \dots, n\}$, that expend resources to win a prize of value $V > 0$. Contestant i 's payoff in the n -player Hirshleifer contest with parameter $\alpha > 0$ is given as

$$\Pi_i^{N,\alpha}(x_1, \dots, x_n) = \frac{V}{\sum_{j=1}^n \exp(\alpha(x_j - x_i))} - x_i, \quad (1)$$

where $x_j \geq 0$, for $j \in N$, denotes contestant j 's expenses (or bid). The parameter α measures the decisiveness of the contest technology. It is easy to see that, as $\alpha \rightarrow 0$, the contest converges to the limit case of a pure lottery, where decisions about expenses do not matter and the winner is determined by chance alone. As $\alpha \rightarrow \infty$, however, the vector of payoffs approximates that of the standard all-pay auction, where the highest bidder wins with certainty (Baye et al., 1996).⁶ Thus, intuitively, chance plays a larger role in the determination of the winner when α is small. We rewrite relationship (1) and obtain the equivalent logit representation of contestant i 's payoffs as

$$\Pi_i^{N,\alpha}(x_1, \dots, x_n) = \frac{\exp(\alpha x_i)V}{\sum_{j=1}^n \exp(\alpha x_j)} - x_i. \quad (2)$$

It is noteworthy that the impact function $x_i \mapsto X_i \equiv \exp(\alpha x_i)$ exhibits strictly increasing returns, i.e., it is strictly convex. Moreover, as mentioned in the Intro-

⁶In fact, the case of homogeneous valuations, which corresponds more closely to our analysis, is analyzed in the working paper version (Baye et al., 1990).

duction, the Hirshleifer contest may be characterized by imposing a collection of simple axioms on its CSF.

In the non-cooperative n -player game defined above, a *pure-strategy Nash equilibrium* (PSNE) is a vector of bids, $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}_{\geq 0}^n$, such that

$$\Pi_i^{N,\alpha}(x_i^*, x_{-i}^*) \geq \Pi_i^{N,\alpha}(x_i, x_{-i}^*) \quad (3)$$

holds for any $i \in N$ and $x_i \in \mathbb{R}_{\geq 0}$, where we adhere to the usual convention that $x^* = (x_i^*, x_{-i}^*)$, etc. A mixed strategy for contestant i is defined as a probability measure μ_i on (the Borel subsets of) the interval $[0, V]$.⁷ Let M_i denote the set of mixed strategies for contestant i , where pure strategies $x_i \in [0, V]$ are understood as Dirac probability measures, as usual. Contestant i 's expected payoff from a mixed-strategy profile $\mu \in M \equiv M_1 \times \dots \times M_n$ will be written as $E_{(\mu_i, \mu_{-i})}[\Pi_i^{N,\alpha}(x_i, x_{-i})]$. A *mixed-strategy Nash equilibrium* (MSNE) is then a tuple $\mu^* = (\mu_1^*, \dots, \mu_n^*) \in M$ such that

$$E_{(\mu_i^*, \mu_{-i}^*)}[\Pi_i^{N,\alpha}(x_i, x_{-i})] \geq E_{(\mu_i, \mu_{-i}^*)}[\Pi_i^{N,\alpha}(x_i, x_{-i})] \quad (4)$$

for any $i \in N$ and $\mu_i \in M_i$. General results guarantee the existence of a MSNE for all parameter constellations.

Lemma 1. (Existence) *The n -player Hirshleifer contest with parameter α admits a MSNE for any $n \geq 2$ and $\alpha > 0$.*

Proof. See the Appendix.⁸ \square

In all that follows, we will normalize the value of the contested prize to one, so that $V = 1$. This normalization is without loss of generality, as may be easily seen

⁷Clearly, the upper bound is without loss of generality. Indeed, any bid weakly exceeding the value of the prize is strictly dominated by the zero bid.

⁸In Section 3, we will show that there always exists a symmetric MSNE.

by expressing expenses in units of the prize.⁹

2.2 Initial observations

We prepare the main analysis by collecting some general facts regarding the distribution of equilibrium bids in any equilibrium. The first is that any optimal mixed strategy in the n -player Hirshleifer contest has finite support.¹⁰

Lemma 2.¹¹ **(Finite support)** *Let $i \in N$. Then, for any profile of mixed strategies $\mu_{-i} = (\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_n)$, the set of pure strategies that maximize contestant i 's expected payoff against μ_{-i} ,*

$$X_i(\mu_{-i}) = \arg \max_{x_i \in [0,1]} E_{\mu_{-i}}[\Pi_i^{N,\alpha}(x_i, x_{-i})], \quad (5)$$

is finite. Moreover, any mixed strategy $\mu_i \in M_i$ maximizing $E_{(\mu_i, \mu_{-i})}[\Pi_i^{N,\alpha}(x_i, x_{-i})]$ has finite support.

Proof. See the Appendix. \square

This lemma has the useful implication that, in any MSNE, each contestant $i \in N$ randomizes over a finite set of bids

$$y_i^{(1)} > \dots > y_i^{(L_i)} \geq 0, \quad (6)$$

for some $L_i \geq 1$, such that $y_i^{(l_i)}$ is selected with probability $q_i^{(l_i)} > 0$, for $l_i \in \{1, \dots, L_i\}$. In particular, the expectation over a profile of equilibrium strategies reduces to a finite sum, which of course simplifies the analysis in many ways.

Illustrations of MSNE with finite support will be provided later in the paper.

⁹It suffices to note that a bid vector $\hat{x}^* \in \mathbb{R}_{\geq 0}^n$ is a PSNE in the unnormalized contest with prize of value V and parameter $\hat{\alpha}$ if and only if the rescaled vector $x^* = \hat{x}^*/V$ is a PSNE in the normalized contest with parameter $\alpha = \hat{\alpha} \cdot V$. An analogous relationship holds for MSNE.

¹⁰The *support* of a mixed strategy $\mu_i \in M_i$, denoted by $\text{supp}\{\mu_i\} \subseteq [0, 1]$, is defined as usual as the intersection of all closed sets to which the probability measure μ_i assigns probability one.

¹¹This result is essentially known (see, e.g., Ewerhart and Sun, 2018, Lemma 1). For the reader's convenience, however, we include a derivation in the Appendix.

Lemma 2 implies a difference to the standard model of the all-pay auction (Baye et al., 1996), where mass points in equilibrium strategies are feasible at the origin only, and where the interior part of the bid distribution is absolutely continuous.¹²

Next, we look at mass points at the zero bid. In fact, we will also consider mass points located in a small neighborhood of the zero bid. An analysis of the second-order condition at any lowest positive bid leads to the following result.

Lemma 3. (Zero bids and small bids)

- (i) *At least $(n - 1)$ of the contestants choose zero with positive probability.*
- (ii) *If $y_i^{(L_i)} > 0$ for some $i \in N$, then there is no mass point in the interval $(0, y_i^{(L_i)}]$ for any contestant $j \neq i$.*

Proof. See the Appendix. \square

Thus, either all contestants choose the zero bid with positive probability (e.g., in a symmetric MSNE), or there is precisely one contestant $i \in N$ whose lowest bid $y_i^{(L_i)}$ is positive, while nobody else bids in the interval $(0, y_i^{(L_i)}]$.

We say that contestant $i \in N$ is *active* (*always active*, *inactive*) in a mixed-strategy profile $\mu \in M$ if her strategy μ_i employs positive bids with positive probability (with probability one, with probability zero). Lemma 3(i) may hence be summarized by saying that at most one contestant is always active.

Lemma 3(i) reveals another difference between the Hirshleifer contest and the standard all-pay auction. Indeed, the symmetric MSNE in the all-pay auction with homogeneous valuations does not feature any mass points (Hillman and Riley, 1989). Furthermore, in any asymmetric MSNE of the all-pay auction with homogeneous valuations, at most $(n - 2)$ contestants bid zero with positive probability (Baye et al., 1996).

¹²Of course, the picture changes once there are resource constraints. See Che and Gale (1998).

3. Symmetric equilibria

As usual, we call a Nash equilibrium (pure or mixed) *symmetric* if all players use the same strategy. In Subsection 3.1, we characterize the set of symmetric PSNE. Then, in Subsection 3.2, we describe the set of symmetric MSNE.

3.1 Pure-strategy Nash equilibrium

We will say that a PSNE x^* reflects *multilateral peace* if $x_1^* = \dots = x_n^* = 0$. Note that the equilibrium payoff under multilateral peace is $\Pi_i^* = \frac{1}{n}$, for any $i \in N$. By Lemma 3(i), multilateral peace is the only symmetric PSNE feasible. To understand the conditions for this to be an equilibrium, suppose that all opponents $j \neq 1$ of contestant 1 choose an expense level of zero. Then, contestant 1's expected payoff is given by

$$\bar{\Pi}_1^N(x_1; \alpha) \equiv \Pi_1^{N,\alpha}(x_1, \mathbf{0}_{n-1}) = \frac{\exp(\alpha x_1)}{\exp(\alpha x_1) + n - 1} - x_1, \quad (7)$$

where $\mathbf{0}_{n-1} = (0, \dots, 0) \in \mathbb{R}^{n-1}$. Figure 1 outlines the graph of the function $\bar{\Pi}_1^N(\cdot; \alpha)$ for $n = 3$ and selected values of α . As the illustration suggests, multilateral peace is sustainable as an equilibrium between three contestants only with a sufficient amount of noise, i.e., only if α is small enough. This fact holds analogously for any number of contestants, as will be explained below.

