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Cost of transformation: a measure on matchings

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Cost of transformation: a measure on matchings

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Abstract

This paper constructs a normative framework to quantify the difference (distance) between outcomes of market mechanisms in matching markets. We investigate the “cost of transformation” from one market mechanism to another, based on the differences in the outputs of these mechanisms, i.e., the matchings. Several conditions are introduced to ensure that this cost reflects the welfare effect of the transformation on individuals. We find a class of measures called *scaled Borda measures*, which is characterized by these conditions. Several possible applications of these measures in different markets are also discussed, such as measuring how unstable, how unfair, or how inefficient a mechanism (or a matching) is.

Keywords: matching markets, distance function, metrics, cost of stability

JEL Classifications: C78, D61, D63.

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1 Introduction

Matching theory analyzes markets where agents, e.g., buyers and sellers, hospitals and interns, high schools and students, are matched according to their preferences, and thereby conduct some transactions within the relevant context. Some of the well-known mechanisms are the deferred acceptance (introduced by [Gale and Shapley \(1962\)](#), characterized by [Kojima and Manea \(2010\)](#) and [Morrill \(2013a\)](#)), the Boston mechanism (characterized by [Afacan \(2013\)](#) and [Kojima and Ünver \(2014\)](#)), and the top trading cycle (introduced by [Shapley and Scarf \(1974\)](#) and characterized by [Morrill \(2013b\)](#) and [Pycia and Ünver \(2017\)](#)). These mechanisms produce matchings with various normative features¹, e.g., stability, Pareto efficiency, fairness, etc. They also have different computational complexity².

Given two mechanisms with different features and complexity, a measure on matchings can be used to compare the outcomes of the mechanisms and hence quantify the *cost of transformation* from one mechanism to the other. Such a measure can be interpreted in various ways. For instance, it can be interpreted as the *cost of stability* if one mechanism is not stable and the other is, or as the *cost of simplicity* if one mechanism is complex and the other is not, e.g., in terms of computation.

The most intuitive way to compare two matchings is by simply looking at the number of individuals who are matched differently. This measure³ would assign zero if the matchings are identical in all pairs, and would be maximal if they have nothing in common, i.e., the matchings are disjoint. However, this method neglects individuals' preferences in the market. That is, it does not matter how individuals rank their partners in the corresponding matchings. Therefore, it is not sensible to use it as a measure with the interpretation of the cost of transformation in relation to individual preferences.

This paper explores metric (distance) functions⁴ on matchings. We introduce intuitive conditions and endogenize individual preferences in quantifying the dissimilarity (distance) between two matchings, and hence between two mechanisms in roommate markets⁵. The conditions characterize an intuitive class of positional measures that behave like Borda scoring rules in the context of voting. Since roommate markets are very generic one-to-one matching problems, the results apply to the marriage markets as well. We show how the results can further be extended to object allocation

¹For comparisons of some of these methods, see [Abdulkadiroglu and Sönmez \(2003\)](#); [Abdulkadiroğlu et al. \(2005\)](#); [Ergin and Sönmez \(2006\)](#); [Chen and Sönmez \(2006\)](#); [Erdil and Ergin \(2008\)](#); [Kesten \(2010\)](#); [Abdulkadiroğlu et al. \(2011\)](#); [Kesten and Ünver \(2015\)](#).

²See [Irving \(1985\)](#), [Irving et al. \(2000\)](#), [Manlove et al. \(2002\)](#)

³To the best of our knowledge, the first reference to such measures can be found in [Klaus et al. \(2010\)](#) for stochastic markets.

⁴A metric is a function which satisfies non-negativity, identity of indiscernibles, symmetry, and triangular inequality.

⁵A roommate market is a one-sided one-to-one matching market.

problems in a much easier fashion. In addition, we also show how the measures can further be generalized to many-to-one markets, such as the school choice problems.

Our first condition *Betweenness* is a well-known additivity condition which requires that if every individual ranks a matching between two other matchings then the measure must be additive on these three matchings⁶. This allows for an additive treatment of welfare analysis, in case matchings are ordered properly and the transformation from one market mechanism (matching) to another implies welfare improvement. *Monotonicity* implies that if from one market to another, the set of agents ranked between two matchings weakly expand, the distance should be responsive to this and should also weakly increase. *Anonymity* requires that relabelling of individuals must not effect the measure. *Independence of irrelevant newcomers*, implies that when an “irrelevant” newcomer⁷ joins the market (like a dummy variable), the measure is unchanged if he or she remains single in both matchings. This typical invariance condition allows the measures to be applicable to variable population settings and brings about comparability across markets of varying sizes. Finally, *standardization* sets the minimal possible distance for any two disjoint matchings, i.e., matchings that have nothing in common, to be standard (fixed) across all markets with the same set of agents.

We investigate the behavior of measures that satisfy these conditions. We find that the measures are positional, in the sense that they assign distances based on ranks of agents’ partners. Having this intuition in mind, we introduce a class of Borda-like measures, which we call scaled Borda measures. Given a market, these measures scale the sum of absolute differences in Borda scores⁸ of agents’ partners in two matchings. We find that a measure satisfies the aforementioned conditions, if and only if it belongs to this class, i.e., it is a scalar of the Borda measure⁹. We formulate our result on the domain of roommate markets since we are also interested in markets that are not necessarily solvable, i.e., markets in which there are no stable matchings. In fact, the measures work on the full domain, i.e., they can compare any two matchings. This creates richness in the way these measures can be employed under different interpretations¹⁰. In case the measure is applied on the set of stable matchings only, it can serve as a utility to find a “fair” compromise among stable matchings, e.g., between men-optimal and women-optimal stable matchings in a marriage market.

⁶This is a standard method for strengthening the triangular inequality for cases where the weak inequality becomes equality, e.g., when three points are “on a line” in the Euclidian sense (see [Kemeny \(1959\)](#)).

⁷We consider a newcomer irrelevant if the market expands such that the incumbents preference does not change and they prefer being matched among themselves to being matched with the newcomer.

⁸The Borda score of a matching for an individual is the number of alternatives that are ranked strictly below the partner of the individual in that matching.

⁹Note that this class comprises only of measures with scoring vectors that are a scalar of Borda scores, and not linear transformations thereof.

¹⁰Note however that certain restricted domains lead to very natural interpretations of the measures. For instance, in markets where where all matchings are individually rational, i.e., being single is the worst option for everyone, the measures have a very straightforward interpretation as social welfare attained by any given matching. This could simply be done by comparing this matching with the matching wherein all agents are single.

Furthermore, it can also be used to quantify the level of positive discrimination or favoritism in the choice of stable matchings in a marriage market.

The paper proceeds as follows. In Section 2, we present the basic notation for the model. Section 3 introduces the model; a metric framework and the conditions on measures. Section 4 is devoted to the analysis of the structure of measures satisfying those conditions and eventually bringing forth a complete characterization. Section 5 concludes the paper with discussion and possible applications, while most proofs and the logical independence of the characterizing conditions are presented in the appendix.

2 Notation

We consider a countable and infinite set of potential individuals, denoted by \mathcal{N} , with a non-empty and finite subset $N \subsetneq \mathcal{N}$ interpreted as a set of agents. For each $i \in N$, let R_i denote the preference of agent i , that is a complete, transitive and antisymmetric binary relation over N , while $R \equiv (R_i)_{i \in N}$ is the preference profile. We say agent j is “at least as good as” agent k for agent i whenever $j R_i k$. We denote the position of agent j in the preference R_i , by $rank(j, R_i) = |\{k \in N : k R_i j\}|$. A generic market (also referred to as a roommate problem) is denoted by $P = (N, R)$, and the set of all possible roommate problems over a particular set of agents N by $\mathcal{P}(N)$. We denote the domain of all roommate problems by $\mathcal{D} = \langle \mathcal{P}(N) \rangle_{N \subsetneq \mathcal{N}}$, i.e., the set of all possible roommate problems over all possible sets of agents.

A *matching* μ is a permutation on N such that for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu(j) = i$. We refer to j as the *partner* (roommate) of i at matching μ , and in case $\mu(i) = i$, i is said to be *single* at matching μ . A matching in which every agent is single is referred to as the *identity matching* and is denoted by μ^I . We denote the set of all possible matchings on N by $\mathcal{M}(N)$. Given any problem $P = (N, R)$ and any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$, the set of agents that are preferred (nested) between $\mu(i)$ and $\bar{\mu}(i)$ according to R_i forms an *interval* denoted by $[\mu, \bar{\mu}]_{R_i}$. Formally,

$$[\mu, \bar{\mu}]_{R_i} = \{j \in N : \mu(i) R_i j R_i \bar{\mu}(i) \text{ or } \bar{\mu}(i) R_i j R_i \mu(i)\}.$$

The *length* of an interval is denoted by $|\mu, \bar{\mu}|_{R_i} = \#[\mu_1, \mu_2]_{R_i} - 1$, i.e., the cardinality of the interval minus 1. As an example in Figure 4, $[\mu^1, \mu^3]_{R_1} = \{2, 4, 3\}$ and $|\mu^1, \mu^3|_{R_1} = 2$.

We say a matching $\bar{\mu}$ is *between* matchings μ and $\bar{\mu}$, if $\bar{\mu}(i) \in [\mu, \bar{\mu}]_{R_i}$ for all $i \in N$. Given any sequence of matchings μ^1, \dots, μ^t in $\mathcal{M}(N)$ we say μ^1, \dots, μ^t are “on a line”, denoted by $[\mu^1 - \mu^2 - \dots - \mu^t]$, if μ^j is between μ^i and μ^k for all $1 \leq i \leq j \leq k \leq t$. We say a matching μ is *weakly above* $\bar{\mu}$ whenever $\mu(i) R_i \bar{\mu}(i)$ for all $i \in N$. In addition, we say μ and $\bar{\mu}$ are *adjacent* whenever $|\mu, \bar{\mu}|_{R_i} = 1$ for all $i \in N$, we say μ and $\bar{\mu}$ are *disjoint* whenever $\mu(i) \neq \bar{\mu}(i)$ for all $i \in N$.

Consider problem $P = (N, R)$. Let π be a permutation over the set of agents N . We denote the *permuted preference profile* by R^π where for all $i, j, k \in N$, $j R_i k$ if and only if $\pi(j) R_{\pi(i)}^\pi \pi(k)$. Define the *permuted problem* $P^\pi = (N, R^\pi)$ accordingly. Given a matching $\mu \in \mathcal{M}(N)$, we denote the *permuted matching* by μ^π where for all $i, j \in N$, $\mu(i) = j$ if and only if $\mu^\pi(\pi(i)) = \pi(j)$.¹¹ The permutations are denoted by the cycle notation, e.g., $\pi = (123)(45)$ denotes $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$, $\pi(5) = 4$ and $\pi(i) = i$ for all $i \in N \setminus \{1, 2, 3, 4, 5\}$.

Let N be a set of agents and consider a *newcomer* $a \in \mathcal{N} \setminus N$. A problem $P^* = (N \cup \{a\}, R^*)$ is called an *extension* of the problem $P = (N, R)$ by a whenever preferences of agents' in N over agents in N does not change from R to R^* , and the *newcomer* is ranked at the bottom in R^* by everyone in N . Formally:

1. $\text{rank}(j, R_i) = \text{rank}(j, R_i^*)$, for all $i, j \in N$,
2. $\text{rank}(j, R_i^*) = \#N + 1$, for all $i, j \in N$.

Similarly, we say $\bar{\mu} \in \mathcal{M}(N \cup \{a\})$, is the extension of a matching $\mu \in \mathcal{M}(N)$ by agent $a \in \mathcal{N} \setminus N$, whenever $\bar{\mu}(i) = \mu(i)$ for all $i \in N$, and $\bar{\mu}(a) = a$. In such extensions, we call $a \in \mathcal{N} \setminus N$, an *irrelevant newcomer*.

Finally, let $A = \{a_1, a_2, \dots, a_k\}$, be a set of agents such that $N \cap A = \emptyset$. Consider the sequence $P^0, P^1, P^2, \dots, P^k$ of problems such that $P^0 = P$ and P^t is an extension of P^{t-1} by agent $a_t \in A$. Then we say P^k is an extension of P by the set of agents A . Similarly, we can define the extension of a matching with a set of agents. It should be noted that, the order of adding agents in the set results in different problems.

3 Model

We use metric functions as our main framework for comparing matchings. Given a set of agents N , and a problem $P \in \mathcal{P}(N)$, a function on matchings $\delta_P : \mathcal{M}(N) \times \mathcal{M}(N) \rightarrow \mathbb{R}$ is called a metric (or a distance function) function if and only if it satisfies the regular metric conditions¹². Hence, a distance function δ_P assigns every pair of matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ a non-negative real number depending on the structure of the problem P . We consider *measures* on matchings, i.e., collections of distance functions on all possible problems in the domain, denoted by

$$\delta = \langle \delta_P \rangle_{P \in \mathcal{D}}.$$

¹¹This is a typical definition for permutations in roommate markets, as examples of this see Klaus (2017); Özkal-Sanver (2010); Sasaki and Toda (1992).

¹²i) Non-negativity: $\delta_P(\mu, \bar{\mu}) \geq 0$, ii) identity of indiscernibles: $\delta_P(\mu, \bar{\mu}) = 0$ if and only if $\mu = \bar{\mu}$, iii) symmetry: $\delta_P(\mu, \bar{\mu}) = \delta_P(\bar{\mu}, \mu)$, and iv) triangular inequality: $\delta_P(\mu, \bar{\bar{\mu}}) \leq \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}})$.

The first condition requires that if three matchings are ordered “on a line” then the measure should be additive on these matchings.

Condition 1 (Betweenness): δ satisfies *betweenness* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ such that $\bar{\mu}$ is between $\mu, \bar{\bar{\mu}}$

$$\delta_P(\mu, \bar{\bar{\mu}}) = \delta_P(\mu, \bar{\mu}) + \delta_P(\bar{\mu}, \bar{\bar{\mu}}).$$

Anonymity condition is straightforward and requires that the relabeling of the agents should not matter.

Condition 2 (Anonymity): δ satisfies *anonymity* if for all problems $P = (N, R) \in \mathcal{D}$ and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ and permutation $\pi : N \rightarrow N$

$$\delta_P(\mu, \bar{\mu}) = \delta_{P^\pi}(\mu^\pi, \bar{\mu}^\pi).$$

Monotonicity condition requires that if from one problem to another, the two matchings fall further apart from one another, then the measure should reflect that by an increase in the distance.

