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On L2-stability of solutions of linear stochastic delay differential equations

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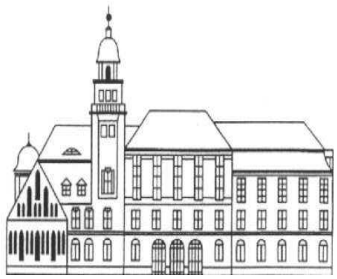

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On $\mathcal{L}^2$-stability of solutions of Linear Stochastic Delay Differential Equations*

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Abstract

Stochastic Delay Differential Equations (SDDE) are Stochastic Functional Differential Equations with important applications. It is of interest to characterize the $\mathcal{L}^2$-stability (stability of second moments) of solutions of SDDE. For the class of linear, scalar SDDE we can show that second comoment function of the solution satisfies a partial differential equation (PDE) with time delay and derive a characteristic equation from it determining the asymptotic behaviour of the second moments. Additionally we derive a necessary criterion for weak stationarity of solutions of linear SDDE.

Keywords: SDDE, SFDE, stochastic delay equations, stability, characteristic equation

AMS 2000 Subject Classification: 34K50, 34F05, 60Hxx

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space provided with an increasing right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$. Stochastic Delay Differential Equations (SDDE)

$$\begin{array}{lcll} dX(t) & = & F(t, X_t)\,dt + G(t, X_t)\,dW(t), & t \in [0,\infty)\ , \\ X(t) & = & \xi(t) & t \in [-\tau, 0]\ . \end{array} \tag{1.1}$$

where

$(W(t), \mathcal{F}_t)_{t\in[0,\infty)}$	is a standard Brownian motion,	
$\tau \in [0,\infty)$	is the non-negative delay-time,	
$X_t = \{X(s) \mid s \in [t-\tau, t]\} \subset \mathcal{C}([-\tau,0],\mathbb{R}), t \in [0,\infty)$	is a segment at t,	(1.2)
$\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$	is a $\mathcal{F}_0$-measurable random variable,	
$F, G \in \mathcal{C}(\mathbb{R} \times \mathcal{C}([-\tau,0],\mathbb{R}), \mathcal{C}([-\tau,0],\mathbb{R}))$	parameter funcions.	

are a standard tool in the modelling of realistic phenomena in biology, physics, economics in particular in presence of time-delayed feedback processes.

It is of interest to investigate the long term behaviour of solutions of SDDE in form of the time evolution of the second moment of solutions, that is $m_2^X(t) = \mathbb{E}[\,|X(t)|^2\,]$ for $t \mapsto \infty$. For general SDDE like (1.1) it is hard to achieve exact results due to possible non-linearity of the parameter functions F, G and due to the memory. However, in the case of linear, time-homogeneous SDDE one can consider the so-called comoment function $K_2^X(t,s) = \mathbb{E}[\,X(t)\,X(s)\,]$, into which m_2^X is embedded. Using Fubini theorems and the linearity of integrals we can show that K_2^X satisfies a certain PDE with time delay which is called (time-continuous) amplified system. In addition to this we will show that solutions of amplified systems have some special properties. One of the important properties is that its solutions are completely determined by its values on a diagonal „strip“ $D_\tau^2 = \{(t,s) \mid s,t \in [-\tau,\infty), |t-s| \le \tau\}$. This allows to use results from the semigroup theory to derive a characteristic equation in (complex)

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eigenvalues and corresponding eigenfunctions. The real part of the eigenvalues determines the behaviour of m_2^X for $t \mapsto \infty$. A second result from the investigation of the amplified system is a necessary criterion for weak stationarity of solutions of linear, scalar SDDE.

As an example we want to consider a particular linear, scalar SDDE :

$$dX(t) = (a_0 X(t) + a_1 X(t-\tau))\, dt + (b_0 X(t) + b_1 X(t-\tau))\, dW(t), \qquad t \in [0,\infty), \tag{1.3}$$

with real parameters a_0, a_1, b_0, b_1 and its discretizations. This will give us an idea how the amplified system for (1.3) looks like. Furthermore this provides us with an important link to the time-evolution of those stochastic recurrence relations which are the result from the application of numerical methods to linear SDDE. Let us choose here the numerical method „Euler explicit“ and set the discretization step width to $h = \tau$. Then we obtain a numerical solution $Y = \{Y_n\}_{n\in\mathbb{N}}$ satisfying:

$$Y_{n+1} = Y_n + h\,(a_0 Y_n + a_1 Y_{n-1}) + \sqrt{h}\,(b_0 Y_n + b_1 Y_{n-1})\,\epsilon_{n+1}, \qquad n \in \mathbb{N}, \tag{1.4}$$

where $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. $N(0,1)$-distributed random variables. As in [4] was pointed out the second moments of $\{Y_n\}_{n\in\mathbb{N}}$ satisfy a deterministic recurrence relation - the so-called (time-discrete) amplified system. Following the notations in [4] it is:

$$Z_{n+1} = A\, Z_n \tag{1.5}$$

where

$$\begin{aligned} Z_{n+1} &= (\mathbb{E}[Y_{n+1}^2], \mathbb{E}[Y_{n+1}Y_n], \mathbb{E}[Y_n^2])^\top, \\ A &= \begin{pmatrix} (1+a_0h)^2 + b_0^2 h & 2(1+a_0h)a_1h + 2b_0b_1h & (a_1h)^2 + b_1^2 h \\ 1 + a_0 h & a_1 h & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{1.6}$$

The first row of this recursion reads as

$$\begin{aligned} \mathbb{E}[Y_{n+1}^2] \;=\; & \mathbb{E}[Y_n^2] + h\,(\,(2a_0 + b_0^2)\,\mathbb{E}[Y_n^2] + 2(a_1 + b_0 b_1)\,\mathbb{E}[Y_n Y_{n-1}] + b_1^2\,\mathbb{E}[Y_{n-1}^2]\,) \\ & + h^2\,(\,(a_0^2 + a_1^2)\,\mathbb{E}[Y_n]^2 + 2a_0a_1\,\mathbb{E}[Y_nY_{n-1}] + a_1^2\,\mathbb{E}[Y_{n-1}^2]\,) \end{aligned} \tag{1.7}$$

Summing up both sides over the first n recurrence elements results in

$$\begin{aligned} \mathbb{E}[Y_{n+1}^2] \;=\; \mathbb{E}[Y_0^2] + & \sum_{k=0}^{n} (\,2a_0\,\mathbb{E}[Y_k^2] + 2a_1\,\mathbb{E}[Y_kY_{k-1}] + b_0^2\,\mathbb{E}[Y_k^2] + 2b_0b_1\,\mathbb{E}[Y_kY_{k-1}] + b_1^2\,\mathbb{E}[Y_{k-1}^2]\,)\,h \\ & + h \sum_{k=0}^{n} (\,(a_0^2 + b_0^2)\,\mathbb{E}[Y_k^2] + 2a_0a_1\,\mathbb{E}[Y_kY_{k-1}] + a_1^2\,\mathbb{E}[Y_{k-1}^2]\,)\,h \;. \end{aligned} \tag{1.8}$$

Assuming the equidistant lattice $(t_0 - \tau) + \tau\,\mathbb{N}$ and considering the step functions $\sum_{n=0}^{\infty} Y_n\, 1_{[t_n, t_{n+1})}$ equation (1.8) is equivalent to

$$\begin{aligned} \mathbb{E}[X_h(t_{n+1})^2] \;=\; & \mathbb{E}[X_h(t_0)^2] \\ & + 2\sum_{k=0}^{n} (\,a_0\mathbb{E}[X_h(t_k)^2] + a_1\mathbb{E}[X_h(t_k)X_h(t_{k-1})]\,)\,(t_{k+1} - t_k) \\ & + \sum_{k=0}^{n} (b_0^2\,\mathbb{E}[X_h(t_k)^2] + 2b_0b_1\,\mathbb{E}[X_h(t_k)X_h(t_k-\tau)] + b_1^2\,\mathbb{E}[X_h(t_k-\tau)^2]\,)\,(t_{k+1} - t_k) \\ & + h\sum_{k=0}^{n} ((a_0^2 + b_0^2)\,\mathbb{E}[X_h(t_{k-1})^2] + 2a_0a_1\,\mathbb{E}[X_h(t_k)X_h(t_{k-1})] + a_1^2\mathbb{E}[X_h(t_{k-1})^2]\,)\,(t_{k+1} - t_k)\,) \\ \;=\; & \mathbb{E}[X_h(t_0)^2] \\ & + 2\int_{t_0}^{t_{n+1}} (\,a_0\mathbb{E}[X_h(s)^2] + a_1\mathbb{E}[X_h(s)X_h(s-\tau)]\,)ds \\ & + \int_{t_0}^{t_{n+1}} (\,b_0^2\,\mathbb{E}[X_h(s)^2] + 2b_0b_1\,\mathbb{E}[X_h(s)X_h(s-\tau)] + b_1^2\,\mathbb{E}[X_h(s-\tau)^2]\,)ds \\ & + h\int_{t_0}^{t_{n+1}} (a_0^2 + b_0^2)\,\mathbb{E}[X_h(s-\tau)^2]) + 2a_0a_1\,\mathbb{E}[X_h(s)X_h(s-\tau)] + a_1^2\,\mathbb{E}[X_h(s-\tau)^2])\,ds \;. \end{aligned} \tag{1.9}$$

If we let h converge down to zero in (1.9) and take into account that then step-function X_h converges to the exact solution X of (1.3) in the Delfour-Mitter space $\mathcal{M}([-\tau,T],\mathbb{R})$ (see [6]) we expect X to satisfy:

$$\begin{aligned}\mathbb{E}[X(t)^2] &= \mathbb{E}[X(0)^2] + \int_0^t 2a_0\mathbb{E}[X(s)^2] + 2a_1\mathbb{E}[X(s)X(s-\tau)]\,)\,ds \\ &\quad + \int_0^t b_0^2\,\mathbb{E}[X(s)^2] + 2b_0b_1\,\mathbb{E}[X(s-\tau)^2] + b_1^2\,\mathbb{E}[X(s-\tau)^2]\,)\,ds.\end{aligned} \tag{1.10}$$

The second row of the recurrence relation (1.5) reads as

$$\mathbb{E}[Y_{n+1}Y_n] = \mathbb{E}[Y_n^2] + h(a_0\,\mathbb{E}[Y_n^2] + a_1\,\mathbb{E}[Y_nY_{n-1}])\,). \tag{1.11}$$

Here summing up does not lead to the cancellation of summands on both sides. Furthermore, increasing the dimension of the discrete problem by letting h tending to 0 leads to possibly infinite sums. However we observe that (1.12) is equivalent to

$$\frac{1}{h}(\,\mathbb{E}[Y_{n+1}Y_n] - \mathbb{E}[Y_n^2]\,) = (\,a_0\,\mathbb{E}[Y_n^2] + a_1\,\mathbb{E}[Y_nY_{n-1}]\,) \tag{1.12}$$

or

$$\tfrac{1}{h}(\,\mathbb{E}[X(t_n+h)X(t_n)] - \mathbb{E}[X(t_n)X(t_n)]\,) \;=\; (\,a_0\mathbb{E}[X(t_n)^2\,] + a_1\mathbb{E}[X(t_n)X(t_n-\tau)]\,)\;. \tag{1.13}$$

Revisiting the first row of the recurrence scheme (1.5) from that point of view we see that we can restate it similarly as

$$\begin{aligned}\tfrac{1}{h}(\mathbb{E}[X(t_n+h)^2] - \mathbb{E}[X(t_n)^2]) &= 2(a_0\mathbb{E}[X(t_n)^2] + a_1\mathbb{E}[X(t_n)X(t_n-\tau)]) \\ &\quad + (b_0^2\,\mathbb{E}[X(t_n)^2] + 2b_0b_1\,\mathbb{E}[X(t_n)X(t_n-\tau)] + b_1^2\,\mathbb{E}[X(t_n-\tau)^2] \\ &\quad + h(2a_0a_1)\mathbb{E}[X(t_n)X(t_n-\tau)] + a_1^2\mathbb{E}[X(t_n-\tau)^2]\;.\end{aligned} \tag{1.14}$$

The equations (1.13) and (1.14) show, that for the numerical method „Euler explicit“ the recurrence relation (1.5) of the second order moments of the generated numerical solution Y can be understood as a system of equations involving particular difference quotients. Like (1.10) this leads to a conjecture for a (time-continues) amplified system of the linear SDDE: a PDE with time-delay satisfied by the comoment function of the exact solution X of (1.7).

In this paper we do not want to consider the relation between time-discretized and time-continuous amplified systems. Instead our focus lies on the time-continuous amplified systems. In section 2 we derive the PDE with time-delay which is satisfied by the comoment function K_2^X of the exact solution. We also consider existence and uniqueness of solutions of this PDE and its smoothness properties. In section 3 we introduce a strong continuous semigroup associated to the (time-continuous) amplified system as well as a characteristic equation in eigenvalues and eigenfunctions of its generator. In section 4 we derive necessary criteria for the weak stationarity of solutions of scalar, linear SDDE. Section 5 provides some additional knowledge which is used. Furthermore it contains the proofs of statements which are too long or not instructive in their respective sections.

2 The amplified system embedding the second moments of a linear SDDE

As usual we assume a probability space $(\Omega,\mathcal{F},\mathbb{P})$ which is provided with an increasing, right continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$. Let us consider the linear SDDE

$$\begin{aligned} dX(t) &= \int_{-\tau}^0 X(s+u)\,da(u)\,dt + \int_{-\tau}^0 X(s+u)\,db(u)\,dW(s), && t\in(0,\infty), \\ X(t) &= \xi(t), && t\in[-\tau,0]. \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &(W(t),\mathcal{F}_t)_{t\in[0,\infty)} && \text{is a standard Brownian motion,} \\ &\xi\in\mathcal{C}([-\tau,0],\mathbb{R}) && \text{is a } \mathcal{F}_0\text{-measurable random variable,} \\ &a,b\in([-\tau,0],\mathbb{R}) && \text{are functions of bounded variation,} \\ &\tau\in[0,\infty) && \text{is the non-negative delay-time.} \end{aligned} \tag{2.2}$$

Here we are interested in the asymptotic behaviour of the second moments of solutions of (2.1). That is we are interested in behaviour of the **second moment's function of X**

$$\begin{array}{rccl} m_2^X\colon & [-\tau,\infty) & \longrightarrow & \mathbb{R} \\ & t\in[-\tau,\infty) & \mapsto & m_2^X(t) := \mathbb{E}[\,X(t)^2\,]. \end{array} \tag{2.3}$$

for $t \longrightarrow \infty$. In the second subsection it turns out that the time derivative of m_2^X on $[0,\infty)$ does not depend on the values of m_2^X and time only. Due to the memory in (2.1) it depends on the values of the **comoment function (of order 2) of X**

$$\begin{array}{rccl} K_2^X\colon & [-\tau,\infty)^2 & \longrightarrow & \mathbb{R} \\ & t,r\in[-\tau,\infty) & \mapsto & K_2^X(t) := \mathbb{E}[\,X(t)\,X(r)\,]. \end{array} \tag{2.4}$$

So instead of considering m_2^X we investigate the comoment function K_2^X and will see in a later subsection that K_2^X satisfies a PDE with time delay. In the following this PDE is referred to as the **amplified system** as it describes the dynamics of K_2^X and as m_2^X is „embedded" into K_2^X by $m_2^X(t) = K_2^X(t,t)$. In a further subsection we describe the existence and uniqueness of solutions of the respective PDE. In the last subsection we collect further properties of the comoment function K_2^X.

But before starting to consider m_2^X and K_2^X, in the next subsection we provide basic results on existence and uniqueness of solutions of (2.1) and existence of its moments in a first subsection.

2.1 Existence and uniqueness of solutions of linear SDDE

There are some standard references about existence and uniqueness theorems. We present here basic results from [6]. Recall, that (1.1) in fact is an integral equation

$$\begin{array}{rcll} X(t) & = & X(0) + \int_0^t F(s,X_s)\,ds + \int_0^t G(s,X_s)\,dW(s), & t\in[0,\infty)\ , \\ X(t) & = & \xi(t) & t\in[-\tau,0]\ . \end{array} \tag{2.5}$$

with the stochastic Itô-integral and satisfying (1.2).

Definition 2.1
A **solution of (1.1) and (2.5)**, respectively, is then a stochastic process $(X,[-\tau,\infty))$ which is adapted to $\mathcal{F}_{[0,\infty)}$ and satisfies (1.1).

Let us introduce the following spaces.

Definition 2.2
Let $t_0, T\in[-\tau,\infty)$, $\mathbb{F}_{t_0,T} := \{\mathcal{F}_t\}_{t\in[0\vee t_0,T]}$. Define:
(i) Let $f\in([t_0,T]\times\Omega,\mathbb{R})$ be adapted to $\mathbb{F}_{t_0,T}$.

$$\begin{array}{rll} \text{Define} & \|f\|_{\mathcal{M}^2([t_0,T],\mathbb{R})} & := \mathbb{E}[\int_{t_0}^T |f(s)|^2\,ds\,] \\ & \|f\|_{\mathcal{M}^2_{sup}([t_0,T],\mathbb{R})} & := \mathbb{E}[\sup_{s\in[t_0,T]} |f(s)|^2\,] \end{array}$$

(ii) $\mathcal{M}^2([t_0,T],\mathbb{R}) := \{\, f\in[t_0,T]\times\Omega \mid f \textit{ adapted to } \mathbb{F}_{t_0,T} \textit{ and } \|f\|_{\mathcal{M}^2([t_0,T],\mathbb{R})}<\infty\,\}$,
(iii) $\mathcal{M}^2_{sup}([t_0,T],\mathbb{R}) := \{\, f\in[t_0,T]\times\Omega \mid f \textit{ adapted to } \mathbb{F}_{t_0,T} \textit{ and } \|f\|_{\mathcal{M}^2_{sup}([t_0,T],\mathbb{R})}<\infty\,\}$.

Then the following theorem provides conditions for the existence and uniqueness of solutions of (1.1).

Theorem 2.3
Let the initial function ξ be in $\mathcal{C}([-\tau,0],\mathbb{R})$, the parameter functions F, G satisfy:
(i) *(Lipschitz condition)*
$\forall\, T\in(0,\infty)\,\exists\, K_{T,n}\in(0,\infty)$ *such that* $\forall\,\varphi,\psi\in\mathcal{C}([-\tau,0],\mathbb{R})$ *with* $\|\varphi\|_{\mathcal{C}([-\tau,0],\mathbb{R})}\vee\|\psi\|_{\mathcal{C}([-\tau,0],\mathbb{R})}$ *and* $\forall\, t\in[0,T]$*:*

$$|F(t,\varphi)-F(t,\psi)|^2 \vee |G(t,\varphi)-G(t,\psi)|^2 \le K_{T,n}\,\|\varphi-\psi)\|^2_{\mathcal{C}([-\tau,0],\mathbb{R})}$$

(ii) *(Growth condition)*
$\forall\, T\in(0,\infty)\,\exists\, K_T\in(0,\infty)$ *such that* $\forall\,\varphi,\psi\in\mathcal{C}([-\tau,0],\mathbb{R})$, $\forall\, t\in[0,T]$*:*

$$|F(t,\varphi)|^2 \vee |G(t,\varphi)|^2 \le K_T\,(1+\|\varphi\|^2_{\mathcal{C}([-\tau,0],\mathbb{R})})$$

Then it holds:
(1) *The SDDE (1.1) has a unique solution* X.
(2) $\forall\, T \in (0,\infty)$*:* $X \in \mathcal{M}^2([-\tau,T],\mathbb{R})$ *and* $X \in \mathcal{M}^2_{sup}([-\tau,T],\mathbb{R})$.