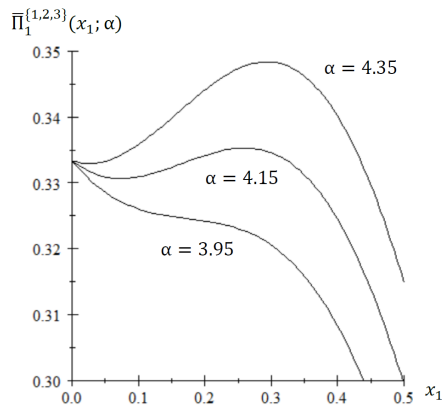


Figure 1. Payoff against inactive contestants

A straightforward examination of marginal payoffs shows that $\bar{\Pi}_1^N(\cdot; \alpha)$ is strictly declining for $\alpha \leq 4$, regardless of n .¹³ On the other hand, for $\alpha > 4$, there is a unique interior local maximum $\tilde{x}_1(\alpha) > 0$, where we suppress the dependence on n in the notation. In that case, an application of the envelope theorem shows that the payoff at the local maximum, $\bar{\Pi}_1^N(\tilde{x}_1(\alpha), \alpha)$, is strictly increasing in α . Indeed, we have

$$\frac{\partial \bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha)}{\partial \alpha} = \left. \frac{\partial \bar{\Pi}_1^N(x_1; \alpha)}{\partial \alpha} \right|_{x_1=\tilde{x}_1(\alpha)} \quad (8)$$

$$= \frac{\alpha \tilde{x}_1(\alpha) \exp(\alpha \tilde{x}_1(\alpha))(n-1)}{(\exp(\alpha \tilde{x}_1(\alpha)) + n-1)^2} \quad (9)$$

$$> 0. \quad (10)$$

Further, it should be intuitively clear that $\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha)$ strictly exceeds the fair share $\Pi_i^* = \frac{1}{n}$ for any sufficiently large α . Arguing along these lines, we arrive at the following result.

Proposition 1. (Symmetric PSNE) *Consider the n -player Hirshleifer contest with parameter α , where $n \geq 2$. Then, there is a threshold value $\alpha^*(n) \geq 4$ such that:*

(i) *a symmetric PSNE exists if and only if $\alpha \in (0, \alpha^*(n)]$;*

(ii) *in this case, the PSNE is unique and corresponds to multilateral peace.*

Proof. See the Appendix. \square

Thus, for any $n \geq 2$, there is a nonempty and half-open interval of values for the decisiveness parameter α such that multilateral peace is a PSNE. Proposition 1 illustrates one important dimension in which the Hirshleifer contest differs dramatically from the Tullock contest. Indeed, it is well-known that any equilibrium in the standard Tullock contest requires activity of at least two contestants.

¹³For $\alpha = 4$, there is a saddle point. Of course, $\bar{\Pi}_1^N(\cdot; \alpha)$ is strictly declining also in that case.

The threshold $\alpha^*(n)$ may be characterized as the unique solution of the indifference relationship

$$\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha) = \bar{\Pi}_1^N(0; \alpha) = \frac{1}{n}. \quad (11)$$

For the special case $n = 2$, solving this equation delivers $\alpha^*(2) = 4$, so that we retrieve Hirshleifer's (1989) classic result that bilateral peace is a PSNE if and only if $\alpha \leq 4$. For $n \geq 3$, however, the threshold value $\alpha^*(n)$ does not allow a simple analytical expression. E.g., $\alpha^*(3) \approx 4.12$. Nonetheless, one can show that $\alpha^*(n)$ is strictly increasing in n , as well as unbounded.¹⁴ Intuitively, the strict monotonicity of the threshold says that the advent of an additional contestant always makes it easier to sustain peace. We will elaborate on this point in Sections 6 and 7 below.

3.2 Symmetric mixed-strategy Nash equilibrium

In this subsection, we offer a characterization of symmetric MSNE in the n -player Hirshleifer contest. From Lemma 2, the support of any symmetric equilibrium strategy is necessarily a finite set including, in particular, the zero bid. Denote by L the number of mass points in the equilibrium distribution of bids. For any number $n \geq 2$ of contestants, we establish the following result.

Proposition 2. (Symmetric MSNE) *Consider the n -player Hirshleifer contest with parameter α . Then, the following holds true:*

- (i) *there exists a symmetric MSNE with $L \geq 2$ if and only if $\alpha > \alpha^*(n)$;*
- (ii) *the number L respects the lower bound given by*

$$L \geq \left(\frac{(n-1)\alpha}{n^2} \right)^{\frac{1}{n-1}}. \quad (12)$$

Proof. See the Appendix. \square

¹⁴The proof can be found in the Appendix.

This proposition characterizes the structure of symmetric MSNE in the n -player Hirshleifer contest, and clarifies how the parameter α relates to the cardinality of the support of the symmetric equilibrium strategy. Part (i) establishes existence of MSNE with $L \geq 2$ if and only if α strictly exceeds the threshold $\alpha^*(n)$ introduced in the previous subsection. It should be noted that, combined with Proposition 1(i), this entails a general existence result for symmetric MSNE. The more interesting implication, however, is that multilateral peace (where $L = 1$) is the unique symmetric MSNE whenever multilateral peace is a PSNE, i.e., for $\alpha \leq \alpha^*(n)$.¹⁵ Part (ii) puts a general lower bound on the number of mass points in the equilibrium bid distribution. As can be seen, the lower bound on L is strictly increasing and unbounded in α . Thus, the number of mass points will ultimately surpass any finite bound as $\alpha \rightarrow \infty$.

The following example illustrates the symmetric MSNE in the simplest case where the support of the symmetric equilibrium strategy has precisely two elements.

Example 1 (Symmetric MSNE with $L = 2$). Let $n \geq 2$. Consider a symmetric equilibrium strategy that places probability $q^{(1)} > 0$ on the positive bid $y^{(1)} > 0$, and a complementary probability $q^{(2)} = 1 - q^{(1)} > 0$ on the zero bid $y^{(2)} = 0$. Then, we have two equations that jointly characterize $y^{(1)}$ and $q^{(1)}$, viz. the first-order condition at the interior bid $y^{(1)}$, and the indifference condition between $y^{(1)}$ and $y^{(2)}$. E.g., for $n = 3$, this type of equilibrium exists for $\alpha \in (4.12, 6.98)$. For smaller values of α , each contestant would prefer to become inactive. On the other hand, for larger values of α , each contestant would wish to deviate to a bid level strictly between zero and $y^{(1)}$.

¹⁵To prove this, we first show that for $\alpha \leq \alpha^*(n)$ and $n \geq 3$, any contestant strictly prefers bidding zero over matching the bid of a single active opponent. The claim then follows by noting that bids are strategic substitutes in the relevant domain.

In the example, each of the n contestants randomizes between a positive and the zero bid. It follows from Proposition 2(ii) that similar examples may be constructed for values of L that exceed any given finite bound.

4. Asymmetric equilibria

In this section, we characterize the set of asymmetric equilibria in the n -player Hirshleifer contest. Subsection 4.1 deals with PSNE, while Subsection 4.2 discusses MSNE.

4.1 Asymmetric pure-strategy Nash equilibria

We will say that a PSNE x^* reflects *one-sided dominance* if $x_i^* > 0$ is positive for some contestant $i \in N$, while $x_j^* = 0$ holds for all other contestants $j \in N \setminus \{i\}$. In that case, player i will be referred to as the dominant contestant. As mentioned before, the two-player Hirshleifer contest with homogeneous valuations does not admit one-sided dominance as a PSNE, let alone any other type of asymmetric equilibrium. However, as our next result shows, one-sided dominance is feasible as a PSNE outcome for any number $n \geq 3$ of contestants.

Proposition 3. (Asymmetric PSNE) *Consider the n -player Hirshleifer contest with parameter α . Suppose that $n \geq 3$. Then:*

- (i) *there is a threshold value $\alpha^{**}(n)$ satisfying $\alpha^{**}(n) > \alpha^*(n)$ such that one-sided dominance is a PSNE if and only if $\alpha \in [\alpha^*(n), \alpha^{**}(n)]$;*
- (ii) *whenever it exists, one-sided dominance is the unique PSNE up to renaming of the dominant contestant;*
- (iii) *for $\alpha > \alpha^{**}(n)$, there does not exist any PSNE.*

Proof. See the Appendix. \square

Part (i) says that, for any $n \geq 3$, there is a non-empty, compact, and nondegenerate interval of values for the decisiveness parameter α with the property that one-sided dominance is a PSNE if and only if α lies in that interval. For $\alpha < \alpha^*(n)$, the dominant contestant would prefer to back off by bidding zero. For $\alpha > \alpha^{**}(n)$, however, any of the other contestants would find it profitable to overbid the dominant contestant. In the proof, we use again the envelope theorem to show that $\alpha^{**}(n)$ is well-defined, in analogy to the argument used in Section 2. E.g., $\alpha^{**}(3) \approx 4.66$. To verify that $\alpha^*(n) < \alpha^{**}(n)$, we check that none of the inactive players has an incentive to deviate at $\alpha = \alpha^*(n)$. It may be noted that $\lim_{n \rightarrow \infty} \alpha^{**}(n) = \infty$, as an immediate implication of the unboundedness of $\alpha^*(n)$.

Regarding part (ii), we remark that the necessary first-order condition for the unique interior optimum delivers the resource commitment of the dominant contestant as $x_1^* = \tilde{x}_1(\alpha)$, where

$$\tilde{x}_1(\alpha) = \frac{1}{\alpha} \ln \left(\frac{n-1}{2} \left\{ \alpha - 2 + \sqrt{\alpha(\alpha-4)} \right\} \right). \quad (13)$$

A numerical exercise reveals that, in the relevant range where the equilibrium exists, $\tilde{x}_1(\alpha)$ is strictly increasing in α for $n \in \{3, 4\}$, hump-shaped in α for $n \in \{5, 6\}$, and strictly declining $n \geq 7$.¹⁶ Thus, contrary to intuition, if the number of contestants is small, then a sharper sword may actually require higher expenses to dominate the population.