Condition 3 (Monotonicity): δ satisfies *monotonicity* if for all problems $P = (N, R) \in \mathcal{D}$ and $\hat{P} = (N, \hat{R}) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ such that $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\hat{R}_i}$ for all $i \in N$

$$\delta_P(\mu, \bar{\mu}) \leq \delta_{\hat{P}}(\mu, \bar{\mu}).$$

Remark 1. *Immediate implication of monotonicity is that if for two matchings μ and $\bar{\mu}$, the intervals remain the same across two problems on the same set of agents, then the distance should not change. Furthermore changing the relative order of $\mu, \bar{\mu}$ in individual preferences, does not alter the distance as long as the intervals remain the same.*

The next condition is an invariance axiom which states that if a problem and two matchings are extended by a dummy agent which essentially does not change the matchings, then the distance between these matchings should be the same in the extended problem.

Condition 4 (Independence of irrelevant newcomers): δ satisfies *independence of irrelevant newcomers* if for all problem $P = (N, R) \in \mathcal{D}$ and any extension $P^* = (N^*, R^*) \in \mathcal{D}$ and all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$ with the extension $\mu^*, \bar{\mu}^* \in \mathcal{M}(\hat{N})$ by some agent $a \in \mathcal{N} \setminus N$

$$\delta_P(\mu, \bar{\mu}) = \delta_{P^*}(\mu^*, \bar{\mu}^*).$$

Remark 2. *An immediate implication of independence of irrelevant newcomers is that if $P^*, \mu^*, \bar{\mu}^*$ are an extension of the $P, \mu, \bar{\mu}$, by a set of agents A , then $\delta_P(\mu, \bar{\mu}) = \delta_{P^*}(\mu^*, \bar{\mu}^*)$.*

Our final condition, standardization, requires that given a set of agents, the minimal distance must be the same across all pairs of disjoint matchings.

Condition 5 (Standardization): δ satisfies *standardization* if there exists a function $\kappa : 2^N \rightarrow \mathbb{R}$ such that for all N and for all disjoint matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$,

$$\min_{P \in \mathcal{P}(N)} \delta_P(\mu, \bar{\mu}) = \kappa(N).$$

4 Results

In what follows, we restrict our attention only to measures that satisfy the five conditions laid out in Section 3, i.e., Betweenness, Anonymity, Monotonicity, Independence of irrelevant newcomers and Standardization. We first introduce Lemma 1 (Decomposition lemma) which proves that the measures we seek, decompose the distance into sums of distances between pairs of matchings that look like components of the original matchings (see Figure 1). Thereafter, the results are presented in two subsections. In Section 4.1 we analyze the behavior of these measures specifically when they compare a matching with the identity matching, i.e., the matching where every agent is single, and in Section 4.2, the results are extended to cases where any two matchings are compared.

In Section 4.1, we first use Lemmas 2, 3, and 4 to show the distances between one-couple matchings (matchings in which everyone is single except one couple) and the identity matching is the same across all problems so long as the interval lengths are the same. These lemmas also quantify how different interval lengths relate to one another. Proposition 1 shows that the distances of such one-couple matchings to the identity matching should be based on positions of the partners. Finally, Theorem 1 combines the aforementioned results and extends Proposition 1 to conclude that the measure we seek is equivalent to a class of positional measures.

In Section 4.2, we extend the findings of Section 4.1 to any two matchings using two more building blocks, i.e., Propositions 2 and 3, to generalize Theorem 1 for any two matchings. Hence, Theorem 2 provides a complete characterization of a class of positional measures which we call as scaled Borda measures. We also show, in Appendix C, that the conditions in the characterization results are indeed logically independent.

To state the first lemma, let $\mu, \bar{\mu} \in \mathcal{M}(N)$ be two matchings and $S \subseteq N$ be a subset of agents that are matched among themselves in μ and $\bar{\mu}$, i.e., $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$. Based on the set S , we define two matchings, μ^S and $\mu^{\bar{S}}$, as follows:

1. for all $i \in S$, let $\mu^S(i) = \mu(i)$ and for all $i \in N \setminus S$, let $\mu^S(i) = \bar{\mu}(i)$,
2. for all $i \in S$, let $\mu^{\bar{S}}(i) = \bar{\mu}(i)$ and for all $i \in N \setminus S$, let $\mu^{\bar{S}}(i) = \mu(i)$.

In the following lemma, we show that the distance between $\mu, \bar{\mu}$ can be decomposed into the sum of the distances from μ^S and $\mu^{\bar{S}}$ to μ (or $\bar{\mu}$). Figure 1 shows a demonstration of this decomposition.

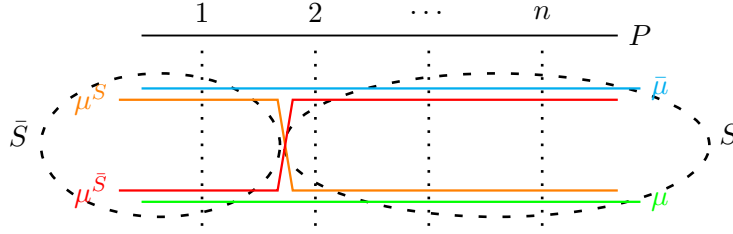


Figure 1: The general view of the Decomposition Lemma.

Lemma 1. (Decomposition Lemma) Let $\mu, \bar{\mu} \in \mathcal{M}(N)$. Then, for all $S \subseteq N$ such that $\mu(i), \bar{\mu}(i) \in S$ for all $i \in S$, we have

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu, \mu^S) = \delta_P(\mu^{\bar{S}}, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu}).$$

Proof. By definition of $\mu^{\bar{S}}$ and μ^S , both are between μ and $\bar{\mu}$, hence betweenness yields

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu^{\bar{S}}, \bar{\mu}) \text{ and,} \quad (1)$$

$$\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}). \quad (2)$$

Since μ and $\bar{\mu}$ are both between μ^S and $\mu^{\bar{S}}$ betweenness results in

$$\delta_P(\mu^S, \mu^{\bar{S}}) = \delta_P(\mu^S, \mu) + \delta_P(\mu, \mu^{\bar{S}}) \text{ and,}$$

$$\delta_P(\mu^S, \mu^{\bar{S}}) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \mu^{\bar{S}}).$$

The four equations above yield

$$\delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu^{\bar{S}}, \bar{\mu}) = \delta_P(\mu, \mu^S) + \delta_P(\mu^S, \bar{\mu}) \text{ and,}$$

$$\delta_P(\mu^S, \mu) + \delta_P(\mu, \mu^{\bar{S}}) = \delta_P(\mu^S, \bar{\mu}) + \delta_P(\bar{\mu}, \mu^{\bar{S}}).$$

Subtracting the latter equation from the former we have $\delta_P(\mu^{\bar{S}}, \bar{\mu}) - \delta_P(\mu^S, \mu) = \delta_P(\mu, \mu^S) - \delta_P(\bar{\mu}, \mu^{\bar{S}})$ which reduces to

$$\delta_P(\mu^{\bar{S}}, \bar{\mu}) = \delta_P(\mu^S, \mu). \quad (3)$$

Plugging Equation 3 into Equation 1, and as δ is a symmetric function, yields $\delta_P(\mu, \bar{\mu}) = \delta_P(\mu, \mu^{\bar{S}}) + \delta_P(\mu, \mu^S)$, and plugging Equation 3 into Equation 2 results in $\delta_P(\mu, \bar{\mu}) = \delta_P(\mu^{\bar{S}}, \bar{\mu}) + \delta_P(\mu^S, \bar{\mu})$. ■

4.1 Comparing any matching with the identity matching

We now focus on the distance between any matching and the identity matching. By monotonicity, as long as the intervals between the two matchings remain the same the distance will be unchanged.

Therefore, in order to keep the figures simple, we draw the identity matching below the other matchings and we often denote the matchings as straight lines whenever possible.

Consider a matching in which everyone is single except one couple, say $\mu(i) = j$ with $i \neq j$. We call such a matching a *one-couple matching* (see Figure 3) and denote it by μ^{ij} . Given a problem $P = (N, R)$, we say a one-couple matching μ^{ij} is of length (x, y) whenever $(|\mu^{ij}(i), i|_{R_i}, |\mu^{ij}(j), j|_{R_j}) = (x, y)$.

Remark 3. Consider any matching μ with k distinct couples. Then, by Decomposition Lemma, and letting $S = \{i, j\}$ and $\bar{S} = N \setminus S$ for each couple of μ , the distance between μ and μ^I can be decomposed as the sum of distances of each of these k one-couple matchings, and the identity matching.

According to Remark 3, to compute the distance between any matching and the identity matching, we only need to focus on the distance between a one-couple matching and the identity matching. Then the total distance equals the sum of each of these one-couple matchings. In the sequel, we show that the distance between a one-couple matching and identity matching is the same for all problems whenever the interval lengths are the same. In Lemma 2, we show this for the case where the interval length is $(x, 1)$, see Figure 2. Then in Lemma 3, we extend this to any interval length (x, y) , see Figure 3.

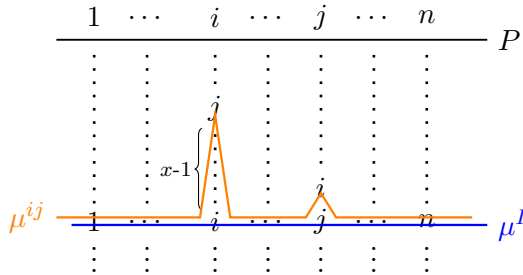


Figure 2: A one-couple matching μ^{ij} of length $(x, 1)$.

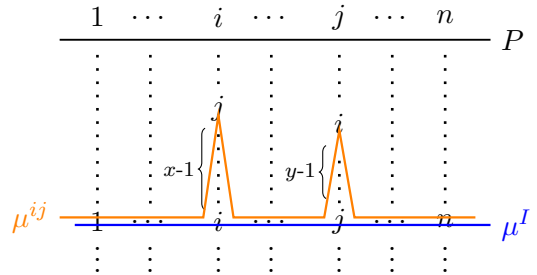


Figure 3: A one-couple matching μ^{ij} of length (x, y) .

Lemma 2. Consider any $N, N' \subsetneq \mathcal{N}$ and a strictly positive integer x . Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length $(x, 1)$ in P . Similarly consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length $(x, 1)$ in P' . Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}).$$

Proof. See Appendix A.1. ■

Lemma 2, shows that the distance between the identity matching and any one-couple matching of length $(x, 1)$ is the same across all the problems in the domain, i.e., regardless of the set of agents. To simplify notation we denote this distance by α_{x1} . The next lemma extends Lemma 2 to any one-couple matching of length (x, y) .

Lemma 3. *Consider any $N, N' \subsetneq \mathcal{N}$ and two strictly positive integers x and y . Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, y) in P . Similarly consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length (x, y) in P' . Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then*

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

Proof. See Appendix A.2. ■

Lemma 3, shows that the distance between the identity matching and any one-couple matching of length (x, y) is the same across all the problems in the domain, i.e., regardless of the set of agents. To simplify notation we denote this distance by α_{xy} . Next as a particular case of Lemma 3, we show that for any strictly positive integer x , $\alpha_{xx} = x\alpha_{11}$, i.e., a one-couple matching of length (x, x) has x times the distance that a one-couple matching of length $(1, 1)$ has (to the identity matching).

Lemma 4. *Consider any $N \subsetneq \mathcal{N}$ and a strictly positive integer x . Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any problem $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, x) in P . Let μ^I denote the identity matching, then*

$$\delta_P(\mu^{ij}, \mu^I) = x \times \alpha_{11}.$$

Proof. See Appendix A.3. ■

Now we introduce Proposition 1. When all five conditions in Section 3 are imposed on a distance function δ , Proposition 1 states that the distance between the identity matching and a one-couple matching μ^{ij} must equal to a scalar function of the sum of absolute changes in the position of each agents' partners in these matchings.

Proposition 1. *For any problem $P = (N, R)$ and any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$ we have*

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{xy} = \frac{1}{2}\alpha_{11} \sum_{k \in \{i, j\}} |\text{rank}(\mu^{ij}(k), R_k) - \text{rank}(\mu^I(k), R_k)|.$$

Proof. Let μ^{ij} be any one-couple matching of length (x, y) . By Lemma 3, $\alpha_{xy} = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. On the other hand by Lemma 3, $\alpha_{xx} = \alpha_{x1} + \alpha_{x1} - \alpha_{11} = 2\alpha_{x1} - \alpha_{11}$, and by Lemma 4, $\alpha_{xx} = x\alpha_{11}$.

Combining the two implies $\alpha_{x1} = \frac{(x+1)}{2}\alpha_{11}$. Setting $\alpha_{x1} = \frac{(x+1)}{2}\alpha_{11}$ and $\alpha_{y1} = \frac{(y+1)}{2}\alpha_{11}$ into $\alpha_{xy} = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$ simplifies to

$$\alpha_{xy} = \frac{1}{2}\alpha_{11}(x + y).$$

Note that $x = |\mu^{ij}, \mu^I|_{R_i}$ and $y = |\mu^{ij}, \mu^I|_{R_j}$. Then we have $\alpha_{xy} = \frac{1}{2}\alpha_{11}(|\mu^{ij}, \mu^I|_{R_i} + |\mu^{ij}, \mu^I|_{R_j})$ which can be rearranged as

$$\alpha_{xy} = \frac{1}{2}\alpha_{11} \sum_{k \in \{i,j\}} |\text{rank}(\mu^{ij}(k), R_k) - \text{rank}(\mu^I(k), R_k)|. \quad (4)$$

■

Proposition 1 is fundamental in that it compares one-couple matchings and their distance to identity matching, also draws a clear picture of how the class of measures we are looking for should behave. In fact, the right hand side of Equation 4 in the proof above is very similar to a positional voting concept known as the Borda rule in voting literature¹³. For each candidate in a voting problem, Borda rule defines a score that candidates gets from each voter as follows:

$$\text{BordaScore}(j, R_i) = |N| - \text{rank}(j, R_i) \quad (5)$$

which is interpreted as the score candidate j gets from voter i . A very straightforward application of this scoring concept to comparing two matchings, is one where we compare the Borda scores of partners of an agent i , in these two matchings (in absolute value):

$$|\text{BordaScore}(\mu(i), R_i) - \text{BordaScore}(\bar{\mu}(i), R_i)|$$

Finally, summing all the Borda score differentials for each individual's partners in the two matchings would properly define a new measure which we call the Borda measure.

