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On Reward Sharing in Blockchain Mining Pools

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On Reward Sharing in Blockchain Mining Pools

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Abstract

This paper proposes a conceptual framework for the analysis of reward sharing schemes in mining pools, such as those associated with Bitcoin. The framework is centered around the reported shares in a pool instead of agents and results in two new fairness criteria, absolute and relative redistribution. These criteria impose that the addition of a share to the pool affects all previous shares in the same way, either in absolute amount or in relative ratio. We characterize two large classes of economically viable reward sharing schemes corresponding to each of these fairness criteria in turn. We further show that the intersection of these classes brings about a generalization of the the well-known proportional scheme, which also leads to a new characterization of the proportional scheme as a corollary.

Keywords: Blockchain, Bitcoin, fairness, mining pools, resource allocation, mechanism design.

JEL Classification: D63, G20, L86, D31.

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1 Introduction

The invention of Bitcoin (Nakamoto (2008)) sparked a huge interest in decentralized networks with distributed trust, both academically and businesswise. By August 2020, a single bitcoin is traded around 11500 USD. The Bitcoin market value exceeds 215 billion USD, which is comparable to the GDPs of various countries¹. The daily number of transfers/transactions on Bitcoin is around 350,000, which is worth a volume of 12 Billion USD.

The technology that runs Bitcoin, and all the other cryptocurrencies for that matter, is called the Blockchain. Loosely speaking, a blockchain is a ledger composed of an immutable chain of transactions organized in *blocks*. Each block is synchronized across the network users (nodes) through a distributed consensus protocol which ensures that all the nodes in the network agree on the latest status of the ledger. To agree on the latest status of the ledger, Blockchains utilize various different consensus protocols². The protocol which Bitcoin is built upon is called the Proof-of-Work (PoW). To achieve consensus under this protocol, some nodes, called *miners*, compete with one another to solve a cryptographic puzzle³. The process of searching for a solution is called *mining* and the solution to this aforementioned puzzle is called a *full solution*, for which the successful miner is given a financial reward.

In theory, the probability of finding a full solution is proportional to the computational power of the miner⁴. However, the computational power of individual miners are negligible in comparison to that of the network⁵. Therefore, mining alone leads to a highly unstable income and incentivizes miners to *pool* their resources and split the resulting rewards. Such cooperative actions result in lower income variation for miners (Romiti et al. (2019); Rosenfeld (2011)). In fact, as of 2020, almost 90% of the total computational power in Bitcoin blockchain is provided by the top ten mining pools⁶. Consequently, mining pools are probably the most important actors in the Bitcoin ecosystem.

Typically, a mining pool is maintained and coordinated by a *pool manager*. The success of

¹For example, Hungary, Kenya, Luxembourg and many others. See the following link for a ranking: <https://worldpopulationreview.com/countries/countries-by-gdp>. For other financial statistics, see <https://bitinfocharts.com/bitcoin/>.

²For a technical survey on different consensus protocols, see Nguyen and Kim (2018); Mingxiao et al. (2017).

³Miners search for an integer (a nonce) such that when combined (hashed) together with the list of transactions (the block header), produces another integer known as the hash value. In case, this hash value is less than a predefined number (network target value) set by the network, the puzzle is solved.

⁴For an extensive demonstration of these concepts see Anders Brownworth's blog at <https://andersbrownworth.com/blockchain/hash>.

⁵At the time of writing this paper, the total hashing power of the Bitcoin blockchain is 120^{18} KH/s, while a state-of-the-art CPU has about 11KH/s. Therefore, the probability of finding a full solution as a solo miner is roughly $\frac{1}{10^{19}}$.

⁶<https://btc.com/stats/pool>

a pool depends on its computational power, therefore the miners must commit their resources to find a full solution through mining. To estimate the computing power of each miner, the pool manager sets an easier puzzle to solve and requests the solutions to this puzzle, which are called *partial solutions*⁷. Miners submit these partial solutions to the manager, each of which forms a *share*. In case one of these shares is a full solution, the pool manager gets the reward and distributes it among the pool members based on their submitted shares and a predefined *reward sharing scheme*. The choice of reward sharing scheme is a crucial design element in a mining pool. The scheme must ensure that the pool is economically viable and must provide miners with the right incentives to act in ways beneficial for the common good of the pool.

In this paper, we analyze and design *reward sharing schemes* in mining pools by proposing a comprehensive axiomatic perspective. We depart from existing literature in terms of framework in various ways. First, our axiomatic framework is not on the consensus protocols but on the mining pools in any of these protocols. Second, our model is not restricted to a static single block, since various schemes in practice pay the miners repetitively over time in various blocks. Third, we propose reward sharing schemes and *allocations* not on the miners in a pool but instead on the shares submitted by these miners. This is particularly enriching, because defining allocations on shares (rather than on miners), enables more granular and relevant parameters such as the submission time or order of the shares. Therefore, we can practically formulate any of the existing schemes under our unified axiomatic framework, e.g., the Slush and PPLNS⁸, and allow designers to propose new ones.

We formulate several desirable axioms for reward sharing schemes. In particular, we propose two axioms concerning fairness, i.e., how the awards for the shares should be redistributed⁹ when the round is delayed by an additional share. We show that, together with other axioms, each of these fairness axioms, *absolute redistribution* and *relative redistribution*, characterize two distinct classes of reward sharing schemes, the class of absolute fair and relative fair schemes. Thereafter, we characterize the generalized class of proportional reward schemes, i.e., *k*-pseudo proportional schemes, which satisfies both of these axioms simultaneously. Finally, by imposing an additional *strict positivity* requirement, we single out the well-known proportional reward scheme.

⁷This is done by setting an internal pool target value higher (easier) than the network target value for the original puzzle (see footnote 3). Note that the set of partial solutions for this easier target value is a superset of the full solutions. Therefore it is a very good approximation of the computational commitment of a miner.

⁸See [Rosenfeld \(2011\)](#) for a comprehensive list of reward sharing schemes, and Section 5 for formal definitions.

⁹This redistribution could be either in absolute amount or in relative ratio. Given the symmetric nature of the shares in a pool round, these fairness axioms represent a natural (stronger) versions of the celebrated “population monotonicity” axiom in the literature on fair allocation (e.g. [Thomson \(2016\)](#)).

As such, our paper relates to the literature on miner’s general incentives under the PoW protocol. The seminal paper by [Rosenfeld \(2011\)](#) provides one of the earliest comprehensive mathematical analysis on the reward sharing schemes in mining pools. [Schrijvers et al. \(2016\)](#) discuss strategic behavior of miners under “incentive compatibility”. [Zolotavkin et al. \(2017\)](#) investigate the interplay between incentive compatibility and the distribution of computational power among miners in the pool. [Lewenberg et al. \(2015\)](#) model mining pools from a cooperative game perspective while [Qin et al. \(2018\)](#) and [Chatzigiannis et al. \(2019\)](#) investigate competition among pools¹⁰.

In terms of methodology, our paper utilizes tools from the literature on economic design (see e.g., [Moulin \(1991\)](#); [Young \(1994\)](#); [Thomson \(2018\)](#)) and is inspired by welfare economics, in particular the literature on distributional fairness (see e.g., [Roemer \(1996\)](#); [Moulin \(1987\)](#); [Young \(1994\)](#)) and fair allocation in networks (see e.g., [Hougaard \(2018\)](#); [Moulin \(2018\)](#)) and take an axiomatic approach to the analysis of reward sharing schemes: a list of desirable properties of generic schemes (axioms) are identified, and individual schemes are uniquely characterized by different sets of axioms, which allows to qualitatively compare various reward sharing schemes based on this axiomatic foundation.

There has been a steadily growing literature, pointing towards utilization of mechanism design in the context of blockchains. [Can \(2019\)](#) proposes the use of economic design on the consensus protocol, while [Hougaard et al. \(2020\)](#) provides a different rationale for PoW by decentralized socially optimal reward schemes. Concerning an axiomatic approach to PoW, [Leshno and Strack \(2019\)](#) and [Chen et al. \(2019\)](#) propose characterizations (on the consensus protocols) with properties such as anonymity, collusion-proofness and sybil-proofness the last two of which are analogous to merging and splitting¹¹.

To sum up: our contribution is at least two-fold. First, we provide a conceptual framework directly related to reward sharing in pools rather than the consensus protocol. Our modeling framework emphasizes the potential richness of the design universe: a mechanism designer can construct schemes in which shares are rewarded based on any combination of their order, time stamps, round size, history, etc. Second, we identify two large classes of economically viable reward sharing schemes illustrating the flexibility in design when the pool manager favors fair payments to miners of the pool.

The remaining part of the paper is organized as follows. In [Section 2](#) we propose our mod-

¹⁰For a list of further readings regarding miners’ behaviour, see [Eyal and Sirer \(2014\)](#); [Biais et al. \(2019\)](#); [Babaioff et al. \(2012\)](#); [Sapirshstein et al. \(2016\)](#); [Fisch et al. \(2017\)](#); [Carlsten et al. \(2016\)](#); [Kiayias et al. \(2016\)](#); [Cong et al. \(2019\)](#).

¹¹Both papers propose characterizations of the proportional reward scheme which is well studied in the economic design literature and in bankruptcy models (see [Banker \(1981\)](#); [Moulin \(1987\)](#); [Chun \(1988\)](#); [de Frutos \(1999\)](#)).

eling framework and notation. Section 3 defines and comments on various axioms including the two fairness conditions. Section 4 presents the characterization results. Finally, Section 5 closes with a discussion of well-known schemes under our proposed framework and logical independence of the characterizing axioms.

2 Model and notation

Let T denote the set of all possible *time signatures*. Let H denote the set of all possible *hashes* and $\mathcal{S} \subsetneq H$ denote the set of all admissible hashes, i.e., hash values that satisfy the difficulty of the pool. A (partial solution) *share* is then a two-tuple $s = (h, t) \in \mathcal{S} \times T$ consisting of an admissible hash submitted to the pool at time¹² t . We denote the time signature of a share s by a function $\tau(s)$, i.e., for $s = (h, t)$, we have $\tau(s) = t$. The set of all ordered shares submitted in a pool is denoted by $S = \{s_1, s_2, s_3, \dots, s_m\} \subsetneq \mathcal{S} \times T$ where the order is defined by the shares' time signatures. We do not associate shares with individual miners since all schemes that are used in practice are neutral towards the “identity” of the miner.

A *pool round* is an ordered set of shares ending with a share, submitted by the pool, which is a full solution on the blockchain. So everytime a share submitted by the pool is a full solution on the blockchain, a pool round ends and a new round begins. We consider the partitioning of the submitted set of shares by (pool) rounds, and denote it by $\mathcal{P}(S)$: for instance, let $l = |\mathcal{P}(S)|$ and let P_1, P_2, \dots, P_l denote the set of shares submitted in rounds 1, 2, \dots , l respectively. Let $H = (S, \mathcal{P}(S))$ denote the *history* of the pool.

Given the partition of a history $\mathcal{P}(S)$ and any share $s \in S$, we denote the set of all shares in the same round as s by $P(s) = X \in \mathcal{P}(S)$ such that $s \in X$. Given a round $P \in \mathcal{P}(S)$, we denote the relative rank of a share s in P by $\rho(s) = |\{s' \in P | \tau(s') < \tau(s)\}| + 1$, i.e., the number of shares that are submitted in round P up to, and including, s . We define the *length* of a round P by the number of shares submitted in that round, i.e. $|P|$.

Example 1. Equation 1 below, illustrates an example of a history: shares in bold are full solutions.

$$\begin{aligned}
 S &= \{s_1, \dots, \mathbf{s}_{10}, s_{11}, \dots, \mathbf{s}_{150}, \dots, s_{564}, \dots, \mathbf{s}_{589}, \dots, \mathbf{s}_m\} \\
 \mathcal{P}(S) &= \left\{ \underbrace{\{s_1, s_2, \dots, \mathbf{s}_{10}\}}_{P_1}, \underbrace{\{s_{11}, \dots, \mathbf{s}_{150}\}}_{P_2}, \dots, \underbrace{\{s_{564}, \dots, \mathbf{s}_{589}\}}_{P_r}, \dots, \underbrace{\{\dots, \mathbf{s}_m\}}_{P_l} \right\} \quad (1)
 \end{aligned}$$

Considering, for instance, the share s_{564} we see that it is the first share of round r , so $P(s_{564}) = P_r = \{s_{564}, \dots, s_{589}\}$ and $\rho(s_{564}) = 1$. The length of round r is $|P_r| = 26$.