In the equation (2.1) the parameter functions F and G are defined by:

$$\forall\, \varphi \in \mathcal{C}([-\tau,0],\mathbb{R}),\, t \in [0,\infty)\colon F(t,\varphi) = \int_{-\tau}^{0} \varphi(u)\, da(u) \text{ and } G(t,\varphi) = \int_{-\tau}^{0} \varphi(u)\, db(u)\ .$$

As a, b are functions of bounded variation the conditions 2.3.(i) and 2.3.(ii) are satisfied.
Let $K^2_{a,b} = 2\,(\,|a|^2 \vee |b|^2\,)$. Then $\forall\, \varphi,\psi \in \mathcal{C}([-\tau,0],\mathbb{R})$, $t \in [0,\infty)$ it holds:

$$|\int_{-\tau}^{0} \varphi(u) - \psi(u)\, da(u)|^2 \vee |\int_{-\tau}^{0} \varphi(u) - \psi(u)\, db(u)|^2 \le 2\,(|a|^2 \vee |b|^2)([-\tau,0])\, \|\varphi-\psi\|^2_{\mathcal{C}([-\tau,0],\mathbb{R})}\ ,$$

$$|\int_{-\tau}^{0} \varphi(u)\, da(u)|^2 \vee |\int_{-\tau}^{0} \varphi(u)\, db(u)|^2 \le (1 + 2\,(|a|^2 \vee |b|^2)\,([-\tau,0])\,)\, \|\varphi\|^2_{\mathcal{C}([-\tau,0],\mathbb{R})}\ .$$

Hence by 2.3.(1), 2.1 has a unique solution X. Due to 2.3.(2), any solution X as second moments and comoments.

2.2 m_2^X, K_2^X and the amplified system

When investigating the time evolution of second moments of solutions X of (2.1) then it is natural to ask whether $m_2^X(t) = \mathbb{E}[\,X(t)^2\,]$ satisfies a kind of differential equation. Following this approach we have a closer look at the difference quotient

$$\frac{1}{h}(\, m_2^X(t+h) - m_2^X(t)\,) = \frac{1}{h}(\,\mathbb{E}[\,X(t+h)^2 - X(t)^2\,]\,) \tag{2.6}$$

where $h \in [0,\infty)$, $t \in (0,\infty)$ and try to determine its limit for $h \downarrow 0$. The main reason why the existence of a limit of (2.6) for h can be expected is that $X(t+h)^2 - X(t)^2$ can be represented as a quadratic polynomial of $X(t)$ and of the state increments $\Delta_X(t,h) = X(t+h) - X(t)$. However, as X is a solution of the SDDE (2.1) $\Delta_X(t,h)$ can be represented as a sum of integrals resulting from the SDDE (2.6). More precisely it holds:

$$\begin{aligned}
&\tfrac{1}{h}(\,\mathbb{E}[\,X(t+h)^2 - X(t)^2\,]\,)\\
&\quad= \tfrac{1}{h}\mathbb{E}[\,X(t+h) + X(t)\,)(\,X(t+h) - X(t)\,)\,]\\
&\quad= \tfrac{1}{h}\mathbb{E}[\,(\,2X(t) + \int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds + \int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s)\,)\\
&\qquad\qquad (\int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds + \int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s)\,)\,]\\
&\quad= \tfrac{1}{h}\,(\quad 2\mathbb{E}[\,X(t)\int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds\,]\\
&\qquad + 2\mathbb{E}[\,X(t)\int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s)\,]\\
&\qquad + \mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds\,]\\
&\qquad + 2\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s)\,]\\
&\qquad + \mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s) \int_t^{t+h}\int_{-\tau}^{0} X(s+u)db(u)\, dW(s)\,]\ )\,.
\end{aligned} \tag{2.7}$$

In the following we consider each of the five summands of the last equation in (2.7) separately and compute the respective limits for $h \downarrow 0$.

$$
\begin{aligned}
(i)\quad \tfrac{1}{h}\mathbb{E}[\, X(t) \int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds\,] &= \tfrac{1}{h}\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(t)X(s+u)da(u)\, ds\,] \\
&\underset{Fubini}{=} \tfrac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\mathbb{E}[\, X(t)X(s+u)\,]\, da(u)\, ds \\
\lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}[\, X(t) \int_t^{t+h}\int_{-\tau}^{0} X(s+u)da(u)\, ds\,] &= \int_{-\tau}^{0}\mathbb{E}[\, X(t)X(t+u)\,]\, da(u), \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad \tfrac{1}{h}\mathbb{E}[\, X(t) \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, db(u)\, dW(s)\,] &\underset{Fubini}{=} \tfrac{1}{h}\mathbb{E}[\, X(t)\int_{-\tau}^{0}\int_t^{t+h} X(s+u)\, dW(s)\, db(u)\,] \\
&= \tfrac{1}{h}\mathbb{E}[\int_{-\tau}^{0} X(t)\int_t^{t+h} X(s+u)\, dW(s)\, db(u)\,] \\
&\underset{Fubini}{=} \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\, X(t)\int_t^{t+h} X(s+u)\, dW(s)\,]\, db(u) \\
&= \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\, X(t)\,\mathbb{E}[\int_t^{t+h} X(s+u)\, dW(s)\ |\ \mathcal{F}_t\,]\,]\, db(u) \\
&\underset{Itô-integral}{=} \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\, X(t)\, 0\,]\, db(u) \\
&= 0 \\
\lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}[\, X(t) \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, db(u)\, dW(s)\,] &= 0\ , \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
(iii)\ |\ \tfrac{1}{h}\,\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds\,]\ | & \\
&\underset{Fubini}{\leq} \tfrac{1}{h}\int_t^{t+h}\int_t^{t+h}\ |\ \mathbb{E}[\int_{-\tau}^{0} X(s+u)\, da(u)\int_{-\tau}^{0} X(r+u)\, da(u)\,]\ |\ ds\, dr \\
&\underset{Hölder}{\leq} \tfrac{1}{h}\int_t^{t+h}\int_t^{t+h}\mathbb{E}[\,|\int_{-\tau}^{0} X(s+u)\, da(u)\ |^2\,]^{1/2}\,\mathbb{E}[\,|\int_{-\tau}^{0} X(s+u)\, da(u)\ |^2\,]^{1/2}\, ds\, dr, \\
\lim_{h\downarrow 0}|\ \tfrac{1}{h}\,\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds\,]\ | & \\
&\leq \lim_{h\downarrow 0}\tfrac{1}{h}\int_t^{t+h}\int_t^{t+h}\mathbb{E}[\,|\int_{-\tau}^{0} X(s+u)\, da(u)\ |^2\,]^{1/2}\,\mathbb{E}[\,|\int_{-\tau}^{0} X(s+u)\, da(u)\ |^2\,]^{1/2}\, ds\, dr, \\
&\leq 0, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
(iv)\ |\ \tfrac{1}{h}\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, db(u)\, dW(s)\,]\ | & \\
&\underset{Fubini}{=}\ |\ \tfrac{1}{h}\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0} X(s+u)\int_t^{t+h} X(r+v)\, dW(r)\, db(v)\, da(u)\, ds\,]\ | \\
&\underset{Fubini}{\leq}\ |\ \tfrac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\, X(s+u)\int_t^{t+h} X(r+v)\, dW(r)\,]\, db(v)\, da(u)\, ds\ | \\
&\underset{Jensen}{\leq} \tfrac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0}\ |\ \mathbb{E}[\, X(s+u)\int_t^{t+h} X(r+v)\, dW(r)\,]\ |\ d|b|(v)\, d|a|(u)\, ds \\
&\underset{Hölder}{\leq} \tfrac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\, X(s+u)^2\,]^{1/2}\,(\int_t^{t+h}\mathbb{E}[\, X(r+v)^2\,]\, dr)^{1/2}\, d|b|(v)\, d|a|(u)\, ds, \\
\lim_{h\downarrow 0}|\ \tfrac{1}{h}\mathbb{E}[\int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, da(u)\, ds \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\, db(u)\, dW(s)\,]\ | & \\
&\leq \lim_{h\downarrow 0}\tfrac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\, X(s+u)^2\,]^{1/2}\,(\int_t^{t+h}\mathbb{E}[\, X(r+v)^2\,]\, dr)^{1/2}\, d|b|(v)\, d|a|(u)\, ds, \\
&\leq 0\ , \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
(v)\ \mathbb{E}[\ \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\,db(u)\,dW(s)\ \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\,db(u)\,dW(s)\ ] \\
&\underset{Itô-integral}{=} \frac{1}{h}\int_t^{t+h}\mathbb{E}[\ \int_{-\tau}^{0} X(s+u)\,db(u)^2\ ]\,ds \\
&\underset{Fubini}{=} \frac{1}{h}\int_t^{t+h}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\ X(s+u)X(s+v)\ ]\,db(u)\,db(v)\ ]\,ds\ , \\
\lim_{h\to 0}\mathbb{E}[\ \int_t^{t+h}\int_{-\tau}^{0} X(s+u)\,db(u)\,dW(s)\ \int_{t-\tau}^{t+h}\int_{-\tau}^{0} X(s+u)\,db(u)\,dW(s)\ ] \\
&= \int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\ X(s+u)X(s+v)\ ]\,db(u)\,db(v)\ ]\,ds\ . \qquad (2.12)
\end{aligned}
$$

The same considerations can be taken for the difference quotient

$$
\frac{1}{-h}(\ m_2^X(t-h) - m_2^X(t)\ ) = \frac{1}{-h}(\ \mathbb{E}[\ X(t-h)^2 - X(t)^2\ ]\ ). \qquad (2.13)
$$

where $t \in (0,\infty)$, $h \in [0,t)$ and we obtain:

LEMMA 2.4
Let X be a solution of (2.1), $t \in [0,\infty)$.
Then it holds:

$$
\begin{aligned}
(i)\ \lim_{h\downarrow 0}\tfrac{1}{h}(\mathbb{E}[X(t+h)^2] - \mathbb{E}[X(t)^2]) &= 2\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t)\ ]\,da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t+v)\ ]\,db(u)\,db(v), \\
(ii)\ \lim_{h\downarrow 0}\tfrac{1}{h}(\mathbb{E}[X(t-h)^2] - \mathbb{E}[X(t)^2]) &= 2\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t)\ ]\,da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t+v)\ ]\,db(u)\,db(v). \\
(iii)\ \lim_{h\to 0}\tfrac{1}{h}(\mathbb{E}[X(t+h)^2] - \mathbb{E}[X(t)^2]) &= 2\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t)\ ]\,da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\ X(t+u)\,X(t+v)\ ]\,db(u)\,db(v).
\end{aligned}
$$

Proof:
(i) Follows from (2.8) - (2.12).
(ii) Deferred to the appendix.
(iii) Combine (i) and (ii). □

The previous lemma shows, that the derivative of the function m_2^X at $t \in (0,\infty)$ does not depend only on $\{\ \mathbb{E}[\ X(r)^2\ ]\}_{r\in[t-\tau,t]}$ (m_2^X on $[t-\tau,t]$) but as well on all comoments $\{\ \mathbb{E}[\ X(t)\,X(r)\ ]\}_{r\in[t-\tau,t]}$ $r \in [\,t-\tau,t\,]$ (on K_2^X). So in order to characterize m_2^X one necessarily has to consider K_2^X.

REMARK 2.5
(i) This result is natural. The time evolution of second moments of sequences $\{\ X_n\ \}_{n\in -k(h)+\mathbb{N}}$, where $k(h) = \tau/h \in \mathbb{N}$ represented the length of memory and $h \in (0,\infty)$ a step width, generated by a certain class of numerical schemes (see [4]) can be described exactly by the recurrence relation (1.5). This recurrence relation is a recurrence relation which does not include only $\{\ \mathbb{E}[\ X_n\ ]\ \}_{n\in -k(h)+\mathbb{N}}$. Instead it is a recurrence relation on the vector of all comoments $\{\ \mathbb{E}[\ X_{n-i}\,X_{n-j}\ ]\ \}_{i,j=0,\ldots,k(h)}$. So the previous lemma generalizes this result to the time-continuous case.
(ii) In the introductory section it was observed that for the numerical scheme „Euler explicit“ the linear recurrence relation can be restated as a representation of difference quotients $(\ \mathbb{E}[\ X_{n+1}^2\ ] - \mathbb{E}[\ X_n^2\ ])/h$ and $(\ \mathbb{E}[\ X_{n+1}\,X_{n-r}\ ] - \mathbb{E}[\ X_n\,X_{n-r}\ ])/h$, $r \in \{0,\ldots,k(h)\}$, in terms of a function of $\{\ \mathbb{E}[\ X_{n-r}\ \}_{r=0,\ldots,k(h)}$.

Hence following the observations of remark 2.5.(ii) we consider now in addition to the previous difference quotients in m_2^X or equivalently of K_2^X in direction $(1,1)$ the difference quotients

$$
\frac{1}{h}(\ K_2^X(t+h,r) - K_2^X(t,r)\ ) = \frac{1}{h}(\ \mathbb{E}[\ X(t+h)\,X(r)\ ] - \mathbb{E}[\ X(t)\,X(r)\ ]\ ) \qquad (2.14)
$$

where $t \in [0,\infty)$, $r \in [-\tau,t)$, $h \in (-t,\infty)$. In order to show the existence of the limit of (2.14) for $h \downarrow 0$ we proceed as before and represent $\Delta_X(t,h)$ with integrals coming from the SDDE (2.1). It is easy to see that for h with $|h| < |t-r|$ it holds:

$$
\begin{aligned}
&\tfrac{1}{h}(\,\mathbb{E}[\,(X(t+h)\,X(r)\,]-\mathbb{E}[\,X(t)\,X(r)\,]\,)\\
&\quad= \tfrac{1}{h}\mathbb{E}[\,(\int_t^{t+h}\int_{-\tau}^{0}X(s+u)\,da(u)\,ds+\int_t^{t+h}\int_{-\tau}^{0}X(s+u)\,db(u)\,dW(s)\,)\,X(r)\,]\\
&\underset{Fubini}{=}\ \tfrac{1}{h}(\int_t^{t+h}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)\int_t^{t+h}X(s+u)\,dW(s)\,]\,db(u)\,)\\
&\quad= \tfrac{1}{h}(\int_t^{t+h}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)\mathbb{E}[\int_t^{t+h}X(s+u)\,dW(s)\ \mid\ \mathcal{F}_t\,]\,]\,db(u)\,)\\
&\underset{Itô-integral}{=}\ \tfrac{1}{h}(\int_t^{t+h}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)0\,]\,db(u)\,)\\
&\quad= \tfrac{1}{h}(\int_t^{t+h}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds,
\end{aligned}
$$

$$
\lim_{h\downarrow 0}\tfrac{1}{h}(\,\mathbb{E}[\,(X(t+h)-X(t))X(r)\,)=\int_{-\tau}^{0}\mathbb{E}[\,X(t+u)X(r)\,]\,da(u). \tag{2.15}
$$

The same procedure can be applied to show the existence and to determine the limit of the difference quotient

$$
\frac{1}{-h}(\,K_2^X(t-h)-K_2^X(t)\,)=\frac{1}{-h}(\,\mathbb{E}[\,(X(t-h)-X(t))X(r)\,] \tag{2.16}
$$

where $t\in(0,\infty)$, $r\in[-\tau,t)$, $h\in[0,t-r)$. This allows us to conclude:

LEMMA 2.6
Let X be a solution of (2.1).
Then it holds:

$$
\begin{aligned}
&(i)\quad \lim_{h\downarrow 0}\tfrac{1}{h}(\,\mathbb{E}[\,X(t+h)\,X(r)\,]-\mathbb{E}[\,X(t)\,X(r)\,])=\int_{-\tau}^{0}\mathbb{E}[\,X(t+u)\,X(r)\,]\,da(u),\quad t\in[0,\infty),\ r\in[-\tau,t),\\
&(ii)\quad \lim_{h\downarrow 0}\tfrac{1}{h}(\,\mathbb{E}[\,X(t-h)\,X(r)\,]-\mathbb{E}[\,X(t)\,X(r)\,])=\int_{-\tau}^{0}\mathbb{E}[\,X(t+u)\,X(r)\,]\,da(u),\quad t\in(0,\infty),\ r\in[-\tau,t),\\
&(iii)\quad \lim_{h\to 0}\tfrac{1}{h}(\,\mathbb{E}[\,X(t+h)\,X(r)\,]-\mathbb{E}[\,X(t)\,X(r)\,])=\int_{-\tau}^{0}\mathbb{E}[\,X(t+u)\,X(r)\,]\,da(u),\quad t\in(0,\infty),\ r\in[-\tau,t).
\end{aligned}
$$

Proof:
(i) Follows from (2.15).
(ii) Deferred to the appendix.
(iii) Follows from (i) and (ii).

□

The properties of comoments of $\mathbb{E}[\,X(t)\,X(s)\,]$, $t,s\in[-\tau,\infty)$, which have been presented in lemma 2.4 and lemma 2.6 can be extended by the observation that

$$
\begin{aligned}
&\forall t_1,t_2\in[-\tau,\infty):\quad \mathbb{E}[\,X(t_1)\,X(t_2)\,]=\mathbb{E}[\,X(t_2)\,X(t_1)\,]\ ,\\
&\forall t_1,t_2\in[-\tau,0]:\quad \mathbb{E}[\,X(t_1)\,X(t_2)\,]=\mathbb{E}[\,\xi(t_1)\,\xi(t_2)\,]\ .
\end{aligned} \tag{2.17}
$$

So collecting the results of lemma 2.4, lemma 2.17 and (2.17) we see that K_2^X satisfies the system

$$
\begin{aligned}
&\tfrac{d}{dx}K(x,x)_{|x=t} &&= 2\int_{-\tau}^{0}K(t+u,t)\,da(u)+\int_{-\tau}^{0}\int_{-\tau}^{0}K(t+u,t+v)\,db(u)\,db(v),\ t\in(0,\infty)\ ,\\
&\tfrac{d}{dx}K(x,x)_{|x=0+} &&= 2\int_{-\tau}^{0}K(u,t)\,da(u)+\int_{-\tau}^{0}\int_{-\tau}^{0}K(u,v)\,db(u)\,db(v),\\
&\tfrac{d}{dx}K(x,r)_{|x=t} &&= \int_{-\tau}^{0}K(t+u,r)\,da(u), && t\in(0,\infty),r\in[-\tau,t)\ ,\\
&\tfrac{d}{dx}K(x,r)_{|x=0+} &&= \int_{-\tau}^{0}K(u,r)\,da(u),\\
&K(t_1,t_2) &&= K(t_2,t_1), && t_1,t_2\in[-\tau,\infty)\ ,\\
&K(t_1,t_2) &&= k(t_1,t_2), && t_1,t_2\in[-\tau,0]
\end{aligned} \tag{2.18}
$$

with the initial function $k(t_1,t_2) = \mathbb{E}[\,X(t_1),X(t_2)\,]$, $t_1,t_2 \in [-\tau,0]$.

DEFINITION 2.7
The system (2.18) of equations is called **amplified system**.
A function $K \in [-\tau,\infty)^2,\mathbb{R})$ satisfying (2.18) is called **solution of (2.18)**.
A function $k = K_{|[-\tau,0]^2} \in ([-\tau,0]^2,\mathbb{R})$ is called **initial function (of the amplified system)**.

2.3 Solutions of the amplified system

In the previous subsection we considered the comoment function K_2^X of a solution X of (2.1) and observed that it satisfies the amplified system (2.18). In this subsection we want to consider amplified systems (2.18) and collect some properties their solutions. First of all we see that the solutions of (2.18) are symmetric functions. This makes it sometimes convenient to consider solutions of (2.18) on only selected subsets of $\mathbb{R}^2$. That is why we introduce the set $\mathbb{K}^2_{+,\geq}(c)$ along with further helpful sets.