Borda Measure: A measure is called the *Borda Measure*, denoted by δ^{Borda} , if for all $P = (N, R) \in \mathcal{D}$, and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$

$$\delta_P^{Borda}(\mu, \bar{\mu}) = \sum_{i \in N} |\text{BordaScore}(\mu(i), R_i) - \text{BordaScore}(\bar{\mu}(i), R_i)|. \quad (6)$$

$$= \sum_{i \in N} |\text{rank}(\mu(i), R_i) - \text{rank}(\bar{\mu}(i), R_i)|. \quad (7)$$

¹³See Borda (1781); Saari (1990).

Remark that Equation 7 shows a clear resemblance to Equation 4 in Proposition 1. In fact, the latter is just a scalar transformation of the former with some constant. In what follows, we formally define these scalar transformations of the Borda measure, and call them scaled Borda measures. Formally:

Scaled Borda Measures: A measure is called a *Scaled Borda Measures*, denoted by $\delta^{\sigma-Borda}$, if for some $\sigma \in \mathbb{R}_{++}$, for all $P = (N, R) \in \mathcal{D}$, and for all matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$

$$\delta_P^{\sigma-Borda}(\mu, \bar{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \bar{\mu}) \quad (8)$$

The reader can verify that the Borda measure and all the scalar transformations of it satisfy the conditions introduced in Section 3, and therefore the lemmata in this section. We can now introduce our first theorem which expands Proposition 1. Theorem 1 states that when all five conditions in Section 3 are imposed on a distance function δ , the distance between the identity matching and any other matching must equal to a scaled Borda measure for some positive constant σ . Formally:

Theorem 1. *For any problem $P = (N, R)$ and any $\mu \in \mathcal{M}(N)$ we have*

$$\delta_P(\mu, \mu^I) = \sigma \times \delta_P^{Borda}(\mu, \mu^I) \text{ for some } \sigma \in \mathbb{R}_{++}.$$

Proof. Considering the distance between any one-couple matching μ^{ij} of any length (x, y) and the identity matching μ^I , we can plug Equation 7 into Proposition 1 which yields:

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{xy} = \frac{1}{2}\alpha_{11} \times \delta_P^{Borda}(\mu^{ij}, \mu^I).$$

Note that by the Decomposition Lemma, both for δ_P and δ_P^{Borda} , the distance between any μ and the identity matching μ^I is the sum of distances between the identity matching and all one-couple matchings induced by μ . Therefore:

$$\delta_P(\mu, \mu^I) = \frac{1}{2}\alpha_{11} \times \delta_P^{Borda}(\mu, \mu^I)$$

Finally, as δ_P is a metric function, $\alpha_{11} > 0$ (nonnegativity and identity of indiscernibles). Setting $\sigma = \frac{1}{2}\alpha_{11} > 0$, we conclude the distance is a scaled Borda measure with $\sigma = \frac{1}{2}\alpha_{11}$.

$$\delta_P(\mu, \mu^I) = \sigma \times \delta_P^{Borda}(\mu, \mu^I) = \delta_P^{\sigma-Borda}(\mu, \mu^I).$$

■

Theorem 1 proves that a measure satisfying the desired conditions must behave like a scaled Borda measure when comparing the distance between a matching and the identity matching. Next

section shows that this is in fact the case for any two matchings, hence providing a complete characterization.

4.2 Comparing any two non-identity matching

In this section, we generalize Theorem 1 to any two matchings. That is, under the imposed conditions, given any problem and any two matchings, a measure which compares these two matchings must be equivalent to a scalar Borda measure. To do so, first we propose two propositions for four-agents problems, and use these two propositions as building blocks to construct Theorem 2.

Proposition 2. *Consider a problem P over four agents with the preference profile and the matchings shown in Figure 4. Note that one singleton is nested between μ^1 and μ^2 and another is nested between μ^2 and μ^3 . In such specific cases,*

1. $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$,
2. $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

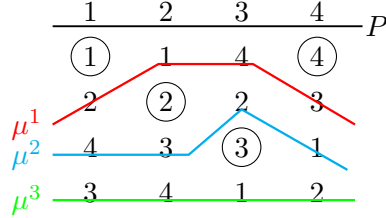


Figure 4: A problem over four agent with one singleton agent between the matchings.

Proof. See Appendix B.1. ■

Proposition 3. *Consider a problem P over four agents with the preference profile and the matchings shown in Figure 5. Note that two singletons are nested between μ^1 and μ^2 and another two are nested between μ^2 and μ^3 . In such specific cases,*

1. $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$,
2. $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

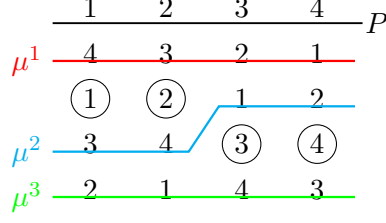


Figure 5: Problem P over four agents with two singleton agents between the matchings.

Proof. See Appendix B.2. ■

Next we propose our main characterization. A measure satisfies Betweenness, Anonymity, Monotonicity, Independence of irrelevant newcomers and Standardization if and only if it is a scalar Borda measure for some $\sigma > 0$.

Theorem 2. *For any problem $P = (N, R)$ and $\mu, \tilde{\mu} \in \mathcal{M}(N)$, we have*

$$\delta_P(\mu, \tilde{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \tilde{\mu}) \text{ for some } \sigma \in \mathbb{R}_{++}.$$

Proof. We leave the “if” part to the reader and provide the “only if” part of the theorem. Without loss of generality, let $N = \{1, 2, \dots, n\}$ be the set of agents and consider any $P \in \mathcal{P}(N)$. In case $\mu = \tilde{\mu}$, as δ is a metric function we have $\delta_P(\mu, \tilde{\mu}) = 0$ which equals $\sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ for any $\sigma > 0$. In case $\mu = \mu^I$ (or $\tilde{\mu} = \mu^I$), by Theorem 1, we have $\delta_P(\mu, \tilde{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$. Next we shall prove that for any other possible pairs of matchings $\mu, \tilde{\mu} \in \mathcal{M}(N) \setminus \{\mu^I\}$ such that $\mu \neq \tilde{\mu}$, the measure also equals a scaled Borda measure with $\sigma = \frac{1}{2}\alpha_{11}$, i.e., $\delta_P(\mu, \tilde{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that if the number of agents is odd, we can use extensions of P, μ , and $\tilde{\mu}$ by one irrelevant newcomer. By independence of irrelevant newcomers, the distance would be unchanged. So without loss of generality we can assume that the number of agents to be even. Furthermore, by monotonicity, we can assume that μ is weakly above $\tilde{\mu}$.

Let $N' = \{1', 2', \dots, n'\}$ be a set of agents such that $|N| = |N'|$ and $N \cap N' = \emptyset$. Let $\bar{N} = N \cup N'$. Let $P^*, \mu^*, \tilde{\mu}^*$ be an extension of $P, \mu, \tilde{\mu}$ by the set N' . By Remark 2, $\delta_P(\mu, \tilde{\mu}) = \delta_{P^*}(\mu^*, \tilde{\mu}^*)$. For simplicity, we abuse the notation and write P, μ and $\tilde{\mu}$ instead of writing P^*, μ^* and $\tilde{\mu}^*$, respectively. Let us define two additional matchings $\mu^B, \mu^T \in \mathcal{M}(\bar{N})$ such that: (1) for all $i \in N$, $\mu^B(i) = i' \in N'$, and (2) for all odd $i \in N$, $\mu^T(i) = (i + 1)' \in N'$ and for all even $i \in N$, $\mu^T(i) = (i - 1)' \in N'$.

Next we construct another problem $\bar{P} = (\bar{N}, \bar{R})$ on the same set of agents \bar{N} (see Figure 6 for a general view of the structure of this problem) such that

1. $[\mu, \tilde{\mu}]_{R_i} = [\mu, \tilde{\mu}]_{\bar{R}_i}$ for all $i \in \bar{N}$, i.e., the intervals of μ and $\tilde{\mu}$ in \bar{P} are the same as those in P ,
2. μ^T is weakly above μ^B , μ^B is weakly above μ (and they are adjacent), and μ is weakly above $\tilde{\mu}$,
3. if $i \in [\mu, \tilde{\mu}]_{\bar{R}_i}$ then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), \mu^B(i)\}$, i.e., if i is nested between $\mu(i)$ and $\tilde{\mu}(i)$ then no other agent is nested between $\mu^T(i)$ and $\mu^B(i)$,
4. if $i \notin [\mu, \tilde{\mu}]_{\bar{R}_i}$ then $[\mu^T, \mu^B]_{\bar{R}_i} = \{\mu^T(i), i, \mu^B(i)\}$, i.e., if i is not nested between $\mu(i)$ and $\tilde{\mu}(i)$ then i is the only other agent nested between $\mu^T(i)$ and $\mu^B(i)$.

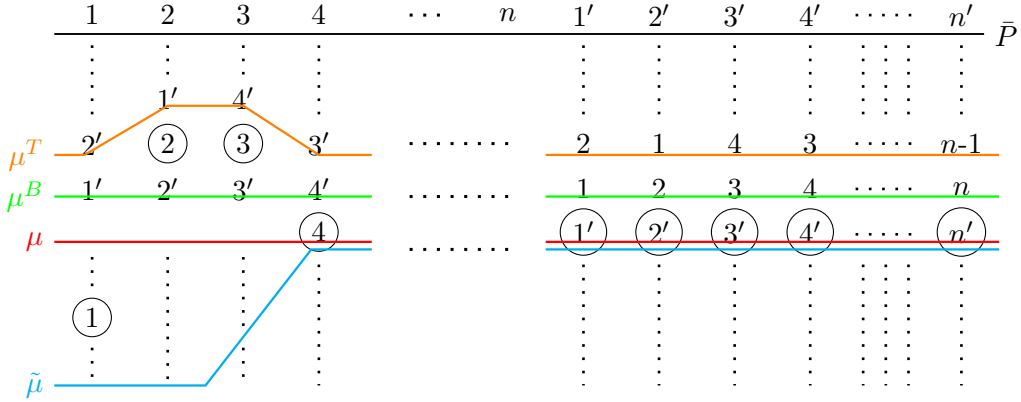


Figure 6: General structure for \bar{P} .

Note that by monotonicity for problems P and \bar{P} , we have $\delta_P(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu, \tilde{\mu})$. Also, $\delta_P^{Borda}(\mu, \tilde{\mu}) = \delta_{\bar{P}}^{Borda}(\mu, \tilde{\mu})$. Hence it is sufficient to show $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that $[\mu^T - \mu^B - \mu - \tilde{\mu}]$ are on a line in problem \bar{P} , therefore betweenness implies $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \mu^B) + \delta_{\bar{P}}(\mu^B, \mu) + \delta_{\bar{P}}(\mu, \tilde{\mu})$ and hence

$$\delta_{\bar{P}}(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu^T, \tilde{\mu}) - \delta_{\bar{P}}(\mu^T, \mu^B) - \delta_{\bar{P}}(\mu^B, \mu). \quad (9)$$

In the next three steps we show that the distance between each of the three pairs of matchings on the right-hand side of Equation 9 equals the scaled Borda measure for some $\sigma > 0$. By betweenness of scaled Borda measures, this in return shall imply $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu, \tilde{\mu})$ for some $\sigma > 0$.

Step 1. (Proving that $\delta_{\bar{P}}(\mu^T, \tilde{\mu})$ equals the scaled Borda measure for some $\sigma > 0$.) By construction of \bar{P} , $[\mu^T - \mu^I - \tilde{\mu}]$ are on a line. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^T, \tilde{\mu}) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^T, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Step 2. (Proving that $\delta_{\bar{P}}(\mu^B, \mu)$ equals the scaled Borda measure for some $\sigma > 0$.) By construction of \bar{P} we can consider any problem $\bar{\bar{P}}$ where $[\mu^B - \mu - \mu^I]$ are on a line, and the intervals

of μ^B and μ are unchanged, i.e., $[\mu^B, \mu]_{\bar{R}_i} = [\mu^B, \mu]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$, therefore by monotonicity the distance is unchanged. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^B, \mu) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^B, \mu)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Step 3. (Proving that $\delta_{\bar{P}}(\mu^T, \mu^B)$ equals the scaled Borda measure for some $\sigma > 0$.) Consider the partition of \bar{N} into the following subsets of agents $T_1 = \{1, 2, 1', 2'\}$, $T_2 = \{3, 4, 3', 4'\}$, \dots , $T_{\frac{n}{2}} = \{n-1, n, (n-1)', n'\}$ where $\bar{N} = \bigcup_{l=1}^{\frac{n}{2}} T_l$. Let μ^{T_l} denote a matching where $\mu^{T_l}(i) = \mu^T(i)$ for all $i \in T_l$, and $\mu^{T_l}(i) = \mu^B(i)$ for all $i \in \bar{N} \setminus T_l$. By construction for all $l \in \{1, \dots, \frac{n}{2}\}$, μ^{T_l} is between μ^T and μ^B . By Decomposition Lemma, we have:

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sum_{l=1}^{\frac{n}{2}} \delta_{\bar{P}}(\mu^{T_l}, \mu^B). \quad (10)$$

To simplify notation, we denote a generic μ^{T_l} simply by μ^S . Based on the construction of μ^T and μ^B , each of these matchings, μ^S , will have one of the following three structures: (1) **no singleton** is nested between μ^S and μ^B (see Figure 7), or (2) **one singleton** is nested between μ^S and μ^B (see Figure 8 and 11), or (3) **two singletons** are nested between μ^S and μ^B (see Figure 12). In the sequel, we shall show that for each of the three possible structures, $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$, i.e., the distance is a scaled Borda measure.

- **Case 1. (no singleton)** Consider the case in which no singleton is nested between μ^S and μ^B (Figure 7).

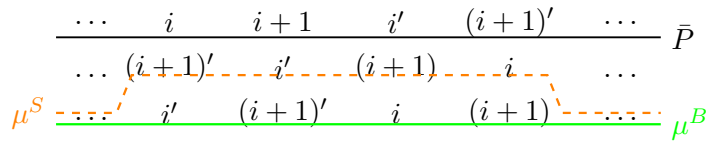


Figure 7: The no singleton structure.