As for the equilibrium payoffs, the dominant contestant, say player 1, receives a payoff weakly exceeding the fair share, i.e., $\Pi_1^* \geq \frac{1}{n}$. Indeed, contestant 1 must find it weakly profitable to depart from multilateral peace, which yields $\Pi_i^* = \frac{1}{n}$ for any $i \in N$. Moreover, as the dominant contestant's expenses and raised probability of winning bite into the total cake available for distribution, less than the fair share is left for each of the subdued contestants. Thus, $\Pi_j^* < \frac{1}{n}$ for any $j \neq 1$. We remark

¹⁶Details on the numerical results reported in this paper are available from the authors upon request.

that, because the identity of the dominant contestant is undetermined, the present discussion implies the existence of a multiplicity of payoff-inequivalent PSNE.¹⁷

Finally, part (iii) of the proposition says that there is no PSNE whatsoever for $\alpha > \alpha^{**}(n)$. Given that multilateral peace and one-sided dominance are the only feasible types of PSNE by Lemma 3(i), this conclusion should be immediate in view of part (i) and Proposition 1.

4.2 Asymmetric mixed-strategy Nash equilibria

In this subsection, we turn to the analysis of asymmetric MSNE. Thus, we consider equilibria in which (i) at least one contestant randomizes and (ii) not all contestants use the same strategy. As discussed, there are no asymmetric MSNE for $n = 2$. For $n \geq 3$, however, the set of asymmetric equilibria turns out to be quite large.

Proposition 4. (Continuum of payoff-inequivalent MSNE) *Suppose there are $n \geq 3$ contestants. Then, at $\alpha = \alpha^*(n)$, there is a continuum of MSNE in which contestant 1 randomizes, choosing $y_1^{(1)} = \tilde{x}_1(\alpha) > 0$ with probability $q_1^{(1)} \in [0, 1]$, and the zero bid otherwise, while the other $(n - 1)$ contestants all remain inactive.*

Proof. See the Appendix. \square

Thus, a continuum of MSNE exists when the parameter α lies precisely at the threshold value, i.e., when $\alpha = \alpha^*(n)$. On a purely technical level, this possibility is due to the fact, familiar from the theory of bimatrix games, that the set of beliefs to which a pure strategy is a best response is convex. However, a similar multiplicity does not occur in the case of $n = 2$ contestants. Indeed, one-sided dominance is not possible with only two players.

¹⁷The comparative statics of equilibrium payoffs is intuitive. Indeed, as seen before, the equilibrium payoff of the dominant contestant, $\Pi_1^* = \bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha)$, is strictly increasing in α . On the other hand, the equilibrium payoff of any subdued contestant $j \neq 1$ is strictly declining in α , as may be easily seen by plugging the explicit solution (13) into j 's payoff function.

Below, we present examples of robust asymmetric MSNE. To keep oversight, it is useful to recall from Lemma 3(i) that at most one contestant can be always active. In Example 2 below, two contestants randomize in an identical way, while a third contestant remains inactive. In Example 3, one contestant is always active, one contestant randomizes, and all others remain inactive. Finally, in Example 4, one contestant is always active, while two other contestants randomize identically. Jointly, these examples illustrate the richness of the set of asymmetric MSNE in the n -player Hirshleifer contest with $n \geq 3$.

Example 2. (Two identically randomizing players and one inactive player) Let $n = 3$. Suppose that contestants 1 and 2 use an identical mixed strategy that selects $y_1^{(1)} = y_2^{(1)} > 0$ with probability $q_1^{(1)} = q_2^{(1)} \in (0, 1)$ and the zero bid otherwise, while contestant 3 remains inactive. Then, similar to Example 1, a first-order condition and an indifference relation jointly characterize $y_1^{(1)}$ and $q_1^{(1)}$. The resulting MSNE exists for $\alpha \in (4.12, 7.01)$.¹⁸ For smaller values of α , contestants 1 and 2 would wish to reduce expenses to zero. For larger values of α , however, both contestants would prefer some bid strictly between $y_1^{(1)}$ and zero.

Example 3. (One always active player, one randomizing player, and $(n - 2)$ inactive players) Let $n \geq 3$ be arbitrary. Suppose that contestant 1 chooses a positive bid $y_1^{(1)} > 0$ with probability one, while contestant 2 randomizes between $y_2^{(1)} > 0$ and the zero bid. Suppose also that contestants 3, \dots , n all remain inactive. In this case, we have three equilibrium conditions, viz. the respective first-order conditions for $y_1^{(1)}$ and $y_2^{(1)}$, and the indifference relation for contestant 2. E.g., for $n = 3$, the MSNE exists for $\alpha \in (4.58, 4.66)$.¹⁹ For smaller values of α , contestant 1 has an incentive to deviate to zero. For larger values of α , however,

¹⁸E.g., for $\alpha = 4.5$, one finds $y_1^{(1)} = 0.348$ and $q_1^{(1)} = 0.36$.

¹⁹At $\alpha = 4.6$, for instance, we find $y_1^{(1)} = 0.300$, $y_2^{(1)} = 0.513$, and $q_2^{(1)} = 0.163$.

contestant 2 would prefer placing all probability weight on the zero bid.

Example 4. (One always active and two identically randomizing players)

Let $n = 3$. Suppose that contestant 1 plays a pure strategy $y_1^{(1)} > 0$, while contestants 2 and 3 identically randomize between $y_2^{(1)} = y_3^{(1)} > 0$ and the zero bid. This MSNE exists for $\alpha \in (4.66, 4.86)$. For smaller values of α , contestants 2 and 3 would prefer to become inactive, while for larger values of α , contestant 1 would wish to withdraw.²⁰

5. The case of arbitrarily small noise

In this section, we study the structure of MSNE for both very small and vanishing noise. We start by characterizing equilibrium bid distributions for α arbitrarily large but still finite.

Proposition 5. (Arbitrarily small noise) *Let $\varepsilon > 0$. Then, for any α sufficiently large, any MSNE μ^* of the n -player Hirshleifer contest with parameter α has the following properties:*

- (i) $y_i^{(1)} > 1 - \varepsilon$, for at least two contestants $i \in N$;
- (ii) $\Pi_i^* < \varepsilon$, for any $i \in N$;
- (iii) $\text{pr}_{\mu_i^*}\{x_i = 0\} < \varepsilon$, for at least two contestants $i \in N$,²¹
- (iv) either $y_i^{(1)} > 1 - \varepsilon$ or $y_i^{(1)} < \varepsilon$, for any $i \in N$.

Proof. See the Appendix.

The proposition characterizes equilibrium strategies and expected payoffs in the case where the noise in the contest technology becomes arbitrarily small.

²⁰It may be noted that all our examples of asymmetric MSNE are semi-mixed, i.e., at least one contestant uses a pure strategy. However, it seems that, in the set-up of Example 4, at $\alpha = 4.86$, there is an asymmetric equilibrium in which all three contestants randomize.

²¹We use $\text{pr}_{\mu_i^*}\{x_i = 0\}$ to denote the probability that contestant i uses the zero bid when adhering to the mixed strategy μ_i^* .

Part (i) says that at least two contestants bid arbitrarily close to one. This cut-throat competition is intuitively necessary because without it, every contestant could earn a substantial rent by overbidding the entire population. Indeed, this possibility would be open even to those $(n - 1)$ bidders that occasionally bid zero, as follows from Lemma 3(i). However, rents from bidding zero are very small for large α , intuitively because such rents could be easily taken away by any other contestant that bids zero with positive probability. In fact, as we show more specifically in the Appendix, for any contestant $i \in N$ such that $0 \in \text{supp}\{\mu_i^*\}$, the equilibrium payoff satisfies

$$\Pi_i^* \leq \frac{n}{(n - 1)\alpha}. \quad (14)$$

Next, part (ii) says that any rent is dissipated as $\alpha \rightarrow \infty$. Given what we have said above, this claim reduces to showing that even the rent of any always active contestant goes to zero. Here the argument is that any bidder earning substantial rents cannot be part of the cut-throat competition, i.e., would not bid arbitrarily close to the value of the prize. But then, as we show, one of the bidders involved in the cut-throat competition would find it strictly beneficial to marginally overbid the always active contestant, which is impossible.

Part (iii) says that, for at least two contestants, the probability of bidding zero vanishes as $\alpha \rightarrow \infty$.²² The argument here is simple and related to one that has been used above. Specifically, too much weight on the zero bid would allow others to profitably overbid.

Finally, part (iv) shows that, in the limit, everyone either bids up to one or becomes ultimately inactive. Like in part (ii), this claim relies on the idea that one of the contestants bidding nearly up to the value of the prize would prefer to slightly overbid any contestant whose highest bid lies in the interval $[\varepsilon, 1 - \varepsilon]$.

²²Judging on the basis of the all-pay auction, all of these contestants should bid up to one in the limit.

However, compared to the proof of part (ii), the proof of part (iv) requires an additional argument, as the rent of the lower bidder cannot be assumed to be substantial.²³

Thus, overall, the structure of equilibria in the Hirshleifer contest as $\alpha \rightarrow \infty$ is similar to those of the n -bidder all-pay auction with homogeneous valuations. And indeed, in the Appendix, we outline the proof of the following limit result.

Proposition 6. (Robustness of the APA) *Let $\{\alpha_m\}_{m=0}^\infty$ be an increasing and unbounded sequence of parameters in $\mathbb{R}_{>0}$, and let $\{\mu^m\}_{m=0}^\infty$ be a sequence of mixed-strategy profiles in M , such that μ^m is a MSNE in the n -player Hirshleifer contests with parameter α_m , for any $m \geq 0$. If $\{\mu^m\}_{m=0}^\infty$ converges in distribution to some limit $\mu^* \in M$, then μ^* is a MSNE of the all-pay auction.*

Proof. See the Appendix. \square

6. The case of large n

In this section, we change the perspective by letting the number of contestants grow indefinitely while keeping the decisiveness parameter of the contest technology fixed.