DEFINITION 2.8
Let $c \in \mathbb{R}$.

$$\begin{array}{lrcl}
\text{Define} & \Delta(\mathbb{R}^2) & := & \{\,(t,t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\,\}\ , \\
& \mathbb{K}^2_+(c) & := & \{\,(t_1,t_2) \in [c,\infty)^2\,\}\ , \\
& \mathbb{K}^2_{+,\Delta}(c) & := & \mathbb{K}^2_+(c) \setminus \Delta(\mathbb{R}^2)\ , \\
& \mathbb{K}^2_{+,>}(c) & := & \{\,(t_1,t_2) \in [c,\infty)^2 \mid t_1 > t_2\,\}\ , \\
& \mathbb{K}^2_{+,\geq}(c) & := & \{\,(t_1,t_2) \in [c,\infty)^2 \mid t_1 \geq t_2\,\}\ .
\end{array}$$

In addition to the symmetry, we see that solutions of the amplified system (2.18) have a special structure. This special structure consists in that the dynamics of solutions of (2.18) on $\mathbb{K}^2_+(-\tau)$ can be decomposed into a dynamics along the ray into direction $(1,1)$ and starting in $(-\tau,-\tau)$ and a dynamics along rays into directions $(1,0)$ and starting points on the diagonal $\Delta(\mathbb{R}^2) \cap \mathbb{K}^2_+(-\tau)$. In order to make this rigorous we introduce the functions:

DEFINITION 2.9

$$\begin{array}{rcll}
y_s(t) & = & K(t,s), & s \in [-\tau,\infty),\ t \in [s,\infty), \\
z(t) & = & K(t,t), & t \in [-\tau,\infty).
\end{array} \tag{2.19}$$

and reformulate the amplified system (2.18) as an integral equation, that is for all $(t,s) \in \mathbb{K}^2_+(-\tau)$:

$$\begin{array}{lll}
y_s(t) = z(s) + \int_s^{t\wedge(s+\tau)} \int_{-\tau}^{s-r} y_{r+u}(s)\,da(u)\,dr + \int_s^t \int_{-\tau\vee(s-r)}^0 y_s(r+u)\,da(u)\,dr, & & s \in (0,\infty) \\
y_s(t) = \int_0^t \int_{-\tau}^{-\tau\vee-r} y_{(r+u)\wedge s}((r+u)\vee s)\,da(u)\,dr + \int_0^t \int_{-\tau\vee-r}^0 y_s(r+u)\,da(u)\,dr, & & s \in [-\tau,0], t \in [0,\infty) \\
y_s(t) = k(t,s), & & t \in [-\tau,0], \\
z(t) = z(s) + 2\int_{s-\tau}^t \int_{-\tau}^0 y_{r+u}(r)\,da(u)\,dr + \int_{s-\tau}^t \int_{-\tau}^0 \int_{-\tau}^0 y_{r+(u\wedge v)}(r+(u\vee v))\,db(u)\,db(v), & & s \in (0,\infty), \\
z(t) = k(t,t), & & t \in [-\tau,0].
\end{array} \tag{2.20}$$

This shows that for all $s \in [-\tau,\infty)$ the function y_s satisfies the deterministic DDE

$$\begin{array}{rcll}
x'(t) & = & \int_{-\tau}^0 x(t+u)\,da(u), & t \in [s\vee 0,\infty), \\
x(t) & = & \begin{cases} z(t), & s \in (0,\infty),\ t = s, \\ y_{t\wedge s}(t\vee s), & s \in (0,\infty),\ t \in [s-\tau,s), \\ k(t,s) & s \in [-\tau,0],\ t \in [-\tau,0]. \end{cases}
\end{array} \tag{2.21}$$

In particular we observe that for all $s \in [-\tau,\infty)$ the functions y_s solve a linear DDE with the same linear parameter function a but with specific start functions. For $s \in (0,\infty)$ the start function of y_s is y_s on

the interval $[s-\tau, s]$ and is a function which takes the values $y_r(s)$, $r \in [s-\tau, s)$ and $z(s)$ at $r = s$. For $s \in [-\tau, 0]$ however, the start function of y_s is y_s on the interval $[-\tau, \tau]$, where it is completely determined by the values of the initial function k.

Furthermore, the function z satisfies the equation

$$\begin{aligned} z'(t) &= \int_{-\tau}^{0} y_{t+u}(t)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} y_{t+(u\wedge v)}(t+(u\vee v))\, db(u)\, db(v), \quad t \in [0,\infty), \\ z(0) &= k(0,0). \end{aligned} \tag{2.22}$$

Next we consider the question of existence and uniqueness of solutions of the amplified system. In there previous subsection we have seen, that if the functions $a, b \in ([-\tau, 0], \mathbb{R})$ are suitable parameter functions and if $k = \mathbb{E}[\,\xi(\_)\,\xi(\_)\,] \in ([-\tau, 0]^2$ is a suitable initial function ξ of (2.1) that then the comoment function of the corresponding solution X of (2.1) provides a solution of (2.20). Here we want to find conditions on a, b, k which do not rely on an SDDE and which allow the system (2.20) to have a unique solution.

We start with a simple but helpful lemma.

LEMMA 2.10
Let K_1, K_2 be two solutions of the amplified system (2.20).

Then it holds:
(i) $\forall\, \lambda_1, \lambda_2 \in \mathbb{R}$: $\lambda_1 K_1 + \lambda_2 K_2$ is a solution of (2.20).
(ii) Let $\Delta := K_2 - K_1$. Then Δ satisfies the system:

$$\begin{aligned} &\forall t \in [0,\infty): \\ &\quad \Delta K(t,t) = \Delta K(0,0) + 2\int_{0}^{t}\int_{-\tau}^{0} \Delta K(u+r, r)\, da(u)\, dr + \int_{0}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0} \Delta K(u+r, v+r)\, db(u)\, db(v)\, dr, \\ &\forall s \in [-\tau,\infty), t \in (s\vee 0, \infty): \\ &\quad \Delta K(t,s) = \Delta K(s\vee 0, s) + \int_{s\vee 0}^{t}\int_{-\tau}^{0} \Delta K(u+r, s)\, da(u)\, dr, \end{aligned} \tag{2.23}$$

Proof:
(i) Obvious.
(ii) (2.23) is the integral form of (2.1). □

2.10.(ii) indicates that a growth bound for the difference of two solutions of (2.20) in terms of the difference of their initial functions can be gained by the application of a Gronwall-type lemma. However, as Δ is two dimensional, this requires a suitable choice of a one-dimensional „error"-function which quantifies the difference Δ and to which the Gronwall lemma can be applied. The following lemma provides the result.

DEFINITION 2.11
Let ΔK solution of (2.20). Define
(i) $C(a,b,\tau) := 2\int_{-\tau}^{0} d|a|(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} d|b|(u)\, d|b|(v) < \infty$,
(ii) $\mathcal{K}_{sup}([-\tau,0]^2, \mathbb{R}) := \{\, k \in ([-\tau,0]^2, \mathbb{R}) \mid \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} < \infty \,\}$, the class of bounded initial functions,
(iii) $\delta_{sup}(t) := \sup\{\, \mid \Delta K(u,v) \mid \mid \; u, v \in [-\tau, t] \,\}$, an error function.

LEMMA 2.12
Let K_1, K_2 be solutions of the amplified system (2.18) with respective initial functions k_1, k_2,
$k_2 - k_1 \in \mathcal{K}_{sup}([-\tau,0]^2, \mathbb{R})$.
Then it holds:

$$\forall t, s \in [-\tau,\infty): \qquad |K_2(t,s) - K_1(t,s)| \;\le\; \| k_2 - k_1 \|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}\, e^{C(a,b,\tau)\,(t\vee s\vee 0)}$$

Proof:

(1) By lemma 2.10.(ii) it follows for all $t \in (0,\infty)$:

$$|\Delta K(t,t)| \leq \delta_{sup}(0) + 2 \int_{0}^{t} \int_{-\tau}^{0} d|a|(u)\, \delta_{sup}(r)\, dr + \int_{0}^{t} \int_{-\tau}^{0} \int_{-\tau}^{0} d|b|(u)\, d|b|(v)\, \delta_{sup}(r)\, dr$$
$$\leq \delta_{sup}(0) + \int_{0}^{t} C(a,b,\tau)\, \delta(r)\, dr \quad .$$

Hence with lemma 2.10.(ii) it also follows for all $t \geq s \vee 0 \geq -\tau$:

$$\begin{aligned}
|\Delta K(t,s)| &\leq |\Delta K(s \vee 0, s)| + \int_{s\vee 0}^{t} \int_{-\tau}^{0} d|a|(u)\, \delta_{sup}((r+u) \vee s \vee 0))\, dr \\
&\leq \delta_{sup}(0) + \int_{0}^{s\vee 0} C(a,b,\tau)\, \delta_{sup}(r)\, dr + \int_{s\vee 0}^{t} \int_{-\tau}^{0} d|a|(u)\, \delta_{sup}((r+u) \vee s \vee 0)\, dr \\
&\leq \delta_{sup}(0) + \int_{0}^{s\vee 0} C(a,b,\tau)\, \delta_{sup}(r)\, dr + \int_{s\vee 0}^{t} \int_{-\tau}^{0} d|a|(u)\, \delta_{sup}(r)\, dr \\
&\leq \delta_{sup}(0) + \int_{0}^{t} C(a,b,\tau)\, \delta_{sup}(r)\, dr.
\end{aligned}$$

(3) From (2) we can conclude that for all $t \in [0,\infty)$:

$$\sup\{\, |\, \Delta K(t,r)\, |\, |\ r \in [-\tau,t]\,\} \leq \delta_{sup}(0) + \int_{0}^{t} C(a,b,\tau)\, \delta_{sup}(r)\, dr$$

(4) Let $t \in [0,\infty)$, $s \in [-\tau,t)$. From (3) we can conclude

$$\begin{aligned}
\sup\{\, |\, \Delta K(s,r)\, |\ |\ r \in [-\tau,s]\,\} &\leq \delta_{sup}(0) + \int_{0}^{s\vee 0} C(a,b,\tau)\, \delta_{sup}(r)\, dr \\
&\leq \delta_{sup}(0) + \int_{0}^{t} C(a,b,\tau)\, \delta_{sup}(r)\, dr
\end{aligned}$$

(5) Let $t \in [0,\infty)$. From (3), (4) we obtain:

$$\delta_{sup}(t) \leq \delta_{sup}(0) + \int_{0}^{t} C(a,b,\tau)\, \delta_{sup}(r)\, dr$$

The lemma of Gronwall shows then that $\forall t \in [0,\infty) : \delta_{sup}(t) \leq \delta_{sup}(0)\, e^{C(a,b,\tau)\, t}$.

But then it holds for all $t \geq (s \vee 0) \geq -\tau$: $|\Delta K(t,s)| \leq \delta_{sup}(0)\, e^{C(a,b,\tau)\, t}$.

As for all $t,s \in [-\tau,0]$ it holds: $|\Delta K(t,s)| \leq \delta_{sup}(0)$, as ΔK is symmetric and as $K_{1\,|[-\tau,0]} = k_1$, $K_{2\,|[-\tau,0]} = k_2$, the proof is complete.

□

A reconsideration of (1) in the above proof reveals that the supremum in the factor $\| k_2 - k_1 \|_{C([-\tau,0]^2,\mathbb{R})}$ is the result of the estimations:

$$\begin{aligned}
&|\Delta K(0,0) + \int_{0}^{t\wedge\tau} \int_{-\tau}^{-r} \int_{-\tau}^{-r} \Delta K(u+r, v+r)\, db(u)\, db(v)\, dr| \\
&\qquad \leq |\Delta K(0,0)| + \int_{0}^{t\wedge\tau} \int_{-\tau}^{-r} \int_{-\tau}^{-r} |\Delta K(u+r, v+r)|\, d|b|(u)\, d|b|(v)\, dr \\
&|\Delta K(s,0) + \int_{0}^{s\wedge\tau} \int_{-\tau}^{-r} \Delta K(u+r, s)\, da(u)\, dr|, \\
&\qquad \leq |\Delta K(s,0)| + \int_{0}^{s\wedge\tau} \int_{-\tau}^{-r} |\Delta K(u+r, s)|\, d|a|(u)\, dr,\ s \in [-\tau,0].
\end{aligned}$$

Hence the result of lemma 2.12 can be improved by a more careful choice of the error function and a refined analysis.

DEFINITION 2.13
Let ΔK solution of (2.20). Define:

(i) $\forall k \in (\,[-\tau,0]^2, \mathbb{R}) \;:$

$$\begin{aligned}\|k\|_{\mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})} \;=\; & \sup_{t,s\in[-\tau,0],t\vee s=0} \{\,|\,k(t,s))\,|\,\} \\ & + 2 \sup_{s\in[-\tau,0]} \{ \int_0^{\tau} \int_{-\tau}^{-r} |\,k(r+u,s)\,|\,d|a|(u)\,dr\,\} \\ & + \int_0^{\tau} \int_{-\tau}^{-r} \int_{-\tau}^{-r} |\,k(r+u,r+v)\,|\,d|b|(u)\,d|b|(v)\,dr\,\} \;.\end{aligned}$$

(ii) $\mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})$

$$\begin{aligned}:= \;\{\, k \in ([\,-\tau,0\,]^2,\mathbb{R}) \;\mid\; & \forall\, 0 \geq t \geq s \geq -\tau\colon \quad k(t,s) = k(s,t), \\ & \forall\, 0 \geq s \geq -\tau\colon \quad k(\_,s) \in \mathcal{L}^1([-\tau,0],\,\mathbb{R},\,d|a|\,), \\ & k \in \mathcal{L}^1([-\tau,0]^2,\,\mathbb{R},\,d|b| \otimes d|b|\,), \\ & \|k\|_{\mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})} < \infty\,\},\end{aligned}$$

(iii)

$$\delta(t) = \begin{cases} \sup\{\,|\Delta K(t,s)| \;\mid\; (t,s) \in [-\tau,\infty)^2 \setminus [-\tau,0]^2\,\}, & t \in (0,\infty), \\ \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} & t = 0. \end{cases}$$

LEMMA 2.14

Let K_1, K_2 be solutions of system (2.18) with the initial function k_1 and k_2, respectively,

$k_2 - k_2 \in \mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})$.

Then it holds:

$$\forall\, t,s \in [-\tau,\infty) : \quad |\,K_2(t,s) - K_1(t,s)\,| \leq \|\,k_2 - k_1\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} \;\; e^{C(a,b,\tau)(t\vee s\vee 0)} \;.$$

Proof:

(1) For the following estimations we use the set

$$A_\tau := \{\,(t,u) \in \mathbb{R}^2 \mid t \in [-\tau,\tau], u \in [-t \wedge -\tau, -t]\,\} \;.$$

A_τ describes tuples (t,u) for which $t+u \in [-\tau,0]$.

Furthermore, for denotational convenience define

$$S^0_\tau := \sup_{t,s\in[-\tau,0],t\vee s=0} \{\,|\,\Delta K(t,s)\,|\,\} \;,$$

$$S^1_\tau := \sup_{s\in[-\tau,0]} \{ \int_0^{\tau} \int_{-\tau}^{-r} |\,k(r+u,s)\,|\,d|a|(u)\,dr\,\} \;,$$

$$S^2_\tau := \int_0^{\tau} \int_{-\tau}^{-r} \int_{-\tau}^{-r} |\,k(r+u,r+v)\,|\,d|b|(u)\,d|b|(v)\,dr \;.$$

(2) For all $t \in (0,\infty)$ it holds:

$$\Delta K(0,0) \quad \leq \quad S^0_\tau \;,$$

$$|\int_0^t\int_{-\tau}^0 \Delta K(r+u,r)\,da(u)\,dr\,| \;\le\; \int_0^t\int_{-\tau}^0 |\,\Delta K(r+u,r)|\,d|a|(u)\,dr$$
$$\le\; \int_0^t\int_{-\tau}^0 |\,\delta(r)\,d|a|(u)\,dr \;\le\; \int_{-\tau}^0 d|a|(u)\int_0^t \delta(r)\,dr\;,$$

$$|\int_0^t\int_{-\tau}^0\int_{-\tau}^0 \Delta K(r+u,r+v)\,db(u)\,db(v)\,dr\,|$$
$$\le\; \int_0^t\int_{-\tau}^0\int_{-\tau}^0 |\,\Delta K(r+u,r+v)\,|\,d|b|(u)\,d|b|(v)\,dr$$
$$\le\; S_\tau^2 + \int_0^t\int_{-\tau}^0\int_{-\tau}^0 \delta(r)\,d|b|(u)\,d|b|(v)\,dr$$
$$\le\; S_\tau^2 + (\int_{-\tau}^0 d|b|(u)\,)^2\int_0^t \delta(r)\,dr\;.$$

By 2.10.(ii) then it follows:

$$|\Delta K(t,\dots,t)\,| \;\le\; S_\tau^0 + S_\tau^2 + C(a,b,\tau)\int_0^t \delta(r)\,dr \;\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr\;.$$

(3) Let $\infty > s \ge -\tau$. Then for all $t \in (0 \vee s, \infty)$ it holds:

$$|\,\Delta K(t,s)\,| \;\le\; |\,\Delta K(0\vee s,s)\,| + |\int_{0\vee s}^t\int_{-\tau}^0 \Delta K(r+u,s)\,da(u)\,dr\,|$$
$$\le\; (\,\|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^s \delta(r)\,dr + \int_s^t\int_{-\tau}^0 \delta(r)\,d|a|(u)\,dr\,)\,1_{(0,\infty)}(s)$$
$$+(\,S_\tau^0 + \int_0^t\int_{-\tau}^0 \delta(r)\,d|a|(u)\,(1-1_{A_\tau}((t,u)))\,dr + S_\tau^1\,)\,1_{[-\tau,0]}(s)$$
$$\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;1_{(0,\infty)}(s) + (\,S_\tau^0 + S_\tau^1\,)\,1_{[-\tau,0]}(s)$$
$$+\,C(a,b,\tau)\int_0^{0\vee s}\delta(r)\,dr + \int_{-\tau}^0 d|a|(u)\int_{0\vee s}^t \delta(r)\,dr$$
$$\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr.$$

(4) Let $t \in (0,\infty)$. Due to (2), (3), $\forall\,\bar s \ge \tilde s \ge -\tau$, $t > \bar s \ge 0$, $t > s \ge -\tau$ it holds:

$$|\,\Delta K(\bar s,\tilde s)\,| \;\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^{\bar s}\delta(r)\,dr$$
$$\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr,$$
$$|\,\Delta K(t,t)\,| \;\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr\;,$$
$$|\,\Delta K(t,s)\,| \;\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr\;.$$

Hence, using the definition of $\delta(0)$, for all $t \in [0,\infty)$ it holds:

$$\delta(t) \;=\; \sup\{\,|\Delta K(\bar s,\tilde s)|\;\mid\;(\bar s,\tilde s) \in [-\tau,\infty)^2 \setminus [-\tau,0]^2\,\}$$
$$\le\; \|\,\Delta K\,\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + C(a,b,\tau)\int_0^t \delta(r)\,dr.$$

Applying the lemma of Gronwall to δ on $[0,\infty)$ we get the desired result. □

This growth bound for the difference of solutions is used now to show a criterion for uniqueness of solutions.

COROLLARY 2.15
Let K_1, K_2 be two solutions of the amplified system (2.18) with the same initial function $k \in \mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})$. Then it holds:

$$K_1 = K_2.$$

Proof:
$\forall t,s \in \mathbb{K}_+^2(-\tau) \setminus [-\tau,0]^2$ it holds:

$$|K_2(t,s) - K_1(t,s)| \;\le\; \|\,k-k\,\|_{\mathcal{K}_{a,b}(\,[-\tau,0]^2,\mathbb{R})}\,e^{\,C(a,b,\tau)\,(t\vee s\vee 0)} \;=\; 0.$$

As $K_{1\,|[-\tau,0]} = k = K_{2\,|[-\tau,0]}$, the proof is complete.

□

Furthermore we obtain trivially:

COROLLARY 2.16
Let K be a solution of the amplified system (2.20) with the initial function $k = 0$.
Then it holds:
$K \equiv 0$.

Proof:
$K_0 \equiv 0$ is a solution of the amplified system with the zero initial function.
Corollary 2.15 shows that K_0 is the unique solution.

□

Combining lemma 2.12 and corollary 2.16 we obtain a growth bound for solutions of (2.18)

COROLLARY 2.17
Let K be a solution of the amplified system (2.18) with the initial function $k \in \mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})$.
Then it holds:
$\forall t,s \in [-\tau,\infty): \quad |K(t,s)| \leq \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\, e^{C(a,b,\tau)\,(t\vee s\vee 0)}$.

Proof:
Let $K_2 = K$, $K_1 = 0$ and apply lemma 2.12 .

□

Finally we consider the existence of solutions of the amplified system. As a „tool“ we use the well-known method of Picard-iterates.

DEFINITION 2.18
$\forall (t,s) \in [-\tau,\infty)^2$:
$K_0(t,s) := k(0\wedge t, 0\wedge s)$
$\forall\, n \in \mathbb{N}$ define:

$$0 \geq t,\ s \geq -\tau: \qquad K_{n+1}(t,s) = k(t,s)\ ,$$

$$t \geq 0: \qquad K_{n+1}(t,t) = K_n(0,0) + 2\int_0^t\int_{-\tau}^0 K_n(r+u,r)\,da(u)\,dr + \int_0^t\int_{-\tau}^0\int_{-\tau}^0 K_n(r+u,r+v)\,db(u)\,db(v)\,dr,$$

$$s \geq -\tau,\ t \geq 0\vee s: \quad K_{n+1}(t,s) = K_n(0\vee s,s) + \int_{0\vee s}^t\int_{-\tau}^0 K_n(r+u,s)\,da(u)\,dr.$$

The sequence $\{K_n\}_{n\in\mathbb{N}}$ is called **Picard-iterations** and the K_n, $n \in \mathbb{N}$, are called the **Picard-iterates**.