By construction of \bar{P} we can consider any problem $\bar{\bar{P}}$ where $[\mu^S - \mu^B - \mu^I]$ are on a line, and the intervals of μ^S and μ^B are unchanged, i.e., $[\mu^S, \mu^B]_{\bar{R}_i} = [\mu^S, \mu^B]_{\bar{\bar{R}}_i}$ for all $i \in \bar{N}$, therefore by monotonicity the distance is unchanged. Then by betweenness and Theorem 1, $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

- **Case 2. (one singleton)** Consider the case in which one singleton is nested between μ^S and μ^B . By construction of μ^T and μ^B the singleton is either i or $i+1$. Therefore, two situations are plausible,

I. i is the singleton nested (see Figure 8).

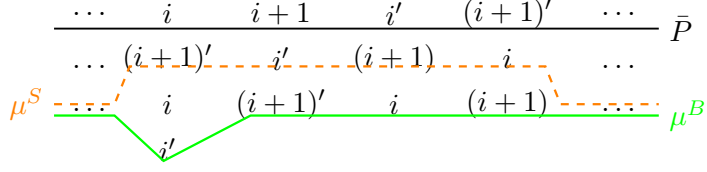


Figure 8: The one singleton structure with i as the singleton.

Consider the four agent problem P in Proposition 2, and rename the agents as $2 = i, 4 = i + 1, 3 = i'$ and $1 = (i + 1)'$. Let \hat{P} be an extension of this problem P , by the set of agents $A = \bar{N} \setminus \{i, i', (i + 1), (i + 1)'\}$, and $\hat{\mu}^1$ and $\hat{\mu}^2$ be the extension of μ^1 and μ^2 by the set A , respectively (see Figure 9). By Remark 2,

$$\delta_P(\mu^1, \mu^2) = \delta_{\hat{P}}(\hat{\mu}^1, \hat{\mu}^2). \quad (11)$$

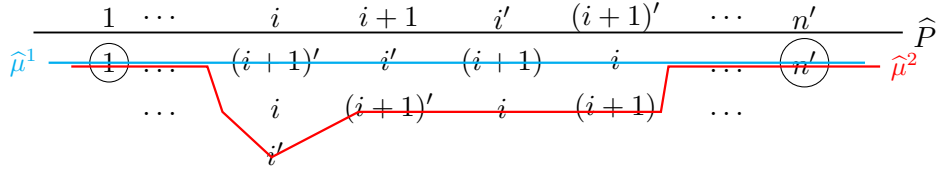


Figure 9: The four agents problem P of Proposition 2 after adding the set of agents $A = \bar{N} \setminus \{i, i', (i + 1), (i + 1)'\}$, as irrelevant newcomers.

Now, consider another problem P' shown in Figure 10. Monotonicity implies

$$\delta_{\hat{P}}(\hat{\mu}^1, \hat{\mu}^2) = \delta_{P'}(\hat{\mu}^1, \hat{\mu}^2). \quad (12)$$

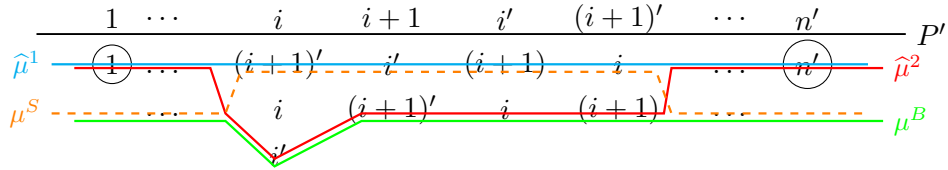


Figure 10: The problem P' .

Note that the structure of the four matchings, $\hat{\mu}^1, \hat{\mu}^2, \mu^S, \mu^B$, in problem P' corresponds to the four matchings in Figure 1 (to $\bar{\mu}, \mu^{\bar{S}}, \mu^S, \mu^B$ respectively). Therefore by Equation 3 in Decomposition Lemma we have

$$\delta_{P'}(\hat{\mu}^1, \hat{\mu}^2) = \delta_{P'}(\mu^S, \mu^B). \quad (13)$$

Putting Equations 11, 12, and 13 together results in $\delta_P(\mu^1, \mu^2) = \delta_{P'}(\mu^S, \mu^B)$. Note that by monotonicity for problems P' and \bar{P} , we have $\delta_{P'}(\mu^S, \mu^B) = \delta_{\bar{P}}(\mu^S, \mu^B)$. Combining these two equations yields

$$\delta_P(\mu^1, \mu^2) = \delta_{\bar{P}}(\mu^S, \mu^B). \quad (14)$$

By Proposition 2, we have $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which also equals $\sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Plugging the last term back into the left-hand side of Equation 14 yields $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

II. $(i+1)$ is the singleton nested (see Figure 11).

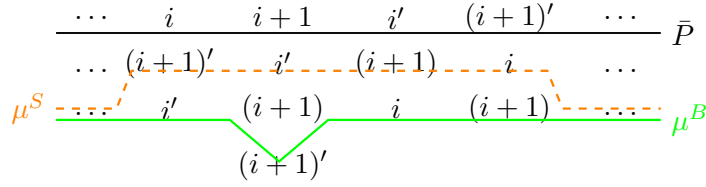


Figure 11: The one singleton structure with $i+1$ as the singleton.

Renaming the agents in Proposition 2 as $4 = i, 2 = i+1, 1 = i'$ and $3 = (i+1)'$ and using a similar argument as above yields $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

- **Case 3. (two singleton)** Consider the case in which two singletons are nested between μ^S and μ^B . By the construction of μ^T and μ^B only i and $i+1$ can be the singletons (see Figure 12).

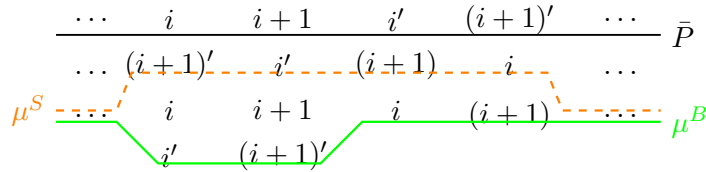


Figure 12: The two singleton structure with both i and $i+1$ as the singleton agents.

Renaming the agents in Proposition 3 as $1 = i, 2 = i+1, 3 = i', 4 = (i+1)'$ and using a similar argument as above, where only i was single, yields $\delta_{\bar{P}}(\mu^S, \mu^B) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^S, \mu^B)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Plugging the results of the three cases above into Equation 10 yields:

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sigma \times \sum_{l=1}^{\frac{n}{2}} \delta_{\bar{P}}^{Borda}(\mu^{T_l}, \mu^B) \text{ for } \sigma = \frac{1}{2}\alpha_{11}. \quad (15)$$

As Borda measure satisfies the conditions, and by Decomposition Lemma, the right-hand side of Equation 15 can be rearranged as:

$$\delta_{\bar{P}}(\mu^T, \mu^B) = \sigma \times \delta_P^{Borda}(\mu^T, \mu^B) \text{ for } \sigma = \frac{1}{2}\alpha_{11}.$$

Finally, combining all the three steps for $\delta_{\bar{P}}(\mu^T, \tilde{\mu})$, $\delta_{\bar{P}}(\mu^T, \mu^B)$, and $\delta_{\bar{P}}(\mu^B, \mu)$ into Equation 9 yields:

$$\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \left(\delta_P^{Borda}(\mu^T, \tilde{\mu}) - \delta_P^{Borda}(\mu^T, \mu^B) - \delta_P^{Borda}(\mu^B, \mu) \right) \text{ for } \sigma = \frac{1}{2}\alpha_{11}.$$

By betweenness of scaled Borda measures and symmetry, the right-hand side of the equation above reduces to $\sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ and hence $\delta_{\bar{P}}(\mu, \tilde{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Note that by monotonicity for problems P and \bar{P} , we have $\delta_P(\mu, \tilde{\mu}) = \delta_{\bar{P}}(\mu, \tilde{\mu})$. Also, $\delta_P^{Borda}(\mu, \tilde{\mu}) = \delta_{\bar{P}}^{Borda}(\mu, \tilde{\mu})$. Therefore, with respect to the previous equation we have $\delta_P(\mu, \tilde{\mu}) = \sigma \times \delta_P^{Borda}(\mu, \tilde{\mu})$ for $\sigma = \frac{1}{2}\alpha_{11}$. ■

5 Conclusion

Different mechanisms exhibit various desirable (or undesirable) features. In case a social planner decides to transform the design of a system by changing the mechanism employed, the question is how much change this will mean for the entire society. This paper proposes a way to quantify this difference based on the outputs mechanisms produce, i.e., matchings. We interpret this difference as the cost of transformation in general. We quantify the cost of transformation by introducing normative conditions on functions, which are shown to be logically independent. These conditions mostly address the effect of the transformation on individuals from one matching to another, instead of merely looking at the number of disjoint matches. We introduce the class of scaled Borda measures and show this class to be the only one satisfying these conditions on the domain of roommate markets.

There are multiple ways these measures can be extended. For instance, our framework assumes no indifference in preferences, i.e., there are no ties. The measures could possibly be extended to domains where ties are allowed. In such cases, one may end up with multiple disjoint matchings in which all agents are indifferent between their partners in these matchings. Then, in terms of positions these matchings are identical, and one approach is to assign zero value to the distance between such matchings. This, however, violates the metric condition, in particular the identity

of indiscernibles. So the metric condition must be modified into a pseudometric¹⁴ condition to accomodate this violation.

Another direction to extend the measures could be differentiating on the relative importance of the positions. In the proposed framework, we assume the weight across the matching partners of each agent is the same and hence the difference in being matched to the last two partners at the bottom of the preferences is the same as that of being matched to the two partners at top of the preferences. Therefore, one might be looking for measures in which there is a weight on the position of each agent, similar to that of Kemeny distance as it is characterized in [Can \(2014\)](#). In [Can \(2014\)](#) It is shown that by milding the betweenness condition one can achieve such a measure. However in our case, as it is shown in example [C.3](#) the only condition that restricts these weighted measures is the monotonicity condition. Hence, to define such measures one need to come up with an alternative or weaker version of the monotonicity condition.

Lastly, one can consider extending these measures to other interesting markets/domains in matching theory. After all, scaled Borda measures are attempts to answer “how much?” and depending on the context, the measures can be interpreted as parameters that quantify different concepts. Next we discuss, in detail, some of these possible applications and interpretations. Although the conditions may need minor modifications for different domains, the measures themselves are straightforward to implement on each of these domains.

5.1 Applications and Interpretations

Consider cases when a designer needs a refinement from a set of matchings, perhaps induced by a solution concept for a market. A scaled Borda measure can act as a mechanism to refine this set. In many two sided markets, the interpretation is very exciting. For instance, the core of the marriage markets forms a lattice structure with men-optimal and women-optimal matchings as the two extremes. It is not difficult to see that, within the core, these two matchings are the farthest pair according to these measures. A mechanism on these markets can pick the matching(s) from the core with the minimal total distance to all other stable matchings, acting as a tool to find the “median stable matching(s)”. This mechanism is in fact analogous to the use of the Kemeny distance¹⁵ in Kemeny-Young method¹⁶ which finds the median ranking(s) for a given ranking profile. Furthermore, given any choice among the set of stable matchings, one can immediately measure,

¹⁴A pseudometric is a function where all metric conditions are satisfied except the identity of indiscernibles. Instead a weaker version is applied, i.e., for all $\mu, \bar{\mu}$, if $\mu = \bar{\mu}$, then $\delta(\mu, \bar{\mu}) = 0$.

¹⁵[Kemeny \(1959\)](#) proposed and characterized this metric, albeit with a redundant axiom. For a recent characterization with logically independent conditions, see [Can and Storcken \(2018\)](#).

¹⁶This aggregation method, a.k.a the maximum likelihood method, was suggested in [Kemeny \(1959\)](#) and characterized in [Young and Levenglick \(1978\)](#).

how “close” this outcome is to the men-optimal (or women-optimal) stable matchings, leading to a fairness analysis. Next, we demonstrate some domain specific use cases of scaled Borda measures.

5.1.1 Marriage markets (a measure of gender bias)

Marriage markets are well studied two-sided one-to-one matching problems. The domain of marriage markets are known to be a subdomain of the roommate markets, i.e., a marriage market is a roommate market with gender wherein preferences are such that each man (woman) prefers to be single rather than matching with other men (women). Our results, therefore, are immediately applicable to this subdomain of roommate markets. Consider the following marriage market with three men m_1, m_2, m_3 and three women w_1, w_2, w_3 with the following preferences:

m_1	m_2	m_3
w_1	w_2	w_3
w_2	w_3	w_1
w_3	w_1	w_2
m_1	m_2	m_3

Figure 13: Men’s preferences.

w_1	w_2	w_3
m_2	m_3	m_1
m_3	m_1	m_2
m_1	m_2	m_3
w_1	w_2	w_3

Figure 14: Women’s preferences.

Consider the men-optimal matching μ^{MO} in which $\mu^{MO}(m_1) = w_1$, $\mu^{MO}(m_2) = w_2$, and $\mu^{MO}(m_3) = w_3$. Additionally, let μ^{WO} be the women-optimal matching in which $\mu^{WO}(w_1) = m_2$, $\mu^{WO}(w_2) = m_3$, and $\mu^{WO}(w_3) = m_1$ (see Figure 15).

	m_1	m_2	m_3	w_1	w_2	w_3
μ^{MO}	w_1	w_2	w_3	m_2	m_3	m_1
	w_2	w_3	w_1	m_3	m_1	m_2
μ^{WO}	w_3	w_1	w_2	m_1	m_2	m_3
	m_1	m_2	m_3	w_1	w_2	w_3

Figure 15: The preferences of men and women together.

In case a policy designer wants to choose a mechanism which assigns a matching, our measure can be used as a *measure of gender bias* which compares the distance of the output to men-optimal and women-optimal matchings. In such cases, note that, with respect to our measure, the matching μ^A has the same distance from both μ^{MO} and μ^{WO} which can be interpreted as a fair matching with respect to gender bias.

5.1.2 Object Allocation (a measure of welfare)

Object allocation is a one-sided market. An interesting example of object allocation is house allocation with existing tenants (Abdulkadiroğlu and Sönmez (1999)). In these markets each agent has an initial endowment and a preference over all endowments. In case our measure is used to compare the outcome of any individually rational mechanism and the initial endowment, the result can be interpreted as the social welfare improvement of implementing that mechanism.

Consider the following example. Let $N = \{1, 2, \dots, 6\}$ be the set of agents and $H = \{h_1, \dots, h_6\}$ be the set of houses. Let the initial endowments σ of the agents be $\sigma(i) = h_i$ for all $i \in \{1, \dots, 6\}$. The preference of each agent is shown in Figure 16. It can be verified that after applying the top trading cycle algorithm the final allocation μ^{TTC} will be as follows: $\mu^{TTC}(1) = h_1, \mu^{TTC}(2) = h_3, \mu^{TTC}(3) = h_4, \mu^{TTC}(4) = h_2, \mu^{TTC}(5) = h_5$ and $\mu^{TTC}(6) = h_6$.