¹²Since some known reward sharing schemes use time signature as input we denote the shares with (h, t) . We shall drop the time signature when it is redundant for the schemes we analyze.

Definition 1. Given a history $H = (S, \mathcal{P}(S))$, we define a reward sharing scheme, α , by awards

$$\alpha(s, H) \in \mathbb{R}_+$$

to every share $s \in S$.

In practice, the payments to miners are made after a certain amount of time, therefore when considering the allocation of rewards, we analyze a round that is terminated and already confirmed on the Blockchain¹³. For each pool round, the pool manager charges a fee to compensate the costs of running the pool. For a history $H = (S, \mathcal{P}(S))$ we denote this fee by the mapping $f : 2^S \setminus \emptyset \rightarrow \mathbb{R}$. A generic fee for a round P_r in that case would be $f(P_r)$. We denote the award to be distributed in this round by $R_r = B - f(P_r)$, where B is the block reward assigned to the pool when finding a full solution. Next, we introduce one of the most intuitive schemes, also known as the *Proportional scheme* below.

Example 2. The *Proportional scheme* is one of the most straightforward and intuitive reward sharing schemes. For any given round, the pool manager gets a fixed fraction of the reward and then it assigns to every share $s \in P(s)$, a proportion of the reward relative to the length of the corresponding round $|P(s)|$. Formally, for any history $H = (S, \mathcal{P}(S))$ and any share $s \in S$:

$$\alpha(s, H) = \frac{R_r}{|P(s)|}.$$

In Section 5.1 we provide a brief discussion of various reward sharing schemes that are used in practice or proposed to improve existing practices.

For our axiomatic analysis below, we are inspired by the framework in Schrijvers et al. (2016). In particular, we consider situations in which full solutions are delayed, i.e., when the pool round ends with the submission of additional share(s) at the end. In the simplest of such cases, we consider histories where a pool round is extended by one additional share at the end of the round. We therefore need some additional definitions.

Formally, let $H = (S, \mathcal{P}(S))$ be the history of a pool. Let $s^* \notin S$ be the so-called additional last share (into the r^{th} round), i.e., $\tau(\underline{s}) < \tau(s^*) < \tau(\bar{s})$ for all $\underline{s} \in P_r$ and for all $\bar{s} \in P_{r+1}$. The history $H' = (S', \mathcal{P}'(S'))$ is an *extension of H at the r^{th} round* whenever $S' = S \cup \{s^*\}$ and:

- $P'_k = P_k$ for all $k \neq r$,
- $P'_r = P_r \cup \{s^*\}$.

¹³The pool manager should wait for at least 6 block confirmation to make sure the block found by the pool ends up on the longest chain.

Note that, one can consider the extension of a round at any position and not only at the end of the round. However, this will be a stronger version and it is more natural to extend a round by adding one share at the end.

Next, we define the restriction of a history to a single round. Let $H = (S, \mathcal{P}(S))$ be any history. The *restriction of H to the r^{th} round* is denoted as $H|_r = (P_r, \{P_r\})$. That is, the set of shares in the history only consists of those at the r^{th} round and the only partition of the history is P_r .

Example 3. Recall the situation in Example 1. Below is the example of an extension of the history in Equation 1 at the r^{th} round by the full solution (share) s^* .

$$S = \{s_1, \dots, \mathbf{s}_{10}, s_{11} \dots, \mathbf{s}_{150}, \dots, s_{564}, \dots, s_{589}, \mathbf{s}^*, \dots, \mathbf{s}_m\}$$

$$\mathcal{P}(S) = \left\{ \underbrace{\{s_1, s_2, \dots, \mathbf{s}_{10}\}}_{P_1}, \underbrace{\{s_{11} \dots, \mathbf{s}_{150}\}}_{P_2}, \dots, \underbrace{\{s_{564}, \dots, s_{589}, \mathbf{s}^*\}}_{P_r}, \dots, \underbrace{\{\dots, \mathbf{s}_m\}}_{P_l} \right\}$$

Moreover, an example of restriction of the history in Equation 1 to the r^{th} round is

$$H|_r = \left(\{s_{564}, \dots, \mathbf{s}_{589}\}, \left\{ \underbrace{\{s_{564}, \dots, \mathbf{s}_{589}\}}_{P_r} \right\} \right)$$

3 Axioms

In the following we discuss five desirable properties of generic reward sharing schemes. Two of these reflect different aspects of fairness related to how the rewards must be re-distributed in case there is one additional share in the round¹⁴.

The first property ensures a *fixed total reward* to the miners for any pool round in a history. That implies that the fee charged by the pool manager is the same for any two rounds in a history. This guarantees that the pool manager can not take advantage (or be harmed) from shorter (longer) rounds, as the pool must distribute the same amount to the miners in any round. Formally,

• **Fixed Total Reward:** A scheme α satisfies fixed total reward whenever, for any history $H = (S, \mathcal{P}(S))$, and any two rounds $P, P' \in \mathcal{P}(S)$, we have

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H).$$

¹⁴In this sense our fairness axioms can be viewed as relational axioms in case of a variable populations framework, see e.g., Thomson (2016).

Next, we consider situations where the submission time of shares may change. To analyze such situations, we create a *ceteris paribus* case where the time signature of only a single share in a history changes in a minimal way. Formally, let $H = (S, \mathcal{P}(S))$ be any history, and consider any round $P_r \in \mathcal{P}(S)$, and any share $s_i \in P_r$ in this round. A *time-shift* $H' = (S', \mathcal{P}'(S'))$ of H at s_i is defined as:

1. $P'_j = P_j$ for all $j \neq r$, and
2. $S' = (S \setminus \{s_i\}) \cup \{s'_i\}$ for some $s'_i \in S \setminus S$ such that $\tau(s_{i-1}) < \tau(s'_i) < \tau(s_{i+1})$,

The next condition, dubbed *ordinality*, requires that time-shifts should not affect the reward distribution, so long as the order of shares is preserved.

• **Ordinality:** A scheme α satisfies ordinality if, for any $H = (S, \mathcal{P}(S))$, and for any time-shift $H' = (S', \mathcal{P}'(S'))$ of H at any s_i , we have:

$$\alpha(s, H) = \alpha(s, H') \text{ for all } s \in S \setminus \{s_i\}$$

The next condition, dubbed *budget limit*, requires that the pool manager charges a non-negative fee. This ensures that the pool does not go bankrupt.

• **Budget Limit:** A scheme α satisfies budget limit whenever, for any history $H = (S, \mathcal{P}(S))$, and any round $P \in \mathcal{P}(S)$, we have:

$$\sum_{s \in P} \alpha(s, H) \leq B.$$

We now turn to our two main fairness conditions. The first is dubbed *absolute redistribution*, and requires that in case a round is extended (delayed) by one additional share, the award assigned to any existing share in the corresponding round decreases by the same *amount*¹⁵. Formally,

• **Absolute redistribution:** A scheme α satisfies absolute redistribution whenever, for any history $H = (S, \mathcal{P}(S))$, any round P_r with $|P_r| > 1$, and any extension $H' = (S', \mathcal{P}'(S'))$ at the r^{th} round we have for any $s_i, s_j \in P_r$:

$$\alpha(s_i, H) - \alpha(s_i, H') = \alpha(s_j, H) - \alpha(s_j, H').$$

In the same spirit, the next condition, dubbed *relative redistribution*, requires that in case a round is extended (delayed) by one additional share, the reward of each existing share in the corresponding round is decreased by the same *ratio*. Formally,

¹⁵This is, in fact, a strong version of Population monotonicity in Thomson (2016) and resembles the spirit of Myerson's fairness axiom Myerson (1977) stating that agents should be affected equally from entering mutual agreements.

• **Relative redistribution:** A scheme α satisfies relative redistribution whenever for any history $H = (S, \mathcal{P}(S))$, any round P_r with $|P_r| > 1$, and any extension $H' = (S', \mathcal{P}'(S'))$ at the r^{th} round, we have for any $s_i, s_j \in P_r$ with $\alpha(s_i, H) \neq 0$ and $\alpha(s_j, H) \neq 0$:

$$\frac{\alpha(s_i, H')}{\alpha(s_i, H)} = \frac{\alpha(s_j, H')}{\alpha(s_j, H)}.$$

The final condition, dubbed *round based rewards*, requires that the distribution of the reward only depends on the round itself and it is not affected by any other rounds in the history. Formally,

• **Round based rewards:** A scheme α satisfies round based rewards whenever, for any history H , and any round P_r , we have for all $s \in P_r$:

$$\alpha(s, H) = \alpha(s, H|_r).$$

We first observe that round based rewards strengthens the fixed total reward condition, such that the latter imposes the fixed total rewards for any rounds in any history.

Lemma 1. *If a scheme α satisfies fixed total reward and round based rewards, then for any two histories $H = (S, \mathcal{P}(S))$ and $H' = (S', \mathcal{P}'(S'))$ and any two rounds $P \in \mathcal{P}(S)$ and $P' \in \mathcal{P}'(S')$ we have*

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H').$$

Proof. See Appendix A. ■

Remark 1. *An immediate consequence of Lemma 1 is that, if a scheme satisfies fixed total rewards and round based rewards then the fee must be the same for all histories and all rounds in these histories. Therefore, these two axioms, together with budget limit imply a stronger version of the budget limit¹⁶ in the sense that for a scheme which satisfies fixed total rewards and round based rewards and for any history $H = (S, \mathcal{P}(S))$ and any round $P \in \mathcal{P}(S)$, we have $\sum_{s \in P} \alpha(s, H) = R$, with $R = B - f$ for some $f \in [0, B]$.*

Second, we observe that joint with ordinality, fixed total reward and round based rewards imply that shares with same relative rank in rounds of equal length get identical awards. Formally,

Lemma 2. *If a scheme α satisfies fixed total reward, round based rewards and ordinality, then for any two histories $H = (S, \mathcal{P}(S))$ and $\bar{H} = (\bar{S}, \bar{\mathcal{P}}(\bar{S}))$ and any two rounds $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ such that $|P| = |\bar{P}|$, we have for all $s \in P$ and for all $\bar{s} \in \bar{P}$ such that $\rho(s) = \bar{\rho}(\bar{s})$:*

$$\alpha(s, H) = \alpha(\bar{s}, \bar{H}).$$

¹⁶This is conceptually similar to the *strong budget balance* in Chen et al. (2019).

Proof. See Appendix B. ■

4 Fair reward sharing schemes

4.1 Absolute Fairness

In this section we single out a particular class of schemes, called the class of *absolute fair schemes*. We show that this is the only such class that satisfies absolute redistribution together with fixed total reward, budget limit, round based rewards, and ordinality.

Specifically, a scheme belongs to this class if there exists $\varepsilon : N \rightarrow [0, 1]$ with $\varepsilon(1) = 1$ and $\varepsilon(\rho(s)) \geq \sum_{i=\rho(s)+1}^{\infty} \frac{\varepsilon(i)}{i-1}$ such that,

$$\alpha^{**}(s, H) = R \left(\varepsilon(\rho(s)) - \sum_{i=\rho(s)+1}^{|\rho(s)|} \frac{\varepsilon(i)}{i-1} \right) \quad (2)$$

Note that for the last share $\rho(s) = |\rho(s)|$, therefore $\sum_{i=|\rho(s)|+1}^{|\rho(s)|} \frac{\varepsilon(i)}{i-1}$ is an empty sum and by convention it equals 0.

We can interpret absolute fair schemes in an iterative fashion: if there is only one share s , then $\rho(s) = 1$ so $\varepsilon(1) = 1$ and the share receives the full net reward R . If we add a share s' , then $\rho(s') = 2$ and $\varepsilon(2) < 1$ so the first share s gets $R(\varepsilon(1) - \varepsilon(2))$ and the second share s' gets $R\varepsilon(2)$. Now, adding a third share s'' the award to both the first and second share should be reduced by the same amount δ . Thus, $\varepsilon(3) = 2\delta$ and consequently the first share is awarded $R(\varepsilon(1) - \varepsilon(2) - \frac{\varepsilon(3)}{2})$ and the second share is awarded $R(\varepsilon(2) - \frac{\varepsilon(3)}{2})$, and so forth, for any share in the round.

Next, we present our first main result.