Theorem 2.19
Let K be a solution of (2.18) with the initial function $k \in \mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})$.
Then it holds:
(i) $\forall T \in [0,\infty)$: *$\{K_n\}_{n\in\mathbb{N}}$ forms a Cauchy-sequence on $[-\tau,T]^2$ with norm $\|\ \|_{\mathcal{C}([-\tau,T],\mathbb{R})}$.*
(ii) The system (2.18) has a solution K.
(iii) $\forall\, t,s \in [-\tau,T]: \ \|K(t,s) - K_n(t,s)\| \leq \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\ 2\,(e^{C(a,b,\tau)\,T} - \sum_{k=0}^{n+1} \frac{(C(a,b,\tau)\,T)^k}{k!})$.

Proof: (of theorem 2.22)
Deferred to section 5.5 .

□

COROLLARY 2.20
Let $k \in \mathcal{K}_{sup}$.
Then it holds:
(i) $\|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} \leq (1 + \tau\, C(a,b,\tau))\,\|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$,
(ii) $k \in \mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})$.

Proof:

(i) $\quad \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} \leq \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} + 2\tau \int_{-\tau}^{0} d|a|(u)\|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} + \tau \left(\int_{-\tau}^{0} d|b|(u)\right)^2 \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$

$\leq (1 + \tau C(a,b,\tau)) \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$.

(ii) As $\|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} < \infty$ and due to (i), $\|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} < \infty$.

□

We have characterized conditions for the unique existence of solutions of the system (2.18). We also know that comoment functions K_2^X of solutions X of linear SDDE (2.1) are solutions of the system (2.18). This raises the question whether all solutions K of the system (2.18) are a comoment function K_2^X of a linear SDDE (2.1). This cannot be confirmed.

LEMMA 2.21
Let $k \in \mathcal{C}([-\tau,0]^2,\mathbb{R})$ with $k(t,t) < 0$ for all $t \in [-\tau,0]$.
Then it holds:

The solution K of system (2.18) with initial function k is not a comoment function of a solution of the linear SDDE (2.1).

Proof:
k is negative on $[-\tau,0]^2 \cap \Delta(\mathbb{R}^2)$ whereas comoment functions are non-negative on $[-\tau,0]^2 \cap \Delta(\mathbb{R}^2)$.
□

2.4 Some properties of comoment functions satisfying the amplified system

In this subsection we consider some properties of comoment functions satisfying the amplified system. We start with the continuity.

Theorem 2.22
Let K be a solution of (2.18) with the initial function $k \in \mathcal{C}([-\tau,0]^2,\mathbb{R})$.
Then it holds:

The system (2.18) has a continuous solution.

Proof:

(1) As in theorem 2.14 we consider the Picard-iterates introduced in (2.18).
Then $\forall (t,s) \in [-\tau,\infty)^2$ it holds:

$$K_{n+1}(t,s) = K_n(0\wedge t, 0\wedge s) + 2\int_0^{0\vee s}\int_{-\tau}^{0} K_n(r+u,r)\,da(u)\,dr + \int_0^{0\vee s}\int_{-\tau}^{0}\int_{-\tau}^{0} K_n(r+u,r+v)\,db(u)\,db(v)\,dr + \int_{0\vee s}^{0\vee t}\int_{-\tau}^{0} K_n(r+u,s)\,da(u)\,dr.$$

(2) By the assumption on k, K_0 is continuous on $[-\tau,\infty)^2$.

(3) Fix $n \in \mathbb{N}$ and assume, that K_n is continuous. Let $t \geq s \geq -\tau$, $\bar{t} \geq \bar{s} \geq -\tau$ w.o.l.g.. Then it holds:
$K_{n+1}(t,s) - K_{n+1}(\bar{t},\bar{s})$

$$\begin{aligned}
&= K_n(0\wedge t, 0\wedge s) - K_n(0\wedge \bar{t}, 0\wedge \bar{s}) \\
&\quad 2\Big(\int_0^{0\vee s}\int_{-\tau}^{0} K_n(r+u,r)\,da(u)\,dr - \int_0^{0\vee \bar{s}}\int_{-\tau}^{0} K_n(r+u,r)\,da(u)\,dr\Big) \\
&\quad + \int_0^{0\vee s}\int_{-\tau}^{0}\int_{-\tau}^{0} K_n(r+u,r+v)\,db(u)\,db(v)\,dr - \int_0^{0\vee \bar{s}}\int_{-\tau}^{0}\int_{-\tau}^{0} K_n(r+u,r+v)\,db(u)\,db(v)\,dr \\
&\quad + \int_{0\vee s}^{0\vee t}\int_{-\tau}^{0} K_n(r+u,0\vee s) - K_n(r+u,0\vee \bar{s})\,da(u)\,dr \\
&\quad + \int_{0\vee s}^{0\vee t}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr - \int_{0\vee(s\wedge\bar{s})}^{0\vee t}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr \\
&\quad + \int_{0\vee(s\wedge\bar{s})}^{0\vee t}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr - \int_{0\vee(s\wedge\bar{s})}^{0\vee \bar{t}}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr \\
&\quad + \int_{0\vee(s\wedge\bar{s})}^{0\vee \bar{t}}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr - \int_{0\vee \bar{s}}^{0\vee \bar{t}}\int_{-\tau}^{0} K_n(r+u,0\vee \bar{s})\,da(u)\,dr
\end{aligned}$$

Hence:

$$\begin{aligned}
&|K_{n+1}(t,s) - K_{n+1}(\bar{t},\bar{s})| \\
&\leq |K_n(0\wedge t, 0\wedge s) - K_n(0\wedge \bar{t}, 0\wedge \bar{s})| \\
&\quad + C(a,b,\tau) \sup_{r\in[0\vee(s\wedge\bar{s}),0\vee(s\vee\bar{s})],u\in[-\tau,0]} \{|K_n(r+u,r)|\} + \sup_{u\in[0\vee s-\tau,0\vee t]} \{|K_n(u,s) - K_n(u,\bar{s})|\} \\
&\quad + 2 \sup_{r\in[0\vee(s\wedge\bar{s}),0\vee(s\vee\bar{s})],u\in[-\tau,0]} \{|K_n(r+u,\bar{s})|\} + \sup_{r\in[0\vee(0\vee t)\wedge(0\vee\bar{t})),0\vee(t\vee\bar{t})],u\in[-\tau,0]} \{|K_n(r+u,\bar{s})|\} \\
&\quad \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

as K_n is continuous.
But then K_{n+1} is also continuous. By complete induction one obtains that $\forall n \in \mathbb{N} : K_n$ is continuous.

(4) As $\|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} \leq C(a,b,\tau)\,\|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} < \infty$, we know from lemma 2.22, that for all $T \in [0,\infty)$ $\{K_n\}_{n\in\mathbb{N}}$ converges in the $\mathcal{C}([-\tau,0]^2,\mathbb{R})$ to a limit K on each compact set $[-\tau,T]$. In (11) of the proof of lemma 2.22 we derived, that for a all $T \in [0,\infty)$, $t,s \in [-\tau,T]^2 - [-\tau,0]^2$, $n,m \in \mathbb{N}$:

$$\begin{aligned}
|K_{n+1}(t,s) - K_{n+m+1}(t,s)| &\leq \sum_{k=n+1}^{n+m+1} |\Delta K_k(t,s)| \\
&\leq \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\; 2\,(e^{C(a,b,\tau)\,T} - \sum_{k=0}^{n+1} \tfrac{(C(a,b,\tau)\,T)^k}{k!}) \;.
\end{aligned}$$

Hence for all $T \in [0,\infty)$ $\{K_n\}_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\mathcal{C}([-\tau,0]^2,\mathbb{R})$. As $\mathcal{C}([-\tau,0]^2,\mathbb{R})$ is complete, K is continuous. □

In the previous theorem the continuity of a solution K of the amplified system (2.18) could be proved by relying merely on the properties of the initial functions and on the amplified system. For more general than continuous initial functions this is not so easy to achieve anymore. The reason is that the amplified system (2.18) is a system of Gateaux-derivatives and not of Frechet-derivatives. Hence it provides information about the solutions K on rays. However, local properties like continuity of K depend not only on values of K on rays but on full-dimensional environments.
As we here are rather interested in comoment functions K_2^X of solutions of SDDE (2.1) than in general solutions of the amplified system (2.18), we limit in the following to these functions. The comoment functions K_2^X however have additional properties which are a consequence of the SDDE the process X satisfies.

Theorem 2.23
Let $\forall\, t_1,t_2 \in \mathbb{K}^2_+(0) : A(t_1,t_2) := \{\, h=(h_1,h_2) \mid (t_1+h_1,t_2+h_2) \in \mathbb{K}^2_+(0)\,\}$,
X *be a solution of (2.1) with comoment function* K_2^X,
$\forall\, T \in [0,\infty) : \sup_{t\in[-\tau,T]} \{\, \mathbb{E}[\,|X(t)|^2\,]\,\} \le C_T < \infty.$
Then it holds:

(i) $\forall\,\gamma\in(0,\frac{1}{2}), \forall\, t_1,t_2 \in \mathbb{K}^2_+(0) :$ $\lim_{h\in A(t_1,t_2),|h|_2\mapsto 0} \frac{1}{\|h\|_2^\gamma} |K_2^X(t_1+h_1,t_2+h_2) - K_2^X(t_1,t_2)| = 0,$

(ii) *For* $\gamma=\frac{1}{2}, \forall\, t_1,t_2 \in \mathbb{K}^2_+(0) :$ $\overline{\lim_{h\in A(t_1,t_2),|h|_2\mapsto 0}} \frac{1}{\|h\|_2^{\frac{1}{2}}} |K_2^X(t_1+h_1,t_2+h_2) - K_2^X(t_1,t_2)| \le \int_{-\tau}^{0} d|b|(u)\, C_T,$

(iii) K_2^X *is continuous on* $\mathbb{K}^2_+(0)$.

Proof: (of theorem 2.23)
(1) Let $t_1,t_2 \in [0,\infty)$, $h=(h_1,h_2) \in A(t_1,t_2)$. For any $t,h\in\mathbb{R}$ define $M(t,h) := (t+h)\vee t$ and $m(t,h) := (t+h)\wedge t$. Then it holds:

$$
\begin{aligned}
&|\, K_2^X(t_1+h_1,t_2+h_2) - K_2^X(t_1,t_2)\,| \\
\le\;& |\, K_2^X(t_1+h_1,t_2+h_2) - K_2^X(t_1+h_1,t_2)\,| + |\, K_2^X(t_1,t_2+h_2) - K_2^X(t_1,t_2)\,| \\
\le\;& |\, \mathbb{E}[\,(X(t_1+h_1)-X(t_1))\,X(t_2+h_2)\,]\,| + |\, \mathbb{E}[\,X(t_1)\,(X(t_2+h_2)-X(t_2))\,]\,| \\
\le\;& |\, \mathbb{E}[\int_{m(t_1,h_1)\wedge\tau}^{M(t_1,h_1)}\int_{-\tau}^{0} X(u+r)\,da(u)\,dr\,X(t_2+h_2)\,]\,| + |\, \mathbb{E}[\int_{m(t_1,h_1)\wedge\tau}^{M(t_1,h_1)}\int_{-\tau}^{0} X(u+r)\,db(u)\,dW(r)\,X(t_2+h_2)\,]\,| \\
&+|\, \mathbb{E}[\,X(t_1)\int_{m(t_2,h_2)\wedge\tau}^{M(t_2,h_2)}\int_{-\tau}^{0} X(u+r)\,da(u)\,dr\,]\,| + |\, \mathbb{E}[\,X(t_1)\int_{m(t_2,h_2)\wedge\tau}^{M(t_2,h_2)}\int_{-\tau}^{0} X(u+r)\,db(u)\,dW(r)\,]\,| \\
\underset{Fubini}{\le}\;& |\int_{m(t_1,h_1)\wedge\tau}^{M(t_1,h_1)}\int_{-\tau}^{0} \mathbb{E}[\,X(u+r)\,X(t_2+h_2)\,]\,da(u)\,dr\,| + |\int_{-\tau}^{0}\mathbb{E}[\int_{m(t_1,h_1)}^{M(t_1,h_1)} X(u+r)\,dW(r)\,X(t_2+h_2)\,]\,db(u)| \\
&+|\int_{m(t_2,h_2)\wedge\tau}^{M(t_2,h_2)}\int_{-\tau}^{0} \mathbb{E}[\,X(t_1)\,X(u+r)\,]\,da(u)\,dr\,| + |\int_{-\tau}^{0}\mathbb{E}[\,X(t_1)\int_{m(t_2,h_2)}^{M(t_2,h_2)} X(u+r)\,dW(r)\,]\,db(u)\,| \\
\underset{Jensen}{\le}\;& \int_{m(t_1,h_1)\wedge\tau}^{M(t_1,h_1)}\int_{-\tau}^{0} \mathbb{E}[\,|\,X(u+r)\,X(t_2+h_2)\,|\,]\,d|a|(u)\,dr + \int_{-\tau}^{0}\mathbb{E}[\,|\int_{m(t_1,h_1)}^{M(t_1,h_1)} X(u+r)\,dW(r)\,X(t_2+h_2)\,|\,]\,d|b|(u) \\
&+\int_{m(t_2,h_2)\wedge\tau}^{M(t_2,h_2)}\int_{-\tau}^{0} \mathbb{E}[\,|\,X(t_1)\,X(u+r)\,|\,]\,d|a|(u)\,dr + \int_{-\tau}^{0}\mathbb{E}[\,|\,X(t_1)\int_{m(t_2,h_2)}^{M(t_2,h_2)} X(u+r)\,dW(r)\,|\,]\,d|b|(u) \\
\underset{H\ddot{o}lder}{\le}\;& \int_{m(t_1,h_1)\wedge\tau}^{M(t_1,h_1)}\int_{-\tau}^{0} \mathbb{E}[\,|X(u+r)|^2\,]^{\frac{1}{2}}\mathbb{E}[\,|X(t_2+h_2)|^2\,]^{\frac{1}{2}}\,d|a|(u)\,dr \\
&+\int_{-\tau}^{0}(\int_{m(t_1,h_1)}^{M(t_1,h_1)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr)^{\frac{1}{2}}\,\mathbb{E}[\,|X(t_2+h_2)|^2\,]^{\frac{1}{2}}\,d|b|(u) \\
&+\int_{m(t_2,h_2)\wedge\tau}^{M(t_2,h_2)}\int_{-\tau}^{0} \mathbb{E}[\,|X(t_1)|^2\,]^{\frac{1}{2}}\,\mathbb{E}[\,X(u+r)|^2\,]^{\frac{1}{2}}\,d|a|(u)\,dr \\
&+\int_{-\tau}^{0}\mathbb{E}[\,|X(t_1)|^2\,]^{\frac{1}{2}}(\int_{m(t_2,h_2)}^{M(t_2,h_2)} \mathbb{E}[\,|X(u+r)|^2]\,dr)^{\frac{1}{2}}\,d|b|(u) \\
\underset{Fubini}{\le}\;& (\int_{-\tau}^{0}\int_{m(t_1,h_1)}^{M(t_1,h_1)} \mathbb{E}[\,|X(u+r)|^2\,]^{\frac{1}{2}}\,dr\,d|a|(u) + \int_{-\tau}^{0}(\int_{m(t_1,h_1)}^{M(t_1,h_1)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,)^{\frac{1}{2}}\,d|b|(u)\,)\,\mathbb{E}[\,|X(t_2+h_2)|^2\,]^{\frac{1}{2}} \\
&+(\int_{-\tau}^{0}\int_{m(t_2,h_2)}^{M(t_2,h_2)} \mathbb{E}[\,X(u+r)|^2\,]^{\frac{1}{2}}\,dr\,d|a|(u) + (\int_{m(t_2,h_2)}^{M(t_2,h_2)} \mathbb{E}[\,|X(u+r)|^2]\,dr\,)^{\frac{1}{2}}\,d|b|(u)\,)\,\mathbb{E}[\,|X(t_1)|^2\,]^{\frac{1}{2}}
\end{aligned}
$$

(2) For $\gamma \in (0, \frac{1}{2})$, $t_i \in [0, \infty)$, $i = 1, 2$, it holds:

$$\overline{\lim_{h \in A(t_1,t_2), h \mapsto 0}} \frac{1}{|h_i|^{\frac{1}{2}}} \Big(\int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,\Big)^{\frac{1}{2}} \;\leq\; C_{t_i}^{\frac{1}{2}} \;,$$

$$\overline{\lim_{h \in A(t_1,t_2), h \mapsto 0}} \frac{1}{|h_i|^{\frac{1}{2}}} \int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]^{\frac{1}{2}}\,dr \quad\leq\; \overline{\lim_{h \in A(t_1,t_2), h \mapsto 0}} |h_i|^{\frac{1}{2}} C_{t_i}^{\frac{1}{2}} = 0 \;,$$

$$\frac{|h_i|}{\|h\|_2} \in [0, 1] \;.$$

(3) Hence it follows ($i = 1, 2$)

for $\gamma \in (0, \frac{1}{2})$:

$$\lim_{h \in A(t_1,t_2), h \mapsto 0} \frac{1}{\|h\|_2^{\gamma}} \Big(\int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,\Big)^{\frac{1}{2}}$$

$$= \lim_{h \in A(t_1,t_2), h \mapsto 0} |h_i|^{\frac{1}{2}-\gamma} \frac{|h_i|^{\gamma}}{\|h\|_2^{\gamma}} \frac{1}{|h_i|^{\frac{1}{2}}} \Big(\int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,\Big)^{\frac{1}{2}} \;=\; 0 \;,$$

for $\gamma = \frac{1}{2}$:

$$\overline{\lim_{h \in A(t_1,t_2), h \mapsto 0}} \frac{1}{\|h\|_2^{\frac{1}{2}}} \Big(\int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,\Big)^{\frac{1}{2}}$$

$$= \overline{\lim_{h \in A(t_1,t_2), h \mapsto 0}} \frac{|h_i|^{\frac{1}{2}}}{\|h\|_2^{\frac{1}{2}}} \frac{1}{|h_i|^{\frac{1}{2}}} \Big(\int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]\,dr\,\Big)^{\frac{1}{2}} \;\leq\; C_T^{\frac{1}{2}} \;,$$

for $\gamma \in (0, \frac{1}{2}]$:

$$\lim_{h \in A(t_1,t_2), h \mapsto 0} \frac{1}{\|h\|_2^{\gamma}} \int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]^{\frac{1}{2}}\,dr$$

$$= \lim_{h \in A(t_1,t_2), h \mapsto 0} |h_i|^{1-\gamma} \frac{|h_i|^{\gamma}}{\|h\|_2^{\gamma}} \frac{1}{|h_i|} \int_{m(t_i,h_i)}^{M(t_i,h_i)} \mathbb{E}[\,|X(u+r)|^2\,]^{\frac{1}{2}}\,dr \;=\; 0 \;.$$

This proves the theorem. □

In the previous theorem we have proven the continuity of a comoment function K_2^X of a solution X of (2.1). We have proven even more: a Hölder-continuity. The Hölder-continuity indicates that the comoment functions K_2^X have a certain smoothness. So next we want to investigate the Frechet-differentiability of comoment functions K_2^X.

In a first step we investigate whether the trajectories of the SDDE posses already a smoothness property. The motivation is that for a deterministic linear DDE

$$x'(t) = \int_{-\tau}^{0} x(t+u)\,da(u), \qquad t \in [0, \infty)\,, \tag{2.24}$$

it can be shown that the trajectories become increasingly smoother with the time evolving ([2]). However, for solutions X of (2.1) this is not true anymore as the following example of the scalar geometric Brownian Motion shows:

$$\begin{aligned} dX(t) &= \alpha X(t)\,dt + \beta X(t) dW(t), \qquad t \in [0, \infty) \\ X(0) &= \xi \in \mathbb{R} \end{aligned} \tag{2.25}$$

It is a well-known fact that the trajectories of solutions $X(t) = \xi\, e^{(\alpha - \frac{\beta^2}{2})t + \beta W(t)}$ are almost sure nowhere differentiable on $[0, \infty)$.