	1	2	3	4	5	6	
	h_2	h_3	h_1	h_3	h_3	h_1	P
σ	h_1	h_3	h_4	h_1	h_1	h_2	
	h_3	h_1	h_2	h_5	h_6	h_3	
	h_4	h_5	h_3	h_4	h_4	h_4	
	h_5	h_4	h_6	h_2	h_5	h_6	μ^{TTC}
	h_6	h_6	h_5	h_6	h_2	h_5	

Figure 16: Housing allocation problem.

In case a policy designer wants to investigate the social welfare improvement, our measure can be used in a straight forward manner. In the example above, the measure can compare the distance between σ and μ^{TTC} , which is the social welfare gain from this mechanism.

5.1.3 College admissions (cost of efficiency)

College admissions problem is also a two-sided market, which is typically many-to-one, i.e., multiple students can match with a college. The following example is from Roth (1982). Let there be three student i_1, i_2, i_3 and three schools s_1, s_2, s_3 each of which has only one seat. The preferences of the colleges and students are as follows:

s_1	s_2	s_3
i_1	i_2	i_2
i_3	i_1	i_1
i_2	i_3	i_3
\emptyset	\emptyset	\emptyset

Figure 17: University's priority.

i_1	i_2	i_3
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3
\emptyset	\emptyset	\emptyset

Figure 18: Student's preferences.

Note that the student proposing deferred acceptance mechanism for the associated college admissions problem is as follows $\mu(i_1) = s_1$, $\mu(i_2) = s_2$, and $\mu(i_3) = s_3$. It can be verified, however, that this matching is student-Pareto-dominated by $\bar{\mu}$ in which $\bar{\mu}(i_1) = s_2$, $\bar{\mu}(i_2) = s_1$, and $\bar{\mu}(i_3) = s_3$.

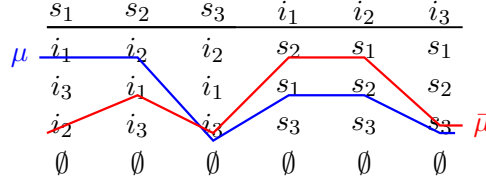


Figure 19: The preference of students and colleges

In case a policy designer wants to transform the student-proposing deferred acceptance mechanism which produces μ into the student-Pareto efficient mechanism which produces $\bar{\mu}$, our proposed measure can be interpreted as *the cost of efficiency*.

5.1.4 School choice (cost of stability)

A similar problem to college admissions is introduced in [Abdulkadiroglu and Sönmez \(2003\)](#) with the following example. There are eight students i_1, \dots, i_8 , and four schools s_1, \dots, s_4 . Schools s_1 and s_2 have two seats each while s_3 and s_4 have three seats each. The priorities of the schools and the preference of the students are as follow:

s_1	s_2	s_3	s_4
i_1	i_3	i_5	i_6
i_2	i_5	i_3	i_8
i_3	i_4	i_1	i_7
i_4	i_8	i_7	i_4
i_5	i_7	i_2	i_2
i_6	i_2	i_8	i_3
i_7	i_1	i_6	i_5
i_8	i_6	i_4	i_1
\emptyset	\emptyset	\emptyset	\emptyset

Figure 20: Universities priorities.

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
s_2	s_1	s_3	s_3	s_1	s_4	s_1	s_1
s_1	s_2	s_2	s_4	s_3	s_1	s_2	s_2
s_3	s_3	s_1	s_1	s_4	s_2	s_3	s_4
s_4	s_4	s_4	s_2	s_2	s_3	s_4	s_3

Figure 21: Student's preferences.

Note that the outcome of the Top Trading Cycle mechanism μ^{TTC} for the associated problem is as follows: $\mu^{TTC}(i_1) = s_2$, $\mu^{TTC}(i_2) = s_1$, $\mu^{TTC}(i_3) = s_3$, $\mu^{TTC}(i_4) = s_3$, $\mu^{TTC}(i_5) = s_1$, $\mu^{TTC}(i_6) = s_4$, $\mu^{TTC}(i_7) = s_2$, and $\mu^{TTC}(i_8) = s_4$. However, it can be verified that μ^{TTC} is not stable. That is, i_8 prefers to be at s_2 , and s_2 prefers i_8 to both i_1 and i_7 .

It can be verified that the outcome of the Gale Shapley student optimal stable mechanism μ^{GS} , is as follows: $\mu^{GS}(i_1) = s_1$, $\mu^{GS}(i_2) = s_1$, $\mu^{GS}(i_3) = s_3$, $\mu^{GS}(i_4) = s_3$, $\mu^{GS}(i_5) = s_3$, $\mu^{GS}(i_6) = s_4$, $\mu^{GS}(i_7) = s_2$, and $\mu^{GS}(i_8) = s_2$.

In order to use our measure, we clone each university based on its quota. That is, we consider each quota as a separate university with the same priority as the original university. This is shown in Figure 22.

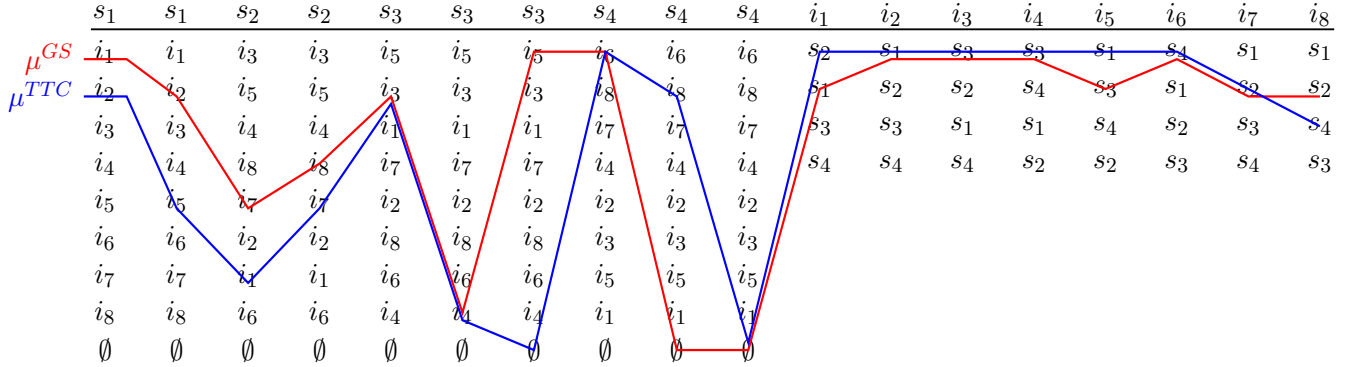


Figure 22: Cloning the universities priorities.

In case a policy designer wants to transform the top trading cycle mechanism which produces μ^{TTC} into the Gale Shapley student optimal stable mechanism which produces μ^{GS} , our proposed measure can be interpreted as *the cost of stability*.

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Appendix

A Proofs of Section 4.1

A.1 Proof of Lemma 2

Lemma 2. Consider any $N, N' \subseteq \mathcal{N}$ and a strictly positive integer x . Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any $P \in \mathcal{P}(N)$ such that μ^{ij} is of length $(x, 1)$ in P . Similarly consider any one-couple matching $\mu^{i'j'} \in \mathcal{M}(N')$, and any $P' \in \mathcal{P}(N')$ such that $\mu^{i'j'}$ is of length $(x, 1)$ in P' . Let μ^I and $\mu^{I'}$ denote the identity matchings in corresponding problems, then

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'}).$$

Proof. Consider an extension \bar{P} of P and the extension $\bar{\mu}^{ij}$ and $\bar{\mu}^I$ of matchings μ^{ij} and μ^I by the set of agents $N' \setminus N$, respectively. By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\bar{P}}(\bar{\mu}^{ij}, \bar{\mu}^I)$. For simplicity, we abuse the notation and write P , μ^{ij} and μ^I instead of \bar{P} , $\bar{\mu}^{ij}$ and $\bar{\mu}^I$, respectively. Also, consider an extension \bar{P}' of P' and the extension $\bar{\mu}^{i'j'}$ and $\bar{\mu}^{I'}$ of matchings $\mu^{i'j'}$ and $\mu^{I'}$ by the set of agents $N \setminus N'$, respectively. By Remark 2, $\delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \delta_{\bar{P}'}(\bar{\mu}^{i'j'}, \bar{\mu}^{I'})$. For simplicity, we abuse the notation and write P' , $\mu^{i'j'}$, and $\mu^{I'}$ instead of \bar{P}' , $\bar{\mu}^{i'j'}$ and $\bar{\mu}^{I'}$, respectively. Note that now both P and P' (as well as the matchings) are defined on the same set of agents $\bar{N} = N' \cup N$.

Let $Z = \{z_1, \dots, z_{x-1}\}$ be the set of other agents nested between j and i in R_i , and $Z' = \{z'_1, \dots, z'_{x-1}\}$ be the set of other agents nested between j' and i' in $R'_{i'}$. There are two possible situations; either $Z = Z'$ or $Z \neq Z'$.

Case 1. $Z = Z'$: Consider permutation $\pi = (ii')(jj')$. Applying this permutation on P , and using anonymity yields $\delta_P(\mu^{ij}, \mu^I) = \delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi)$. Since by this permutation, $(\mu^{ij})^\pi = \mu^{i'j'}$ and $(\mu^I)^\pi = \mu^{I'}$, then $\delta_{P^\pi}((\mu^{ij})^\pi, (\mu^I)^\pi) = \delta_{P^\pi}(\mu^{i'j'}, \mu^{I'})$. Since, $Z = Z'$ and both problems are defined on the same set of agents monotonicity implies $\delta_{P^\pi}(\mu^{i'j'}, \mu^{I'}) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$. Therefore, $\delta_P(\mu^{ij}, \mu^I) = \delta_{P'}(\mu^{i'j'}, \mu^{I'})$.

Case 2. $Z \neq Z'$: In this case we add the same set of irrelevant newcomers to both problems P and P' , and map the agents in Z and Z' to these newcomers so that the set of agents that are nested between the two matchings in these two problems become the same, then part I implies the result. Formally, let $A = \{a_1, \dots, a_{x-1}\}$, be a set of agents such that $\bar{N} \cap A = \emptyset$. Next, let \hat{P} and \hat{P}' be an extensions of P and P' by the set of agents A , respectively. Also, let $\hat{\mu}^{ij}$ and $\hat{\mu}^I$ be the extensions of μ^{ij} and μ^I , and $\hat{\mu}^{i'j'}$ and $\hat{\mu}^{I'}$ be the extensions of $\mu^{i'j'}$ and $\mu^{I'}$, respectively, all by the same set of agents A . By Remark 2, $\delta_P(\mu^{ij}, \mu^I) = \delta_{\hat{P}}(\hat{\mu}^{ij}, \hat{\mu}^I)$ and $\delta_{P'}(\mu^{i'j'}, \mu^{I'}) = \delta_{\hat{P}'}(\hat{\mu}^{i'j'}, \hat{\mu}^{I'})$. For simplicity, we abuse the notation and write P , μ^{ij} , and μ^I instead of \hat{P} , $\hat{\mu}^{ij}$ and $\hat{\mu}^I$, and we

- $\text{rank}(b, \hat{R}_j) = 1$ and $[\mu^{ij}, \mu^I]_{R_j} = [\mu^{ij}, \mu^I]_{\hat{R}_j}$,
- $\text{rank}(i, \hat{R}_a) = 1$, $\text{rank}(b, \hat{R}_a) = |\hat{N}| + 1$, and $\text{rank}(a, \hat{R}_a) = |\hat{N}| + 2$,
- $\text{rank}(j, \hat{R}_b) = 1$, $\text{rank}(a, \hat{R}_b) = |\hat{N}| + 1$, and $\text{rank}(b, \hat{R}_b) = |\hat{N}| + 2$.

By monotonicity we have

$$\delta_P(\mu^{ij}, \mu^I) = \delta_{\hat{P}}(\mu^{ij}, \mu^I). \quad (16)$$

Therefore it is sufficient to prove that $\delta_{\hat{P}}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$.

Consider the following two matchings in problem \hat{P} such that $\mu \in \mathcal{M}(\hat{N})$ with $\mu(i) = j$, $\mu(a) = b$ and $\mu(t) = t$ for all other agent t and $\mu^T \in \mathcal{M}(\hat{N})$ with $\mu^T(i) = a$, $\mu^T(j) = b$ and $\mu^T(t) = t$ for all other agent t .

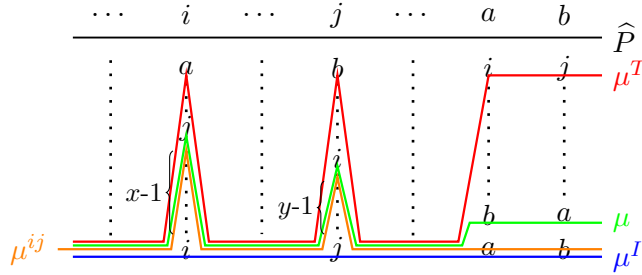


Figure 24: Problem $\hat{P} = (\hat{N}, \hat{R})$.

Claim. $\delta_{\hat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$.

Proof of claim. Consider a new problem \hat{P}^π shown in Figure 25. Problem \hat{P}^π is the permuted problem of \hat{P} with $\pi = (aj)$. By this permutation the identity matching remains the same, hence we write μ^I instead of $(\mu^I)^\pi$ in \hat{P}^π . By anonymity the following equation holds,

$$\delta_{\hat{P}}(\mu^T, \mu) = \delta_{\hat{P}^\pi}((\mu^T)^\pi, \mu^\pi). \quad (17)$$

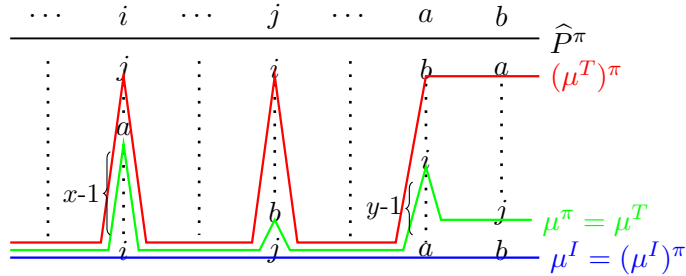


Figure 25: Problem \hat{P}^π after permuting problem \hat{P} in Figure 24 with $\pi = (aj)$.