Theorem 1. *A reward sharing scheme α satisfies round based rewards, budget limit, fixed total reward, ordinality and absolute redistribution if and only if it is an absolute fair scheme in the sense of (2).*

Proof. See Appendix C. ■

A prominent member of the class of absolute fair schemes is the proportional scheme as we shall demonstrate below.

Proposition 1. *The proportional scheme is an absolute fair scheme.*

Proof. Let $H = (S, \mathcal{P}(S))$ be any history. It is easy to see the proportional scheme satisfies the round-baseness condition, hence we only consider a history with a single round, i.e., $H = (\{s_1, \dots, s_k\}, \{\{s_1, \dots, s_k\}\})$. Let $\varepsilon(\rho(s_i)) = \frac{1}{\rho(s_i)}$ for all $i \in \{1, \dots, k\}$. Note that $\rho(s_i) = i$. Therefore, for any share s_j we have:

$$\begin{aligned} \alpha^{**}(s_j, H) &= R \left(\varepsilon(\rho(s_j)) - \sum_{i=\rho(s_j)+1}^{|P(s_j)|} \frac{\varepsilon(i)}{i-1} \right) = R \left(\frac{1}{j} - \sum_{i=j+1}^k \frac{1}{i \times (i-1)} \right) \\ &= R \left(\frac{1}{j} - \sum_{i=j+1}^k \left(\frac{1}{i-1} - \frac{1}{i} \right) \right) = R \left(\frac{1}{j} - \left(\frac{1}{j} - \frac{1}{k} \right) \right) = \frac{R}{k} \end{aligned}$$

Note that for the last share $\sum_{i=k+1}^k \varepsilon(i)$ is an empty sum and by convention it equals 0. ■

4.2 Relative Fairness

In this section we single out a particular class of schemes, called the class of *relative fair schemes*. We show that this is the only such class that satisfies relative redistribution together with fixed total reward, budget limit, round based rewards, and ordinality.

Specifically, a scheme belongs to this class if there exists $\varepsilon(j) : N \rightarrow [0, 1]$ with $\varepsilon(1) = 1$ such that,

$$\alpha^*(s, H) = B\varepsilon(\rho(s)) \prod_{j=\rho(s)+1}^{|P(s)|} (1 - \varepsilon(j)) \quad (3)$$

Note that for the last share $\rho(s) = |P(s)|$, therefore $\prod_{j=|P(s)|+1}^{|P(s)|} (1 - \varepsilon(j))$ is an empty product and by convention it equals 1.

An iterative interpretation of relative fair schemes is as follows: if there is only one share s , $\varepsilon(1) = 1$ and the share is awarded the entire net reward R . If we add a second share s' , then s' will be awarded $R\varepsilon(2)$ and the first share must get $R(\varepsilon(1)(1 - \varepsilon(2))) = R(1 - \varepsilon(2))$. Now, adding third share s'' we must now have that both the award to the first share and the second share change by the same ratio δ so $(1 - \varepsilon(2))\delta + \varepsilon(2)\delta + \varepsilon(3) = 1$ implying that $\delta = 1 - \varepsilon(3)$ yielding the award $R((1 - \varepsilon(2))(1 - \varepsilon(3)))$ to the first share and $R(\varepsilon(2)(1 - \varepsilon(3)))$ to the second share, and finally $R\varepsilon(3)$ to the third share, and so forth for any share in the round.

We now present our second main result characterizing the class of relative fair reward sharing schemes.

Theorem 2. *A reward scheme α satisfies round based rewards, budget limit, fixed total reward, ordinality and relative redistribution if and only if it is a relative fair scheme in the sense of (3).*

Proof. See Appendix D. ■

It turns out that the proportional scheme also takes up a prominent position among the relative fair schemes as recorded below.

Proposition 2. *The proportional scheme is a relative fair scheme.*

Proof. Let $H = (S, \mathcal{P}(S))$ be any history. It is easy to see the proportional scheme satisfies the round-baseness condition, hence we only consider a history with a single round, i.e., $H = (\{s_1, \dots, s_k\}, \{\{s_1, \dots, s_k\}\})$. Let $\varepsilon(\rho(s_i)) = \frac{1}{\rho(s_i)}$ for all $i \in \{1, \dots, k\}$. Note that $\rho(s_i) = i$. Therefore, for any share s_j we have:

$$\begin{aligned} \alpha^*(s_j, H) &= R \times \varepsilon(\rho(s_j)) \times \prod_{i=\rho(s_j)+1}^k (1 - \varepsilon(i)) \\ &= R \times \left(\frac{1}{j}\right) \times \left(1 - \frac{1}{j+1}\right) \times \left(1 - \frac{1}{j+2}\right) \times \dots \times \left(1 - \frac{1}{k}\right) \\ &= R \times \left(\frac{1}{j}\right) \times \left(\frac{j}{j+1}\right) \times \left(\frac{j+1}{j+2}\right) \times \dots \times \left(\frac{k-2}{k-1}\right) \times \left(\frac{k-1}{k}\right) = \frac{R}{k} \end{aligned}$$

Note that for the last share $\prod_{i=k+1}^k (1 - \varepsilon(i))$ is *empty product* and by convention equals 1. ■

4.3 Consensus between absolute and relative redistribution

Next, we search for a possible consensus between the two fairness concepts that we discussed in the previous sections. We find that the two aforementioned classes of absolute and relative fair schemes are not mutually exclusive. Therefore, imposing both fairness axioms together with the others conditions characterize a new class of schemes at the intersection of the former two classes. We call this new class of schemes *k-pseudo proportional schemes* and find that this class is also a generalization of the well-known proportional reward scheme mentioned in Example 2. Formally, given any k and δ , a k -pseudo proportional reward scheme assigns awards to shares in a round identical to the proportional scheme, $\frac{R}{|P(S)|}$, so long as the round is shorter than k , i.e., $|P(S)| < k$. In case the round has more shares than k , then the scheme

assigns an award of i) δ to the k^{th} share, ii) distributes the rest, $R - \delta$, to the first $k - 1$ shares, and iii) awards 0 to any share that is submitted after k . The general structure of the k -pseudo proportional scheme with $k > 1$ is as follows:

$$\alpha^{k,\delta}(s, H) = \begin{cases} \frac{R}{|P(s)|}, & \text{if } |P(s)| < k \\ \frac{R-\delta}{k-1}, & \text{if } |P(s)| \geq k \text{ and } \rho(s) < k \\ \delta, & \text{if } |P(s)| \geq k \text{ and } \rho(s) = k \\ 0, & \text{if } |P(s)| \geq k \text{ and } \rho(s) > k \end{cases} \quad (4)$$

for $0 \leq \delta \leq R$.

The following theorem shows that the class of k -pseudo proportional schemes is the only one at the intersection of relative fair schemes and absolute fair schemes.

Theorem 3. *A reward sharing scheme satisfies round based rewards, budget limit, fixed total reward, ordinality, absolute redistribution, and relative redistribution if and only if it is k -pseudo proportional in the sense of (4).*

Proof. See Appendix E. ■

Next, we show that the proportional reward scheme is the only member of this class that ensures a strictly positive award for all shares for any history. We write this feature as an additional axiom and provide the resulting characterization of the proportional scheme as a corollary to Theorem 3.

• **Strict positivity:** A scheme α satisfies strict positivity whenever, for any history $H = (S, \mathcal{P}(S))$, and any round $P \in \mathcal{P}(S)$, we have:

$$\alpha(s, H) \in \mathbb{R}_{++}.$$

Corollary 1. *The proportional rule is the only rule that satisfies round based rewards, budget limit, fixed total reward, ordinality, absolute redistribution, relative redistribution, and strict positivity.*

Proof. By Theorem 3 any rule that satisfies the first six axioms, is a k -pseudo proportional rule. Suppose for a contradiction, k is finite. Consider any history with a round P_r such that $|P_r| > k$. Obviously $\alpha^{k,\delta}(s_{k+1}) = 0$ which contradicts strict positivity. ■

5 Discussion

This paper provides, for the first time, a rich framework for reward sharing schemes in mining pools through an economic design perspective. To demonstrate the flexibility in the design, we proposed various desirable axioms and put particular emphasis on the fairness concept as an example. We provided three different characterizations of classes within the framework, i.e., absolute fair, relative fair, and k -pseudo proportional schemes.

In what follows, we show that the framework also allows the formalization of various well-known schemes and investigate these schemes axiomatically. The results are summarized in Table 1. We conclude the paper with a discussion of the logical independence of the characterizing axioms and provide a summary in Table 2.

5.1 Known reward sharing schemes

In this section we focus on two of the most popular, and widely applied, reward sharing schemes and examine whether they satisfy the axioms proposed in Section 3. We also comment on several potentially interesting schemes suggested in Rosenfeld (2011) and Schrijvers et al. (2016) respectively.

- **Pay Per Share (PPS):** The strict egalitarian Pay-Per-Share scheme fails to ensure an economically viable mining pool (as noted in Rosenfeld (2011)) but has been applied, for instance by F2Pool¹⁷ and Poolin¹⁸, probably due to its immediate simplicity and transparency. In pay per share, every submitted share receives a fixed reward regardless of when it is submitted, and the round it is submitted in. Formally, $\alpha(s, H) = c$ for some constant c . The payments to the shares are usually adjusted by the network difficult and the length of a round. Trivially, PPS fails fixed total reward and budget limit, but satisfies relative redistribution, absolute redistribution, ordinality and round based rewards.

- **Pay-Per-Last-N-Shares:** The pay-per-last-N-shares (PPLNS for short) has also been popular and is applied, for instance in GHash.IO¹⁹ and P2Pool²⁰. It has been claimed that this scheme (under certain conditions) prevents miners from delay reporting their shares (see Schrijvers et al. (2016)). In PPLNS, the pool manager gets a fixed fee, say f , and the net-reward, $R = (1 - f)B$, is distributed (proportionally) among the N last shares (including the full the solution), regardless of the round boundaries. Therefore, the reward of a share at the time it is submitted depends on the number of full solutions among the next $N - 1$ shares.

¹⁷<https://www.f2pool.com/>

¹⁸<https://www.poolin.com/>

¹⁹<https://ghash.io/>

²⁰<http://p2pool.in/>

That is, if no full solution is found among the next $N - 1$ shares, the share receives no reward, if one full solution is found it is rewarded once with $\frac{1}{N}R$, if two full solutions are found it is rewarded twice with $\frac{1}{N}R$, and so forth.

To formally define the PPLNS, consider a history $H = (S, \mathcal{P}(S))$ and let $\Omega(s)$ denote the index of the pool round that share s belongs to i.e., $\Omega(s) = \{x \leq |\mathcal{P}(S)| \mid s \in P_x\}$. Then the PPLNS scheme is defined as

$$\alpha(s_i, H) = \frac{\Omega(s_{N+i}) - \Omega(s_i)}{N} R.$$

It is trivial that PPLNS satisfies ordinality. However, Example 4 shows that PPLNS fails to satisfy budget limit, fixed total rewards, round based rewards, absolute redistribution and relative redistribution.

Example 4. Consider the PPLNS reward sharing scheme with $N = 3$. Let $H = (S, \mathcal{P}(S))$ be a history as follows,

$$S = \{s_1, s_2, \dots, s_{1000}\}$$

$$\mathcal{P}(S) = \left\{ \underbrace{\{s_1, s_2, s_3, s_4, \mathbf{s}_5\}}_{P_1}, \underbrace{\{\mathbf{s}_6\}}_{P_2}, \underbrace{\{\mathbf{s}_7\}}_{P_3}, \underbrace{\{\mathbf{s}_8\}}_{P_4}, \underbrace{\{s_9, \dots, \mathbf{s}_{20}\}}_{P_5}, \dots, \underbrace{\{\dots, \mathbf{s}_{1000}\}}_{P_{132}} \right\}$$

To show that PPLNS fails to satisfy the budget limit and fixed total rewards, note that the rewards of the shares in the first and second rounds of the aforementioned history are

$$\begin{aligned} \alpha(s_1, H) &= 0 & \alpha(s_2, H) &= 0 & \alpha(s_3, H) &= \frac{1}{3}R \\ \alpha(s_4, H) &= \frac{2}{3}R & \alpha(s_5, H) &= R & \alpha(s_6, H) &= \frac{3}{3}R \end{aligned}$$

Therefore, as $\sum_{s \in P_1} \alpha(s, H) = \frac{6}{3}R$, the PPLNS fails to satisfy the budget limit. Also, as $\sum_{s \in P_1} \alpha(s, H) \neq \sum_{s \in P_2} \alpha(s, H)$ the PPLNS fails to satisfy the fixed total rewards.