On the other hand it is known that the expectations of solutions X of (2.1) satisfy the DDE (2.27). Hence the expectations of the trajectories $\mathbb{E}[\,X(t)\,]$ possess the smoothing property on $[0, \infty)$. So it is reasonable to expect more regularity of comoment functions K_2^X than just continuity.

The following theorem confirms this expectation except a minor limitation. It can be shown that the comoment function K_2^X is Frechet-differentiable on $\mathbb{K}^2_{+,\Delta}(\tau)$.

An essential part of the proof consists in using the linearity of expectations, integrals and of showing that occurring expectations satisfy inhomogeneous affine DDE related to (2.27). The next auxiliary lemma illustrates this. But first we introduce:

DEFINITION 2.24

Let x_0 be the solution of the scalar DDE

$$\begin{aligned} x'(\hat{t}) &= \int_{-\tau}^{0} x(\hat{t}+u)\,da(u), & \hat{t} > 0 \;, \\ x(\hat{t}) &= \delta_0(\hat{t}), & \hat{t} \leq 0 \;. \end{aligned} \tag{2.26}$$

Then x_0 is called the **fundamental solution of (2.26)** .

LEMMA 2.25
Let $s \in [t_0, \infty)$, $v \in [-\tau, 0]$, $h \in \mathbb{R}_+$,

Y a $\mathcal{F}_s$-measurable random variable,

$$G(t;s,h,v) = \mathbb{E}[\, X(t) \int_s^{s+h} X(r+v)\, dW(r)\, Y\,], \qquad t \in [t_0, \infty),$$

$$\overline{G}(t;s,h) = \mathbb{E}[\, X(t) \int_s^{s+h} \int_{-\tau}^{0} X(r+v)\, db(u)\, dW(r)\, Y\,], \quad t \in [t_0, \infty),$$

Then it holds for all $t \in [t_0, \infty)$:

$$G(t;s,h,v) = \int_s^{t\wedge(s+h)} x_0(t-r) \int_{-\tau}^{0} \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, dr \ ,$$

$$\overline{G}(t;s,h) = \int_s^{t\wedge(s+h)} x_0(t-r) \int_{-\tau}^{0}\int_{-\tau}^{0} \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, db(v)\, dr \ .$$

Proof:

(1) For all $t \in (s, \infty)$ it holds:

$$\begin{aligned}
G(t;s,h,v) &= \mathbb{E}[\, X(t) \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,] \\
&= \mathbb{E}[\, X(s) \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,] \\
&\quad + \mathbb{E}[\int_s^t \int_{-\tau}^0 X(r+u)\, da(u)\, dr \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,] \\
&\quad + \mathbb{E}[\int_s^t \int_{-\tau}^0 X(\bar{r}+u)\, db(u)\, dW(\bar{r}) \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,] \\
&\underset{Fubini}{=} \mathbb{E}[\, \mathbb{E}[\int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r}) \mid \mathcal{F}_s\,]\, X(s)\, Y\,] \\
&\quad + \int_s^t \int_{-\tau}^0 \mathbb{E}[\, X(r+u) \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,]\, da(u)\, dr \\
&\quad + \int_{-\tau}^0 \int_s^{t\wedge(s+h)} \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, dr\, db(u) \\
&\underset{Itô-integral}{=} \int_s^t \int_{-\tau}^0 G(r+u;s,h,v)\, da(u)\, dr + \int_{-\tau}^0 \int_s^{t\wedge(s+h)} \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, dr\, db(u) \\
&\underset{Fubini}{=} \int_s^t \int_{-\tau}^0 G(r+u;s,v,h)\, da(u)\, dr + \int_{s-\tau}^t \int_s^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, 1_{[s,s+h]}(r)\, dr \quad .
\end{aligned}$$

For all $t \in [t_0, s]$ it holds:

$$\begin{aligned}
G(t;s,h,v) &= \mathbb{E}[\, X(t) \int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r})\, Y\,] \\
&= \mathbb{E}[\, \mathbb{E}[\int_s^{s+h} X(\bar{r}+v)\, dW(\bar{r}) \mid \mathcal{F}_s\,]\, X(t)\, Y\,] \\
&\underset{Itô-integral}{=} 0 \quad .
\end{aligned}$$

(2) From (1) follows that $G(\_ + s; s, h, v)$ satisfies the affine DDE:

$$\begin{aligned}
x'(\hat{t}) &= \int_{-\tau}^0 x(\hat{t}+u)\, da(u) + g(\hat{t}), \qquad && \hat{t} > 0 \quad , \\
x(\hat{t}) &= 0, && \hat{t} \le 0 \quad ,
\end{aligned} \tag{2.27}$$

where

$$g(\hat{t}) = \int_{-\tau}^0 \mathbb{E}[\, X(\hat{t}+s+u)\, X(\hat{t}+s+v)\, Y\,]\, db(u)\, 1_{[0,h]}(\hat{t}), \qquad \hat{t} \ge 0 \quad .$$

From theory of affine DDE it is known ([2], pp.174), that a solution x of (2.27) can be represented by a variation-of-parameters formula. Using the initial function 0 we conclude:

$$x(\hat{t}) = \int_0^{\hat{t}} x_0(\hat{t}-\hat{r}) g(\hat{r})\, d\hat{r} \qquad \hat{t} \ge 0. \tag{2.28}$$

x_0 is the so-called fundamental solution of (2.27) and is defined above.

(3) From (2) it follows for all $t \in (s,\infty)$:

$$\begin{aligned} G(t;s,h,v) &= \int_s^t x_0(t-r) \int_{-\tau}^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, 1_{[s,s+h]}(\bar{t})\, dr \\ &= \int_s^t x_0(t-r) \int_{-\tau}^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, 1_{[s,s+h]}(\bar{t})\, dr \\ &= \int_s^{t\wedge(s+h)} x_0(t-r) \int_{-\tau}^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, dr \ . \end{aligned} \tag{2.29}$$

Hence it follows for all $t \in (s,\infty)$:

$$\begin{aligned} \overline{G}(t;s,h) &\underset{Fubini}{=} \int_{-\tau}^0 G(t;s,h,v)\, db(v) \\ &= \int_{-\tau}^0 \int_s^{t\wedge(s+h)} x_0(t-r) \int_{-\tau}^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, dr\, db(v) \\ &\underset{Fubini}{=} \int_s^{t\wedge(s+h)} x_0(t-r) \int_{-\tau}^0 \int_{-\tau}^0 \mathbb{E}[\, X(r+u)\, X(r+v)\, Y\,]\, db(u)\, b(v)\ dr \ . \end{aligned} \tag{2.30}$$

□

Now we are prepared to prove:

Theorem 2.26
The comoment function K_2^X of a solution X of (??) is Frechet-differentiable on $\overset{\circ}{\mathbb{K}}{}^2_{+,>}(\tau)$.
The Frechet-derivative of K_2^X is:

$$(K_2^X)'(t,s) = (\int_{-\tau}^0 K_2^X(t+u,s)\, da(u),\ \int_{-\tau}^0 K_2^X(t,s+v)\, da(v) + x_0(t-s) \int_{-\tau}^0 \int_{-\tau}^0 K_2^X(s+u,s+v)\, db(u)\, db(v)\,)^\top$$

and is continuous on $\mathbb{K}^2_{+,\Delta}(\tau)$.

Proof:
Deferred to section 5.5 .

□

2.5 An example for amplified systems corresponding to comoment functions of order $p = 2$

Consider the geometric Brownian motion defined by

$$\begin{aligned} dX(t) &= \alpha\, X(t)\, dt + \beta\, X(t)\, dW(t), \qquad t \in (0,\infty), \\ X(0) &= \xi\ . \end{aligned} \tag{2.31}$$

It can be easily verified that the exact solution of (2.31) is

$$X(t) = \xi\, e^{(\alpha - \frac{1}{2}\beta^2)t + \beta W(t)}, \qquad t \in (0,\infty)\ .$$

X has the comoment function

$$K_2^X(t,s) = c^2 e^{\alpha t + (\alpha+\beta^2)s}, \qquad \infty > t \geq s \geq 0\,, \quad c^2 = \mathbb{E}[\,\xi^2\,]\,.$$

It holds:

$$\begin{aligned} \tfrac{d}{dx} K_2^X(x,x)_{|x=t} &= \tfrac{d}{dt} c^2 e^{(2\alpha+\beta^2)t} &&= 2\alpha c^2 e^{(2\alpha+\beta^2)t} + \beta^2 c^2 e^{(2\alpha+\beta^2)t} &&= 2\alpha K_2^X(t,t) + \beta^2 K_2^X(t,t)\ , \\ \tfrac{d}{dx} K_2^X(x,s)_{|x=t} &= \tfrac{d}{dt} c^2 e^{\alpha t + (\alpha+\beta^2)s} &&= \alpha c^2 e^{\alpha t + (\alpha+\beta^2)s} &&= \alpha K_2^X(t,s)\ . \end{aligned} \tag{2.32}$$

For any $\tau \in [0,\infty)$, the SDE (2.31) can be considered as an SDDE

$$\begin{aligned} dX(t) &= \int_{-\tau}^0 X(t+u)\, da(u)\, dt + \int_{-\tau}^0 X(t+u)\, d\, b(u)\, dW(t), \qquad t \in (0,\infty) \\ X(t) &= \xi\, \delta_0(t) \qquad\qquad t \in [-\tau, 0] \end{aligned} \tag{2.33}$$

where $a = \alpha\,\delta_0$ and $b = \beta\,\delta_0$.
Then it holds:

$$\begin{aligned} \int_{-\tau}^{0} K_2^X(t+u,s)\,da(u) &= \alpha K_2^X(t,s), && t\in[0,\infty),\ s\in[0,t], \\ \int_{-\tau-\tau}^{0}\int^{0} K_2^X(t+u,t+v)\,db(u)\,db(v) &= \beta^2 K_2^X(t,t), && t\in[0,\infty). \end{aligned} \tag{2.34}$$

Hence (2.32) and (2.34) show that the comoment function K satisfies the amplified system.
Furthermore, the Frechet-derivative of K_2^X does exist for all $t > s \geq 0$ and is

$$(K_2^X)'(t,s) = (c^2\,\alpha\,e^{\alpha t+(\alpha+\beta^2)s}, c^2\,(\alpha+\beta^2)\,e^{\alpha t+(\alpha+\beta^2)s})^\top$$

On the other hand, the fundamental solution of

$$x'(t) = \int_{-\tau}^{0} x(u+t)\,da(u)\,dt$$

is $x_0 = e^{\alpha(t-s)}$. Furthermore, for $t > s \geq 0$ it holds:

$$\begin{aligned} \int_{-\tau}^{0} K_2^X(t+u,s)\,da(u) &= c^2\,\alpha\,e^{\alpha t+(\alpha+\beta^2)s}\ , \\ \int_{-\tau}^{0} K_2^X(t+v,s)\,da(v) + x_0(t-s)\int_{-\tau}^{0} K_2^X(s+u,s+v)\,db(u)\,b(v) &= c^2\,\alpha\,e^{\alpha t+(\alpha+\beta^2)s} + e^{\alpha(t-s)}\,c^2\,\beta^2\,e^{(2\alpha+\beta)s} \\ &= c^2\,(\alpha+\beta^2)\,e^{\alpha t+(\alpha+\beta^2)s}\ . \end{aligned}$$

3 An approach to evaluate the evolution of quadratic moments

In the previous section we have shown that for a solution X of a SDDE (2.1) the second moments function m_2^X depends on and is embedded into the comoment function K_2^X of X. Furthermore we showed that K_2^X satisfies the amplified system (2.3). In this section we want to consider the asymptotic behaviour of K_2^X and derive a criterion for the asymptotic behaviour of m_2^X.
First we start with some preliminary considerations. From (2.20) we know that for all $s \in [0,\infty)$, $t \in [s-\tau,\infty)$ $K_2^X(t,s)$ satisfies the DDE

$$\begin{aligned} x'(t) &= \int_{-\tau}^{0} x(t+u)da(u), && t\in[0,\infty) \\ x(t) &= \xi, && t\in[-\tau,0]\ . \end{aligned} \tag{3.1}$$

where $\xi(t) = \xi_s(t) := K_2^X(t+s,s)$. From the theory about DDE ([5], [2]) it is known that the asymptotic behaviour of K_2^X for $t \longrightarrow \infty$ depends on ξ and on the set

$$\sigma(a) := \{\,\lambda\in\mathbb{C} \mid \lambda = \int_{-\tau}^{0} e^{\lambda u}\,da(u)\,\}\ . \tag{3.2}$$

In particular for $s = 0$ one obtains:

LEMMA 3.1
Let X be solution of (2.18) and K_2^X its comoment function.
Then it holds:
(i) $\max(\,Re(\sigma(a))\,) > 0 \Rightarrow \exists\,\xi$ such that from $K_2^X(\_,0) = \xi$ it follows $\overline{\lim_{t\to\infty}}|K_2^X(t,0)| = \infty$.
(ii) $\max(\,Re(\sigma(a))\,) \leq 0 \Rightarrow \forall\,k\in\mathcal{K}_{sup}$ it follows $\overline{\lim_{t\to\infty}}|K_2^X(t,0)| < \infty$.

For $s > 0$ the start function ξ (in (3.1)) depends on K_2^X. It is not obvious whether for any ξ there exists $k \in \mathcal{K}_{sup}$ such that $K_2^X(s+\_,s) = \xi$. So if $Re(\,\sigma(a)\,) \subset (-\infty,0]$, for all $s > 0$ $K_2^X(\_,s)$ is bounded. However, if $Re(\,\sigma(a)\,) \cap (0,\infty)$ it is not clear whether for all $s > 0$ there exist $k \in \mathcal{K}_{sup}$ such that

$\overline{\lim_{t\to\infty}} |K_2^X(t,0)| = \infty$.
We have seen so far that we can characterize partially the asymptotic behaviour of K_2^X along rays $\mathcal{R}(s) := \{ (t,s) \mid t \in [s,\infty) \}$ by $\sigma(a)$. However, in the focus of our interest is m_2^X. So a natural question is whether it is sufficient to know the behaviour of K_2^X along rays $\mathcal{R}(s)$. The Hölder-inequality leads to the criterion:

LEMMA 3.2
Let X be solution of (2.18) and K_2^X its comoment function,
$\exists\, s \in [0,\infty) : \mathbb{E}[\,|X(s)|^2\,] \neq 0$.
Then it holds:

$$\overline{\lim_{t\to\infty}} |K_2^X(t,s)| = \infty \quad \Longrightarrow \quad \overline{\lim_{t\to\infty}} |K_2^X(t,t)| = \infty$$

Proof:
$\overline{\lim_{t\to\infty}} K_2^X(t,s) \leq \overline{\lim_{t\to\infty}} K_2^X(t,t)^{\frac{1}{2}}\, K_2^X(s,s)^{\frac{1}{2}}$. □

On the other hand, from the boundedness of K_2^X on rays $\mathcal{R}(s)$, $s \in [0,\infty)$, $\overline{\lim_{t\to\infty}} |K_2^X(t,t)| < \infty$ cannot be concluded. This proves the following example:

EXAMPLE 3.3
We continue the example from section 2.5. The comoment function of the geometric Brownian motions is $K(t,s) = c^2\, e^{\alpha t + (\alpha+\beta^2)s}$. Hence if $\alpha < 0$ and $2\alpha + \beta^2 > 0$ it follows: $\lim_{t\to\infty} K(t,t) = \infty$, $\forall\, s \in [0,\infty)$: $\lim_{t\to\infty} K(t,s) = 0$.

This makes it necessary to investigate K_2^X on Δ directly.

In standard references to DDE ([2], [5]) one can find two approaches to derive stability criteria for solutions of the DDE (3.1) The first is to show via integration by parts that a solution x of solves a renewal equation. The change into the frequency space then leads to criteria in terms of characteristic equations (spectral theory). The second is to reformulate (3.1) as a linear ODE in a suitable Banach space, to consider the solution operator of this equation, to show its semigroup properties and to use the theory of semigroups to analyze the asymptotic behaviour (properties of the generator). Here we follow the second approach for the equation (2.1). We start with a helpful observation.

LEMMA 3.4
Define $D_\tau^2 = \{ (u,v) \mid u,v \in [-\tau,\infty), \mid v-u \mid \leq \tau \}$.
Any solution K of (2.18) is uniquely determined by its values on D_τ^2.

Proof:
(1) The function z satisfies the equation (2.22). But this shows that for any $s \in [0,\infty)$ the function value $z(s+\tau)$ is uniquely determined by the function values $z(s)$ and $\{y_u(v) \mid u,v \in [s,s+\tau]^2\} \subset D$.
(2) For any $s \in [0,\infty)$ the function y_s satisfies the equation (2.21), a linear DDE. This shows that for any $s \in [0,\infty)$ the function y_s is uniquely determined by the function values $\{ \Delta y_u(v) \mid u \in [s-\tau,s], v = s \} \cup \{ y_u(v) \mid u = s, v \in [s,s+\tau] \}$ on $[s,\infty)$.
(3) By (1), (2) follows the objective.

□

Hence, although the function K has two dimensions, its decisive dynamics occurs in D_τ^2, one-dimensional along the axis Δ_τ. This one-dimensionality allows us to introduce a time-continuous solution group in an suitable Banach space which describes the dynamics of K_2^X in D_τ^2. The aim is then to show that the solution operator is a strong continuous semigroup and to apply well-known semigroup theory to describe the asymptotic behaviour of the K_2^X along Δ_τ.
The Banach-space we consider is the Banach space $(\mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}), \| \|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})})$.

DEFINITION 3.5
Let $\mathcal{T}_{sup} := \{ T(t) \}_{t\in[0,\infty)}$ be an operator family such that:

$$\begin{aligned}
&(i)\quad \forall t \in [0,\infty): \quad T(t) : \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}) \longrightarrow \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}) \ , \\
&(ii)\quad T(t)\,k = g \ :\Longleftrightarrow \ \text{a)}\ \forall\, u,v \in [-\tau,0]^2 :\ g(u,v) = K_k(t+u,t+v) \ , \\
&\qquad\qquad\qquad\quad \text{b)}\ K_k \text{ solves (2.18) with the initial function } k \ .
\end{aligned} \tag{3.3}$$

The following theorem states the main property of $\mathcal{T}$.

Theorem 3.6
$\mathcal{T}_{sup}$ is a strong continuous semigroup.

Proof:
(1) By theorem 2.22, for any $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$ we know that the solution K of (2.18) with initial function k is continuous on $[-\tau,\infty)^2$.
Hence for any $t \in [0,\infty)$: $T(t) : K_{sup}([-\tau,0]^2,\mathbb{R}) \longrightarrow K_{sup}([-\tau,0]^2,\mathbb{R})$.

(2) For any $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$ and $u,v \in [-\tau,0]$ it holds: $(T(0)\,k)(u,v) = K(0+u,0+v) = k(u,v)$.
Hence $T(0) = id$.

(3) Let $s \in [0,\infty)$, $t \in [0 \vee s,\infty)$. Define $\bar{k} = T(s)\,k$ and $\bar{\bar{k}} = T(t-s)\bar{k}$. Let K_k and $K_{\bar{k}}$ be the solution of (2.18) with initial function k and $\bar{k}$, respectively. Define $K_k^s \in ([-\tau,0]^2,\mathbb{R})$ by: $\forall\, \bar{t},\bar{s} \in [-\tau,\infty)$ $K_k^s(\bar{t},\bar{s}) = K_k(s+\bar{t},s+\bar{s})$.

(4) As K_k solves (2.18), by definition K_k^s solves (2.18) as well. Furthermore, $\forall\, u,v \in [-\tau,0]^2$ it holds: $K_k^s(u,v) = K_k(s+u,s+v) = \bar{k}(u,v)$. Hence $K_k^s = K_{\bar{k}}$.

(5) $\forall\, u,v \in [-\tau,0]^2$ it holds:
$$\begin{aligned}
(T(t-s)\,\bar{k})(u,v) &\underset{definition\ of\ T}{=} K_{\bar{k}}(t-s+u,t-s+v) \\
&\underset{(4)}{=} K_k(s+(t-s+u),s+(t-s+v)) \\
&= K_k(t+u,t+v) \\
&\underset{definition\ of\ T}{=} (T(t)\,k)(u,v) \ .
\end{aligned}$$
But then it holds $T(t-s)\,T(s) = T(t)$. Hence $\mathcal{T}_{sup}$ is a semigroup.