Consider a new problem \tilde{P} shown in Figure 26. Problem \tilde{P} is almost identical to problem \hat{P}^π except that the position of the partners of each agent in $(\mu^T)^\pi$ and μ^π are swapped. By monotonicity for \hat{P}^π and \tilde{P} , $\delta_{\hat{P}^\pi}((\mu^T)^\pi, \mu^\pi) = \delta_{\tilde{P}}((\mu^T)^\pi, \mu^\pi)$. Plugging this into Equation 17 we have,

$$\delta_{\hat{P}}(\mu^T, \mu) = \delta_{\tilde{P}}((\mu^T)^\pi, \mu^\pi). \quad (18)$$

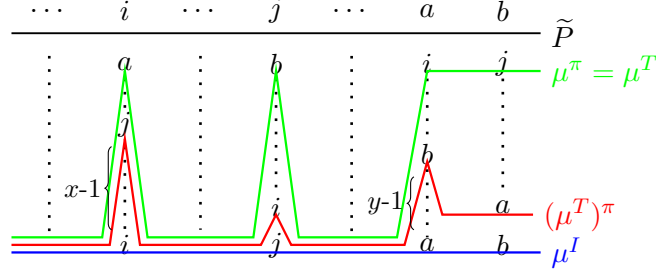


Figure 26: Problem \tilde{P} , after swapping the positions of μ^π and $(\mu^T)^\pi$ in problem \hat{P}^π in Figure 25.

Since μ is between μ^T and μ^I in problem \hat{P} , and $(\mu^T)^\pi$ is between μ^π and μ^I in problem \tilde{P} , betweenness yields

$$\delta_{\hat{P}}(\mu^T, \mu^I) = \delta_{\hat{P}}(\mu^T, \mu) + \delta_{\hat{P}}(\mu, \mu^I) \text{ and,} \quad (19)$$

$$\delta_{\tilde{P}}(\mu^\pi, \mu^I) = \delta_{\tilde{P}}(\mu^\pi, (\mu^T)^\pi) + \delta_{\tilde{P}}((\mu^T)^\pi, \mu^I). \quad (20)$$

Note that by permutation π , $\mu^\pi = \mu^T$ hence $\delta_{\tilde{P}}(\mu^\pi, \mu^I) = \delta_{\tilde{P}}(\mu^T, \mu^I)$. Considering this and the monotonicity for problems \tilde{P} and \hat{P} we have $\delta_{\tilde{P}}(\mu^T, \mu^I) = \delta_{\hat{P}}(\mu^T, \mu^I)$. Therefore the left hand sides of Equations 19 and 20 are equal which yields

$$\delta_{\hat{P}}(\mu^T, \mu) + \delta_{\hat{P}}(\mu, \mu^I) = \delta_{\tilde{P}}(\mu^\pi, (\mu^T)^\pi) + \delta_{\tilde{P}}((\mu^T)^\pi, \mu^I).$$

Combining this with Equation 18 results in $\delta_{\hat{P}}(\mu, \mu^I) = \delta_{\tilde{P}}((\mu^T)^\pi, \mu^I)$. Finally, by Decomposition Lemma and Lemma 2, $\delta_{\tilde{P}}((\mu^T)^\pi, \mu^I) = \alpha_{x1} + \alpha_{y1}$. Hence $\delta_{\hat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$, which concludes the claim. \blacksquare

By Decomposition Lemma, for matching μ in problem \hat{P} , we have $\delta_{\hat{P}}(\mu, \mu^I) = \delta_{\hat{P}}(\mu^{ij}, \mu^I) + \delta_{\hat{P}}(\mu^{ab}, \mu^I)$. By the claim proven above, $\delta_{\hat{P}}(\mu, \mu^I) = \alpha_{x1} + \alpha_{y1}$, and by Lemma 2, we have $\delta_{\hat{P}}(\mu^{ab}, \mu^I) = \alpha_{11}$. So, $\delta_{\hat{P}}(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$, and by Equation 16, $\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}$. Finally, by Lemma 2, the right-hand side of this equation is independent of the set of agents $N \subsetneq \mathcal{N}$. Therefore, for all $N \subsetneq \mathcal{N}$, for all problems $P \in \mathcal{P}(N)$ and for all one-couple matchings μ^{ij} of length (x, y) we conclude that,

$$\delta_P(\mu^{ij}, \mu^I) = \alpha_{x1} + \alpha_{y1} - \alpha_{11}.$$

\blacksquare

A.3 Proof of Lemma 4

Lemma 4. Consider any $N \subsetneq \mathcal{N}$ and a strictly positive integer x . Consider any one-couple matching $\mu^{ij} \in \mathcal{M}(N)$, and any problem $P \in \mathcal{P}(N)$ such that μ^{ij} is of length (x, x) in P . Let μ^I denote the identity matching, then

$$\delta_P(\mu^{ij}, \mu^I) = x \times \alpha_{11}.$$

Proof. By Lemma 3 the distance between the identity matching and any one-couple matching of length (x, y) is the same across all problems in the domain. Therefore, it suffices to prove that the lemma holds for some problem \bar{P} , and some one-couple matching of length (x, x) in \bar{P} . Consider the following specific problem $\bar{P} = (\bar{N}, \bar{R})$ with $\bar{N} = \{1, 2, \dots, 2x\}$ shown in Figure 27. Let $\bar{\mu}^{x, x+1}$ be the one-couple matching of length (x, x) in \bar{P} . Note that the structure of matchings $\bar{\mu}^x, \dots, \bar{\mu}^1$ in problem \bar{P} is such that:

- $\bar{\mu}^x(1) = 2x, \bar{\mu}^x(2) = 2x - 1$, so on and so forth,
- for all $k \in \{2, \dots, x\}$, and for all $i \in \bar{N}$, $\bar{\mu}^{k-1}(i) = \bar{\mu}^k((i + 2) \bmod (2x))$, e.g., $\bar{\mu}^{x-1}(2x - 1) = \bar{\mu}^x(1) = 2x$,
- for all $k \in \{1, \dots, x\}$, $\bar{\mu}^k$ and $\bar{\mu}^{k-1}$ are adjacent.

Next we show that $\delta_{\bar{P}}(\bar{\mu}^{x, x+1}, \bar{\mu}^I) = x \times \alpha_{11}$ which in returns shows that $\alpha_{xx} = x \times \alpha_{11}$. To ease the notation in this problem we denote the identity matching by $\bar{\mu}^0$.

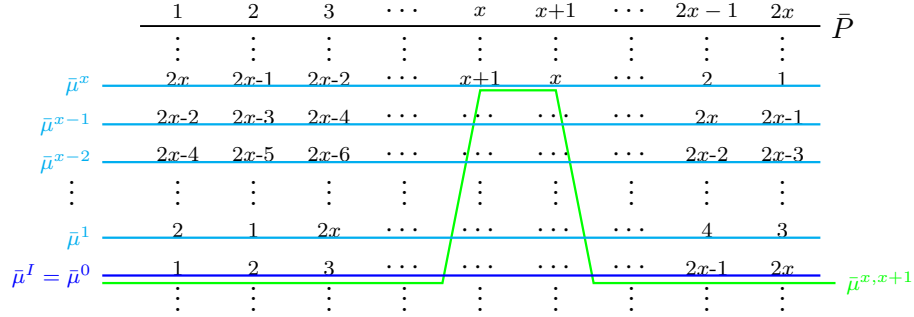


Figure 27: Problem $\bar{P} = (\bar{N}, \bar{R})$.

Claim. $\delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = \frac{|\bar{N}|}{2} \alpha_{11}$ for all $t \in \{0, \dots, x-1\}$

Proof of claim. Note that by construction, the two matchings $\bar{\mu}^0$ and $\bar{\mu}^1$ in \bar{P} are disjoint. By standardization, for $\bar{\mu}^0, \bar{\mu}^1$, there exists a problem $P' = (\bar{N}, R') \in \mathcal{P}(\bar{N})$ such that $\delta_{P'}(\bar{\mu}^0, \bar{\mu}^1) =$

$\kappa(\bar{N})$ and is minimal. Note that for all $i \in \bar{N}$, as $\bar{\mu}^0, \bar{\mu}^1$ have the minimal possible intervals, we have $[\bar{\mu}^0, \bar{\mu}^1]_{\bar{R}_i} \subseteq [\bar{\mu}^0, \bar{\mu}^1]_{R'_i}$. Therefore monotonicity implies that $\delta_{\bar{P}}(\bar{\mu}^0, \bar{\mu}^1) = \kappa(\bar{N})$. By Decomposition Lemma, the distance between $\bar{\mu}^0$ and $\bar{\mu}^1$ can be decomposed as the sum of $\frac{|\bar{N}|}{2}$ one-couple matchings, each of the same length $(1, 1)$. Hence, $\delta_{\bar{P}}(\bar{\mu}^0, \bar{\mu}^1) = \frac{|\bar{N}|}{2}\alpha_{11}$. Together with the previous equation we have $\kappa(\bar{N}) = \frac{|\bar{N}|}{2}\alpha_{11}$. Note that, by monotonicity the distance between $\bar{\mu}^t, \bar{\mu}^{t+1}$ for all $t \in \{1, 2, \dots, x-1\}$, is also minimal, and by standardization this distance also equals $\kappa(\bar{N})$. Hence, $\delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = \frac{|\bar{N}|}{2}\alpha_{11}$ for all $t \in \{0, \dots, x-1\}$ which completes the proof of the claim. ■

Next, we complete the proof of the lemma by showing $\delta_{\bar{P}}(\bar{\mu}^{x,x+1}, \bar{\mu}^I) = x \times \alpha_{11}$. Note that by construction of \bar{P} , the matchings $[\bar{\mu}^x - \bar{\mu}^{x+1} - \dots - \bar{\mu}^1 - \bar{\mu}^0]$ are on a line. Therefore, betweenness -together with the claim above- yields

$$\delta_{\bar{P}}(\bar{\mu}^x, \bar{\mu}^0) = \sum_{t=0}^{x-1} \delta_{\bar{P}}(\bar{\mu}^t, \bar{\mu}^{t+1}) = x \times \frac{|\bar{N}|}{2}\alpha_{11}. \quad (21)$$

By Decomposition Lemma, the distance between $\bar{\mu}^x$ and $\bar{\mu}^0$ can be decomposed as the sum of $\frac{|\bar{N}|}{2}$ one-couple matchings, each of the same length (x, x) . Hence $\delta_{\bar{P}}(\bar{\mu}^x, \bar{\mu}^0) = \frac{|\bar{N}|}{2}\alpha_{xx}$. Together with Equation 21, $\frac{|\bar{N}|}{2}\alpha_{xx} = x \times \frac{|\bar{N}|}{2}\alpha_{11}$ which results in $\alpha_{xx} = x \times \alpha_{11}$. As α_{xx} is the same across all problems in the domain Lemma 3, this completes the proof of the lemma. ■

B Proofs of Section 4.2

B.1 Proof of Proposition 2

Proposition 2. Consider a problem P over four agents with the preference profile and the matchings shown in Figure 4. Note that one singleton is nested between μ^1 and μ^2 and another is nested between μ^2 and μ^3 . In such specific cases,

1. $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$,
2. $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Proof. First we show $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$, and then using this we prove the proposition. Consider the permutation $\pi = (23)$. Applying this permutation on P results in the problem P^π shown on the right-hand side of Figure 28.

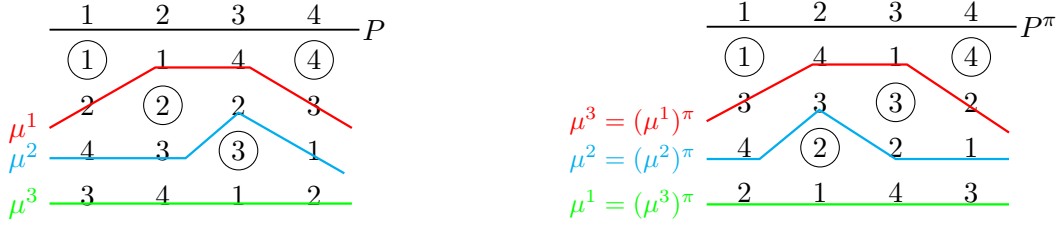


Figure 28: The original problem P in Proposition 2 (on the left) and the permuted problem P^π (on the right) after permuting with $\pi = (23)$.

Note that by anonymity we have, $\delta_P(\mu^1, \mu^2) = \delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi)$. Furthermore, under the permutation π , $(\mu^1)^\pi = \mu^3$ and $(\mu^2)^\pi = \mu^2$, which implies $\delta_{P^\pi}((\mu^2)^\pi, (\mu^1)^\pi) = \delta_{P^\pi}(\mu^2, \mu^3)$. Note that by monotonicity for two problems P and P^π , we have $\delta_{P^\pi}(\mu^2, \mu^3) = \delta_P(\mu^2, \mu^3)$. Combining these equations and the fact that δ is a symmetric function, proves that $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$.

1. Proving $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Let $\bar{P}, \bar{\mu}^1, \bar{\mu}^2$, and $\bar{\mu}^3$ be extensions of P, μ^1, μ^2 , and μ^3 , respectively, by the set of agents $A = \{1', 2', 3', 4'\}$ (see Figure 29). By Remark 2, $\delta_P(\mu^1, \mu^2) = \delta_{\bar{P}}(\bar{\mu}^1, \bar{\mu}^2)$, $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\bar{\mu}^2, \bar{\mu}^3)$, and $\delta_P(\mu^1, \mu^3) = \delta_{\bar{P}}(\bar{\mu}^1, \bar{\mu}^3)$. For simplicity, we abuse the notation and write P, μ^1, μ^2 , and μ^3 instead of $\bar{P}, \bar{\mu}^1, \bar{\mu}^2$ and $\bar{\mu}^3$, respectively.

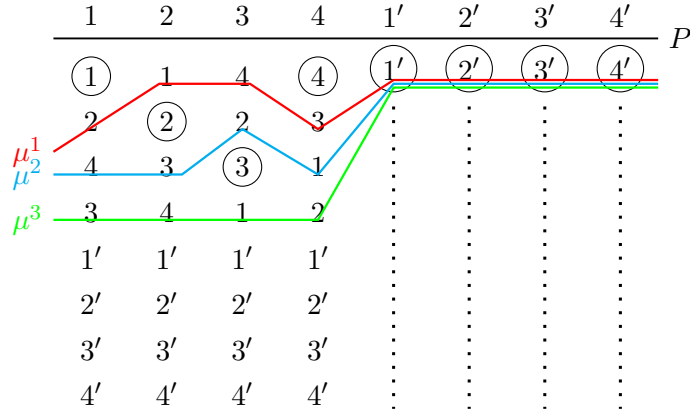


Figure 29: An extension of the problem P in Figure 4 by the set of agents $A = \{1', 2', 3', 4'\}$.