To show that PPLNS fails to satisfy the round based rewards, consider the restriction of H to the first round, i.e., $H|_1 = \left(\{s_1, \dots, \mathbf{s}_5\}, \underbrace{\left\{ \underbrace{\{s_1, \dots, \mathbf{s}_5\}}_{P_1} \right\}}_{P_1} \right)$. The rewards of each share would be $\alpha(s_1, H|_1) = \alpha(s_2, H|_1) = 0$, and $\alpha(s_3, H|_1) = \alpha(s_4, H|_1) = \alpha(s_5, H|_1) = \frac{R}{3}$. Comparing these with the reward of each share at the history H shows that the PPLNS fails to satisfy the round based rewards.

To show that PPLNS fails to satisfy absolute redistribution and relative redistribution, consider $H' = (S', \mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$S' = \{s_1, s_2, \dots, s^*, \dots, s_{1000}\}$$

$$\mathcal{P}'(S') = \left\{ \underbrace{\{s_1, s_2, s_3, s_4, s_5, \mathbf{s}^*\}}_{P_1}, \underbrace{\{\mathbf{s}_6\}}_{P_2}, \underbrace{\{\mathbf{s}_7\}}_{P_3}, \underbrace{\{\mathbf{s}_8\}}_{P_4}, \underbrace{\{s_9, \dots, \mathbf{s}_{20}\}}_{P_5}, \dots, \underbrace{\{\dots, \mathbf{s}_{1000}\}}_{P_{132}} \right\}$$

It is easy to verify that $\alpha(s_1, H') = \alpha(s_2, H') = \alpha(s_3, H') = 0$, $\alpha(s_4, H') = \frac{1}{3}R$, $\alpha(s_5, H') = \frac{2}{3}R$, and $\alpha(s^*, H') = R$.

Since, $\alpha(s_1, H) - \alpha(s_1, H') \neq \alpha(s_3, H) - \alpha(s_3, H')$ then the PPLNS fails to satisfy the absolute redistribution. Also as $\frac{\alpha(s_4, H')}{\alpha(s_4, H)} \neq \frac{\alpha(s_5, H')}{\alpha(s_5, H)}$ the PPLNS fails to satisfy the relative redistribution.

Besides the two popular schemes analyzed above, the academic literature has suggested additional and more sophisticated schemes designed to provide miners with improved incentives. For instance, [Rosenfeld \(2011\)](#) suggests the to use a so-called geometric scheme.

• **Geometric:** In this scheme, the rewards are distributed among the shares in a round using a geometric series based on the order of their submission. Unlike other schemes the fees are variable in this model and they depend on the size of the round. Formally, let $r > 1$. The fee for a round is defined as $f(P(s)) = \frac{1}{r^{|P(s)|}}$ and the reward to each share is

$$\alpha(s, H) = \frac{(r-1)}{r^{|P(s)|-\rho(s)+1}}B.$$

It is straight forward to see that the geometric scheme satisfies relative redistribution, round based rewards and ordinality. The following proposition shows it also satisfies budget limit.

Proposition 3. *The geometric scheme satisfies budget limit.*

Proof. Let $H = (S, \mathcal{P}(S))$ be any history. Take any round r . Let $|P_r| = k$. Without loss of generality, assume $P_r = \{1, 2, \dots, k\}$ so that $\rho(s) = s$. Then

$$\begin{aligned} \sum_{s=1}^k \alpha(s, H) &= \sum_{s=1}^k \frac{(r-1)}{r^{k-s+1}}B = B(r-1) \sum_{s=1}^k \frac{1}{r^s} \\ &= B(r-1) \frac{1}{r} \left(\frac{1 - \frac{1}{r^k}}{1 - \frac{1}{r}} \right) = B(r-1) \left(\frac{1 - \frac{1}{r^k}}{r-1} \right) = B \left(1 - \frac{1}{r^k} \right) \end{aligned} \quad (5)$$

As $r > 1$ then Equation 5 is always less than B . ■

The following example shows the geometric scheme fails to satisfy the fixed total reward and absolute redistribution.

Example 5. *Consider a history $H = (S, \mathcal{P}(S))$ as follows,*

$$\begin{aligned} S &= \{s_1, s_2, \dots, s_{1000}\} \\ \mathcal{P}(S) &= \left\{ \underbrace{\{s_1, s_2\}}_{P_1}, \underbrace{\{s_3\}}_{P_2}, \underbrace{\{s_4, \dots, s_{20}\}}_{P_3}, \dots, \underbrace{\{\dots, s_{1000}\}}_{P_{132}} \right\} \end{aligned}$$

It is easy to verify that $\alpha(s_1, H) = \frac{r-1}{r^2}B$, $\alpha(s_2, H) = \frac{r-1}{r}B$, and $\alpha(s_3, H) = \frac{r-1}{r}B$.

As $\sum_{s \in P_1} \alpha(s, H) \neq \sum_{s \in P_2} \alpha(s, H)$ then geometric scheme fails to satisfy the fixed total rewards.

To show that geometric fails to satisfy absolute redistribution consider $H' = (S', \mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$S' = \{s_1, s_2, \dots, s^*, \dots, s_{1000}\}$$

$$\mathcal{P}'(S') = \left\{ \underbrace{\{s_1, s_2, s^*\}}_{P_1}, \underbrace{\{s_3\}}_{P_2}, \underbrace{\{s_4, \dots, s_{20}\}}_{P_3}, \dots, \underbrace{\{\dots, s_{1000}\}}_{P_{132}} \right\}$$

It is easy to verify that $\alpha(s_1, H') = \frac{r-1}{r^3}B$, $\alpha(s_2, H') = \frac{r-1}{r^2}B$, $\alpha(s^*, H') = \frac{r-1}{r}B$. Since $r > 1$ and $\alpha(s_1, H) - \alpha(s_1, H') \neq \alpha(s_2, H) - \alpha(s_2, H')$ then the geometric fails to satisfy the absolute redistribution.

Note that the Geometric scheme fails to satisfy the fixed total reward axiom. We propose the following modification of this scheme to fix this failure.

• **Constrained Geometric:** This scheme is defined as $\alpha(s, H) = \frac{(r-1)}{r^{|P(s)|-\rho(s)+1}} \times \frac{r^{|P(s)|}}{r^{|P(s)|-1}}$. The constrained geometric scheme is included in the class of relative fairness reward sharing schemes (with $\epsilon_1 = 1$ and $\epsilon_j = \frac{r^{j-1}-1}{r^j-1}$). Therefore, it satisfies budget limit, total fix rewards, ordinality and relative redistribution. To show that it fails absolute re-distribution consider the history H and its extension H' as presented in Example 5. It can be verified that $\alpha(s_1, H) = \frac{r-1}{r^2} \times \frac{r^2}{r^2-1}B$, $\alpha(s_2, H) = \frac{r-1}{r} \times \frac{r^2}{r^2-1}B$, and $\alpha(s_1, H') = \frac{r-1}{r^3} \times \frac{r^3}{r^3-1}B$, $\alpha(s_2, H') = \frac{r-1}{r^2} \times \frac{r^3}{r^3-1}B$, $\alpha(s^*, H') = \frac{r-1}{r} \times \frac{r^3}{r^3-1}B$. Since $r > 1$ and $\alpha(s_1, H) - \alpha(s_1, H') \neq \alpha(s_2, H) - \alpha(s_2, H')$ then the constrained geometric fails to satisfy the absolute redistribution.

Next, we formulate another scheme which approaches the reward sharing problem from the “incentive compatibility” perspective, i.e., it provides miners with the incentive to report shares immediately.

• **IC scheme:** The IC scheme is proposed by Schrijvers et al. (2016) as an *incentive compatible* reward sharing scheme. In words, let $1/D$ denote the probability of a share to be a full solution. Then, if the round length is at least of the same size as D , the scheme distributes the reward proportionally according to the length of the round; if the round length is shorter, every share receives $1/D$ and the residual budget is given to the last (full) share of the round. Formally,

$$\alpha(s, H) = \begin{cases} \frac{R}{D} & \text{if } |P(s)| \leq D \text{ and } \rho(s) < |P(s)| \\ \frac{R}{D} + (1 - \frac{|P(s)|}{D})R, & \text{if } |P(s)| \leq D \text{ and } \rho(s) = |P(s)| \\ \frac{R}{|P(s)|}, & \text{if } |P(s)| \geq D \end{cases}$$

This rule obviously satisfies fixed total rewards, ordinality, and round based rewards, while it fails both absolute and relative redistribution.

We conclude this section by formulating with one more reward sharing scheme, which is interesting. The Slush scheme, named after the Slush mining pool²¹, is the only reward sharing scheme that uses time signatures as a parameter for distributing the rewards. Our framework is rich enough to capture this feature. Formally:

• **Slush:** Consider a history $H = (S, \mathcal{P}(S))$ and let $\Omega(s)$ denote the index of the pool round that share s belongs to i.e., $\Omega(s) = \{x \leq |\mathcal{P}(S)| \mid s \in P_x\}$. Let \bar{s}_j denote the last share in the j^{th} round, i.e., $\bar{s}_j = \{s \in P_j \mid \tau(s) \geq \tau(s'), \forall s' \in P_j\}$. Then

$$\text{Let } score(s, j) = \frac{e^{\frac{\tau(s) - \tau(\bar{s}_j)}{\lambda}}}{\sum_{\tau(s') \leq \tau(\bar{s}_j)} e^{\frac{\tau(s') - \tau(\bar{s}_j)}{\lambda}}} \text{ for any } \Omega(s) \leq j \leq l. \text{ Then}$$

$$\alpha(s, H) = R \sum_{i=\Omega(s)}^l score(s, i).$$

The parameter λ is set to 1200 in the Slush pool. In what follows, we therefore assume $\lambda = 1200$.

In the following example we show that the slush scheme does not satisfy fixed total reward, ordinality, budget limit, round based reward, absolute redistribution and relative redistribution.

Example 6. Consider a history $H = (S, \mathcal{P}(S))$ as follows,

$$S = \{s_1, s_2, s_3\}$$

$$\mathcal{P}(S) = \left\{ \underbrace{\{s_1, s_2\}}_{P_1}, \underbrace{\{s_3\}}_{P_2} \right\}$$

²¹<https://slushpool.com/>

Let $\tau(s_1) = 1, \tau(s_2) = 2$ and $\tau(s_3) = 3$. Therefore, we have:

$$\begin{aligned} \text{score}(s_1, 1) &= \frac{e^{\frac{1-2}{1200}}}{e^{\frac{1-2}{1200}} + e^{\frac{2-2}{1200}}} \approx 0.49 \\ \text{score}(s_1, 2) &= \frac{e^{\frac{1-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.33 \\ \text{score}(s_2, 1) &= \frac{e^{\frac{2-2}{1200}}}{e^{\frac{1-2}{1200}} + e^{\frac{2-2}{1200}}} \approx 0.5 \\ \text{score}(s_2, 2) &= \frac{e^{\frac{2-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.33 \\ \text{score}(s_3, 2) &= \frac{e^{\frac{3-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.33 \end{aligned}$$

Therefore, $\alpha(s_1, H) = 0.82R$, $\alpha(s_2, H) = .83R$ and $\alpha(s_3, H) = .33R$.

As $\sum_{s \in P_1} \alpha(s, H) = 1.65R$, the Slush scheme violates the budget limit. As $\sum_{s \in P_1} \alpha(s, H) \neq \sum_{s \in P_2} \alpha(s, H)$, it fails to satisfy the fixed total rewards. It is easy to see that the Slush scheme also fails the round based reward axiom, e.g., for the second round in this example. In addition, one can verify that changing the time signature of any of the shares will have an effect on the award of shares, therefore the Slush scheme also violates ordinality.