(6) Fix $T > 0$. By (1) K_k is uniformly continuous on $[-\tau,T]^2$. But then for any $\epsilon \in (0,1)$ exists an $\delta \in (0,\infty)$ such that $\forall\, \vec{t} = (t_1,t_2), \vec{s} = (s_1,s_2)$ with $\| \vec{t} - \vec{s} \|_2 \le \delta$ it holds $\| K(t_1,t_2) - K(s_1,s_2) \|_2 \le \epsilon$.
But then for all $h \in (0,\delta)$: $\| T(h)k - k \|_{C([-\tau,0]^2,\mathbb{R})} \le \epsilon$, hence $\lim\limits_{h\to 0}(T(h)k - k) = 0$.
This shows that $\mathcal{T}_{sup}$ is strongly continuous.

□

The time evolution of strong continuous semigroups can be characterized by the properties of its generator. So next we investigate the generator of $\mathcal{T}_{sup}$.

DEFINITION 3.7
Let $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$. Define $\epsilon_\tau(t,s) = (-(t \vee s)) \wedge (\tau + (t \wedge s))s$.
For all $(t,s) \in [-\tau,0]^2$ define:
$$D_G(k)(t,s) := \begin{cases} \lim\limits_{h\mapsto 0,\epsilon_\tau(t,s)} \frac{k(t+h,s+h)-k(t,s)}{h}, & \exists \lim\limits_{h\mapsto 0,\epsilon_\tau(t,s)} \frac{k(t+h,s+h)-k(t,s)}{h}, \\ \infty, & \text{otherwise} \end{cases}$$

LEMMA 3.8
Let A be the generator of $\mathcal{T}_{sup}$.
Then it holds:
$$\begin{aligned}
&(i)\quad \forall\, k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}) : A\,k = D_G(k). \\
&(ii)\quad \mathcal{D}(A) = \{\, k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}) \mid D_G(k) \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}),\ \forall t \in [-\tau,0)\ \exists \tfrac{\partial}{\partial t_1}k(t,0), \\
&\qquad\qquad \forall t \in [-\tau,0) : D_G(k)(t,0) = \tfrac{\partial}{\partial t_1}k(t,0) + \int_{-\tau}^{0} k(t,u)\,da(u), \\
&\qquad\qquad D_G(k)(0,0) = 2\int_{-\tau}^{0} k(u,0)\,da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} k(u,v)\,db(u)\,db(v) \,\}.
\end{aligned}$$

Proof:

(1) For any $k \in \mathcal{K}([-\tau,0]^2,\mathbb{R})$ let K_k be the solution of (2.18) with the initial function k.

(2) Let $k \in \mathcal{D}(A) \subset \mathcal{K}([-\tau,0]^2,\mathbb{R})$. Then it holds:

$$0 = \lim_{h\to 0} \| \tfrac{1}{h}(T(h)k-k) - Ak \|_{C([-\tau,0]^2,\mathbb{R})} = \lim_{h\to 0} \sup_{t,s\in[-\tau,0]} \{ | \tfrac{1}{h}(K(t+h,s+h)-K(t,s)) - (Ak)(t,s) | \}.$$

(3) From (2) it follows that for all $t,s \in [-\tau,0)$: $(Ak)(t,s) = \lim_{h\downarrow 0, h\le -(t\vee s)} \frac{1}{h}\,(\,k(t+h,s+h) - k(t,s)\,)$.

(4) As $k \in \mathcal{D}(A)$, $Ak \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$. But then for all $s \in [-\tau,0]$ Ak is uniformly continuous on $\mathcal{R}(s) = \{\,(u,v) \mid (u,v) = (-\tau,s) + \alpha\,(1,1),\ \alpha \in [0,-s]\,\}$.

(5) For all $t,s \in (-\tau,0]$ and $h \in (0,-\tau-(t\wedge s))$:

$$\begin{aligned} &|\tfrac{1}{-h}\,(\,k(t-h,s-h) - k(t,s)\,) - A(t,s)\,| \\ &\qquad \le |\tfrac{1}{h}\,(\,k(t,s) - k(t-h,s-h)\,) - A(t-h,s-h)\,| + |A(t-h,s-h) - A(t,s)| \ . \end{aligned}$$

By the uniform convergence in (2) and (5) it follows: $\lim_{h\downarrow 0} | \tfrac{1}{-h}\,(\,k(t-h,s-h) - k(t,s)\,) | = A(t,s)$.

(6) (3) and (5) show (i).

(7) For $t \in [-\tau,0)$ and $-\tau - t \ge h \ge -t$ it follows:

$$\begin{aligned} (Ak)(t,0) &= \lim_{h\downarrow 0} \tfrac{1}{h}(K(t+h,h) - K(t,0)) \\ &= \lim_{h\downarrow 0} \tfrac{1}{h}(K(t+h,h) - K(t+h,0) + K(t+h,0) - K(t,0)) \\ &= \lim_{h\downarrow 0} (\tfrac{1}{h}\int_{0-\tau}^{h}\int^{0} K(t+h,r+u)\,da(u)\,dr + \tfrac{1}{h}(K(t+h,0) - K(t,0))\,) \\ &\underset{K\ continuous}{=} \int_{-\tau}^{0} K(t,u)\,da(u) + \lim_{h\downarrow 0} \tfrac{1}{h}(K(t+h,0) - K(t,0)) \\ &\underset{(1)}{=} \int_{-\tau}^{0} k(t,u)\,da(u) + \lim_{h\to 0,\, t+h\in[-\tau,0]} \tfrac{1}{h}(k(t+h,0) - k(t,0)) \ . \end{aligned}$$

It follows:

$$\begin{aligned} (Ak)(0,0) &= \lim_{h\downarrow 0} \tfrac{1}{h}(K(h,h) - K(0,0)) \\ &= \lim_{h\downarrow 0} (\tfrac{1}{h}\int_{0-\tau}^{h}\int^{0} K(r+u,r)\,da(u) + \tfrac{1}{h}\int_{0-\tau-\tau}^{h}\int^{0}\int^{0} K(r+u,r+v)\,db(u)\,db(v)\,dr \\ &\underset{K\ continuous}{=} 2\int_{-\tau}^{0} K(u,0)\,da(u) + \int_{-\tau-\tau}^{0}\int^{0} K(u,v)\,db(u)\,db(v) \\ &= \int_{-\tau}^{0} k(u,0)\,da(u) + \int_{-\tau-\tau}^{0}\int^{0} k(u,v)\,db(u)\,db(v) \ . \end{aligned}$$

(8) (2), (3) show (ii).

□

We now consider the spectrum $\sigma(A)$ of the semigroup $\mathcal{T}_{sup}$. It turns out that it can be determined and characterized by a linear DDE with linear constraints.

DEFINITION 3.9
The equation

$$\begin{aligned} f'(t) &= \lambda f(t) - \int_{-\tau}^{0} f(-|u-t|)\,e^{\lambda(u\vee t)}\,da(u), \qquad t \in [-\tau,0], \\ \lambda f(0) &= 2\int_{-\tau}^{0} f(u)\,da(u) + \int_{-\tau-\tau}^{0}\int^{0} f(-|u-v|)\,e^{\lambda(u\vee v)}\,db(u)\,db(v). \end{aligned} \tag{3.4}$$

with $\lambda \in \mathbb{C}$ is called **characteristic equation of order 2**. Any $\lambda \in \mathbb{C}$, for which (3.4) has a non-zero solution $f \in \mathcal{C}([-\tau,0],\mathbb{R}) \cap \mathcal{C}^1([-\tau,0),\mathbb{R})$, is called an **eigenvalue of order 2**.

Theorem 3.10
Let X be a solution of (2.1) and K_2^X its comoment function.
Then it holds:

$$\sigma(A) = \sigma_p(A) = \{\,\lambda \in \mathbb{C} \mid \lambda \textit{ is eigenvalue of order 2}\,\}.$$

Proof:

(1) As $k \in \mathcal{K}([-\tau,0]^2,\mathbb{R})$, K_2^X is continuous. Let $t \geq \tau$. By corollary (2.17) K_2^X satisfies an exponential growth condition. Hence $T(t)$ maps bounded sets $T(t)\,S$ in $\mathcal{C}([-\tau,0]^2,\mathbb{R})$ into bounded sets $T(t)\,S$ in $\mathcal{C}([-\tau,0]^2,\mathbb{R})$. By theorem (2.23) K_2^X is Hölder-continuous for all $\gamma \in (0,\frac{1}{2})$ and has a uniform finite bound of the Hölder-derivative for $\gamma = \frac{1}{2}$. Hence $T(t)\,S$ is precompact in $\mathcal{C}([-\tau,0]^2,\mathbb{R})$ by the theorem of Arzela-Ascoli. Hence $\{T(t)\}_{t\in[0,\infty)}$ is eventually compact. But then it follows $\sigma(A) = \sigma_p(A)$ (see [2]).

(2) From the definition of the point spectrum we recall that $\lambda \in \sigma(A)$ iff $\exists k \in \mathcal{D}(A) : (\lambda - A)\,k = 0$.

(3) From (2) we conclude that $\forall t \in [-\tau,0), s \in (t,0) : \lim_{h\to 0} \frac{1}{h}\,(\,k(t+h,s+h) - k(t,s)\,) = \lambda k(t,s)$. That is for any $u \in [-\tau,0]$ $x(r) := k(r+u,r)$ satisfies the linear ODE $x'(r) = \lambda x(r)$ for $r \in [-\tau-u,0]$.

(4) Define $\forall u \in [-\tau,0] : f(u) := k(u,0)$.
By lemma 3.8.(ii) we know that f is differentiable on $[-\tau,0)$.
Due to (3) and continuity of k it follows $\forall t,s \in [-\tau,0] : k(t,s) = f(-|t-s|)\,e^{\lambda(t\vee s)}$.

(5) Let $t \in [-\tau,0)$. Then it holds:

$$0 = (\lambda k - Ak)\,(t,0) \underset{lemma\ 3.8.(ii)}{=} \lambda f(t) - f'(t) - \int_{-\tau}^{0} f(-|t-u|)\,e^{\lambda(t\vee u)}\,da(u).$$

Furthermore it holds:

$$0 = (\lambda k - Ak)\,(0,0) \underset{lemma\ 3.4.(ii)}{=} \lambda f(0) - 2\int_{-\tau}^{0} f(u)\,da(u) - \int_{-\tau-\tau}^{0}\int^{0} f(-|u-v|)\,e^{\lambda\,(u\vee v)}\,db(u)\,db(v).$$

(6) The claim follows from the definition of the $\mathcal{D}(A)$.

□

Now as the theorem is proven we try to simplify the characteristic equation of order 2. Consider $g(t) = f(t)\,e^{-\lambda t}$. Then it holds

$$\begin{aligned} g'(t) &= -\lambda f(t)\,e^{-\lambda t} + f'(t)\,e^{-\lambda t} \\ &= -\lambda f(t)\,e^{-\lambda t} + \lambda f(t)\,e^{-\lambda t} - \int_{-\tau}^{0} f(-|u-t|)\,e^{\lambda\,(u\vee t)-\lambda t}\,da(u), \\ &= -\int_{-\tau}^{0} g(-|u-t|)\,e^{\lambda\,(u\wedge t)-\lambda t}\,da(u), \end{aligned}$$

As $f,g \in \mathcal{C}([-\tau,0],\mathbb{R})$, this differential equation can be restated as an integral equation

$$g(t) = g(0) + \int_{t-\tau}^{0}\int^{0} g(-|u-s|)\,e^{\lambda\,((u-s)\wedge 0)}\,da(u)\,ds\ .$$

Furthermore it holds:

$$\begin{aligned} \int_{t-\tau}^{0}\int^{0} g(-|u-s|\,)\,e^{\lambda\,((u-s)\wedge 0)}\,da(u)\,ds &\underset{Fubini}{=} \int_{-\tau}^{0}\int_{t}^{0} g(-|u-s|\,)\,e^{\lambda((u-s)\wedge 0)}\,ds\,da(u) \\ &\underset{v=u-s}{=} \int_{-\tau}^{0}\int_{u}^{u-t} g(-|v|\,)\,e^{\lambda(v\wedge 0)}\,dv\,da(u) \\ &\underset{Fubini}{=} \int_{-\tau}^{-t}\int_{(-\tau)\vee(v+t)}^{0\wedge v} g(-|v|\,)\,e^{\lambda(v\wedge 0)}\,da(u)\,dv \\ &= \int_{-\tau}^{-t} g(-|v|\,)\,e^{\lambda(v\wedge 0)}\,(\,a(0\wedge v) - a(-\tau\vee(v+t))\,)\,dv. \end{aligned}$$

Hence:

$$\begin{aligned} &\int_{t-\tau}^{0}\int^{0} g(-|u-s|\,)\,e^{\lambda\,((u-s)\wedge 0)}\,da(u)\,ds \\ &= \int_{-\tau}^{0} g(v)\,(\,a(v) - a(-\tau\vee(v+t))\,)\,e^{\lambda v}\,dv + \int_{0}^{-t} g(-v)\,(\,a(0) - a(v+t)\,)\,dv \\ &= \int_{-\tau}^{0} g(v)\,(\,a(v) - a(-\tau\vee(v+t))\,)\,e^{\lambda v}\,dv + \int_{t}^{0} g(v)\,(\,a(0) - a(t-v)\,)\,dv \\ &= \int_{-\tau}^{0} (\,a(v) - a(-\tau\vee(v+t))\,e^{\lambda v} + (\,a(0) - a(t-v))\,1_{[t,0]}(v)\,)\,g(v)\,dv \\ &= \int_{-\tau}^{0} k_\lambda(t,v)\,g(v)\,dv\ , \end{aligned}$$

where $k_\lambda(t,v) := (\,a(v) - a(-\tau \vee (v+t))\,)\,e^{\lambda v} + (\,a(0) - a(t-v)\,)\,1_{[t,0]}(v)$.
This allows us to characterize the spectrum of the semigroup $\mathcal{T}$ as solutions of a Volterra-integral equation with constraints.

Theorem 3.11
Let X be a solution of (2.1) and K_2^X its comoment function,

$$\begin{aligned} k_\lambda(t,v) &= (\,a(v) - a(-\tau \vee (v+t))\,)\,e^{\lambda v} + (\,a(0) - a(t-v)\,)\,1_{[t,0]}(v)\ , \quad t,v \in [-\tau,0]\ , \\ g(t) &= g(0) + \int_{-\tau}^{0} k_\lambda(t,v)\,g(v)\,dv\ , \qquad t \in [-\tau,0), \\ \lambda\, g(0) &= 2\int_{-\tau}^{0} g(u)\,e^{\lambda u}\,da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} g(-|u-v|)\,e^{\lambda(u\wedge v)}\,db(u)\,db(v)\ . \end{aligned} \tag{3.5}$$

Then it holds:

$$\sigma(A) = \{\,\lambda \in \mathbb{C} \mid \exists\, g \in \mathcal{C}([-\tau,0],\mathbb{R}) \cap \mathcal{C}^1([-\tau,0),\mathbb{R})\text{: } g \text{ satisfies (3.5) and } g \not\equiv 0\,\}.$$

Proof:
Follows immediately from lemma 3.4, the above computations and the fact that $e^{\lambda t}$ is strictly positive (implying $f \equiv 0 \Leftrightarrow g \equiv 0$).

□

Finally we observe that the τ-dependent systems (3.4) and (3.5) can be rescaled to a standard problem.

LEMMA 3.12
Let $\tau \in (0,\infty)$,
$\bar{a},\bar{b} \in ([-1,0],\mathbb{R})$ defined by $\forall\, u \in [-1,0] : \bar{a}(u) := \tau a(\tau u)$, $\bar{b}(u) := \sqrt{\tau}\, b(\tau u)$.
Consider the systems

$$\begin{aligned} f'(t) &= \lambda f(t) - \int_{-1}^{0} f(-|u-t|)\,e^{\lambda(u\vee t)}\,d\bar{a}(u), \qquad t \in [-1,0), \\ \lambda\, f(0) &= 2\int_{-1}^{0} f(u)\,d\bar{a}(u) + \int_{-1}^{0}\int_{-1}^{0} f(-|u-v|)\,e^{\lambda(u\vee v)}\,d\bar{b}(u)\,d\bar{b}(v). \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} k_\lambda(t,v) &= (\,\bar{a}(v) - \bar{a}(-1 \vee (v+t))\,)\,e^{\lambda v} + (\,\bar{a}(0) - \bar{a}(t-v)\,)\,1_{[t,0]}(v), \quad t,v \in [-1,0], \\ g(t) &= g(0) + \int_{-1}^{0} k_\lambda(t,v)\,g(v)\,dv, \qquad t \in [-1,0), \\ \lambda\, g(0) &= 2\int_{-1}^{0} g(u)\,e^{\lambda u}\,d\bar{a}(u) + \int_{-1}^{0}\int_{-1}^{0} g(-|u-v|)\,e^{\lambda(u\wedge v)}\,d\bar{b}(u)\,d\bar{b}(v). \end{aligned} \tag{3.7}$$

Define $\sigma_f(A) = \{\,\lambda \in \mathbb{C} \mid$ *exists* $f \in \mathcal{C}([-1,0],\mathbb{R}) \cap \mathcal{C}^1([-1,0),\mathbb{R})$: f satisfies (3.6) and $f \not\equiv 0\,\}$,
$\sigma_g(A) = \{\,\lambda \in \mathbb{C} \mid$ *exists* $g \in \mathcal{C}([-1,0],\mathbb{R}) \cap \mathcal{C}^1([-1,0),\mathbb{R})$: g satisfies (3.7) and $g \not\equiv 0\,\}$.
Then it holds:

$$\tau\,\sigma(A) = \sigma_f(A) = \sigma_g(A) \tag{3.8}$$

Proof:
(1) Let $\lambda \in \sigma(A)$ and $f \not\equiv 0$ an eigenfunction corresponding to λ satisfying (3.4).
Define $\bar{f} \in \mathcal{C}([-1,0],\mathbb{R})$ defined by: $\forall\, \bar{u} \in [-1,0] : \bar{f}(\bar{u}) = f(\tau\bar{u})$,
$\bar{\lambda} = \lambda\tau$, $u = \tau\bar{u}$, $v = \tau\bar{v}$, $t = \tau\bar{t}$.
Then for all $\bar{u} \in [-1,0]$ it holds:

$$\begin{aligned} \bar{f}'(\bar{u}) = \tau f'(\tau\bar{u}) &= \lambda\tau f(\tau\bar{u}) - \int_{-\tau}^{0} f(-|\tau\tfrac{u}{\tau} - \tau\tfrac{t}{\tau}|)\,e^{\lambda\tau\,((\frac{u}{\tau}-\frac{t}{\tau})\vee 0)}\,d(\tau a(\tau\tfrac{u}{\tau})) \\ &= \bar{\lambda}\bar{f}(\bar{u}) - \int_{-1}^{0} \bar{f}(-|\bar{u}-\bar{t}|)\,e^{\bar{\lambda}\,((\bar{u}-\bar{t})\vee 0)}\,d\bar{a}(\bar{u})\ , \\ \bar{\lambda}\bar{f}(0) = \lambda\tau f(0) &= 2\int_{-\tau}^{0} f(\tau\tfrac{u}{\tau})\,d(\tau a(\tau\tfrac{u}{\tau})) + \int_{-\tau}^{0}\int_{-\tau}^{0} f(-|\tau\tfrac{u}{\tau} - \tau\tfrac{v}{\tau}|)\,e^{\lambda(\tau\frac{u}{\tau}\vee\tau\frac{v}{\tau})}\,d(\sqrt{\tau}\,b(\tau\tfrac{u}{\tau}))\,d(\sqrt{\tau}\,b(\tau\tfrac{v}{\tau})) \\ &= 2\int_{-1}^{0} \bar{f}(\bar{u})\,d\bar{a}(\bar{u}) + \int_{-1}^{0}\int_{-1}^{0} \bar{f}(-|\bar{u}-\bar{v}|)\,e^{\bar{\lambda}(\bar{u}\vee\bar{v})}\,d\bar{b}(\bar{u})\,d\bar{b}(\bar{v}). \end{aligned}$$

As $\bar{f} \not\equiv 0$, $\bar{\lambda}$ belongs to σ_f. Hence $\tau\sigma(A) \subseteq \sigma_f(A)$.