Consider also another problem \hat{P} shown in Figure 30.

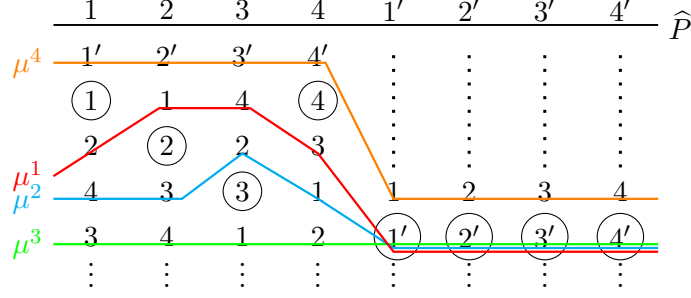


Figure 30: Problem \hat{P} .

Note that by monotonicity for two problems P and \hat{P} we have $\delta_P(\mu^1, \mu^2) = \delta_{\hat{P}}(\mu^1, \mu^2)$, and $\delta_P(\mu^2, \mu^3) = \delta_{\hat{P}}(\mu^2, \mu^3)$. Therefore, using the first part of the proposition, $\delta_{\hat{P}}(\mu^1, \mu^2) = \delta_{\hat{P}}(\mu^2, \mu^3)$. As in problem \hat{P} , $[\mu^4 - \mu^1 - \mu^2 - \mu^3]$ are on a line, by betweenness $\delta_{\hat{P}}(\mu^4, \mu^3) = \delta_{\hat{P}}(\mu^4, \mu^1) + \delta_{\hat{P}}(\mu^1, \mu^2) + \delta_{\hat{P}}(\mu^2, \mu^3)$. Combining this with the previous equation implies

$$\begin{aligned} \delta_{\hat{P}}(\mu^4, \mu^3) &= \delta_{\hat{P}}(\mu^4, \mu^1) + 2\delta_{\hat{P}}(\mu^1, \mu^2) \\ \Rightarrow \delta_{\hat{P}}(\mu^1, \mu^2) &= \frac{1}{2}(\delta_{\hat{P}}(\mu^4, \mu^3) - \delta_{\hat{P}}(\mu^4, \mu^1)) \end{aligned} \quad (22)$$

Next we show that the right hand side of Equation 22 equals $\sigma \times \delta_{\hat{P}}^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. We do this by proving two claims for each of the terms on the right-hand side of Equation 22.

Claim 1. $\delta_{\hat{P}}(\mu^4, \mu^3) = \sigma \times \delta_{\hat{P}}^{Borda}(\mu^4, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Proof of claim 1. As in \hat{P} , the identity matching is between μ^4 and μ^3 , betweenness implies $\delta_{\hat{P}}(\mu^4, \mu^3) = \delta_{\hat{P}}(\mu^4, \mu^I) + \delta_{\hat{P}}(\mu^I, \mu^3)$. Using Theorem 1, $\delta_{\hat{P}}(\mu^4, \mu^I) = \sigma \times \delta_{\hat{P}}^{Borda}(\mu^4, \mu^I)$ for $\sigma = \frac{1}{2}\alpha_{11}$ and $\delta_{\hat{P}}(\mu^I, \mu^3) = \sigma \times \delta_{\hat{P}}^{Borda}(\mu^I, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. With respect to this, and as δ^{Borda} satisfies betweenness, we have $\delta_{\hat{P}}(\mu^4, \mu^3) = \sigma \times \delta_{\hat{P}}^{Borda}(\mu^4, \mu^3)$ where $\sigma = \frac{1}{2}\alpha_{11}$. ■

Claim 2. $\delta_{\hat{P}}(\mu^4, \mu^1) = \sigma \times \delta_{\hat{P}}^{Borda}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Proof of claim 2. To show this, consider the problem \tilde{P} shown in Figure 31. Note that, by monotonicity for two problems \hat{P} and \tilde{P} we have $\delta_{\hat{P}}(\mu^4, \mu^1) = \delta_{\tilde{P}}(\mu^4, \mu^1)$, and $\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1) = \delta_{\hat{P}}^{Borda}(\mu^4, \mu^1)$. Hence it is sufficient to show $\delta_{\tilde{P}}(\mu^4, \mu^1) = \sigma \times \delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

To proceed, we show that $\delta_{\tilde{P}}(\mu^4, \mu^1) = \delta_{\tilde{P}}(\mu^1, \mu^5)$. Applying permutation $\pi = (12)(34)$, on \tilde{P} results in problem \tilde{P}^π which is shown in Figure 32.

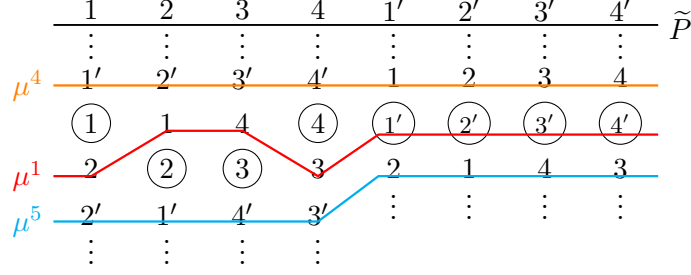


Figure 31: Problem \tilde{P} .

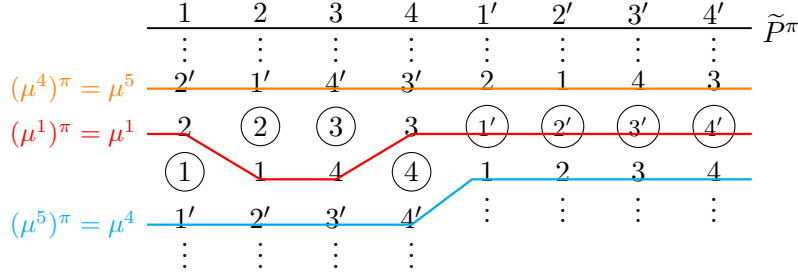


Figure 32: Problem \tilde{P}^π after permuting \tilde{P} in Figure 33 with $\pi = (12)(34)$.

Note that by anonymity we have, $\delta_{\tilde{P}}(\mu^1, \mu^4) = \delta_{\tilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi)$. Furthermore, under the permutation π , $(\mu^1)^\pi = \mu^1$ and $(\mu^4)^\pi = \mu^5$, which implies $\delta_{\tilde{P}^\pi}((\mu^1)^\pi, (\mu^4)^\pi) = \delta_{\tilde{P}^\pi}(\mu^1, \mu^5)$. Note that by monotonicity for two problems \tilde{P} and \tilde{P}^π we have $\delta_{\tilde{P}^\pi}(\mu^1, \mu^5) = \delta_{\tilde{P}}(\mu^1, \mu^5)$, which shows $\delta_{\tilde{P}}(\mu^1, \mu^4) = \delta_{\tilde{P}}(\mu^1, \mu^5)$. Considering this and as in problem \tilde{P} matching μ^1 is between μ^4 and μ^5 , we have

$$\delta_{\tilde{P}}(\mu^4, \mu^1) = \delta_{\tilde{P}}(\mu^1, \mu^5) = \frac{\delta_{\tilde{P}}(\mu^4, \mu^5)}{2}. \quad (23)$$

On the other hand, betweenness of μ^I in problem \tilde{P} yields $\delta_{\tilde{P}}(\mu^4, \mu^5) = \delta_{\tilde{P}}(\mu^4, \mu^I) + \delta_{\tilde{P}}(\mu^I, \mu^5)$. By Theorem 1, $\delta_{\tilde{P}}(\mu^4, \mu^I) = \sigma \times \delta_{\tilde{P}}^{Borda}(\mu^4, \mu^I)$ for $\sigma = \frac{1}{2}\alpha_{11}$ and $\delta_{\tilde{P}}(\mu^I, \mu^5) = \sigma \times \delta_{\tilde{P}}^{Borda}(\mu^I, \mu^5)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Therefore, $\delta_{\tilde{P}}(\mu^4, \mu^5) = \frac{1}{2}\alpha_{11}(\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^I) + \delta_{\tilde{P}}^{Borda}(\mu^I, \mu^5))$. It can be verified that $\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^I) + \delta_{\tilde{P}}^{Borda}(\mu^I, \mu^5) = 2\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1)$, which with the previous equation implies $\delta_{\tilde{P}}(\mu^4, \mu^5) = \alpha_{11}\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1)$. Plugging this into Equation 23 results in $\delta_{\tilde{P}}(\mu^4, \mu^1) = \frac{1}{2}\alpha_{11}\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1)$. Note that by monotonicity for two problems \hat{P} and \tilde{P} we have $\delta_{\tilde{P}}(\mu^4, \mu^1) = \delta_{\hat{P}}(\mu^4, \mu^1)$, and $\delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1) = \delta_{\hat{P}}^{Borda}(\mu^4, \mu^1)$. Hence $\delta_{\tilde{P}}(\mu^4, \mu^1) = \sigma \times \delta_{\tilde{P}}^{Borda}(\mu^4, \mu^1)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which completes the proof of the claim. \blacksquare

Having proven the claims, we plug these back into Equation 22 and,

$$\begin{aligned}
\delta_{\hat{P}}(\mu^1, \mu^2) &= \frac{1}{2} \left(\frac{1}{2} \alpha_{11} \delta_{\hat{P}}^{Borda}(\mu^4, \mu^3) - \frac{1}{2} \alpha_{11} \delta^{Borda} \delta_{\hat{P}}(\mu^4, \mu^1) \right) \\
&= \frac{1}{4} \alpha_{11} \left(\delta_{\hat{P}}^{Borda}(\mu^4, \mu^3) - \delta^{Borda} \delta_{\hat{P}}(\mu^4, \mu^1) \right) \\
&= \frac{1}{4} \alpha_{11} \left(\delta_{\hat{P}}^{Borda}(\mu^1, \mu^3) \right) \\
&= \frac{1}{4} \alpha_{11} \left(2 \delta_{\hat{P}}^{Borda}(\mu^1, \mu^2) \right) \\
&= \frac{1}{2} \alpha_{11} \left(\delta_{\hat{P}}^{Borda}(\mu^1, \mu^2) \right)
\end{aligned}$$

where the third and the fourth equations are due to betweenness of δ^{Borda} . Finally, by monotonicity for two problems P and \hat{P} we have $\delta_P(\mu^1, \mu^2) = \delta_{\hat{P}}(\mu^1, \mu^2)$ and $\delta_P^{Borda}(\mu^1, \mu^2) = \delta_{\hat{P}}^{Borda}(\mu^1, \mu^2)$. Therefore, $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2} \alpha_{11}$, which completes the proof of the first part of the proposition.

2. Proving $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2} \alpha_{11}$. As we proved $\delta_P(\mu^1, \mu^2) = \delta_P(\mu^2, \mu^3)$, and by the first part of this proposition and the fact that $\delta_P^{Borda}(\mu^1, \mu^2) = \delta_P^{Borda}(\mu^2, \mu^3)$, it can be easily concluded.

■

B.2 Proof of Proposition 3

Proposition 3. Consider a problem P over four agents with the preference profile and the matchings shown in Figure 5. Note that two singletons are nested between μ^1 and μ^2 and another two are nested between μ^2 and μ^3 . In such specific cases,

1. $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2} \alpha_{11}$,
2. $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2} \alpha_{11}$.

Proof. Consider another problem \bar{P} shown on the right-hand side of Figure 33. Note that by monotonicity for two problems P and \bar{P} we have $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3)$ and $\delta_P^{Borda}(\mu^2, \mu^3) = \delta_{\bar{P}}^{Borda}(\mu^2, \mu^3)$. Next we show that for problem \bar{P} , $\delta_{\bar{P}}(\mu^3, \mu^2) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$ for $\sigma = \frac{1}{2} \alpha_{11}$.

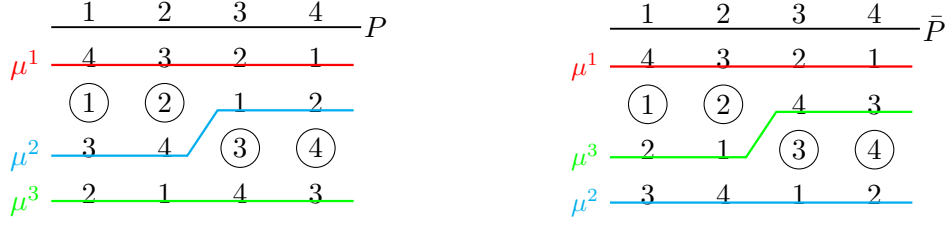


Figure 33: The original problem P of Proposition 3 (on the left) and problem \bar{P} (on the right) after swapping the positions of μ^2 and μ^3 in problem P .

Claim. $\delta_{\bar{P}}(\mu^3, \mu^2) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$.

Proof of claim. Consider the permutation $\pi = (1324)$. Applying this permutation on \bar{P} results in the problem \bar{P}^π , which is shown in Figure 34.

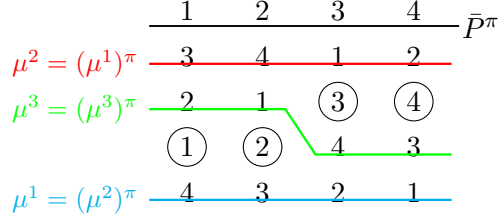


Figure 34: Permuted problem \bar{P}^π after permuting problem \bar{P} of Figure 33 with $\pi = (1324)$.