Now consider $H' = (S', \mathcal{P}'(S'))$ as an extension of H at the first round as follows:

$$\begin{aligned} S' &= \{s_1, s_2, s^*, s_3\} \\ \mathcal{P}'(S') &= \left\{ \underbrace{\{s_1, s_2, s^*\}}_{P_1}, \underbrace{\{s_3\}}_{P_2} \right\} \end{aligned}$$

such that $\tau(s^*) = 2.5$ Then the award of each share would be:

$$\text{score}(s_1, 1) = \frac{e^{\frac{1-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$

$$\text{score}(s_1, 2) = \frac{e^{\frac{1-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$

$$\text{score}(s_2, 1) = \frac{e^{\frac{2-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$

$$\text{score}(s_2, 2) = \frac{e^{\frac{2-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$

$$\text{score}(s^*, 1) = \frac{e^{\frac{2.5-2.5}{1200}}}{e^{\frac{1-2.5}{1200}} + e^{\frac{2-2.5}{1200}} + e^{\frac{2.5-2.5}{1200}}} \approx 0.33$$

$$\text{score}(s^*, 2) = \frac{e^{\frac{2.5-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$

$$\text{score}(s_3, 2) = \frac{e^{\frac{3-3}{1200}}}{e^{\frac{1-3}{1200}} + e^{\frac{2-3}{1200}} + e^{\frac{2.5-3}{1200}} + e^{\frac{3-3}{1200}}} \approx 0.25$$

Therefore, $\alpha(s_1, H') = 0.58R$ and $\alpha(s_2, H') = .58R$. Comparing these to $\alpha(s_1, H) = 0.82R$ and $\alpha(s_2, H) = .83R$, shows that the Slush scheme fails both absolute and fair redistribution axioms.

The results for the well-known schemes are summarized in Table 1 below.

Table 1: Summary of the well-known schemes.

Scheme	Fixed total reward	Relative re-distribution	Absolute re-distribution	Round based rewards	Budget limit	Ordinality
PPS	-	+	+	-	-	+
PPLNS	-	-	-	-	-	+
Geometric	-	+	-	+	+	+
Modified Geometric	+	+	-	+	+	+
IC	+	-	-	+	+	+
Slush	-	-	-	-	-	-

5.2 Logical Independence

We define six reward sharing schemes in order to demonstrate logical independence of the axioms in Section 3. The results are summarized in Table 2 below. Defining schemes 1-6, let $R = B - f$ for some fixed $f \in [0, B]$.

- **Scheme 1:**

$$\alpha(s, H) = \begin{cases} \frac{R}{|P(s)|}, & \text{if } |P(s)| \text{ is odd} \\ \frac{R}{2|P(s)|}, & \text{if } |P(s)| \text{ is even} \end{cases}$$

This scheme fails *fixed total rewards*, but meets all the other axioms.

- **Scheme 2:**

$$\alpha(s, H) = \begin{cases} R, & \text{if } |P(s)| = 1 \\ (R - \lambda) + \frac{\lambda}{|P(s)|}, & \text{if } |P(s)| > 1 \text{ and } \rho(s) = 1 \\ \frac{\lambda}{|P(s)|}, & \text{if } |P(s)| > 1 \text{ and } \rho(s) \neq 1 \end{cases}$$

where $0 < \lambda < R$ is a constant number.

This scheme fails *relative redistribution*, but meets all the other axioms.

- **Scheme 3:**

$$\alpha(s, H) = \begin{cases} \frac{R}{2^{|P(s)|-1}}, & \text{if } \rho(s) = 1 \\ \frac{2^{\rho(s)} - 2R}{2^{|P(s)|-1}}, & \text{if } \rho(s) \neq 1 \end{cases}$$

This scheme fails *absolute redistribution*, but meets all the other axioms.

- **Scheme 4:**

$$\alpha(s, H) = \begin{cases} \frac{R}{|P(s)|}, & \text{if the number of shares in the first round of the history is odd} \\ \frac{R}{2|P(s)|}, & \text{if the number of shares in the first round of the history is even} \end{cases}$$

This scheme fails *round based rewards*, but meets all the other axioms.

- **Scheme 5:**

$$\alpha(s, H) = \frac{2R}{|P(s)|}$$

This scheme fails *budget limit*, but meets all the other axioms.

- **Scheme 6:**

$$\alpha(s, H) = \begin{cases} R, & \text{if } |P(s)| = 1 \\ \alpha^{2, \frac{R}{2}}(s, H), & \text{if } \tau(\rho(s) = 1) - \tau(\rho(s) = 2) < T \\ \alpha^{2, \frac{R}{3}}(s, H), & \text{if } \tau(\rho(s) = 1) - \tau(\rho(s) = 2) \geq T \end{cases}$$

where T is a time threshold.

This scheme fails *ordinality*, but meets all the other axioms.

Table 2: Logical independence of the axioms.

Scheme	Fixed total reward	Relative re- distribution	Absolute re- distribution	Round based rewards	Budget limit	Ordinality
Scheme 1	-	+	+	+	+	+
Scheme 2	+	-	+	+	+	+
Scheme 3	+	+	-	+	+	+
Scheme 4	+	+	+	-	+	+
Scheme 5	+	+	+	+	-	+
Scheme 6	+	+	+	+	+	-

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Appendix

A Proof of Lemma 1

Lemma 1. If a scheme α satisfies fixed total reward and round based rewards, then for any two histories $H = (S, \mathcal{P}(S))$ and $H' = (S', \mathcal{P}'(S'))$ and any two rounds $P \in \mathcal{P}(S)$ and $P' \in \mathcal{P}'(S')$ we have

$$\sum_{s \in P} \alpha(s, H) = \sum_{s \in P'} \alpha(s, H').$$

Proof. Take any two histories H, H' and P, P' as in the lemma. Let r (and r') denote the round number of P (and P') in history H (and H'). As α is fixed total reward, there exist fixed rewards for both histories, say K and K' :

$$\sum_{s \in P} \alpha(s, H) = K \text{ and } \sum_{s \in P'} \alpha(s, H') = K' \quad (6)$$

Consider the restriction of H to r^{th} round (and of H' to r'^{th} round). As α satisfies round based rewards, for all $s \in P$ (and for all $s' \in P'$) we have: $\alpha(s, H) = \alpha(s, H|_r)$ and $\alpha(s', H') = \alpha(s', H'|_{r'})$. This implies:

$$\sum_{s \in P} \alpha(s, H|_r) = K \text{ and } \sum_{s \in P'} \alpha(s, H'|_{r'}) = K' \quad (7)$$

Next we consider two cases. We say two rounds in different histories *overlap* when there are some shares in the these two rounds which have overlapping time signatures. Formally, we say P and P' are overlapping whenever: $\min_{s \in P'} \tau(s) \leq \min_{s \in P} \tau(s) \leq \max_{s \in P'} \tau(s)$ or $\min_{s \in P'} \tau(s) \leq \max_{s \in P'} \tau(s) \leq \max_{s \in P} \tau(s)$. Based on whether these rounds P and P' in histories H and H' overlap or not, we shall separately prove the equality of rewards, $K = K'$, in Equation 6.

Case 1. No overlap. Without loss of generality, assume all the shares in P have an earlier time signature than those in P' . In this case, consider a history with two rounds $H'' = (P \cup P', \{P, P'\})$. Note that H'' is a history consisting only of two rounds, P as the first and P' as the second round. Note also that $\max_{s \in P} \tau(s) < \min_{s \in P'} \tau(s)$.

As α is fixed total reward, there exist a fixed reward for this history, say K'' :

$$\sum_{s \in P} \alpha(s, H'') = \sum_{s \in P'} \alpha(s, H'') = K'' \quad (8)$$

Consider the restriction of H'' to the 1st and the 2nd rounds. As α satisfies round based rewards for all $s \in P$ we have $\alpha(s, H'') = \alpha(s, H''|_1)$, and for all $s \in P'$ we have $\alpha(s, H'') = \alpha(s, H''|_2)$. This implies:

$$\sum_{s \in P} \alpha(s, H''|_1) = K'' \text{ and } \sum_{s \in P'} \alpha(s, H''|_2) = K''. \quad (9)$$

Note that the restriction of H'' to the 1st round is equivalent to the restriction of H to the r^{th} round. Similarly the restriction of H'' to the 2nd round is equivalent to the restriction of H' to the r^{th} round. That is, $H''|_1 = (P, \{P\}) = H|_r$ and $H''|_2 = (P', \{P'\}) = H'|_{r'}$, therefore we have:

$$\sum_{s \in P} \alpha(s, H''|_1) = \sum_{s \in P} \alpha(s, H|_r) \text{ and } \sum_{s \in P'} \alpha(s, H''|_2) = \sum_{s \in P'} \alpha(s, H'|_{r'})$$

The former equality implies $K'' = K$ and the latter implies $K'' = K'$. This completes the proof for Case 1.

Case 2. Overlap. In this case we consider two additional histories with two rounds each: $\overline{H} = (P \cup \overline{P}, \{P, \overline{P}\})$ and $\overline{H}' = (P' \cup \overline{P}', \{P', \overline{P}'\})$ such that

$$\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}} \tau(s) < \max_{s \in \overline{P}'} \tau(s) < \min_{s \in \overline{P}'} \tau(s).$$

Note that \overline{H} is a history consisting only of two rounds, P as the first and \overline{P} as the second round. Similarly, \overline{H}' is a history consisting only of two rounds, P' as the first and \overline{P}' as the second round.

Since $\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}} \tau(s)$ there is no overlaps between P and \overline{P} , therefore applying Case 1 on H and \overline{H} yields:

$$i) \sum_{s \in P} \alpha(s, H) = \sum_{s \in \overline{P}} \alpha(s, \overline{H}) \quad (10)$$

Since $\max_{s \in P \cup P'} \tau(s) < \min_{s \in \overline{P}'} \tau(s)$ there is no overlaps between P' and \overline{P}' , therefore applying Case 1 on H' and \overline{H}' yields:

$$ii) \sum_{s \in P'} \alpha(s, H') = \sum_{s \in \overline{P}'} \alpha(s, \overline{H}'). \quad (11)$$

Since $\max_{s \in \overline{P}} \tau(s) < \min_{s \in \overline{P}'} \tau(s)$ there is no overlaps between \overline{P} and \overline{P}' , therefore applying Case

1 on \bar{H} and \bar{H}' yields:

$$\sum_{s \in \bar{P}} \alpha(s, \bar{H}) = \sum_{s \in \bar{P}'} \alpha(s, \bar{H}').$$

which -combined with Equations 10 and 11- completes the proof. \blacksquare

B Proof of Lemma 2

Lemma 2. If a scheme α satisfies fixed total reward, round based rewards and ordinality, then for any two histories $H = (S, \mathcal{P}(S))$ and $\bar{H} = (\bar{S}, \bar{\mathcal{P}}(\bar{S}))$ and any two rounds $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ such that $|P| = |\bar{P}|$, we have for all $s \in P$ and for all $\bar{s} \in \bar{P}$ such that $\rho(s) = \bar{\rho}(\bar{s})$:

$$\alpha(s, H) = \alpha(\bar{s}, \bar{H}).$$

Proof. Take any scheme α that satisfies these two axioms. Let $H = (S, \mathcal{P}(S))$ and $\bar{H} = (\bar{S}, \bar{\mathcal{P}}(\bar{S}))$ be two histories and let $P \in \mathcal{P}(S)$ and $\bar{P} \in \bar{\mathcal{P}}(\bar{S})$ be any two rounds with $|P| = |\bar{P}| = k$ for some k . Let $P = P_r$ and $\bar{P} = \bar{P}_{\bar{r}}$, i.e., P is the r^{th} round in H and \bar{P} is the \bar{r}^{th} round in \bar{H} . As α satisfies round based rewards we have i) $\alpha(s, H) = \alpha(s, H|_r)$ for all $s \in S$ and ii) $\alpha(\bar{s}, \bar{H}) = \alpha(\bar{s}, \bar{H}|_{\bar{r}})$ for all $\bar{s} \in \bar{S}$. Let $H|_r = (\{s_1, \dots, s_k\}, \{\{s_1, \dots, s_k\}\})$ and $\bar{H}|_{\bar{r}} = (\{\bar{s}_1, \dots, \bar{s}_k\}, \{\{\bar{s}_1, \dots, \bar{s}_k\}\})$ denote these histories. In what follows we show for all $i \leq k$,

$$\alpha(s_i, H) = \alpha(\bar{s}_i, \bar{H})$$

Consider a time-shift \hat{H}^1 of H at s_1 (first share) with $\hat{S}^1 = (S \setminus \{s_1\}) \cup \{\hat{s}_1\}$ such that $\tau(\hat{s}_1) = \min\{\tau(s_1), \tau(\bar{s}_1)\}$. Similarly consider a time-shift $\hat{\bar{H}}^1$ of \bar{H} at \bar{s}_1 (first share) with $\hat{\bar{S}}^1 = (\bar{S} \setminus \{\bar{s}_1\}) \cup \{\hat{\bar{s}}_1\}$ such that $\tau(\hat{\bar{s}}_1) = \min\{\tau(s_1), \tau(\bar{s}_1)\}$. Then by ordinality of α , we have for all $s \in S \setminus \{s_1\}$ and for all $\bar{s} \in \bar{S} \setminus \{\bar{s}_1\}$:

$$\begin{aligned} \alpha(s, H|_r) &= \alpha(\hat{s}, \hat{H}^1) \\ \alpha(\bar{s}, \bar{H}|_{\bar{r}}) &= \alpha(\hat{\bar{s}}, \hat{\bar{H}}^1) \end{aligned}$$