It turns out that, for any given $\alpha > 0$, there exists a threshold value $n^\#(\alpha)$ for the number of contestants such that, for $n \geq n^\#(\alpha)$, multilateral peace is the unique PSNE. To see this, take any PSNE $x^* = (x_1^*, \dots, x_n^*)$. Suppose that some contestant $i \in N$ is active, i.e., $x_i^* > 0$. Then, clearly, noting that the prize has normalized value one, and that inactivity guarantees a positive payoff, it follows that $x_i^* < 1$. Moreover, from the second-order necessary condition for an interior

²³For completeness, we mention that the conclusions become a bit more specific when attention is restricted to symmetric MSNE. In that case, as easily follows from Proposition 5, all bidders enter the cut-throat competition, and give less and less weight to the mass point at the zero bid.

solution to contestant i 's optimization problem,

$$0 \geq \frac{\partial^2 \Pi_i^{N,\alpha}(x_i^*, x_{-i}^*)}{\partial x_i^2} \quad (15)$$

$$= \frac{\alpha^2 X_i^* \left(\sum_{j \neq i} X_j^* \right)}{\left(\sum_{j \in N} X_j^* \right)^3} \left\{ -X_i^* + \sum_{j \neq i} X_j^* \right\} \quad (16)$$

$$> \frac{\alpha^2 X_i^* \left(\sum_{j \neq i} X_j^* \right)}{\left(\sum_{j \in N} X_j^* \right)^3} \{-\exp(\alpha) + n - 1\}, \quad (17)$$

where we used the shorthand notation $X_i^* = \exp(\alpha x_i^*)$, etc. The point to note now is that, for $n \geq n^\#(\alpha) \equiv 1 + \exp(\alpha)$, the term in the curly brackets in (17) is weakly positive, which is impossible. The contradiction shows that, for $n \geq n^\#(\alpha)$, there is no PSNE other than multilateral peace. Intuitively, as n increases, it gets harder for a single contestant to keep the dominant position. As the following result shows, this observation extends to MSNE.

Proposition 7. *For any $\alpha > 0$ and $n > n^\#(\alpha)$, multilateral peace is the unique MSNE in the n -player Hirshleifer contest with parameter α .*

Proof. See the Appendix. \square

7. Concluding remarks

In this paper, we have conducted a thorough examination of the equilibrium set of the n -player Hirshleifer contest with homogeneous valuations. Both PSNE and MSNE have been considered.

A PSNE need not exist in general. However, if it exists, then there is either a unique symmetric PSNE in the form of multilateral peace, or there is a total of n asymmetric PSNE in the form of one-sided dominance.²⁴ The intuition for

²⁴As discussed, symmetric and asymmetric PSNE coexist in the borderline case where $\alpha = \alpha^*(n)$ with $n \geq 3$.

this result is as follows. If α is small, then the contest technology is not very sharp, and incentives to become active are weak. As a result, there is multilateral peace. However, once α exceeds the threshold value $\alpha^*(n)$, every contestant has an incentive to become active. Moreover, once a single contestant grasps this profitable opportunity, the incentives for the other contestants are moderated, so that they optimally choose to remain inactive. But if α grows further, exceeding a second threshold value $\alpha^{**}(n)$, then each of those formerly inactive contestants again has an incentive to overbid the dominant contestant. When this happens, a bidding war evolves, and the PSNE ceases to exist.

The set of MSNE turned out to be very diverse. The main common elements are that the respective supports of equilibrium bid distributions are finite, and that the zero bid is chosen with positive probability by at least $(n - 1)$ contestants. We studied both symmetric and asymmetric MSNE. A symmetric MSNE always exists. For $\alpha < \alpha^*(n)$, we could even prove that the MSNE is unique and necessarily degenerate.²⁵ For $\alpha \geq \alpha^*(n)$, we identified a variety of asymmetric MSNE. In addition to a continuum of such equilibria in the borderline case $\alpha = \alpha^*(n)$, we characterized several robust examples of asymmetric payoff-inequivalent MSNE. Finally, we derived a collection of necessary properties of the equilibrium for the case of arbitrarily small noise, and showed that sequences of MSNE that correspond to an unbounded sequence of α and that converge in distribution ultimately approach an equilibrium of the standard all-pay auction.

The unique prediction of peace in the presence of sufficiently many contestants raises conceptual issues. It is at odds with anecdotal evidence in a multi-sided armed conflict (such as the contemporary Syrian civil war, for instance). But the result is driven by very low returns from military engagement that are predicted

²⁵We conjecture that, more generally, the symmetric MSNE between $n \geq 3$ contestants is unique. This interesting question must be left for future work, however.

by the Hirshleifer technology when there are many parties involved. We conclude that, while Hirshleifer's assumptions have much appeal as a model of military conflict in the case of two contestants (Hirshleifer, 2000), this may be less so for contests with many contestants. Therefore, there is a need for alternative contest technologies that are more suitable as a model of multilateral military conflict.

While we have focused on the Hirshleifer technology, we conjecture that our main results extend to any analytic contest technology that exhibits increasing returns against higher bids.²⁶

Appendix. Proofs

This Appendix contains technical proofs omitted from the body of the paper. We start by introducing some additional notation.

Additional notation used in the proofs. For bids $x_i \geq 0$ and $x_{-i} \in \mathbb{R}_{\geq 0}^{n-1}$, let

$$p_i^{N,\alpha}(x_i, x_{-i}) = \frac{\exp(\alpha x_i)}{\exp(\alpha x_i) + \sum_{j \neq i} \exp(\alpha x_j)} \quad (18)$$

denote contestant i 's *probability of winning*. Further, for a bid vector $x \in \mathbb{R}_{\geq 0}^n$ entering as an argument of either $\Pi_i^{N,\alpha}$ or $p_i^{N,\alpha}$, we will alternatively write $x = (x_i, x_j, x_{-i,j})$, where

$$x_{-i,j} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}_{\geq 0}^{n-2}.$$

Proof of Lemma 1. As discussed, we may assume without loss of generality that bids are chosen from $[0, V]$, which is a compact and nonempty subset of \mathbb{R} . Moreover, payoff functions are continuous in pure strategies. Therefore, with

²⁶Maybe surprisingly, analyticity is crucial for our results. Indeed, the n -player serial contest with increasing returns (Alcalde and Dahm, 2007) is not analytic. Still, that contest is known to always admit a MSNE in which precisely two of n contestants are active.

mixed strategies corresponding to probability measures on the Borel subsets of $[0, V]$, the existence of a MSNE is immediate from Glicksberg's (1952) theorem. \square

Proof of Lemma 2. For any fixed bid vector $x_{-i} \in \mathbb{R}_{\geq 0}^{n-1}$, contestant i 's payoff $\Pi_i^{N,\alpha}(x_i, x_{-i})$ may be written as

$$f_K(x_i) = \frac{\exp(\alpha x_i)}{\exp(\alpha x_i) + K} - x_i \quad (x_i \in \mathbb{R}_{\geq 0}), \quad (19)$$

where $K = \sum_{j \neq i} \exp(\alpha x_j) > 0$. Denote the measure of the probability distribution induced by μ_{-i} on values of K by ν . Then, i 's expected payoff $E_{\mu_{-i}}[\Pi_i^{N,\alpha}(x_i, x_{-i})]$ from a pure bid $x_i \geq 0$ becomes

$$f(x_i) = \int_0^{(n-1)\exp(\alpha)} f_K(x_i) d\nu(K) \quad (x_i \in \mathbb{R}_{\geq 0}). \quad (20)$$

Let $\mathbb{C} = \{\xi + \theta\sqrt{-1} : \xi, \theta \in \mathbb{R}\}$ denote the field of complex numbers. Define the strip $S \subseteq \mathbb{C}$ as

$$S = \{\xi + \theta\sqrt{-1} : \xi \in (-\varepsilon, \infty), \theta \in (-\varepsilon, \varepsilon)\} \supseteq \mathbb{R}_{\geq 0}, \quad (21)$$

where $\varepsilon > 0$ is small. We claim that, for any $K > 0$, the function (19) admits the complex-analytic extension

$$\tilde{f}_K(z_i) = \frac{\exp(\alpha z_i)}{\exp(\alpha z_i) + K} - z_i \quad (z_i \in S). \quad (22)$$

Indeed, by Euler's formula,

$$\exp(\alpha z_i) = \exp(\alpha x_i + \sqrt{-1}\alpha y_i) \quad (23)$$

$$= \exp(\alpha x_i) \exp(\sqrt{-1}\alpha y_i) \quad (24)$$

$$= \exp(\alpha x_i) \{ \cos(\alpha y_i) + \sqrt{-1} \sin(\alpha y_i) \}, \quad (25)$$

so that for any $z_i \in S$, the real part of $\exp(\alpha z_i) + K$ is positive, i.e.,

$$\operatorname{Re}(\exp(\alpha z_i) + K) = \exp(\alpha x_i) \cos(\alpha y_i) + K > 0. \quad (26)$$

In particular, the denominator of the ratio in equation (22) does not vanish on S , which proves the claim. Hence, we may argue as in Ewerhart (2015a) to see that the integral

$$\tilde{f}(z_i) = \int_0^{(n-1)\exp(\alpha)} \tilde{f}_K(z_i) d\nu(K) \quad (z_i \in S) \quad (27)$$

is a complex-analytic extension of the expected payoff function (20). As $\tilde{f}(z_i)$ is unbounded on S , it is not constant. Hence, the complex-analytic function $\tilde{f}(z_i)$ attains the same value over the unit interval only at finitely many points. In particular, there is only a finite set of solutions of contestant i 's optimization problem. This proves the first claim. The second claim regarding the finite support of any optimal mixed strategy is now immediate. \square