Note that by anonymity we have, $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi)$. Furthermore, under the permutation π , $(\mu^1)^\pi = \mu^2$ and $(\mu^3)^\pi = \mu^3$, which implies $\delta_{\bar{P}^\pi}((\mu^1)^\pi, (\mu^3)^\pi) = \delta_{\bar{P}^\pi}(\mu^2, \mu^3)$. Note that by monotonicity for two problems \bar{P} and \bar{P}^π , we have $\delta_{\bar{P}^\pi}(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3)$. Combining these equations and the fact that δ is a symmetric function proves that $\delta_{\bar{P}}(\mu^1, \mu^3) = \delta_{\bar{P}}(\mu^3, \mu^2)$. As in \bar{P} , μ^3 is between μ^1 and μ^2 , betweenness implies $\delta_{\bar{P}}(\mu^1, \mu^2) = \delta_{\bar{P}}(\mu^1, \mu^3) + \delta_{\bar{P}}(\mu^3, \mu^2)$, this with the previous equation implies

$$\delta_{\bar{P}}(\mu^3, \mu^2) = \frac{\delta_{\bar{P}}(\mu^1, \mu^2)}{2} \quad (24)$$

Now, as in problem \bar{P} the identity matching is between μ^1 and μ^2 , betweenness yields $\delta_{\bar{P}}(\mu^1, \mu^2) = \delta_{\bar{P}}(\mu^1, \mu^I) + \delta_{\bar{P}}(\mu^I, \mu^2)$. By Theorem 1, $\delta_{\bar{P}}(\mu^1, \mu^I) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^1, \mu^I)$ for $\sigma = \frac{1}{2}\alpha_{11}$ and $\delta_{\bar{P}}(\mu^I, \mu^2) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^I, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Therefore,

$$\delta_{\bar{P}}(\mu^1, \mu^2) = \sigma \left(\delta_{\bar{P}}^{Borda}(\mu^1, \mu^I) + \delta_{\bar{P}}^{Borda}(\mu^I, \mu^2) \right) \text{ for } \sigma = \frac{1}{2}\alpha_{11}. \quad (25)$$

It can be verified that $\delta_{\bar{P}}^{Borda}(\mu^1, \mu^I) + \delta_{\bar{P}}^{Borda}(\mu^I, \mu^2) = \delta_{\bar{P}}^{Borda}(\mu^1, \mu^3) + \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$, and that $\delta_{\bar{P}}^{Borda}(\mu^1, \mu^3) = \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$. Replacing this into Equation 25 implies $\delta_{\bar{P}}(\mu^1, \mu^2) = 2\sigma \times \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$

for $\sigma = \frac{1}{2}\alpha_{11}$. Plugging this into Equation 24 yields $\delta_{\bar{P}}(\mu^3, \mu^2) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^3, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. This concludes the claim. \blacksquare

1. Proving that $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Note that by monotonicity for two problems P and \bar{P} we have $\delta_P(\mu^2, \mu^3) = \delta_{\bar{P}}(\mu^2, \mu^3)$. Replacing the latter using the above claim we have $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_{\bar{P}}^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. As $\delta_P^{Borda}(\mu^2, \mu^3) = \delta_{\bar{P}}^{Borda}(\mu^2, \mu^3)$, we have $\delta_P(\mu^2, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$, which concludes the first part of the proposition.
2. Proving that $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. As in problem P the identity matching is between μ^1 and μ^3 , betweenness implies $\delta_P(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \delta_P(\mu^2, \mu^3)$. Using Theorem 1, $\delta_P(\mu^1, \mu^3) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. As in problem P , μ^2 is between μ^1 and μ^3 , betweenness implies $\delta_P(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \delta_P(\mu^2, \mu^3)$. Together with the previous equation, we have $\sigma \times \delta_P^{Borda}(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \delta_P(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Replacing $\delta_P(\mu^2, \mu^3)$ with the first part of the proposition results in $\sigma \times \delta_P^{Borda}(\mu^1, \mu^3) = \delta_P(\mu^1, \mu^2) + \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$. Rearranging, will result in $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^3) - \sigma \times \delta_P^{Borda}(\mu^2, \mu^3)$ for $\sigma = \frac{1}{2}\alpha_{11}$ which by betweenness of Borda measure equals $\delta_P(\mu^1, \mu^2) = \sigma \times \delta_P^{Borda}(\mu^1, \mu^2)$ for $\sigma = \frac{1}{2}\alpha_{11}$. This concludes the second part of the proposition. \blacksquare

C Logical Independence of the Conditions

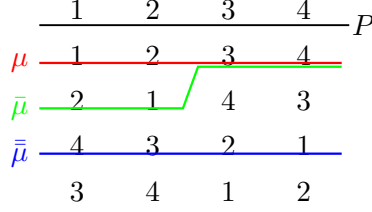
In this section we discuss the logical independence of the conditions used in the characterization, by presenting different measures which satisfy every condition except one of them.

C.1 Betweenness

For any N and for any $P \in \mathcal{P}(N)$ the following rule satisfies everything except betweenness.

$$\delta_P^B(\mu, \bar{\mu}) = |\{i \in N : \mu(i) \neq \bar{\mu}(i)\}|$$

The following example shows δ^B violates the betweenness condition.



It is easy to verify that $\delta_P^B(\mu, \bar{\mu}) = 2$, $\delta_P^B(\bar{\mu}, \bar{\mu}) = 4$ and $\delta_P^B(\mu, \bar{\mu}) = 4$, however according to betweenness we must have $\delta_P^B(\mu, \bar{\mu}) = 6$. It is easy to see that this rule satisfies anonymity, monotonicity, independence of irrelevant newcomers and standardization.

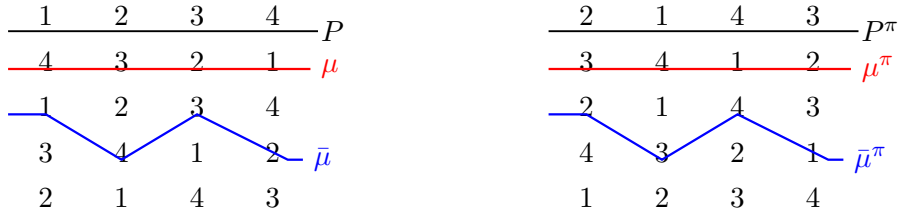
C.2 Anonymity

Let the set of potential agents \mathcal{N} equal to natural numbers that is \mathbb{N} , then for any N and for any $P \in \mathcal{P}(N)$ the following rule satisfies everything except anonymity.

$$\delta_P^A(\mu, \bar{\mu}) = 2 \times \sum_{i \in O} |\mu, \bar{\mu}|_{R_i} + \sum_{i \in E} |\mu, \bar{\mu}|_{R_i}$$

where O denotes the set of odd numbered agents and E denotes the set of even numbered agents.

The following example shows δ^A violates the anonymity condition.



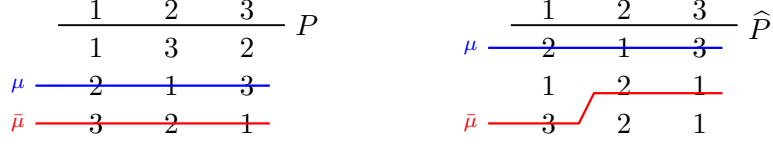
It is easy to check that $\delta_P^A(\mu, \bar{\mu}) = 2(1+1) + (2+2) = 8$, however after applying the permutation $\pi = (12)(34)$, we have $\delta_{P^\pi}^A(\mu, \bar{\mu}) = 2(2+2) + (1+1) = 10$. It is easy to see that this rule satisfies betweenness, monotonicity, independence of irrelevant newcomers and standardization.

C.3 Monotonicity

For any N and for any $P \in \mathcal{P}(N)$ the following rule satisfies everything except monotonicity.

$$\delta_P^M(\mu, \bar{\mu}) = \sum_{i \in N} \left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right|$$

The following example shows δ^A violates the monotonicity condition.



It is obvious that $[\mu, \bar{\mu}]_{R_i} \subseteq [\mu, \bar{\mu}]_{\hat{R}_i}$, for $i \in \{1, 2, 3\}$, however $\delta_P^M(\mu, \bar{\mu}) = 3 \times |2^2 - 2^3| = 12$ and $\delta_{\hat{P}}^M(\mu, \bar{\mu}) = |2^1 - 2^3| + |2^1 - 2^2| + |2^1 - 2^2| = 10$ which violates monotonicity.

To see that δ^M satisfies betweenness, let P be a problem and $\bar{\mu}$ be such that it is between μ and $\bar{\bar{\mu}}$. We have,

$$\begin{aligned}
\delta_P^M(\mu, \bar{\mu}) + \delta_P^M(\bar{\mu}, \bar{\bar{\mu}}) &= \sum_{i \in N} \left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right| + \sum_{i \in N} \left| 2^{\text{rank}(\bar{\mu}(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \\
&= \sum_{i \in N} \left(\left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\mu}(i), R_i)} \right| + \left| 2^{\text{rank}(\bar{\mu}(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \right) \\
&= \sum_{i \in N} \left(\left| 2^{\text{rank}(\mu(i), R_i)} - 2^{\text{rank}(\bar{\bar{\mu}}(i), R_i)} \right| \right) \\
&= \delta_P^M(\mu, \bar{\bar{\mu}})
\end{aligned}$$

Note that the third equality is due to the fact that for each $i \in N$ we have either $\mu(i)R_i\bar{\mu}(i)$ or $\bar{\mu}(i)R_i\mu(i)$. Therefore, $\text{rank}(\mu(i), R_i) \leq \text{rank}(\bar{\mu}(i), R_i) \leq \text{rank}(\bar{\bar{\mu}}(i), R_i)$ or $\text{rank}(\mu(i), R_i) \geq \text{rank}(\bar{\mu}(i), R_i) \geq \text{rank}(\bar{\bar{\mu}}(i), R_i)$. Hence, in the second equation both absolute values have the same sign which allows to conclude the third equation. It is easy to see that this rule satisfies anonymity, independence of irrelevant newcomers and standardization.

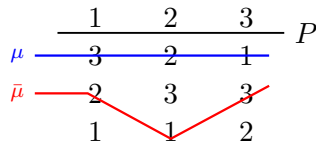
C.4 Independence of irrelevant newcomers

To show that the independence of irrelevant newcomers is logically independent from other conditions, we first define the set of matchings that are between two given matchings.

The following rule satisfies everything except independence of irrelevant newcomers.

$$\delta_P^I(\mu, \bar{\mu}) = \begin{cases} 3, & \text{if } |N| = 3 \text{ and } \mu, \bar{\mu} \text{ are disjoint and there is no matching between } \mu, \bar{\mu} \\ \delta_P^{\text{Borda}}(\mu, \bar{\mu}), & \text{otherwise} \end{cases}$$

The following example shows δ^I violates the independence of irrelevant newcomers condition.



As $|N| = 3$, and μ and $\bar{\mu}$ are disjoint and there is no other matching between them we have $\delta_P^I(\mu, \bar{\mu}) = 3$. Now assume that an irrelevant newcomer joins and hence $|N| = 4$, which results in $\delta_P^I(\mu, \bar{\mu}) = \delta_P^{Borda}(\mu, \bar{\mu}) = 4$. It is easy to verify that the rule satisfies anonymity, monotonicity, betweenness and standardization.

C.5 Standardization

Let N be any set of agents and $P \in \mathcal{P}(N)$. First we define the following sets for any two matchings $\mu, \bar{\mu} \in \mathcal{M}(N)$. Let Γ be the set of agents such that they are single in only one of the matchings, that is $\Gamma(\mu, \bar{\mu}) = \{i \in N \mid [\mu(i) = i \text{ or } \bar{\mu}(i) = i] \text{ and } [\mu(i) \neq \bar{\mu}(i)]\}$. Also let Ω be the set of agents that are single in the strict interval between μ and $\bar{\mu}$. Formally, $\Omega(\mu, \bar{\mu}) = \{i \in N \mid i \in [\mu, \bar{\mu}]_{R_i} \text{ and } i \neq \mu(i) \text{ and } i \neq \bar{\mu}(i)\}$. Consider the following rule

$$\delta_P^S(\mu, \bar{\mu}) = \delta_P^{Borda}(\mu, \bar{\mu}) - \frac{1}{2}|\Gamma(\mu, \bar{\mu})| - |\Omega(\mu, \bar{\mu})|.$$

To show that δ^S violates standardization, let $N = \{1, 2, 3, 4\}$ and consider the matchings $\mu = \{(1, 4), (3, 2)\}$, $\bar{\mu} = \{(1, 2), (3, 4)\}$. By the above measure $\min_{P \in \mathcal{P}(N)} \delta_P^S(\mu, \bar{\mu}) = 4$ and $\min_{P \in \mathcal{P}(N)} \delta_P^S(\mu, \mu^I) = 2$. Hence, standardization fails.

To show that δ^S satisfies betweenness, let $\mu, \bar{\mu}, \bar{\bar{\mu}} \in \mathcal{M}(N)$ be such that $\bar{\mu}$ is between μ and $\bar{\bar{\mu}}$. Let \bar{s} be the number of agents that are single in $\bar{\mu}$ but not in μ or $\bar{\bar{\mu}}$. Then the following equation holds

$$|\Omega(\mu, \bar{\bar{\mu}})| = \bar{s} + |\Omega(\mu, \bar{\mu})| + |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \quad (26)$$

That is the number of agents that are single in the strict interval of μ and $\bar{\bar{\mu}}$ equals to the number of agents that are single in the strict interval of μ and $\bar{\mu}$ plus the number of agents that are single in the strict interval of $\bar{\mu}$ and $\bar{\bar{\mu}}$ plus those agents that are only single in $\bar{\mu}$. Also it is straight forward to see that

$$|\Gamma(\mu, \bar{\mu})| + |\Gamma(\bar{\mu}, \bar{\bar{\mu}})| = |\Gamma(\mu, \bar{\bar{\mu}})| + 2\bar{s} \quad (27)$$

We have

$$\begin{aligned}
& \delta_P^S(\mu, \bar{\mu}) + \delta_P^S(\bar{\mu}, \bar{\bar{\mu}}) \\
&= \delta_P^{Borda}(\mu, \bar{\mu}) - \frac{1}{2}|\Gamma(\mu, \bar{\mu})| - |\Omega(\mu, \bar{\mu})| + \delta_P^{Borda}(\bar{\mu}, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\bar{\mu}, \bar{\bar{\mu}})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&= \delta_P^{Borda}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\mu})| - \frac{1}{2}|\Gamma(\bar{\mu}, \bar{\bar{\mu}})| - |\Omega(\mu, \bar{\mu})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&\stackrel{\text{by}}{\underset{(22)}{=}} \delta_P^{Borda}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - \bar{s} - |\Omega(\mu, \bar{\mu})| - |\Omega(\bar{\mu}, \bar{\bar{\mu}})| \\
&\stackrel{\text{by}}{\underset{(21)}{=}} \delta_P^{Borda}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - \bar{s} - |\Omega(\mu, \bar{\bar{\mu}})| + \bar{s} \\
&= \delta_P^{Borda}(\mu, \bar{\bar{\mu}}) - \frac{1}{2}|\Gamma(\mu, \bar{\bar{\mu}})| - |\Omega(\mu, \bar{\bar{\mu}})| \\
&= \delta_P^S(\mu, \bar{\bar{\mu}})
\end{aligned}$$

where the second equation holds as δ^{Borda} satisfies betweenness. It is easy to see that δ^S satisfies anonymity, monotonicity and independence of irrelevant newcomers.