Note that by Remark 1, the total reward is fixed both in P and in \bar{P} , therefore we conclude that for all $s \in S$ and for all $\bar{s} \in \bar{S}$:

$$\begin{aligned} \alpha(s, H|_r) &= \alpha(\hat{s}, \hat{H}^1) \\ \alpha(\bar{s}, \bar{H}|_{\bar{r}}) &= \alpha(\hat{\bar{s}}, \hat{\bar{H}}^1) \end{aligned}$$

Continuing iteratively and letting \hat{H}^i be the time-shift of H at the i^{th} share and $\hat{\bar{H}}^i$ as the time-shift of \bar{H} at the i^{th} share, the same argument above yields for all $i \leq k$:

$$\begin{aligned}\alpha(s, H|_r) &= \alpha(\hat{s}, \hat{H}^1) = \alpha(\hat{s}, \hat{H}^2) = \dots = \alpha(\hat{s}, \hat{H}^k) \\ \alpha(\bar{s}, \bar{H}|\bar{r}) &= \alpha(\hat{\bar{s}}, \hat{\bar{H}}^1) = \alpha(\hat{\bar{s}}, \hat{\bar{H}}^2) = \dots = \alpha(\hat{\bar{s}}, \hat{\bar{H}}^k)\end{aligned}$$

Note that by construction, the time signature of all shares at \hat{H}^k are the same with those at $\hat{\bar{H}}^k$, i.e., $\hat{H}^k = \hat{\bar{H}}^k$. Therefore, $\alpha(\hat{s}, \hat{H}^k) = \alpha(\hat{\bar{s}}, \hat{\bar{H}}^k)$. This -together with the two equations above- implies $\alpha(s, H|_r) = \alpha(\bar{s}, \bar{H}|\bar{r})$. \blacksquare

C Proof of Theorem 1

Theorem 1. A reward sharing scheme α satisfies round based rewards, budget limit, fixed total reward, ordinality and absolute redistribution if and only if it is absolute fair in the sense of 2.

Proof. If part.

Round based rewards. Let $H = (S, \mathcal{P}(S))$ be a history, and let r be any round. Note that, by restricting a round to the r^{th} round the relative rank of a share s in the round, as well as its time signature and value are the same at both H and $H|_r$. Therefore, $\alpha^{**}(s, H) = \alpha^{**}(s, H|_r)$.

Budget limit. Let $H = (S, \mathcal{P}(S))$ be any history. Take any round r . Let $|P_r| = k$. As the relative fair scheme is based on $\rho(s)$, without loss of generality, we assume $P_r = \{1, 2, \dots, k\}$ so that $\rho(s) = s$. Note that $\sum_{s=1}^k \alpha^{**}(s, H) = R \sum_{s=1}^k \left(\varepsilon(s) - \sum_{i=s+1}^k \frac{\varepsilon(i)}{i-1} \right)$. Therefore,

$$\begin{aligned}\frac{1}{R} \sum_{s=1}^k \alpha^{**}(s, H) &= \sum_{s=1}^k \left(\varepsilon(s) - \sum_{i=s+1}^k \frac{\varepsilon(i)}{i-1} \right) \\ &= \sum_{s=1}^{k-1} \left(\varepsilon(s) - \sum_{i=s+1}^k \frac{\varepsilon(i)}{i-1} \right) + \varepsilon(k) \\ &= \sum_{s=1}^{k-1} \varepsilon(s) - \sum_{s=1}^{k-1} \left(\sum_{i=s+1}^k \frac{\varepsilon(i)}{i-1} \right) + \varepsilon(k) \\ &= \sum_{s=1}^{k-1} \varepsilon(s) - \sum_{s=1}^{k-1} \left(\sum_{i=s+1}^{k-1} \frac{\varepsilon(i)}{i-1} + \frac{\varepsilon(k)}{k-1} \right) + \varepsilon(k)\end{aligned}$$

$$\begin{aligned}
&= \sum_{s=1}^{k-1} \varepsilon(s) - \sum_{s=1}^{k-1} \sum_{i=s+1}^{k-1} \frac{\varepsilon(i)}{i-1} - \sum_{s=1}^{k-1} \frac{\varepsilon(k)}{k-1} + \varepsilon(k) \\
&= \sum_{s=1}^{k-1} \varepsilon(s) - \sum_{s=1}^{k-1} \sum_{i=s+1}^{k-1} \frac{\varepsilon(i)}{i-1} - (k-1) \frac{\varepsilon(k)}{k-1} + \varepsilon(k) \\
&= \sum_{s=1}^{k-1} \varepsilon(s) - \sum_{s=1}^{k-1} \sum_{i=s+1}^{k-1} \frac{\varepsilon(i)}{i-1} \\
&\dots\dots\dots \\
&= \sum_{s=1}^2 \varepsilon(s) - \sum_{s=1}^2 \sum_{i=s+1}^2 \frac{\varepsilon(i)}{i-1} \\
&= (\varepsilon(1) + \varepsilon(2)) - (\varepsilon(2)) = \varepsilon(1)
\end{aligned}$$

By definition of the ε we have $0 \leq \varepsilon(1) \leq 1$, hence $\sum_{s=1}^k \alpha^{**}(s, H) \leq R$ the absolute redistribution scheme satisfies the budget limit.

Fixed total rewards. It follows from a similar proof of the above.

Absolute redistribution. Let $H = (S, \mathcal{P}(S))$ be any history and $P_r = \{s_1, s_2, \dots, s_k\}$ be any round. Let $H' = (S', \mathcal{P}'(S'))$ be any extension of H at the r^{th} round. Take any $s_a \in P_r$. Note that $|P'(s_a)| = |P(s_a)| + 1$.

$$\begin{aligned}
\alpha^{**}(s_a, H) - \alpha^{**}(s_a, H') &= R \left(\varepsilon(\rho(s_a)) - \sum_{i=\rho(s_a)+1}^{|P(s_a)|} \frac{\varepsilon(i)}{i-1} \right) - R \left(\varepsilon(\rho(s_a)) - \sum_{i=\rho(s_a)+1}^{|P'(s_a)|} \frac{\varepsilon(i)}{i-1} \right) \\
&= -R \left(\sum_{i=\rho(s_a)+1}^{|P(s_a)|} \frac{\varepsilon(i)}{i-1} \right) + R \left(\sum_{i=\rho(s_a)+1}^{|P(s_a)|+1} \frac{\varepsilon(i)}{i-1} \right) \\
&= R \frac{\varepsilon(|P(s_a)| + 1)}{|P(s_a)|} = R \frac{\varepsilon(k+1)}{k}
\end{aligned}$$

Only if part. Take any reward sharing scheme α that satisfies the axioms. Take any history $H = (S, \mathcal{P}(S))$ and any $s \in S$. Let $s \in P_r$ for some $1 \leq r \leq l = |\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s, H) = \alpha(s, H|_r)$. Similarly for α^{**} , we have $\alpha^{**}(s, H) = \alpha^{**}(s, H|_r)$. Hence, it suffices to prove $\alpha(s, H|_r) = \alpha^{**}(s, H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r = \{s_1, s_2, \dots, s_n\}$ and the restricted histories as $H^n = H|_r$. Therefore, it suffices to prove $\alpha(s_i, H^n) = \alpha^{**}(s_i, H^n)$ for all n and for all $i \leq n$.

By absolute redistribution, for any $j > 1$, we can let δ_j denote the absolute decrease in the rewards of all shares whilst moving from history H^{j-1} to H^j , i.e., $\alpha(s, H^{j-1}) - \alpha(s, H^j)$. In addition let $\delta_1 = \alpha(s_1, H^1)$. As α is well-defined, for all $n \geq 1$, $\alpha(s_1, H^n) \geq 0$. Therefore, $\delta_1 \geq \sum_{i=2}^{\infty} \delta_i$ (otherwise at some history, s_1 would get a negative reward). Similarly, as α is well-defined, for all $n > 1$, and for all $p \leq n$, $\alpha(s_p, H^n) \geq 0$. Note that by construction and Remark 1, $\alpha(s_p, H^p) = (p-1)\delta_p$. Therefore, $(p-1)\delta_p \geq \sum_{i=p+1}^{\infty} \delta_i$ (otherwise at some history, s_p would get a negative reward). Now, consider a function ε such that $\varepsilon(1) = 1$ and for all $j > 1$,

$$\varepsilon(j) = \frac{(j-1)\delta_j}{R} \quad (12)$$

Note that we have the following two properties for ε :

1. for all $j \geq 1$, $\varepsilon(j) \rightarrow [0, 1]$. By definition δ_j denotes the absolute decrease in the rewards of all shares whilst moving from history H^{j-1} to H^j . Therefore, the last share at H^j gets $(j-1)\delta_j$ (as there are $j-1$ that each payouts δ_j). As the scheme satisfies total fixed rewards and round based rewards then by Remark 1 the total reward at round H^j is R . Therefore, it must be the case that $(j-1)\delta_j \leq R$ so $\delta_j \leq \frac{R}{j-1}$. Therefore, $\varepsilon(j) = \frac{(j-1)\delta_j}{R} \leq \frac{j-1}{R} \times \frac{R}{j-1}$. As $\delta_j \geq 0$ then $\varepsilon(j) \geq 0$. All in all, $\varepsilon(j) \rightarrow [0, 1]$ for all $j \geq 1$.

2. for all $j \geq 1$, $\varepsilon(j) \geq \sum_{i=j+1}^{\infty} \frac{\varepsilon(i)}{i-1}$.

For $j = 1$ we have $\varepsilon(j) = 1$ we have $1 \geq \sum_{i=2}^{\infty} \frac{\varepsilon(i)}{i-1}$. Replacing from Equation 12 we have

$$1 \geq \sum_{i=2}^{\infty} \frac{1}{i-1} \frac{(i-1)\delta_i}{R} \text{ which implies } R \geq \sum_{i=2}^{\infty} \delta_i \dots$$

For $\varepsilon(j)$ we have $\varepsilon(j) = \frac{(j-1)\delta_j}{R} \geq \sum_{i=j+1}^{\infty} \frac{\varepsilon(i)}{i-1}$. Replacing from Equation 12 we have

$$(j-1)\delta_j \geq \sum_{i=j+1}^{\infty} \delta_i.$$

In what follows we shall show that $\alpha(s, H^n) = \alpha^{**}(s, H^n)$ for the aforementioned ε function. We do it first for a single share rounds and then for multi-share rounds.

Single-share round: Let H^1 be any history with a single round with a single share s . By Remark 1, $\alpha(s, H^1) = R$. Setting $\varepsilon(1) = 1$ we have:

$$\alpha(s, H^1) = \alpha^{**}(s, H^1) = R\varepsilon(1) \quad (13)$$

Multi-share round: Let H^n be any history with a single round with multiple shares s_1, \dots, s_n . In what follows, we show that $\alpha(s, H^n) = \alpha^{**}(s, H^n)$ any $s \in P_r$, by induction on n , i.e., the size of the round P_r .

Induction Basis: Let $n = 2$. Let $H^2 = (\{s_1, s_2\}, \{\{s_1, s_2\}\})$. We will show that $\alpha(s_1, H^2) = \alpha^{**}(s_1, H^2)$ and $\alpha(s_2, H^2) = \alpha^{**}(s_2, H^2)$. By Remark 1 for two histories H^1 and H^2 , and by Equation 13, we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = R$. So, $\alpha(s_1, H^2) = R - \delta_2$ and $\alpha(s_2, H^2) = \delta_2$. Setting $\varepsilon(2) = \frac{\delta_2}{R}$, yields $\alpha(s_1, H^2) = R\varepsilon(1) - R\varepsilon(2) = \alpha^{**}(s_1, H^2)$ and $\alpha(s_2, H^2) = R\varepsilon(2) = \alpha^{**}(s_2, H^2)$.

Induction Hypothesis: Let $n = k$ with $k > 1$. Suppose we have $\alpha(s_i, H^k) = \alpha^{**}(s_i, H^k)$ for all $i \leq k$.