Proof of Lemma 3. (i) The result is known for $n = 2$ (Hirshleifer, 1989). Let $n \geq 3$. To provoke a contradiction, suppose that two contestants $i, j \in N$ with $i \neq j$ are always active. Then, without loss of generality, the lowest bid in contestant i 's bid distribution, $\underline{x}_i = y_{L_i}^{(i)}$, satisfies $0 < \underline{x}_i \leq y_{L_j}^{(j)}$. Hence, $p_i^{N,\alpha}(\underline{x}_i, x_{-i}) \in (0, \frac{1}{2})$ for any $x_{-i} \in \text{supp}\{\mu_{-i}^*\}$. But then, the second derivative of contestant i 's equilibrium payoff at \underline{x}_i satisfies

$$\frac{\partial^2 E_{\mu_{-i}^*}[\Pi_i^{N,\alpha}(\underline{x}_i, x_{-i})]}{\partial x_i^2} = \alpha^2 E_{\mu_{-i}^*} \left[p_i^{N,\alpha}(\underline{x}_i, x_{-i})(1 - p_i^{N,\alpha}(\underline{x}_i, x_{-i}))(1 - 2p_i^{N,\alpha}(\underline{x}_i, x_{-i})) \right] \quad (28)$$

$$> 0, \quad (29)$$

which shows that \underline{x}_i cannot be an interior maximum. The contradiction proves the claim. (ii) This follows, similarly, from the second-order condition at any possible positive bid for contestant j . \square

Proof of Proposition 1. (i) Calculating marginal payoff of contestant 1,

$$\frac{\partial \bar{\Pi}_1^N(x_1; \alpha)}{\partial x_1} = \frac{\alpha(n-1)\exp(\alpha x_1)}{(\exp(\alpha x_1) + n - 1)^2} - 1, \quad (30)$$

it may be seen that $\bar{\Pi}_1^N(x_1; \alpha)$ has at most two critical points where its derivative with respect to x_1 vanish. These are characterized by the solutions to

$$X_1^2 + (2 - \alpha)(n - 1)X_1 + (n - 1)^2 = 0, \quad (31)$$

where $X_1 = \exp(\alpha x_1)$. There is no solution for $\alpha < 4$, one solution $X_1 = n - 1$ for $\alpha = 4$, and two solutions for $\alpha > 4$. In the case $\alpha > 4$, contestant 1's payoff function has a unique local maximum at

$$\tilde{x}_1(\alpha) = \frac{1}{\alpha} \ln \left(\frac{n-1}{2} \left\{ \alpha - 2 + \sqrt{\alpha(\alpha - 4)} \right\} \right). \quad (32)$$

The boundary local maximum is globally optimal if and only if $\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha) \leq \frac{1}{n}$.

By the envelope theorem,

$$\frac{\partial \bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left\{ \frac{\exp(\alpha x_1)}{\exp(\alpha x_1) + n - 1} - x_1 \right\} \Big|_{x_1 = \tilde{x}_1(\alpha)} \quad (33)$$

$$= \frac{(n-1)\tilde{x}_1(\alpha) \exp(\alpha \tilde{x}_1(\alpha))}{\alpha(\exp(\alpha \tilde{x}_1(\alpha)) + n - 1)^2} \quad (34)$$

$$> 0. \quad (35)$$

Thus, $\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha)$ is strictly increasing in α . Next, suppose that α grows above all bounds. Then,

$$\frac{\partial \bar{\Pi}_1^N(0; \alpha)}{\partial x_1} = \frac{\alpha(n-1)}{n^2} - 1 \quad (36)$$

turns positive, so that $\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha) > \frac{1}{n}$. Thus, there is indeed a unique threshold value $\alpha^*(n) \in [4, \frac{n^2}{n-1}]$ such that multilateral peace is a PSNE if and only if $\alpha \in (0, \alpha^*(n)]$. This proves the assertion.

(ii) By Lemma 3(i), at least $(n - 1)$ contestants remain inactive in any PSNE. However, as shown in part (i), for $\alpha \leq \alpha^*(n)$, the n -th contestant's best response is the zero bid. The claim follows. \square

The following lemma states properties of the function $\alpha^*(n)$ that have been claimed without proof in the body of the paper.

Lemma A.1 (Properties of $\alpha^*(n)$)

(i) $\alpha^*(2) = 4$, and $\alpha^*(n) \in (4, \frac{n^2}{n-1})$ for any $n > 2$;

(ii) $\alpha^*(n)$ is strictly increasing in n ;

(iii) $\lim_{n \rightarrow \infty} \alpha^*(n) = \infty$.

Proof. (i) As shown in the body of the paper, multilateral peace is a PSNE for any $\alpha \in (0, 4]$. Hence, $\alpha^*(n) \geq 4$ for any $n \geq 2$. Conversely, from the KKT condition in multilateral peace, we have

$$\frac{\partial \Pi_i^{N,\alpha}(0, \mathbf{0}_{n-1})}{\partial x_i} = \frac{\alpha(n-1)}{n^2} - 1 \leq 0, \quad (37)$$

so that necessarily $\alpha \leq \frac{n^2}{n-1}$. Thus, $\alpha^*(2) = 4$. Further, the necessary second-order condition at $\alpha = \frac{n^2}{n-1}$ reads

$$\frac{\partial^2 \Pi_i^{N,\alpha}(0, \mathbf{0}_{n-1})}{\partial x_i^2} = \frac{\alpha^2(n-1)(n-2)}{n^3} \leq 0, \quad (38)$$

which is impossible for $n > 2$. Thus, $\alpha^*(n) < \frac{n^2}{n-1}$ for any $n > 2$. Further, from the strict monotonicity of $\alpha^*(n)$ to be shown in part (ii),²⁷ we may conclude that $\alpha^*(n) > 4$ for any $n > 2$. This proves the claim.

(ii) To see that $\alpha^*(n)$ increases in n , we treat n as a continuous variable, and take the total differential of the indifference relationship $p_i - x_i = \frac{1}{n}$. This yields

$$dp_i - dx_i = -\frac{dn}{n^2}. \quad (39)$$

However, using the first-order condition $\alpha p_i(1 - p_i) = 1$, we get

$$dp_i = \alpha p_i(1 - p_i) \left\{ dx_i - \frac{1}{\alpha(n-1)} dn + \frac{x_i}{\alpha} d\alpha \right\} \quad (40)$$

$$= dx_i - \frac{1}{\alpha(n-1)} dn + \frac{x_i}{\alpha} d\alpha. \quad (41)$$

²⁷There is no circularity here because the proof of part (ii) uses only facts from part (i) that have already been shown.

Plugging this into (39) and rearranging leads to

$$\frac{d\alpha}{dn} = \frac{\alpha^2}{n^2 x_i} \cdot \underbrace{\left(\frac{n^2}{n-1} - \alpha \right)}_{>0 \text{ by part (i)}} > 0, \quad (42)$$

for any continuous value $n > 2$. The claim follows.

(iii) To provoke a contradiction, suppose that the sequence $\{\alpha^*(n)\}_{n=2}^\infty$ is bounded. Then, the corresponding sequence $\{\tilde{x}_i(\alpha^*(n), n)\}_{n=2}^\infty$, with $\tilde{x}_i(\alpha, n)$ defined by

$$\tilde{x}_1(\alpha, n) = \frac{1}{\alpha} \ln \left(\frac{n-1}{2} \left\{ \alpha - 2 + \sqrt{\alpha(\alpha-4)} \right\} \right), \quad (43)$$

is easily seen to be unbounded. However, at $\alpha = \alpha^*(n)$, we have $p_i - x_i = \frac{1}{n}$, hence, $\tilde{x}_i(\alpha, n) \leq 1$, a contradiction. \square

The following lemma is used in the necessity part of the proof of Proposition 2(i). Intuitively, the lemma captures the point already mentioned in the body of the paper that, for $\alpha \leq \alpha^*(n)$ and $n \geq 3$, any Hirshleifer contestant strictly prefers remaining inactive over using the same bid as a single active opponent.

Lemma A.2 *Let $n \geq 3$ and $\alpha \leq \alpha^*(n)$. Then, for any $y > 0$,*

$$\Pi_1^{N,\alpha}(y, y, \mathbf{0}_{n-2}) < \Pi_1^{N,\alpha}(0, y, \mathbf{0}_{n-2}). \quad (44)$$

Proof. Since $\alpha \leq \alpha^*(n)$, we know that $\Pi_1^{N,\alpha}(0, \mathbf{0}_{n-1}) \geq \Pi_1^{N,\alpha}(y, \mathbf{0}_{n-1})$, i.e.,

$$\frac{1}{n} \geq \frac{Y}{Y+n-1} - y, \quad (45)$$

where we used the shorthand notation $Y = \exp(\alpha y)$. Hence,

$$\begin{aligned} & \Pi_1^{N,\alpha}(0, y, \mathbf{0}_{n-2}) - \Pi_1^{N,\alpha}(y, y, \mathbf{0}_{n-2}) \\ &= \frac{1}{Y+n-1} - \frac{Y}{2Y+n-2} + y \end{aligned} \quad (46)$$

$$\geq \frac{1}{Y+n-1} - \frac{Y}{2Y+n-2} + \frac{Y}{Y+n-1} - \frac{1}{n} \quad (47)$$

$$= \frac{(n-2)(Y-1)^2}{n(Y+n-1)(2Y+n-2)} \quad (48)$$

$$> 0. \quad (49)$$

This proves the claim. \square

Proof of Proposition 2. (i) (*Sufficiency*) Suppose that $\alpha > \alpha^*(n)$. In analogy to the proof of Lemma 1, replacing Glicksberg's theorem by Becker and Damianov (2006, Th. 1), there exists a symmetric MSNE. This equilibrium cannot be a PSNE by Proposition 1(i). Hence, $L \geq 2$, as claimed. (*Necessity*) For $n = 2$, the proof follows from the uniqueness result in Ewerhart and Sun (2018). Suppose, therefore, that $n \geq 3$. Take a symmetric MSNE μ^* with support

$$0 = y^{(L)} < y^{(L-1)} < \dots < y^{(1)}, \quad (50)$$

with $L \geq 2$. To provoke a contradiction, suppose that $\alpha \leq \alpha^*(n)$. Focus on the smallest positive bid, $y^{(L-1)} > 0$. By the discussion preceding the statement of Proposition 1,

$$\Pi_1^{N,\alpha}(y^{(L-1)}, \mathbf{0}_{n-1}) \leq \Pi_1^{N,\alpha}(0, \mathbf{0}_{n-1}). \quad (51)$$