(2) Let $\bar\lambda \in \sigma_f$ and $\bar f \neq 0$ an eigenfunction corresponding to $\bar\lambda$ satisfying (3.6).
Define $f \in \mathcal{C}([-\tau,0],\mathbb{R})$ by: $\forall\, u \in [-\tau,0] : f(u) = \bar f(\frac{u}{\tau})$.
$\lambda = \frac{\bar\lambda}{\tau}$, $\bar u = \frac{u}{\tau}$, $\bar v = \frac{v}{\tau}$, $\bar t = \frac{t}{\tau}$,
Then for all $t \in [-\tau,0]$ it holds:

$$\begin{aligned} f'(t) &= \tfrac{1}{\tau}\bar f'(\tfrac{t}{\tau}) = \tfrac{1}{\tau}\lambda\tau\bar f(\tfrac{t}{\tau}) - \int_{-1}^{0} \bar f(-|\tfrac{\tau\bar u}{\tau} - \tfrac{\tau\bar t}{\tau}|)\, e^{\lambda\tau\,((\frac{\tau\bar u}{\tau}-\frac{\tau\bar t}{\tau})\vee 0)}\; d(\tfrac{1}{\tau}\bar a(\tfrac{\tau\bar u}{\tau})) \\ &= \lambda f(u) - \int_{-\tau}^{0} f(-|u-t|)\, e^{\lambda\,((u-t)\vee 0)}\; da(u) \\ \lambda f(0) &= \tfrac{1}{\tau}\bar\lambda\bar f(0) = 2\int_{-1}^{0} \bar f(\tfrac{\tau\bar u}{\tau})\, d(\tfrac{1}{\tau}\bar a(\tfrac{\tau\bar u}{\tau})) + \int_{-1}^{0}\int_{-1}^{0} \bar f(-|\tfrac{\tau\bar u}{\tau} - \tfrac{\tau\bar v}{\tau}|)\, e^{\lambda(\frac{\tau\bar u}{\tau}\vee\frac{\tau\bar v}{\tau})}\, d(\tfrac{1}{\sqrt{\tau}}\,\bar b(\tfrac{\tau\bar u}{\tau}))\, d(\tfrac{1}{\sqrt{\tau}}\,\bar b(\tfrac{\tau\bar v}{\tau})) \\ &= 2\int_{-\tau}^{0} f(u)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} f(-|u-v|)\, e^{\lambda(u\vee v)}\, db(u)\, db(v). \end{aligned}$$

As $f \neq 0$, λ belongs to $\sigma(A)$. Hence $\sigma_f(A) \subseteq \tau\sigma(A)$, and with (1) it follows $\sigma_f(A) = \tau\sigma(A)$.
(3) From lemma 3.11 it follows that $\sigma_f(A) = \sigma_g(A)$.

□

Finally we conclude:

COROLLARY 3.13
Let X be a solution of (2.1) and K_2^X be its comoment function with initial function $k \in \mathbb{C}([-\tau,0]^2,\mathbb{R})$. Then it holds:

(i) If $\exists\,\lambda \in \sigma(A)$ with $Re(\lambda) > 0$, then $\exists\, k \in \mathbb{C}([-\tau,0]^2,\mathbb{R})$ such that $\lim_{t\to\infty} \mathbb{E}[\,|X(t)|^2\,] = \infty$.

(ii) If $\forall\,\lambda \in \sigma(A) : Re(\lambda) < 0$, then $\lim_{t\to\infty} \mathbb{E}[\,|X(t)|^2\,] = 0$.

(iii) If $\forall\,\lambda \in \sigma(A) : Re(\lambda) \leq 0$, then $\overline{\lim_{t\to\infty}}\, \mathbb{E}[\,|X(t)|^2\,] < \infty$.

Proof:
(1) Let $\{(t_n,s_n)\}_{n\in\mathbb{N}} \subset D_\tau^2$ be a sequence such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \infty$, $\forall\, n \in \mathbb{N}\,|t_n - s_n| \leq \tau$ and $\overline{\lim_{n\to\infty}}\,|K(t_n,s_n)| = \infty$. By the Hölder-inequality: $|K(t_n,s_n)| \leq |K(t_n,t_n)|^{\frac12}\,|K(s_n,s_n)|^{\frac12}$. It follows: $\overline{\lim_{n\to\infty}}\,|K(t_n,t_n)| \vee |K(s_n,s_n)| = \infty$.
(2) Let $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$.
If $\lim_{t\to\infty} \|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} = \infty$, by (1) it follows $\lim_{t\to\infty} |(T(t)k)\,(t,t)| = \infty$.
If $\lim_{t\to\infty} \|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} = 0$, by (1) it follows $\lim_{t\to\infty} \|(T(t)k)\,(t,t)\| = 0$.
If $\sup_{t\in[0,\infty}\{\,\|T(t)k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}\,\} \leq \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$: $\sup_{t\in[0,\infty)}\{\,|(T(t)k)\,(t,t)|\,\} \underset{(1)}{\leq} \|k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$.
(3) The semigroup theory relates $\sigma(A)$ to $\|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}$.
Define $m(A) = \max\{\,Re(\lambda)\,|\,\lambda \in \sigma(A)\,\}$.
If $m(A) > 0$ then there exists $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$ such that $\lim_{t\to\infty} \|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} = \infty$.
If $m(A) < 0$ then for all $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$: $\overline{\lim_{t\to\infty}}\, \|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} = 0$.
If $m(A) = 0$ then for all $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$: $\overline{\lim_{t\to\infty}}\, \|T(t)\,k\|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})} < \infty$.
(4) From (1)-(3) follows the claim.

□

EXAMPLE 3.14
Consider the SDE describing a geometric Brownian motion:

$$dX(t) = a_0 X(t)\,dt + b_0 X(t)\,dW(t), \qquad t \in [0,\infty)\,. \tag{3.9}$$

Then the system (3.4) is

$$\begin{aligned} f'(t) &= (\lambda - a_0) f(t), && t \in [-\tau,0], \\ \lambda f(0) &= 2a_0\, f(0) + b_0^2\, f(0). \end{aligned} \tag{3.10}$$

and has the non-zero solutions

$$\begin{array}{lcll} f(0) & = & c_f, & c \neq 0, \\ f(t) & = & c_f\, e^{(\lambda - a_0)\, t}, & t \in [-\tau, 0], \\ \lambda & = & 2a_0 + b_0^2. & \end{array} \tag{3.11}$$

On the other hand system (3.5) is

$$\begin{array}{lcll} k_\lambda(t,v) & = & a_0\, \delta_0(v)\, (1 - \delta_t(0))\, e^{\lambda v} + a_0\, (1 - \delta_t(v))\, 1_{[t,0]}(v), & \\ g(t) & = & g(0) + \int\limits_t^0 a_0\, g(v)\, d(v), & t \in [-\tau, 0], \\ \lambda g(0) & = & 2a_0\, g(0) + b_0^2\, g(0). & \end{array} \tag{3.12}$$

and has the non-zero solutions

$$\begin{array}{lcll} g(0) & = & c_g, & c \neq 0, \\ g(t) & = & c_g\, e^{-a_0\, t}, & t \in [-\tau, 0], \\ \lambda & = & 2a_0 + b_0^2. & \end{array} \tag{3.13}$$

Obviously, if $c_f = c_g$ one confirms $g(t) = f(t)\, e^{-\lambda t}$.
Corollary 3.13 yields then that

$$\begin{array}{lll} (i) & \lim\limits_{t\to\infty} \mathbb{E}[\, X(t)^2\,] = \infty & 2a_0 + b_0^2 > 0\ , \\ (ii) & \lim\limits_{t\to\infty} \mathbb{E}[\, X(t)^2\,] = 0 & 2a_0 + b_0^2 < 0\ , \\ (iii) & \forall\, t \in [0,\infty) : \mathbb{E}[\, X(t)^2\,] \leq \mathbb{E}[\, X(0)^2\,]0 & 2a_0 + b_0^2 = 0\ . \end{array}$$

Example 3.15
Consider the equation

$$dX(t) = (a_0 X(t) + a_1 X(t-\tau))\, dt + (b_0 X(t) + b_1 X(t-\tau))\, dW(t), \qquad t \in [0,\infty) \tag{3.14}$$

Then X is a SDDE (2.1) where the parameter functions are $a(u) = a_1 \delta_{[-\tau,\infty)}(t) + a_0 \delta_{[0,\infty)}(t)$ and $b(u) = b_1 \delta_{[-\tau,\infty)}(t) + b_0 \delta_{[0,\infty)}(t)$. For equation (3.14) the system (3.4) is

$$\begin{array}{lcll} f'(t) & = & (\lambda - a_0)\, f(t) - a_1\, e^{\lambda t} f(-\tau - t), & t \in [-\tau, 0], \\ \lambda f(0) & = & 2a_0 f(0) + 2a_1 f(-\tau) + (b_0^2 + b_1^2 e^{-\lambda\tau})\, f(0) + 2b_0 b_1\, f(-\tau). & \end{array} \tag{3.15}$$

The system (3.5) is

$$\begin{array}{lcll} k_\lambda(t,v) & = & a_0 1_{(t,0]}(v) + a_1 1_{[-\tau,-\tau-t)}(v), & t, v \in [-\tau, 0], \\ g(t) & = & g(0) + a_0 \int\limits_t^0 g(v)\, dv + a_1 \int\limits_{-\tau}^{-\tau-t} g(v)\, e^{\lambda v}\, dv, & t \in [-\tau, 0], \\ \lambda g(0) & = & 2a_0 g(0) + 2a_1\, e^{-\lambda\tau} g(-\tau) + (b_0^2 + b_1^2\, e^{-\lambda\tau})\, g(0) + 2b_0 b_1\, e^{-\lambda\tau}\, g(-\tau). & \end{array} \tag{3.16}$$

Let $\bar{a}_0 = a_0\tau$, $\bar{a}_1 = a_1\tau$. Then the normalized coefficient functions $\bar{a}, \bar{b}$ are defined by $\bar{a}(u) = \bar{a}_1 \delta_{[-1,\infty)}(t) + \bar{a}_0 \delta_{[0,\infty)}(t)$ and $\bar{b}(u) = \bar{b}_1 \delta_{[-1,\infty)}(t) + \bar{b}_0 \delta_{[0,\infty)}(t)$. Hence the normalized characteristic systems are

$$\begin{array}{lcll} f'(t) & = & (\lambda - \bar{a_0})\, f(t) - \bar{a}_1\, e^{\lambda t} f(-1 - t), & t \in [-1, 0], \\ \lambda f(0) & = & 2\bar{a}_0 f(0) + 2\bar{a}_1 f(-1) + (\bar{b}_0^2 + \bar{b}_1^2 e^{-\lambda})\, f(0) + 2\bar{b}_0 \bar{b}_1\, f(-1). & \end{array} \tag{3.17}$$

and

$$\begin{array}{lcll} k_\lambda(t,v) & = & \bar{a}_0 1_{(t,0]}(v) + \bar{a}_1 1_{[-1,-1-t)}(v), & t, v \in [-1, 0], \\ g(t) & = & g(0) + \bar{a}_0 \int\limits_t^0 g(v)\, dv + \bar{a}_1 \int\limits_{-1}^{-1-t} g(v)\, e^{\lambda v}\, dv, & t \in [-1, 0], \\ \lambda g(0) & = & 2\bar{a}_0 g(0) + 2\bar{a}_1\, e^{-\lambda} g(-1) + (\bar{b}_0^2 + \bar{b}_1^2\, e^{-\lambda})\, g(0) + 2\bar{b}_0 \bar{b}_1\, e^{-\lambda}\, g(-1). & \end{array} \tag{3.18}$$

4 Weak stationarity

A side effect of considering the amplified system (2.18) is a characterization of weak stationary solutions of (2.1).

DEFINITION 4.1
Let $(Y, \mathbb{T})$, $\mathbb{T} \subset \mathbb{R}$, be a stochastic process with finite first order moments.
The function $m_1^Y\colon \begin{array}{ccc} \mathbb{T} & \longrightarrow & \mathbb{R} \\ t & \mapsto & m_1^Y(t) := \mathbb{E}[\,Y(t)\,] \end{array}$
is called the **first order moment** or **mean function**.
If m_1^Y is constant in $\mathbb{T}$ then m_1^Y is called **stationary mean function**.

DEFINITION 4.2
Let $(Y, \mathbb{T})$, $\mathbb{T} \subset \mathbb{R}$, be a stochastic process with finite second order moments.
The function $K_2^Y\colon \begin{array}{ccc} \mathbb{T} & \longrightarrow & \mathbb{R} \\ t,s & \mapsto & K_2^Y(t,s) := \mathbb{E}[\,Y(t)\,Y(s)\,] \end{array}$
is called the **second order moment** or **comoment function**.
If $\exists\, \bar{K} \in (\mathbb{R}, \mathbb{R})$ such that $\forall, t, s \in \mathbb{T} : K_2^Y(t,s) = \bar{K}(t-s)$, then K_2^Y is called **stationary comoment function**.

DEFINITION 4.3
Let $(Y, [0,\infty)\,)$ be a stochastic process with finite first and second order moments.
If m_1^Y and K_2^Y are stationary, then Y is called a **weakly stationary process**.

We consider now solutions X of (2.1). Then X has the following representation

$$\begin{array}{rcll} X(t) & = & X(0) + \int\limits_{0-\tau}^{t}\int\limits^{0} X(u+r)\,da(u)\,dr + \int\limits_{0-\tau}^{t}\int\limits^{0} X(u+r)\,db(u)\,dW(r)\ , & t \in (0,\infty)\ , \\ X(t) & = & \xi(t)\ , & t \in [-\tau, 0]\ . \end{array} \tag{4.1}$$

We use this representation to derive a criterion for a constant first moments function of solutions X of (2.1).

Theorem 4.4
Let X be a solution of (2.1) with the first moments function m_1^X.
Then it holds: m_1^X *is stationary on* $[-\tau,\infty)$ $\Leftrightarrow$ $\begin{array}{l} (i)\ \ \forall\, t \in [-\tau, 0) :\ m_1^X(t) = m_1^X(0)\ , \\ (ii)\ \ 0 = m_1^X(0)\, a(\,[-\tau, 0]\,)\ . \end{array}$

Proof:
(1) Let $\forall\, t \in [-\tau,\infty) :\ m_1^X(t) = m_1^X(0)$. Then (i) is obvious. For $t \in [0,\infty)$ it follows:

$$\begin{array}{rcll} m_1^X(t) = \mathbb{E}[\,X(t)\,] & \underset{(4.1),Fubini}{=} & \mathbb{E}[\,X(0)\,] + \int\limits_{0-\tau}^{t}\int\limits^{0} \mathbb{E}[\,X(u+r)\,]\,da(u)\,dr & \\ & = & m_1^X(0) + \int\limits_{0-\tau}^{t}\int\limits^{0} m_1^X(u+r)\,da(u)\,dr\ , & \\ m_1^X(0) & = & m_1^X(0) + \int\limits_{0-\tau}^{t}\int\limits^{0} m_1^X(0)\,da(u)\,dr\ , & \\ 0 & = & m_1^X(0)\, a(\,[-\tau, 0]\,)\ . & \end{array}$$

This shows (ii).

(2) Let (i) and (ii) be satisfied. By (4.1) for $t \in [0,\infty)$ it follows that m_1^X satisfies the DDE (here in the equivalent integral representation):

$$\begin{array}{rcll} x(t) & = & x(0) + \int\limits_{0-\tau}^{t}\int\limits^{0} x(u+r)\,da(u)\,dr\ , & t \in [0,\infty)\ , \\ x(t) & = & m_1^X(0)\ , & t \in [-\tau, 0]\ . \end{array}$$

The above DDE has a unique solution. However as $a(\,[-\tau, 0]\,) = 0$, $x(t) \equiv m_1^X(0)$ solves the above DDE. Hence $m_1^X \equiv m_1^X(0)$ on $[-\tau,\infty)$. □

Next we use the representation (4.1) to derive a criterion for stationary comoment functions K_2^X of solutions X of (2.1).

Theorem 4.5

Let X *be a solution of (2.1) with the comoment function* K_2^X,
K_2^X *satisfy (2.18) with the initial condition* $k \in \mathcal{K}_{a,b}([-\tau,0]^2, \mathbb{R})$.

Then it holds:

(i) K_2^X *is stationary on* $[-\tau,\infty) \Leftrightarrow \forall\, u,v \in [-\tau,0]^2 : k(u,v) = f(-|u-v|)$ *where* $f \in ([-\tau,0], \mathbb{R})$ *satisfies*

$$0 = 2\int_{-\tau}^{0} f(u)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} f(-|u-v|)\, db(u)\, db(v) \ ,$$

$$f(t) = f(0) + \int_{t}^{0}\int_{-\tau}^{0} f(-|u-r|)\, da(u)\, dr \ , \qquad t \in [-\tau,0] \ .$$

(ii) *If* K_2^X *is stationary on* $[-\tau,\infty)$, *then* $\forall\, s \in [-\tau,\infty)$, $\bar{t} \in [0,\infty)$: $K_2^X(t+s,s) = x_f(\bar{t})$ *where* x_f *solves (3.1) with* $\xi = k(\_,0)$.

Proof:

(1) If K_2^X is stationary, then there exists $\bar{K}_2^X$ such that $\forall\, t,s \in [-\tau,0]:\ K_2^X(t,s) = \bar{K}_2^X(t-s)$.
Hence:
$\forall\, t,s \in [-\tau,\infty)$: $K_2^X(t,t) = \bar{K}_2^X(t-t) = \bar{K}_2^X(0) = \bar{K}_2^X(s-s) = K_2^X(s,s)$.
$\forall\, t \geq s \geq -\tau$: $K_2^X(t,s) = \bar{K}_2^X(t-s) = K_2^X(t-s,0)$.

(2) Define $\forall\, u \in [-\tau,0]:\ f(u) = K_2^X(u,0)$ and $\forall\, t \in [-\tau,0]:\ x_f(t) = K_2^X(t,0)$. (1) implies:

$$\begin{aligned}
\forall\, t \in [0,\infty): \quad 0 &= \tfrac{d}{dx} K_2^X(x,x)_{|x=t} \ , \\
\text{for } t=0: \quad 0 &= \tfrac{d}{dx} K_2^X(x,x)_{|x=0} \\
&\underset{(2.18)}{=} 2\int_{-\tau}^{0} K_2^X(u,0)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} K_2^X(u,v)\, db(u)\, db(v) \ , \\
&\underset{(2.18)}{=} 2\int_{-\tau}^{0} f(u)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} f(u\wedge v - u\vee v)\, db(u)\, db(v) \ , \\
\forall\, t \in [0,\infty): \quad x_f'(t) &= \tfrac{d}{dx} K_2^X(x,0)_{|x=t} \\
&\underset{(2.18)}{=} \int_{-\tau}^{0} K_2^X(t+u,0)\, da(u) \quad = \quad \int_{-\tau}^{0} x_f(t+u)\, da(u) \ . \\
\forall\, t \in [-\tau,0]: \quad f(t) &\underset{(1)}{=} x_f(-t) \\
&\underset{(2.18)}{=} x_f(0) + \int_{0}^{-t}\int_{-\tau}^{0} x_f(u+r)\, da(u)\, dr \\
f(t) &= f(0) + \int_{0}^{-t}\int_{-\tau}^{0} f(-|u+r|)\, da(u)\, dr
\end{aligned}$$

This shows „⇒“ in (i) and (ii).

(3) Let k and f given by (iii) and define x_f to be the solution of (3.1) with $\xi = f$.
Define $\forall\, t,s \in [-\tau,0]^2$: $\bar{K}(t,s) := k(t,s)$,
$\forall\, s \in [-\tau,\infty),\ \bar{t} \in [0,\infty)$: $\bar{K}(\bar{t}+s,s) := \bar{K}(s,\bar{t}+s) := x_f(\bar{t})$.

(4) Let $u \in [-\tau,0]$, $v = 0$. Then $k(u,0) = f(u\wedge 0 - u\vee 0) = f(u)$.

(5) $\bar{K}$ defined in (3) satisfies by definition $\bar{K}_{|[-\tau,0]^2} = k$.