To prove for $n = k + 1$, consider any $H^{k+1} = (\{s_1, \dots, s_k, s_{k+1}\}, \{\{s_1, \dots, s_k, s_{k+1}\}\})$. We will show that $\alpha(s_i, H^{k+1}) = \alpha^{**}(s_i, H^{k+1})$ for all $i \leq k + 1$. Let $H^k = (\{s_1, \dots, s_k\}, \{\{s_1, \dots, s_k\}\})$.

By construction, $\alpha(s_i, H^{k+1}) = \alpha(s_i, H^k) - \delta_{k+1}$ for all $i \leq k$. By induction hypothesis $\alpha(s_i, H^k) = \alpha^{**}(s_i, H^k)$ for all $i \leq k$ which implies $\alpha(s_i, H^{k+1}) = \alpha^{**}(s_i, H^k) - \delta_{k+1}$. Therefore,

$$\alpha(s_i, H^{k+1}) = R \left(\varepsilon(i) - \sum_{j=i+1}^k \frac{\varepsilon(j)}{j-1} \right) - \delta_{k+1} \text{ for all } i \leq k \quad (14)$$

As $\delta_{k+1} = \frac{\varepsilon(k+1)R}{k}$, the above equation is simplified as:

$$\alpha(s_i, H^{k+1}) = R \left(\varepsilon(i) - \sum_{j=i+1}^{k+1} \frac{\varepsilon(j)}{j-1} \right) \text{ for all } i \leq k \quad (15)$$

Note that $\alpha(s_i, H^{k+1}) = \alpha(s_i, H^k) - \delta_{k+1}$, and by Lemma 1 (on H^k and on H^1), we have

$$\sum_{i=1}^k \alpha(s_i, H^{k+1}) = \sum_{i=1}^k \alpha(s_i, H^k) - k\delta_{k+1} = R - k\delta_{k+1} \quad (16)$$

Note also that $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = \sum_{i=1}^k \alpha(s_i, H^{k+1}) + \alpha(s_{k+1}, H^{k+1})$. Plugging Equation 16 into this yields $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R - k\delta_{k+1} + \alpha(s_{k+1}, H^{k+1})$. By Remark 1, $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R$, therefore we have:

$$\alpha(s_{k+1}, H^{k+1}) = k\delta_{k+1} = R\varepsilon(k+1). \quad (17)$$

Equations 15 and 17 together prove $\alpha(s_i, H^{k+1}) = \alpha^{**}(s_i, H^{k+1})$ for $i \in \{1, \dots, k\}$ and for $i = k + 1$, respectively. ■

D Proof of Theorem 2

Theorem 2. A reward scheme α satisfies round based rewards, budget limit, fixed total reward, ordinality and relative redistribution if and only if it is relative fair in the sense of 3.

Proof. If part.

Round based rewards. To show that the relative fair scheme satisfies $\alpha^*(s, H) = \alpha^*(s, H|_r)$, let $H = (S, \mathcal{P}(S))$ be a history, and let r be any round. Note that, by restricting a history to the r^{th} round, the round and the shares remain intact, hence the relative rank of a share s is the same at both H and $H|_r$. Therefore, the relative fair scheme satisfies round based rewards.

Budget limit. Let $H = (S, \mathcal{P}(S))$ be any history. Take any round r . Let $|P_r| = k$. As the relative fair scheme is based on $\rho(s)$, without loss of generality, we assume $P_r = \{1, 2, \dots, k\}$ so that $\rho(s) = s$. Therefore to show $\sum_{s \in P_r} \alpha^{**}(s, H) \leq R$, we have

$$\begin{aligned}
\sum_{s \in P_r} \alpha(s, H) &= \sum_{s=1}^k \left(R \times \varepsilon(\rho(s)) \times \prod_{i=\rho(s)+1}^k (1 - \varepsilon(i)) \right) \\
&= R \left(\varepsilon(1) \times \prod_{i=2}^k (1 - \varepsilon(i)) \right) + R \sum_{s=2}^k \left(\varepsilon(\rho(s)) \times \prod_{i=\rho(s)+1}^k (1 - \varepsilon(i)) \right) \\
&= R \left((1 - \varepsilon(2)) \left(\prod_{i=3}^k (1 - \varepsilon(i)) \right) \right) + R \sum_{s=2}^k \left(\varepsilon(s) \times \prod_{i=s+1}^k (1 - \varepsilon(i)) \right) \\
&= R \left(\prod_{i=3}^k (1 - \varepsilon(i)) - \varepsilon(2) \prod_{i=3}^k (1 - \varepsilon(i)) \right) + R \sum_{s=2}^k \left(\varepsilon(s) \times \prod_{i=s+1}^k (1 - \varepsilon(i)) \right) \\
&= R \prod_{i=3}^k (1 - \varepsilon(i)) - R \varepsilon(2) \prod_{i=3}^k (1 - \varepsilon(i)) + R \varepsilon(2) \prod_{i=3}^k (1 - \varepsilon(i)) + R \sum_{s=3}^k \left(\varepsilon(s) \times \prod_{i=s+1}^k (1 - \varepsilon(i)) \right) \\
&= R \prod_{i=3}^k (1 - \varepsilon(i)) + R \sum_{s=3}^k \left(\varepsilon(s) \times \prod_{i=s+1}^k (1 - \varepsilon(i)) \right) \\
&= \dots \\
&= R \times \left((1 - \varepsilon(k)) \right) + R \sum_{s=k}^k \left(\varepsilon(s) \times \prod_{i=s+1}^k (1 - \varepsilon(i)) \right) \\
&= R \times \left((1 - \varepsilon(k)) \right) + R \varepsilon(k) = R
\end{aligned}$$

Fixed total rewards. It follows from a similar proof of the above.

Relative redistribution. To show Relative fairness let $H = (S, \mathcal{P}(S))$ be any history and $P_r = \{s_1, s_2, \dots, s_k\}$ be any round. Let $H' = (S', \mathcal{P}'(S'))$ be any extension of H at the r^{th} round. Then for any $s_a, s_b \in P_r$ $\alpha^*(s_a, H) \neq 0$ and $\alpha^*(s_b, H) \neq 0$ we have:

$$\frac{\alpha^*(s_a, H')}{\alpha^*(s_a, H)} = \frac{R \times \varepsilon(\rho(s_a)) \times \prod_{i=\rho(s_a)+1}^{|P(s_a)|+1} (1 - \varepsilon(i))}{R \times \varepsilon(\rho(s_a)) \times \prod_{i=\rho(s_a)+1}^{|P(s_a)|} (1 - \varepsilon(i))} = 1 - \varepsilon(|P(s_a)| + 1) = 1 - \varepsilon(k + 1)$$

$$\frac{\alpha^*(s_b, H')}{\alpha^*(s_b, H)} = \frac{R \times \varepsilon(\rho(s_b)) \times \prod_{i=\rho(s_b)+1}^{|P(s_b)|+1} (1 - \varepsilon(i))}{R \times \varepsilon(\rho(s_b)) \times \prod_{i=\rho(s_b)+1}^{|P(s_b)|} (1 - \varepsilon(i))} = 1 - \varepsilon(|P(s_b)| + 1) = 1 - \varepsilon(k + 1)$$

The above two equations shows that $\frac{\alpha^*(s_a, H')}{\alpha^*(s_a, H)} = \frac{\alpha^*(s_b, H')}{\alpha^*(s_b, H)}$.

Only if part. Take any reward sharing scheme α that satisfies the axioms. Take any history $H = (S, \mathcal{P}(S))$ and any $s \in S$. Let $s \in P_r$ for some $1 \leq r \leq l = |\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s, H) = \alpha(s, H|_r)$. Similarly for α^{**} , we have $\alpha^{**}(s, H) = \alpha^{**}(s, H|_r)$. Hence, it suffices to prove $\alpha(s, H|_r) = \alpha^{**}(s, H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r = \{s_1, s_2, \dots, s_n\}$ and the restricted histories as $H^n = H|_r$. Therefore, it suffices to prove $\alpha(s_i, H^n) = \alpha^{**}(s_i, H^n)$ for all n and for all $i \leq n$.

Single-share round: Let H^1 be any history with a single round with a single share s . As α satisfies fixed total rewards and round based rewards then by Remark 1, $\alpha(s, H^1) = R$. Setting $\varepsilon(1) = 1$ we have:

$$\alpha(s, H^1) = \alpha^*(s, H^1) = R. \quad (18)$$

Multi-share round: Let H^n be any history with a single round with multiple shares s_1, \dots, s_n . In what follows, we show that $\alpha(s, H^n) = \alpha^*(s, H^n)$ any $s \in P_r$, by induction on n , i.e., the size of the round P_r .

Induction Basis: Let $n = 2$. Let $H^2 = (\{s_1, s_2\}, \{\{s_1, s_2\}\})$. We will show that $\alpha(s_1, H^2) = \alpha^*(s_1, H^2)$ and $\alpha(s_2, H^2) = \alpha^*(s_2, H^2)$. By Lemma 1 for two histories H^1 and H^2 , and by Equation 18, we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = R$. So, $\alpha(s_1, H^2) = R(1 - \delta_2)$ and

$\alpha(s_2, H^2) = R\delta_2$ for some $\delta_2 \in [0, 1]$. Setting $\varepsilon(2) = \delta_2$, yields $\alpha(s_1, H^2) = R(1 - \varepsilon(2))$ and $\alpha(s_2, H^2) = R\varepsilon(2)$. Therefore, $\alpha(s_1, H^2) = R\varepsilon(1)(1 - \varepsilon(2)) = \alpha^*(s_1, H^2)$ and $\alpha(s_2, H^2) = R\varepsilon(2) = \alpha^*(s_2, H^2)$. Note that, $\varepsilon(1) \in [0, 1]$, and as $\delta_2 \in [0, 1]$ then $\varepsilon(2) \in [0, 1]$.

Induction Hypothesis: Let $n = k$ with $k > 1$. Suppose we have $\alpha(s_i, H^k) = \alpha^*(s_i, H^k)$ for all $i \leq k$.

To prove for $n = k + 1$, consider any $H^{k+1} = (\{s_1, \dots, s_k, s_{k+1}\}, \{\{s_1, \dots, s_k, s_{k+1}\}\})$. We will show that $\alpha(s_i, H^{k+1}) = \alpha^*(s_i, H^{k+1})$ for all $i \leq k + 1$. Let $H^k = (\{s_1, \dots, s_k\}, \{\{s_1, \dots, s_k\}\})$.

By relative redistribution for all $i, j \in \{1, \dots, k\}$ we have

$$\frac{\alpha(s_i, H^{k+1})}{\alpha(s_i, H^k)} = \frac{\alpha(s_j, H^{k+1})}{\alpha(s_j, H^k)} \quad (19)$$

Let us denote this ratio above by $1 - \delta_{k+1}$. Therefore, $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha(s_i, H^k)$ for all $i \leq k$. By induction hypothesis $\alpha(s_i, H^k) = \alpha^*(s_i, H^k)$ for all $i \leq k$ which implies $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha^*(s_i, H^k)$. Therefore,

$$\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})R\varepsilon(i) \prod_{j=i+1}^k (1 - \varepsilon(j)) \text{ for all } i \leq k \quad (20)$$

By Equation 19, $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha(s_i, H^k)$, and by Lemma 1 (on H^k and on H^1), we have

$$\sum_{i=1}^k \alpha(s_i, H^{k+1}) = (1 - \delta_{k+1}) \sum_{i=1}^k \alpha(s_i, H^k) = (1 - \delta_{k+1})R \quad (21)$$

Note that $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = \sum_{i=1}^k \alpha(s_i, H^{k+1}) + \alpha(s_{k+1}, H^{k+1})$. Plugging Equation 21 into this, yields $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})R + \alpha(s_{k+1}, H^{k+1})$. By Remark 1, $\sum_{i=1}^{k+1} \alpha(s_i, H^{k+1}) = R$, therefore we have:

$$\alpha(s_{k+1}, H^{k+1}) = \delta_{k+1}R. \quad (22)$$

Setting $\varepsilon(k + 1) = \delta_{k+1}$ in Equations 20 and 22, proves $\alpha(s_i, H^{k+1}) = \alpha^*(s_i, H^{k+1})$ for all $i \in \{1, \dots, k\}$ and for $i = k + 1$, respectively.