Next, let $x_{-1}^+ = (y^{(L-1)}, \mathbf{0}_{n-2})$ be such that contestant 2 bids $y^{(L-1)}$ and contestants 3, \dots , n all bid zero. As, for any $i \neq 1$, expenses x_1 and x_i are strategic substitutes with respect to $\Pi_1^{N,\alpha}(x_1, \dots, x_n)$ as long as $\exp(\alpha x_1) \leq \sum_{j \neq 1} \exp \alpha x_j$, we have

$$\Pi_1^{N,\alpha}(y_{L-1}, x_{-1}) - \Pi_1^{N,\alpha}(0, x_{-1}) \leq \Pi_1^{N,\alpha}(y_{L-1}, x_{-1}^+) - \Pi_1^{N,\alpha}(0, x_{-1}^+), \quad (52)$$

for any profile $x_{-1} \neq \mathbf{0}_{n-1}$ in the support of μ_{-1}^* . By Lemma A.2,

$$\Pi_1^{N,\alpha}(y_{L-1}, x_{-1}^+) - \Pi_1^{N,\alpha}(0, x_{-1}^+) < 0. \quad (53)$$

Hence, bidding $y^{(L)} = 0$ strictly dominates bidding $y^{(L-1)} > 0$, a contradiction.

(ii) Take a symmetric MSNE μ^* in the n -player Hirshleifer contest with parameter α . By Lemma 2, there exist bid levels $y^{(1)} > \dots > y^{(L)} \geq 0$, for some $L \geq 1$, with corresponding probabilities $q^{(1)}, \dots, q^{(L)} \in [0, 1]$, such that each contestant $i \in N$ chooses $y^{(l)}$ with probability $q^{(l)}$, for any $l \in \{1, \dots, L\}$. Fix $l \in \{1, \dots, L\}$. By the KKT condition at the optimum $x_1 = y^{(l)}$, we get for contestant 1 that

$$1 \geq \frac{\partial E_{\mu_{-1}^*}[p_1^{N,\alpha}(x_1, x_{-1})]}{\partial x_1} \quad (54)$$

$$= \alpha E_{\mu_{-1}^*}[p_1^{N,\alpha}(x_1, x_{-1})(1 - p_1^{N,\alpha}(x_1, x_{-1}))] \quad (55)$$

$$= \alpha \sum_{l_2=1}^L \dots \sum_{l_n=1}^L \left(\prod_{i=2}^n q^{(l_i)} \right) p_1^{N,\alpha}(x_1, y^{(l_2)}, \dots, y^{(l_n)})(1 - p_1^{N,\alpha}(x_1, y^{(l_2)}, \dots, y^{(l_n)})) \quad (56)$$

$$\geq \alpha (q^{(l)})^{n-1} \frac{1}{n} \left(1 - \frac{1}{n} \right). \quad (57)$$

where the inequality in (57) is obtained by dropping all terms corresponding to scenarios in which at least two contestants use different bid levels. Rewriting yields

$$q^{(l)} \leq \left(\frac{n^2}{\alpha(n-1)} \right)^{\frac{1}{n-1}}, \quad (58)$$

for any $l \in \{1, \dots, L\}$. Since $q^{(1)} + \dots + q^{(L)} = 1$, this implies

$$L \geq \left(\frac{(n-1)\alpha}{n^2} \right)^{\frac{1}{n-1}}, \quad (59)$$

as claimed. \square

Proof of Proposition 3. (i) Suppose first that $\alpha < \alpha^*(n)$. In this case, we have shown in the proof of Proposition 1 that contestant i 's unique pure best

response to $x_{-i} = \mathbf{0}_{n-1}$ is the zero bid $x_i = 0$. Thus, one-sided dominance is not a PSNE for $\alpha < \alpha^*(n)$. Suppose next that $\alpha \geq \alpha^*(n)$. Then, again by the proof of Proposition 1, $\bar{\Pi}_1^N(\tilde{x}_1(\alpha); \alpha) \geq \bar{\Pi}_1^N(x_1; \alpha)$ for any $x_1 \geq 0$. Thus, contestant 1 has no incentive to deviate from $x_1^* = \tilde{x}_1(\alpha)$ if all other contestants remain inactive. To understand the incentives to deviate for any inactive contestant $j \in \{2, \dots, n\}$, say for contestant 2, we consider the payoff function

$$\bar{\Pi}_2^N(x_2; \alpha) = \frac{X_2}{X_1^* + X_2 + n - 2} - x_2, \quad (60)$$

where $X_1^* = \exp(\alpha \tilde{x}_1(\alpha))$ and $X_2 = \exp(\alpha x_2)$. In analogy to the discussion of symmetric PSNE, an examination of the derivative with respect to x_2 shows that $\bar{\Pi}_2^N(\cdot; \alpha)$ is strictly declining for $\alpha \leq 4$. Moreover, for $\alpha > 4$, solving the necessary first-order condition

$$\frac{\alpha X_2 (X_1^* + n - 2)}{(X_1^* + X_2 + n - 2)^2} = 1 \quad (61)$$

for x_2 shows that there is a unique interior maximum at $x_2^* = \tilde{x}_2(\alpha)$, where²⁸

$$\tilde{x}_2(\alpha) = \frac{1}{\alpha} \ln \left(\frac{n - 2 + X_1^*}{2} \left\{ \alpha - 2 + \sqrt{\alpha(\alpha - 4)} \right\} \right). \quad (62)$$

This equation implies $\tilde{x}_2(\alpha) > \tilde{x}_1(\alpha)$. Thus, as informally discussed in the body of the paper, for $\alpha > \alpha^*(n)$, contestant 2 would deviate from one-sided dominance by overbidding contestant 1. Moreover, using the envelope theorem,

$$\frac{\partial \bar{\Pi}_2^N(\tilde{x}_2(\alpha); \alpha)}{\partial \alpha} = \frac{\tilde{x}_2(\alpha) \tilde{X}_2 (X_1^* + n - 2)}{(X_1^* + \tilde{X}_2 + n - 2)^2} - \frac{\tilde{x}_1(\alpha) X_1^* \tilde{X}_2}{(X_1^* + \tilde{X}_2 + n - 2)^2} \quad (63)$$

$$= \frac{\tilde{X}_2 (\tilde{x}_2(\alpha)(n - 2) + (\tilde{x}_2(\alpha) - \tilde{x}_1(\alpha)) X_1^*)}{(X_1^* + \tilde{X}_2 + n - 2)^2} > 0, \quad (64)$$

where $\tilde{X}_2 = \exp(\alpha \tilde{x}_2(\alpha))$. On the other hand,

$$\frac{\partial \bar{\Pi}_2^N(0; \alpha)}{\partial \alpha} = -\frac{\tilde{x}_1(\alpha) X_1^*}{(X_1^* + n - 1)^2} < 0. \quad (65)$$

²⁸Again, we suppress the dependence on n in the notation.

Hence, the payoff difference $\bar{\Pi}_2^N(\tilde{x}_2(\alpha); \alpha) - \bar{\Pi}_2^N(0; \alpha)$ is strictly increasing in α . Moreover, for obvious economic reasons, $\lim_{\alpha \rightarrow \infty} \bar{\Pi}_2^N(\tilde{x}_2(\alpha); \alpha) = 1$. Thus, there is a unique threshold value $\alpha^{**}(n)$ such that

$$\bar{\Pi}_2^N(\tilde{x}_2(\alpha^{**}(n)); \alpha^{**}(n)) = \bar{\Pi}_2^N(0; \alpha^{**}(n)). \quad (66)$$

Finally, we claim that $\alpha^{**}(n) > \alpha^*(n)$. For this, given what we have already shown, it suffices to prove that

$$\alpha = \alpha^*(n) \Rightarrow \bar{\Pi}_2^N(\tilde{x}_2(\alpha); \alpha) - \bar{\Pi}_2^N(0; \alpha) < 0. \quad (67)$$

So assume for the moment that $\alpha = \alpha^*(n)$. Comparing (13) with (62), one obtains

$$X_2 = X_1^* \cdot \frac{n-2+X_1^*}{n-1}, \quad (68)$$

and hence,

$$\frac{X_2}{X_1^* + X_2 + n - 2} - \frac{1}{X_1^* + n - 1} = \frac{X_1^* - 1}{X_1^* + n - 1}. \quad (69)$$

Moreover, rewriting in the indifference relationship at $\alpha = \alpha^*(n)$,

$$\frac{X_1^*}{X_1^* + n - 1} - \tilde{x}_1(\alpha) = \frac{1}{n}, \quad (70)$$

yields

$$\frac{X_1^* - 1}{X_1^* + n - 1} = \frac{n}{n-1} \tilde{x}_1(\alpha). \quad (71)$$

Therefore, at $\alpha = \alpha^*(n)$,

$$\bar{\Pi}_2^N(\tilde{x}_2(\alpha); \alpha) - \bar{\Pi}_2^N(0; \alpha) = \frac{X_2}{X_1^* + X_2 + n - 2} - \frac{1}{X_1^* + n - 1} - \tilde{x}_2(\alpha) \quad (72)$$

$$= \frac{n}{n-1} \tilde{x}_1(\alpha) - \tilde{x}_2(\alpha) \quad (73)$$

$$= \frac{1}{\alpha} \ln \left(\frac{n-1}{n-2+X_1^*} (X_1^*)^{\frac{1}{n-1}} \right). \quad (74)$$

The right-hand side of equation (74) is negative since

$$(X_1^*)^{\frac{1}{n-1}} < \frac{n-2+X_1^*}{n-1}, \quad (75)$$

which is implied by the strict inequality between the geometric and the arithmetic mean as $X_1^* > 1$. The claim follows. \square

(ii) The uniqueness of the optimal dominating bid for $\alpha > \alpha^*(n)$ has been shown in Section 2. For $\alpha = \alpha^*(n)$, the unique interior optimum is payoff-equivalent to the zero bid. Notwithstanding, the dominating bid is unique.

(iii) This is immediate from Lemma 3(i). \square

Proof of Proposition 4. Consider a MSNE candidate with $q_1^{(1)} \in [0, 1]$. From Propositions 1 and 3, this is actually a PSNE for $q_1^{(1)} \in \{0, 1\}$. Thus, contestant 1 is indifferent between the two best responses $x_1 = x_1^*$ and $x_1 = 0$. It remains to be shown that contestant $j > 1$ has no incentive to deviate. The equilibrium payoff of an inactive contestant $j > 1$ is

$$\Pi_j^* = q_1^{(1)} \frac{1}{\exp(\alpha x_1^*) + n - 1} + q_2^{(1)} \frac{1}{n}. \quad (76)$$

A deviation to some positive bid level $x_j > 0$ yields

$$\begin{aligned} \Pi_j(x_j) &= q_1^{(1)} \left(\frac{\exp(\alpha x_j)}{\exp(\alpha x_1^*) + \exp(\alpha x_j) + n - 2} - x_j \right) \\ &\quad + q_2^{(1)} \left(\frac{\exp(\alpha x_j)}{\exp(\alpha x_j) + n - 1} - x_j \right). \end{aligned} \quad (77)$$

From the PSNE property of one-sided dominance, we get

$$\frac{\exp(\alpha x_j)}{\exp(\alpha x_1^*) + \exp(\alpha x_j) + n - 2} - x_j \leq \frac{1}{\exp(\alpha x_1^*) + n - 1}. \quad (78)$$

Similarly, since multilateral peace is a PSNE property, we know that

$$\frac{\exp(\alpha x_j)}{\exp(\alpha x_j) + n - 1} - x_j \leq \frac{1}{n}. \quad (79)$$

Therefore, $\Pi_j(x_j) \leq \Pi_j^*$, as claimed. \square

The following Lemmas A.3 through A.7 all prepare the proof of Proposition 5. Lemma A.3 derives an upper bound on the equilibrium payoff for any contestant

that uses the zero bid in equilibrium. This lemma will be used in the proof of Lemma A.4 below.

Lemma A.3 *Let μ^* be a MSNE in the n -player Hirshleifer contest with parameter α , and let $i \in N$ be such that $0 \in \text{supp}\{\mu_i^*\}$. Then, $\Pi_i^* < \frac{n}{(n-1)\alpha}$.*

Proof. Suppose that the zero bid is contained in the support of contestant i 's equilibrium strategy μ_i^* . Then, as $p_i^{N,\alpha}(0, x_{-i}) \leq \frac{1}{n}$ for any $x_{-i} \in \mathbb{R}_{\geq 0}^{n-1}$, we see that

$$\Pi_i^* = E_{\mu_{-i}^*}[p_i^{N,\alpha}(0, x_{-i})] \quad (80)$$

$$\leq \frac{n}{n-1} E_{\mu_{-i}^*}[p_i^{N,\alpha}(0, x_{-i})(1 - p_i^{N,\alpha}(0, x_{-i}))] \quad (81)$$

$$= \frac{n}{\alpha(n-1)} E_{\mu_{-i}^*} \left[\frac{\partial p_i^{N,\alpha}(0, x_{-i})}{\partial x_i} \right]. \quad (82)$$

However, from the KKT condition at zero,

$$E_{\mu_{-i}^*} \left[\frac{\partial p_i^{N,\alpha}(0, x_{-i})}{\partial x_i} \right] \leq 1. \quad (83)$$

It follows that $\Pi_i^* \leq \frac{n}{\alpha(n-1)}$, as claimed. \square

The next lemma is an existence result identifying, for α sufficiently large, a contestant bidding arbitrarily close to the value of the prize. The lemma is the basis for the proof of Proposition 5(i), i.e., for the fact that at least two contestants engage in cut-throat competition for α sufficiently large.

Lemma A.4 *Let $b \in [0, 1)$. Then, for α sufficiently large, there exists a contestant $i \in N$ such that in any MSNE μ^* of the n -player Hirshleifer contest with parameter α ,*

$$F_i(b) \leq \left(\frac{b+2}{3} \right)^{\frac{1}{n-1}}. \quad (84)$$

Proof. Suppose the claim does not hold. Then, for any $i \in N$, and for an unbounded and increasing sequence of α , there exists a MSNE such that

$$\prod_{j \neq i} F_j(b) > \frac{b+2}{3}. \quad (85)$$

By bidding $x_i = \frac{b+1}{2}$, contestant i wins with a probability arbitrarily close to $\prod_{j \neq i} F_j(b)$, and therefore has a payoff approaching at least $\frac{b+2}{3} - \frac{b+1}{2} = \frac{1-b}{6} > 0$. But this is impossible in view of Lemma 3(i) and Lemma A.3. \square

The next two lemmas, Lemma A.5 and A.6, each capture the intuition that, as the contest becomes increasingly decisive, a contestant's probability of winning does not depend on bids that are substantially lower. Lemma A.5 enters the proof of Lemma A.6, while Lemma A.6 is used in the proof of Proposition 5(ii) to derive the fact that all rents get dissipated for α sufficiently large.

Lemma A.5 *Let $i \in N$, $\delta > 0$, and $\varepsilon > 0$. Then, for α large enough,*

$$p_i^{N,\alpha}(x_i + \varepsilon, 0, x_{-i,j}) - p_i^{N,\alpha}(x_i + \varepsilon, x_i, x_{-i,j}) \leq \delta, \quad (86)$$

holds for any $x_i \geq 0$ and $x_{-i,j} \in \mathbb{R}_{\geq 0}^{n-2}$.

Proof. Writing $X_{-i,j} = \sum_{k \neq i,j} \exp(\alpha x_k)$, we see that

$$\begin{aligned} & p_i^{N,\alpha}(x_i + \varepsilon, 0, x_{-i,j}) - p_i^{N,\alpha}(x_i + \varepsilon, x_i, x_{-i,j}) \\ &= \frac{\exp(\alpha(x_i + \varepsilon))}{\exp(\alpha(x_i + \varepsilon)) + 1 + X_{-i,j}} - \frac{\exp(\alpha(x_i + \varepsilon))}{\exp(\alpha(x_i + \varepsilon)) + \exp(\alpha x_i) + X_{-i,j}} \end{aligned} \quad (87)$$

$$= \frac{\exp(\alpha(x_i + \varepsilon))(\exp(\alpha x_i) - 1)}{(\exp(\alpha(x_i + \varepsilon)) + 1 + X_{-i,j})(\exp(\alpha(x_i + \varepsilon)) + \exp(\alpha x_i) + X_{-i,j})} \quad (88)$$

$$\leq \frac{\exp(\alpha x_i) - 1}{\exp(\alpha(x_i + \varepsilon))} \quad (89)$$

$$\leq \exp(-\alpha \varepsilon). \quad (90)$$

This proves the claim. \square

Lemma A.6 *Let $\delta > 0$. Then, for α large enough, in any MSNE μ^* of the n -player Hirshleifer contest with parameter α , we have*

$$\begin{aligned} p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j}) &\geq p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j}) - \delta \\ &(x_j \in \text{supp}\{\mu_j^*\}, x_i \in \text{supp}\{\mu_i^*\}, x_{-i,j} \in \mathbb{R}_{\geq 0}^{n-2}). \end{aligned} \quad (91)$$

Proof. There are two cases. If $x_i > y_j^{(1)}$, then even

$$p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j}) \geq p_j^{N,\alpha}(y_j^{(1)} + \delta, x_i, x_{-i,j}) \quad (92)$$

$$\geq p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j}). \quad (93)$$

If, however, $x_i \leq y_j^{(1)}$, then by Lemma A.5, for α large enough,

$$p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j}) \geq p_j^{N,\alpha}(y_j^{(1)} + \delta, x_i, x_{-i,j}) - \delta \quad (94)$$

$$\geq p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j}) - \delta. \quad (95)$$

This proves (91). \square

The next and final auxiliary lemma puts a lower bound on the probability that no opponent of some contestant j bids substantially above a bid level used by j with positive probability. Lemma A.7 may be seen as a counterpart of Lemma A.4 for low bids. It entails an important additional argument that is needed to show the bidding dichotomy captured by Proposition 5(iv), i.e., that for α large, bidders either enter the cut-throat competition or ultimately become inactive.

Lemma A.7 *Let $j \in N$, $\varepsilon > 0$, and $\delta > 0$. Then, for α large enough, we have for any MSNE μ^* in the n -player Hirshleifer contest with parameter α that*

$$\prod_{k \in N \setminus \{j\}} F_k(x_j + \delta) > \frac{\varepsilon}{2}, \quad (96)$$

for any $x_j \in \text{supp}\{\mu_j^*\}$ such that $x_j \geq \varepsilon$.