Note that, $\varepsilon(k + 1) \in [0, 1]$ for all $k > 0$. Otherwise, if $\varepsilon(k + 1) < 0$ then $\delta_{k+1} < 0$. However, by Equation 22, this implies $\alpha(s_{k+1}, H^{k+1}) < 0$ which contradicts Definition 1. If $\varepsilon(k + 1) > 1$ then $\delta_{k+1} > 1$ which implies $1 - \delta_{k+1} < 0$. However, by Equation 19, $\alpha(s_i, H^{k+1}) = (1 - \delta_{k+1})\alpha(s_i, H^k)$ for all $i \leq k$, this implies $\alpha(s_i, H^{k+1}) < 0$ for all $i \in \{1, \dots, k\}$ which contradicts Definition 1. ■

E Proof of Theorem 3

Theorem 3. A reward sharing scheme satisfies round based rewards, budget limit, fixed total reward, ordinality, absolute redistribution, and relative redistribution if and only if it is k -pseudo proportional in the sense of 4.

Proof. Take any reward sharing scheme α that satisfies the axioms. Take any history $H = (S, \mathcal{P}(S))$ and any $s \in S$. Let $s \in P_r$ for some $1 \leq r \leq l = |\mathcal{P}(S)|$. As α satisfies round based rewards $\alpha(s, H) = \alpha(s, H|_r)$. Similarly for $\alpha^{k,\delta}$, we have $\alpha^{k,\delta}(s, H) = \alpha^{k,\delta}(s, H|_r)$. Hence, it suffices to prove $\alpha(s, H|_r) = \alpha^{k,\delta}(s, H|_r)$ at the restricted histories, i.e., histories with only a single round.

Note that by Lemma 2, if the rounds in any two restricted histories are of the same size, then the shares are awarded based on their ranks. Hence, without loss of generality, we can denote these rounds $P_r = \{s_1, s_2, \dots, s_n\}$ and the restricted histories as $H^n = H|_r$. Therefore, it suffices to prove that there exist some k and some δ such that $\alpha(s_i, H^n) = \alpha^{k,\delta}(s_i, H^n)$ for all n and for all $i \leq n$. First of all, note that for all single share histories, H^1 , Remark 1 implies that α coincides with $\alpha^{k,\delta}$ regardless of the choice of k and δ . So let $n \geq 2$.

In what follows, we find -by iterating on n - that there exist k and δ such that $\alpha(s_i, H^n) = \alpha^{k,\delta}(s_i, H^n)$ for all $n \geq 2$ and for all $i \leq n$. At each step, we ask if α distributes the awards proportionally, or not. The proof structure is as follows:

1. At step h , if α distributes the awards proportionally, then we move to step $h + 1$.
2. At step h , if α distributes the awards disproportionately (while it was proportional at step $h - 1$), then we set $k = h$, and $\delta = \alpha(s_h, H^h)$, i.e., the award of the last share in the round. Thereafter we show that the scheme α coincides with $\alpha^{k,\delta}$ for all possible rounds and all shares in these rounds.

Note a round of size $n = 2$ is a special case, since a round of size 1 cannot be decided to be proportional or disproportionate. Therefore we first treat such histories.

STEP 2: Let $n = 2$, and $H^2 = (\{s_1, s_2\}, \{\{s_1, s_2\}\})$. By Remark 1, we have $\alpha(s_1, H^2) + \alpha(s_2, H^2) = R$. So $(\alpha(s_1, H^2), \alpha(s_2, H^2)) = (R - \gamma_2, \gamma_2)$ for some $\gamma_2 \in [0, R]$. Note also that $\alpha^{k,\delta}(s_1, H^2) + \alpha^{k,\delta}(s_2, H^2) = R$ for any k and for any δ . Now, there are two cases, either the awards are proportional, or not.

Case 1: If $R - \gamma_2 = \gamma_2$, then $k > 2$, therefore continue to next step, (Step 3, i.e., $n = 3$).

Case 2: If $R - \gamma_2 \neq \gamma_2$, then we set $k = 2$ and $\delta = \gamma_2$. Note that $\alpha^{2,\gamma_2}(s_1, H^2) = \frac{R-\gamma_2}{2-1}$ and $\alpha^{2,\gamma_2}(s_2, H^2) = \gamma_2$. Hence, for $n = 2$, we have $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$ for all $i \leq n$. Next we also show, for any $n > 2$, we have $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$ for all $i \leq n$.

Case 2a. If $\gamma_2 = R$, then $(\alpha(s_1, H^2), \alpha(s_2, H^2)) = (0, R)$. Consider any extension H^3 of H^2 . As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)) = (0, R, 0).$$

Similar argument can be extended to H^4 and further, e.g., $(0, R, 0, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

Case 2b. If $\gamma_2 = 0$, then $(\alpha(s_1, H^2), \alpha(s_2, H^2)) = (R, 0)$. Consider any extension H^3 of H^2 . As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)) = (R, 0, 0).$$

Similar argument can be extended to H^4 and further, e.g., $(R, 0, 0, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

Case 2c. If $\gamma_2 \in (0, R)$, then $(\alpha(s_1, H^2), \alpha(s_2, H^2)) = (R - \gamma_2, \gamma_2)$. Consider any extension H^3 of H^2 . By absolute redistribution we have $\alpha(s_1, H^2) - \alpha(s_1, H^3) = \alpha(s_2, H^2) - \alpha(s_2, H^3)$. By relative redistribution we have

$$\frac{\alpha(s_1, H^3)}{\alpha(s_1, H^2)} = \frac{\alpha(s_2, H^3)}{\alpha(s_2, H^2)} = \theta$$

Combining these equations, we have $\alpha(s_1, H^2) - \alpha(s_1, H^2)\theta = \alpha(s_2, H^2) - \alpha(s_2, H^2)\theta$ which implies $\alpha(s_1, H^2)(1 - \theta) = \alpha(s_2, H^2)(1 - \theta)$. As $R - \gamma_2 \neq \gamma_2$, the previous equation only holds if $\theta = 1$. This results in $\alpha(s_1, H^3) = \alpha(s_1, H^2)$ and $\alpha(s_2, H^3) = \alpha(s_2, H^2)$. Finally, as α does not assign negative awards, Remark 1 implies

$$(\alpha(s_1, H^3), \alpha(s_2, H^3), \alpha(s_3, H^3)) = (R - \gamma_2, \gamma_2, 0).$$

Similar argument can be extended to H^4 and further, e.g., $(R - \gamma_2, \gamma_2, 0, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2,\gamma_2}(s_i, H^n)$.

STEP h: Let $n = h$, and $H^h = (\{s_1, \dots, s_h\}, \{\{s_1, \dots, s_h\}\})$. Reaching to step h implies the awards to shares at $H^{h-1} = (\{s_1, \dots, s_{h-1}\}, \{\{s_1, \dots, s_{h-1}\}\})$ were distributed proportionally and hence all are equal. By relative redistribution for all $i, j \leq h - 1$ we have, $\frac{\alpha(s_i, H^h)}{\alpha(s_i, H^{h-1})} = \frac{\alpha(s_j, H^h)}{\alpha(s_j, H^{h-1})}$. This implies $\alpha(s_i, H^h) = \alpha(s_j, H^h)$ for all $i, j \leq h - 1$. By Remark 1, $\sum_{i=1}^h \alpha(s_i, H^h) = R$, which implies $(h - 1)\alpha(s_i, H^h) + \alpha(s_h, H^h) = R$ for any $i \leq h - 1$. Therefore, $\alpha(s_i, H^h) = \frac{R - \alpha(s_h, H^h)}{h-1}$. Let us denote $\alpha(s_h, H^h) = \gamma_h$ for some $\gamma_h \in [0, R]$, so

$(\alpha(s_1, H^h), \alpha(s_2, H^h), \dots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)) = (\frac{R-\gamma_h}{h-1}, \frac{R-\gamma_h}{h-1}, \dots, \frac{R-\gamma_h}{h-1}, \gamma_h)$. Now, there are two cases, either the awards are proportional, or not.

Case 1: If $\frac{R-\gamma_h}{h-1} = \gamma_h$, then $k > h$, therefore continue to next step, (Step $h + 1$, i.e., $n = h + 1$).

Case 2: If $\frac{R-\gamma_h}{h-1} \neq \gamma_h$, then we set $k = h$ and $\delta = \gamma_h$. Note that $\alpha^{h, \gamma_h}(s_i, H^h) = \frac{R-\gamma_h}{h-1}$ for all $i < h$ and $\alpha^{h, \gamma_h}(s_h, H^h) = \gamma_h$. Hence, for $n = h$, we have $\alpha(s_i, H^n) = \alpha^{h, \gamma_h}(s_i, H^n)$ for all $i \leq n$. Next we also show, for any $n > h$, we have $\alpha(s_i, H^n) = \alpha^{h, \gamma_h}(s_i, H^n)$ for all $i \leq n$.

Case 2a. If $\gamma_h = R$, then $(\alpha(s_1, H^h), \alpha(s_2, H^h), \dots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)) = (0, \dots, 0, R)$. Consider any extension H^{h+1} of H^h . As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$(\alpha(s_1, H^{h+1}), \dots, \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})) = (0, \dots, 0, R, 0)$$

Similar argument can be extended to H^{h+2} and further, e.g., $(0, \dots, 0, R, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{h, \gamma_h}(s_i, H^n)$.

Case 2b. If $\gamma_h = 0$, then $(\alpha(s_1, H^h), \alpha(s_2, H^h), \dots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)) = (\frac{R}{h-1}, \dots, \frac{R}{h-1}, 0)$. Consider any extension H^{h+1} of H^h . As α does not assign negative awards, Remark 1 and absolute redistribution implies

$$(\alpha(s_1, H^{h+1}), \dots, \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})) = (\frac{R}{h-1}, \dots, \frac{R}{h-1}, 0, 0)$$

Similar argument can be extended to H^{h+2} and further, e.g., $(\frac{R}{h-1}, \dots, \frac{R}{h-1}, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{h, \gamma_h}(s_i, H^n)$.

Case 2c. If $\gamma_h \in (0, R)$, then $(\alpha(s_1, H^h), \alpha(s_2, H^h), \dots, \alpha(s_{h-1}, H^h), \alpha(s_h, H^h)) = (\frac{R-\gamma_h}{h-1}, \frac{R-\gamma_h}{h-1}, \dots, \frac{R-\gamma_h}{h-1}, \gamma_h)$. Consider any extension H^{h+1} of H^h . By absolute redistribution for all $i, j \leq h$ we have $\alpha(s_i, H^h) - \alpha(s_i, H^{h+1}) = \alpha(s_j, H^h) - \alpha(s_j, H^{h+1})$. By relative redistribution for all $i, j \leq h$ we have

$$\frac{\alpha(s_i, H^{h+1})}{\alpha(s_i, H^h)} = \frac{\alpha(s_j, H^{h+1})}{\alpha(s_j, H^h)} = \theta$$

Combining these equations, we have $\alpha(s_i, H^h) - \alpha(s_i, H^h)\theta = \alpha(s_j, H^h) - \alpha(s_j, H^h)\theta$ which implies $\alpha(s_i, H^h)(1 - \theta) = \alpha(s_j, H^h)(1 - \theta)$. As $\alpha(s_i, H^h) \neq \alpha(s_j, H^h)$ for all $i, j \leq h$, the previous equation only holds if $\theta = 1$. This results in $\alpha(s_i, H^{h+1}) = \alpha(s_i, H^h)$ for all $i \leq h$. Finally, as α does not assign negative awards, Remark 1 implies

$$(\alpha(s_1, H^{h+1}), \dots, \alpha(s_{h-1}, H^{h+1}), \alpha(s_h, H^{h+1}), \alpha(s_{h+1}, H^{h+1})) = (\frac{R-\gamma_h}{h-1}, \dots, \frac{R-\gamma_h}{h-1}, \gamma_h, 0)$$

Similar argument can be extended to H^4 and further, e.g., $(\frac{R-\gamma h}{h-1}, \dots, \frac{R-\gamma h}{h-1}, \gamma h, 0, \dots, 0)$ which shows $\alpha(s_i, H^n) = \alpha^{2, \gamma^2}(s_i, H^n)$.

Note that in case α never distributes the awards “disproportionately”, then this implies that h goes to infinity and therefore, we set $k = \infty$ and it is clear to see that $\alpha = \alpha^{\infty, \delta}$ for any δ , i.e., α is the proportional scheme, which is an element of k -pseudo proportional class. All in all, this completes the proof. ■