Proof. Clearly, contestant j may ensure herself a positive payoff by bidding zero. Hence, $\Pi_j^* > 0$. Furthermore, as $x_j \in \text{supp}\{\mu_j^*\}$, we get

$$E_{\mu_{-j}^*}[p_j(x_j, x_{-j})] > x_j \geq \varepsilon. \quad (97)$$

Note that $Q_j = \prod_{k \in N \setminus \{i\}} F_k(x_j + \delta)$ is the probability that all opponents of contestant i bid at most $x_j + \delta$. Hence,

$$E_{\mu_{-j}^*}[p_j(x_j, x_{-j})] = E_{\mu_{-j}^*} \left[\frac{\exp(\alpha x_j)}{\exp(\alpha x_j) + \sum_{k \neq j} \exp(\alpha x_k)} \right] \quad (98)$$

$$\leq Q_j \cdot 1 + (1 - Q_j) \exp(-\alpha \delta) \quad (99)$$

$$\leq Q_j + \exp(-\alpha \delta). \quad (100)$$

Choosing now α such that $\exp(-\alpha \delta) \leq \frac{\varepsilon}{2}$ proves the claim. \square

Proof of Proposition 5. (i) By Lemma A.4, for any α sufficiently large, and any MSNE μ^* in the n -player Hirshleifer contest with parameter α , there exists a contestant $i \equiv i(\alpha, \mu^*) \in N$ such that, $y_i^{(1)} \geq 1 - \frac{\varepsilon}{2}$. We claim that, possibly after raising α even further, there always exists another contestant $j \equiv j(\alpha, \mu^*) \in N \setminus \{i(\alpha, \mu^*)\}$ such that $y_j^{(1)} > 1 - \varepsilon$. Suppose not. Then, $y_j^{(1)} \leq 1 - \varepsilon$ for any $j \in N \setminus \{i\}$. But then, by the optimality condition for contestant i 's bid $x_i = y_i^{(1)}$,

$$0 = \frac{\partial}{\partial x_i} E_{\mu_{-i}^*}[\Pi_i^{N,\alpha}(x_i, x_{-i})] \quad (101)$$

$$= \alpha E_{\mu_{-i}^*}[p_i^{N,\alpha}(x_i, x_{-i})(1 - p_i^{N,\alpha}(x_i, x_{-i}))] - 1 \quad (102)$$

$$\leq \alpha E_{\mu_{-i}^*}[1 - p_i^{N,\alpha}(x_i, x_{-i})] - 1 \quad (103)$$

$$\leq \frac{\alpha(n-1)}{\exp(\alpha\varepsilon/2) + (n-1)} - 1. \quad (104)$$

As the right-hand side of inequality (104) turns negative for large α , we arrive at the desired contradiction. The claim follows.

(ii) The idea of the following argument is adapted from Baye et al. (1990). As above, let $i \equiv i(\alpha, \mu^*) \in N$ such that $y_i^{(1)} \geq 1 - \frac{\varepsilon}{2}$. Clearly, this implies $\Pi_i^* < \frac{\varepsilon}{2}$. We claim that, possibly after raising α even further, $\Pi_j^* < \varepsilon$ for any $j \in N \setminus \{i\}$. Suppose not. Then, $\Pi_j^* \geq \varepsilon$ for some $j \in N \setminus \{i\}$. In particular, $y_j^{(1)} < 1 - \varepsilon$. Suppose that contestant i overbids j using the bid $y_j^{(1)} + \delta$, where $\delta > 0$ is small.

Then, from Lemma A.6, taking expectations over μ^* , we see that

$$E_{\mu_{-i}^*}[p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j})] \geq E_{\mu_{-j}^*}[p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j})] - \delta, \quad (105)$$

resulting in

$$\frac{\varepsilon}{2} > \Pi_i^* \geq \Pi_j^* - 2\delta \geq \varepsilon - 2\delta. \quad (106)$$

However, for $\delta \leq \varepsilon/4$, this is impossible. The claim follows.

(iii) By contradiction. Suppose that there is $\varepsilon > 0$ and a contestant $i \in N$ such that for any $j \neq i$, the probability for j being inactive weakly exceeds ε . Then, by bidding slightly above zero, contestant i could guarantee a positive rent even as $\alpha \rightarrow \infty$, which is impossible. The claim follows.

(iv) The argument is a refinement of the proof of part (ii). To provoke a contradiction, suppose that $y_j^{(1)} \in [\varepsilon, 1 - \varepsilon]$ for some contestant $j \in N$. Then, in particular, $y_j^{(1)} + \frac{\varepsilon}{2} \leq 1 - \frac{\varepsilon}{2}$. By Lemma A.4, for α large enough, there is some contestant $i \in N$ such that

$$F_i(y_j^{(1)} + \frac{\varepsilon}{2}) \leq F_i(1 - \frac{\varepsilon}{2}) \leq \left(1 - \frac{\varepsilon}{6}\right)^{\frac{1}{n-1}}. \quad (107)$$

Using Bernoulli's inequality, this delivers

$$1 - F_i(y_j^{(1)} + \frac{\varepsilon}{2}) \geq \frac{\varepsilon}{6(n-1)}. \quad (108)$$

Let $\delta \in (0, \frac{\varepsilon}{2})$ be small. Then, from Lemma A.7, for α large enough,

$$\prod_{k \in N \setminus \{j\}} F_k(y_j^{(1)} + \delta) > \frac{\varepsilon}{2}. \quad (109)$$

Note that $i \neq j$ and $F_i \leq 1$, Hence, inequality (109) implies

$$\prod_{k \in N \setminus \{i,j\}} F_k(y_j^{(1)} + \delta) > \frac{\varepsilon}{2}. \quad (110)$$

Thus, combining (108) with (110), we obtain

$$\left(1 - F_i(y_j^{(1)} + \delta)\right) \cdot \prod_{k \in N \setminus \{i,j\}} F_k(y_j^{(1)} + \delta) > \frac{\varepsilon^2}{12(n-1)} \equiv \hat{\varepsilon}. \quad (111)$$

Note that, even though $\widehat{\varepsilon}$ may be very small, it is positive and exogenous (i.e., it does not depend on α nor on δ). We wish to show that contestant i may win with a substantially higher probability by overbidding contestant j by the arbitrarily small increment δ . Suppose that $x_k \leq y_j^{(1)}$ for all $k \in N \setminus \{i, j\}$. Then

$$p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j}) \geq \frac{1}{1 + (n-1)\exp(-\alpha\delta)}. \quad (112)$$

Suppose also that $x_i \geq y_j^{(1)} + \delta$. Then,

$$p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j}) \leq \exp(-\alpha\delta). \quad (113)$$

Thus, in this case, for α large enough,

$$p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j}) \geq p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j}) + \frac{1}{2}. \quad (114)$$

As shown above, this case happens with probability $\widehat{\varepsilon}$. Therefore, using Lemma A.6, taking expectations over μ^* , we find

$$E_{\mu_{-i}^*}[p_i^{N,\alpha}(y_j^{(1)} + \delta, x_j, x_{-i,j})] \geq E_{\mu_{-j}^*}[p_j^{N,\alpha}(y_j^{(1)}, x_i, x_{-i,j})] - \delta + \frac{\widehat{\varepsilon}}{2}. \quad (115)$$

Thus, by bidding $y_j^{(1)} + \delta$, contestant i achieves a payoff in excess of $\frac{\widehat{\varepsilon}}{4}$, for any sufficiently large α , in conflict with part (ii). This proves the last claim and, hence, the proposition. \square

Proof of Proposition 6 (Sketch). For any $m \geq 0$ and $i \in N$, we denote by F_i^m the cumulative distribution function of contestant i 's mixed strategy μ_i^m in the MSNE μ^m of the n -player Hirshleifer contest with parameter α_m . Clearly, each contestant can ensure a positive expected payoff by bidding zero. Hence, for ε small enough,

$$\prod_{j \neq i} F_j^m(x_i - \varepsilon) - x_i \geq 0 \quad (x_i \in \text{supp}\{\mu_i^m\}). \quad (116)$$

As there cannot be any mass points in the interior of $[0, 1]$ in the limit, this implies

$$\prod_{j \neq i} F_j^m(x_i) - x_i \geq \varepsilon \quad (x_i \in \text{supp}\{\mu_i^m\}). \quad (117)$$

On the other hand, as all rents are dissipated in the limit,

$$\prod_{j \neq i} F_j^m(x_i) - x_i \leq \varepsilon \quad (x_i \in [0, 1]). \quad (118)$$

Letting m go to infinity yields

$$\prod_{j \neq i} F_j(x_i) - x_i = 0 \quad (x_i \in \text{supp} \mu_i^*), \quad (119)$$

where F_j denotes the cumulative distribution function of contest j 's limit strategy μ_j^* . Thus, the limit μ^* is indeed a MSNE of the all-pay auction. \square

Proof of Proposition 7. Assume that $n > n^\#(\alpha) = \exp(\alpha) + 1$. Take a MSNE μ^* in the n -player Hirshleifer contest with parameter α , and let $x_i \in \text{supp}\{\mu_i^*\}$ with $i \in N$. Then, using $x_i \leq 1$,

$$0 < p_i^{N,\alpha} \equiv p_i^{N,\alpha}(x_i, x_{-i}) \leq \frac{\exp(\alpha x_i)}{\exp(\alpha x_i) + n - 1} < \frac{1}{2}, \quad (120)$$

for any $x_{-i} \in \text{supp}\{\mu_{-i}^*\}$. Hence, as in the proof of Lemma 3(i),

$$\frac{\partial^2 E_{\mu_{-i}^*}[\Pi_i^{N,\alpha}(x_i, x_{-i})]}{\partial x_i^2} = \alpha^2 E_{\mu_{-i}^*} \left[p_i^{N,\alpha} (1 - p_i^{N,\alpha}) (1 - 2p_i^{N,\alpha}) \right] > 0, \quad (121)$$

i.e., the second derivative of i 's equilibrium payoff function is strictly positive at x_i . As this implies $x_i = 0$, the claim follows. \square

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