(6) Let $s \in [-\tau,\infty)$, $t \in [0\vee s,\infty)$. Define $\bar{t} := t - s$. Then $\bar{t} \geq 0$ and $t = \bar{t} + s$.
By definition of $\bar{K}$ it follows $\bar{K}(t,s) = \bar{K}(\bar{t}+s,s) = x_f(\bar{t})$.
Hence: $\lim\limits_{h\mapsto 0, h\geq (0\vee s)-t} \frac{1}{h}(\,\bar{K}(t+h,s) - \bar{K}(t,s)\,) = \lim\limits_{h\mapsto 0, h\geq (0\vee s)-t} \frac{1}{h}(\,x_f(\bar{t}+h) - x_f(\bar{t})\,) = x_f'(\bar{t})$.
As x_f satisfies (3.1), it follows: $x_f'(\bar{t}) = \int_{-\tau}^{0} x_f(\bar{t}+u)\, da(u) = \int_{-\tau}^{0} x_f(\bar{t}+u)\, da(u)$.
By definition of $\bar{K}$, $\bar{t}$ it follows $\forall\, u \in [-\tau,0]$: $x_f(\bar{t}+u) = \bar{K}(\bar{t}+u+s,s) = \bar{K}(t+u,s)$.
Hence: $\lim\limits_{h\mapsto 0, h\geq (0\vee s)-t} \frac{1}{h}(\,\bar{K}(t+h,s) - \bar{K}(t,s)\,) = \int_{-\tau}^{0} \bar{K}(t+u,s)\, da(u)$.

(7) By definition of $\bar{K}$, $\forall\, t \in [0,\infty)$ it holds: $\bar{K}(t,t) = x_f(0)$.
Hence $\lim\limits_{h\mapsto 0, h\geq -t} \frac{1}{h}(\,\bar{K}(t+h,t+h) - \bar{K}(t,t)\,) = 0$.
By definition of f, x_f and $\bar{K}$ it follows that:
$\forall\, u \in [-\tau,0]$: $f(u) = x_f(u) = \bar{K}(t+u,t)$,
$\forall\, u,v \in [-\tau,0]$: $f(u\wedge v - u\vee v) = x_f(u\wedge v - u\vee v) = \bar{K}(t+u,t+v)$.
Hence: $\int_{-\tau}^{0} f(u)\, da(u) = \int_{-\tau}^{0} \bar{K}(t+u,t)\, da(u)$.
$\int_{-\tau}^{0}\int_{-\tau}^{0} f(u\wedge v - u\vee v)\, db(u)\, db(v) = \int_{-\tau}^{0}\int_{-\tau}^{0} \bar{K}(t+u,t+v)\, db(u)\, db(v)$.
But then it follows by (iii):

$$\begin{aligned}
\lim_{h\mapsto 0, h\geq -t} \tfrac{1}{h}(\,\bar{K}(t+h,t+h) - \bar{K}(t,t)\,) &= 0 \\
&= 2\int_{-\tau}^{0} f(u)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} f(u\wedge v - u\vee v)\, db(u)\, db(v) \\
&= 2\int_{-\tau}^{0} \bar{K}(t+u,t)\, da(u) + \int_{-\tau}^{0}\int_{-\tau}^{0} \bar{K}(t+u,t+v)\, db(u)\, db(v) \ .
\end{aligned}$$

(8) By (3), (5), (6), (7) $\bar{K}$ solves the amplified system (2.18) with the initial function k.
As K_2^X solves (2.18) with the initial function k, too. By corollary 2.15 it follows $K_2^X = \bar{K}$.
This proves „⇐“ in (i).

□

Now we can state necessary conditions on the existence of weakly stationary solutions of (2.1).

Theorem 4.6
Let X *be a solution of (2.1) with the mean function* m_1^X *and comoment function* K_2^X,
$k \in \mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})$ *defined by:* $\forall\, u,v \in [-\tau,0] :\ \mathbb{E}[\,X(u)\,X(v)\,]$,
$f \in ([-\tau,0],\mathbb{R})$ *defined by* $\forall\, u \in [-\tau,0] :\ f(u) = k(u,0)$,
x_f *solution of (3.1) with* $\xi = f$.
If X *is weakly stationary on* $[-\tau,\infty)$, *then it holds:*

(i) $\quad 0 = m_1^X(0)\, a(\,[-\tau,0]\,)$,

(ii) $\quad \forall\, u,v \in [-\tau,0] :\ k(u,v) = f(-|u-v|)$,

(iii) $\quad 0 \;=\; 2\int\limits_{-\tau}^{0} f(u)\,da(u) + \int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0} f(-|u-v|)\,db(u)\,db(v)$,

$$f(t) \;=\; f(0) + \int\limits_{t}^{0}\int\limits_{-\tau}^{0} f(-|u-r|)\,da(u)\,dr \quad , \qquad\qquad t \in [-\tau,0] \quad ,$$

(iv) $\quad K_2^X(t,s) = x_f(|t-s|)$.

Proof:
Combine the theorems 4.4 and 4.5. □

COROLLARY 4.7
If X is weakly stationary and $k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R})$ then 0 is an eigenvalue of order 2 and $f = k(\_,0)$ is an eigenfunction of order 2 of the generator of $\mathcal{T}_{sup}$. This is as the condition 4.6.(iii) is (3.4) for $\lambda = 0$.

EXAMPLE 4.8
We continue the example from section 2.5 and consider the geometric Brownian Motion.
If X has a stationary mean function by theorem 4.4 it follows:

$$0 = \mathbb{E}[\,X(0)\,]\,\alpha.$$

If X has a stationary comoment function by theorem 4.5 it follows:

$$\begin{aligned} 0 \;&=\; 2\int\limits_{-\tau}^{0} f(u+r)\,da(u) + \int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0} f(u\wedge v - u\vee v)\,db(u)\,db(v) \\ f(t) &= f(0) + \int\limits_{t}^{0}\int\limits_{-\tau}^{0} f(-|u-r|)\,da(u)\,dr \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} 0 \;&=\; (\,2\alpha+\beta^2\,)\,f(0) \\ f(t) &= f(0) + \alpha\int\limits_{t}^{0} f(r)\,dr \end{aligned}$$

hence

$$\begin{aligned} &f(0) = \mathbb{E}[\,|X(0)|^2\,],\ f(t) = f(0)\,e^{-\alpha\,t} \text{ and } x_f(t) = f(0)\,e^{\alpha\,t}, \\ &K_2^X(t,s) = \mathbb{E}[\,|X(0)|^2\,]\,e^{\alpha\,|t-s|} \text{ and } 0 = (\,2\,\alpha+\beta^2\,)\,\mathbb{E}[\,|X(0)|^2\,] \ . \end{aligned}$$

If X is weakly stationary by theorem 4.6 it follows (using the Jensen- and Cauchy-Schwarz-inequality)

$0 \neq 2\alpha+\beta^2\,, \qquad \forall\, t \in [0,\infty)\ \mathbb{E}[\,|X(0)|^2\,] = \mathbb{E}[\,X(0)\,] = 0, \quad$ or
$0 = 2\alpha+\beta^2,\ 0 \neq \alpha\,, \quad \forall\, t \in [0,\infty)\ \mathbb{E}[\,X(t)\,] = 0 \qquad$ or
$0 = \alpha = \beta$

The mean function of X is $m_1^X = \mathbb{E}[\,\xi\,]\,e^{\alpha t}$.
Hence X has a stationary mean function iff $\mathbb{E}[\,\xi\,] = 0$ or $0 = \alpha$.
The comoment function of X is $K_2^X(t,s) = \mathbb{E}[\,\xi^2\,]\,e^{\alpha\,t\vee s+(\alpha+\beta^2)\,t\wedge s} = \mathbb{E}[\,\xi^2\,]\,e^{\alpha\,|t-s|+(2\alpha+\beta^2)\,t\wedge s}$.
Hence X has a stationary comoment function iff $\mathbb{E}[\,\xi^2\,] = 0$ or $0 = 2\alpha+\beta^2$.
Finally, using the Jensen- and Cauchy-Schwarz-inequality, X is weakly stationary if and only if

$$\begin{aligned} &0 \neq 2\alpha+\beta^2, && \mathbb{E}[\,\xi^2\,] = \mathbb{E}[\,\xi\,] = 0, \text{ or} \\ &0 = 2\alpha+\beta^2,\ \alpha \neq 0, && \mathbb{E}[\,\xi\,] = 0, \text{ or} \\ &0 = 2\alpha+\beta^2,\ \alpha = 0. \end{aligned}$$

In this case the necessary conditions are identical with the exact conditions.

5 Appendix

5.1 The theorem of Fubini

Theorem 5.1

Let $(X,\mathcal{A})$, $(Y,\mathcal{B})$ *measurable spaces,*
μ *a* σ*-finite measure on* $(X,\mathcal{A})$, ν *a* σ*-finite measure on* $(Y,\mathcal{B})$.

(i) *Let* $\forall$ *non-negative* $f \in (X\times Y, \bar{\mathbb{R}})$, $(\mathcal{A}_X \times \mathcal{A}_Y, \mathcal{B})$ *-measurable,*

$$f_X(x) := \int_Y f(x,y)\, d\nu(y),\ f_Y(y) := \int_Y f(x,y)\, d\nu(y),$$

Then it holds:

(1) f_X *is non-negative,* $\mathcal{A}$*-measurable,* f_Y *is non-negative,* $\mathcal{B}$*-measurable*

(2) $\int_{X\times Y} f(x,y)\, d\mu\otimes\nu(x,y) = \int_X f_X(x)\, d\mu(x) = \int_Y f_Y(y)\, d\nu(y).$

(ii) *Let* $\forall f \in (X\times Y, \bar{\mathbb{R}})$, $(\mathcal{A}_X \times \mathcal{A}_Y, \mathcal{B})$ *-measurable and* $\mu\otimes\nu$*-integrable.*

$$f_X(x) := \int_Y f(x,y)\, d\nu(y),\ f_Y(y) := \int_Y f(x,y)\, d\nu(y),$$

$$A_X = \{\, x\in X \mid f(x,\_)\, is\, \nu-integrable \,\}, A_Y = \{\, y\in Y \mid f(\_,y)\, is\, \mu-integrable \,\}\ .$$

Then it holds:

(1) $\forall x\in X$*:* $f(x,\_)$ *is* $\mathcal{A}_Y$*-measurable,* ν*-integrable on* A_Y^c *and* $A_Y \in \mathcal{A}_Y$,

(2) $\forall y\in Y$*:* $f(\_,y)$ *is* $\mathcal{A}_X$*-measurable,* μ*-integrable on* A_X^c *and* $A_X \in \mathcal{A}_X$,

(3) $\int_{X\times Y} f(x,y)\, d\mu\otimes\nu(x,y) = \int_X f_X(x)\, d\mu(x) = \int_Y f_Y(y)\, d\nu(y).$

(iii) *Let* $\forall f \in (X\times Y, \bar{\mathbb{R}})$, $(\mathcal{A}_X \times \mathcal{A}_Y, \mathcal{B})$ *-measurable,*

$$\int_{X\times Y} |f|\, d\mu\otimes\nu < \infty \text{ or } \int_X\int_Y |f|\, d\nu\, d\mu < \infty \text{ or } \int_Y\int_X |f|\, d\mu\, d\nu < \infty\ .$$

Then it holds:

(1) $\int_{X\times Y} |f|\, d\mu\otimes\nu = \int_X\int_Y |f|\, d\nu d\mu = \int_Y\int_X |f|\, d\mu d\nu < \infty$,

(2) f *is* $\mu\otimes\nu$*-integrable.*

Proof:
See [3], pp.173. □

5.2 Weakly compact sets on $C(X)$

Theorem 5.2 (Arzelà-Ascoli)

Let (X,d) *be a metric space and* $S\subset\mathcal{C}(X,\mathbb{R})$.

Let (i) S *be bounded,*

(ii) S *be a set of uniformly continuous functions, that is*

$$\forall\epsilon>0\ \ \exists\delta>0\ \ \forall f\in S, x_1,x_2\in X:\ \ d(x_1,x_2)<\delta \implies |f(x)-f(y)|\le\epsilon.$$

Then it holds:

S *is weakly compact.*

If S *is in addition to* (i) *and* (ii) *closed, then it holds:*

S *is compact.*

Proof:
See [7], pp.68-70. □

LEMMA 5.3

Let $S\subset\mathcal{C}(X,\mathbb{R})$, S bounded,

$$\forall f\in S,\ x_1,x_2\in X: |f(x_2)-f(x_1)| \le L_f\, |x_2-x_1|_X^{\gamma_f}\ ,$$

$$\gamma := inf\{\,\gamma_f \mid f\in S\,\} > 0,\ L := sup\{\,L_f \mid f\in S\,\} < \infty.$$

Then it holds:

S is weakly compact.

Proof:
Let $\epsilon\in(0,\infty)$. Choose $\delta = \left(\frac{\epsilon\wedge 0.5}{L\vee 1}\right)^{\frac{1}{\gamma}}$.

Then for any $f \in S$ and $x_2, x_1 \in X$ with $|x_1 - x_2| < \delta$ it holds:

$$|f(x_2) - f(x_1)| < L_f \, |x_2 - x_1|_X^{\gamma_f} \leq L_f \, \delta^{\gamma_f} \leq L_f \, \delta^{\gamma} \leq \epsilon \wedge \tfrac{1}{2} \leq \epsilon \ .$$

Hence, S is uniformly continuous.
As S is bounded by assumption, by the theorem of Arzela-Ascoli S is precompact. □

5.3 Spectral theory of linear operators

The spectral theory of linear operators is the generalization of the theory of eigenvalues of linear, finite dimensional operators. We recall some definitions and results.
Let $(X, |\_|_X)$ be a Banach space.

DEFINITION 5.4
Let $L \in (\mathcal{D}(L), X)$ be a linear operator with $\mathcal{D}(L) \subset X$.
(1) Then $\lambda \in \mathcal{C}$ is called **resolvent of** L if and only if $(\lambda I - K)^{-1}$ exists and is bounded, that is if
   (i) $\lambda I - L$ is injective
   (ii) $Im(\lambda I - L) = X$
   (iii) $(\lambda I - L)^{-1}$ is bounded
(2) The set $\rho(L) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ resolvent of } L \}$ is called **resolvent set of** L.
(3) The set $\sigma(L) = \mathbb{C} \setminus \rho(L)$ is called **spectrum of** L.

The spectrum of infinite linear operators can be more complicated as for finite dimensional linear operators and can be divided into different types classifying how the above conditions $(i) - (iii)$ are not satisfied.

DEFINITION 5.5
Let $L \in (\mathcal{D}(L), X)$ be a linear operator with $\mathcal{D}(L) \subseteq X$.
(1) The set $\sigma_p(L) = \{ \lambda \in \mathbb{C} \mid \exists x \in X \setminus \{0\} : \ (\lambda I - L)\, x = 0 \}$ is called **point spectrum of** L and its elements **eigenvalues of** L.
(2) Let $\lambda \in \sigma_p(L)$.
   Any $x \in X \setminus \{0\}$ with $(\lambda I - L)\, x = 0$ is called an **eigenvector of** L.
   Let $N \subseteq X$ be the kernel (or nullspace) of $(\lambda I - L)$.
   Then N is called **eigenspace of** λ and $dim(N)$ **geometric multiplicity of** λ.
   Let $\bar{N} \subseteq X$ be the smallest closed, linear subspace including $\overset{\infty}{\underset{k=1}{\cup}} N(\lambda I - L)^k$.
   Then $\bar{N}$ is called **generalized eigenspace of** L and $dim(\bar{N})$ **algebraic multiplicity of** λ.
(3) The set $\sigma_c(L) = \{ \lambda \in \mathbb{C} \mid (\lambda I - L)$ injective, $Im(\lambda I - L) \subsetneq X, \overline{Im(\lambda I - L)} = X \}$ is called **continuous spectrum of** L.
(4) The set $\sigma_r(L) = \{ \lambda \in \mathbb{C} \mid (\lambda I - L)$ injective, $\overline{Im(\lambda I - L)} \subsetneq X \}$ is called **residual spectrum of** L.

The spectrum has some associated characteristic values.

DEFINITION 5.6
Let $L \in (\mathcal{D}(L), X)$ be a bounded linear operator with $\mathcal{D}(L) \subseteq X$.
Then $r(L) := \sup\{ |\lambda| \mid \lambda \in \sigma(L) \}$ is called the **spectral radius of** L.
Let $L \in (\mathcal{D}(L), X)$ be a closed linear operator with $\mathcal{D}(L) \subseteq X$.
$s(L) := \sup\{ Re(\lambda) \mid \lambda \in \sigma(L) \}$ is called the **spectral bound of** L.

5.4 Semigroups

Semigroups are related to abstract differential equations. We recall some definitions and results.
Let $(X, |\_|_X)$ be a Banach space.

DEFINITION 5.7
Let $T = \{ T(t) \}_{t \in [0,\infty)}$ be a family of bounded linear operators on X.

(1) If (i) $T(0) = id$,
(ii) for all $t, s \in [0, \infty)$: $T(t)\,T(s) = T(s)\,T(t)$,
then T is called a **semigroup of operators**.
(2) If in addition to (1).(i) and (1).(ii) it holds
(iii) for all $x \in X$: $\lim\limits_{h \downarrow 0} | T(h)x - x |_X = 0$,
then T is called a **strongly continuous semigroup of operators**.

With a strongly continuous semigroup one can associate an abstract differential equation

$$\frac{d}{dt}\,(T(t)\,x) = A(T(t)\,x)\,, \qquad x \in \mathcal{D}(A)\,, \tag{5.1}$$

where we define:

DEFINITION 5.8
Let $A : \mathcal{D}(A) \longrightarrow X$ be an operator, with
(i) $\mathcal{D}(A) := \{\, x \in X \mid \exists \lim\limits_{h \downarrow 0} \frac{1}{h}(T(h)x - x) \in X \,\}$,
(ii) $\forall\, x \in \mathcal{D}(A)$: $Ax := \lim\limits_{h \downarrow 0} \frac{1}{h}\,(T(h)x - x)$.

Then A is called **infinitesimal generator of** T and $\mathcal{D}$ is called **domain of** A.

Strongly continuous semigroups have the following properties:

LEMMA 5.9
(i) $\mathcal{D}(A)$ is dense in X,
(ii) A is closed,
(iii) $\exists c \in \mathbb{R}, M \in [1, \infty)$: $| T(t) | \leq M\, e^{ct}$.

Proof:
See [2], pp.453-454. □

The property 5.9.(iii) is an important property as it states that there exists an exponential growth bound for the norm of $T(t)$ and $t \longrightarrow \infty$. It is natural to ask for the smallest one and to use it as a characteristics of $T(t)$.

DEFINITION 5.10
Let $c_0 = \inf\{\, c \in \mathbb{R} \mid \exists M \in [1, \infty) : \ | T(t) |_X \leq M\, e^{c\,t} \,\}$.
Then c_0 is called the **growth bound of** T.

Reversely it holds:

Theorem 5.11 (Hille-Yosida)
Let $A \in (\mathcal{D}(A), X)$ be a linear operator.
Then the following two statements are equivalent:
(1) *A is an infinitesimal generator of a strongly continuous semigroup $T = \{T(t)\}_{t \in [0, \infty)}$ with $|T(t)| \leq M\, e^{c\,t}$,*
(2) (i) *A is closed, $\mathcal{D}(A)$ is dense in X,*
(ii) $(c, \infty) \subseteq \rho(A)$,
(iii) $\forall \lambda > c, k \in \mathbb{N} \setminus \{0\}$: $|(\lambda I - A)^{-k}| \leq \frac{M}{(\lambda - c)^k}$.

Proof:
See [2], p.455. □

From finite dimensional operators it is known, that there is a relation between the growth of $\{T(t)\}_{t \in [0, \infty)}$ and the eigenvalues of the generator A of T. The relation is described in the so-called **spectral mapping theorem** stating that $\sigma(T(t)) = e^{t\sigma(A)}$ where $\sigma(T(t))$ are the eigenvalues of $T(t)$ and $\sigma(A)$ are the eigenvalues of the infinitesimal generator A of T. For strongly continuous semigroups there are several relations known.

LEMMA 5.12
(*i*) $\sigma(P(L)) = P(\sigma(L)), \qquad P$ polynomial,
(*ii*) $\sigma_p(T(t)) \setminus \{0\} = e^{t\,\sigma_p(A)}$,
(*iii*) $r(T(t)) = e^{c_0\,t}$.
(*iv*) $s(A) \leq c_0$.

Proof:
See [2]. (i): p.470, (ii): p.471, (iii): p.470, (iv): p.470. □

The relation 5.12.(iii) interpretes the growth bound in terms of the linear operators $T(t)$.
The relation 5.12.(iv) provides a characterization of the asymptotic behaviour of T in terms of its infinitesimal generator: the spectrum of A provides a lower bound for the growth bound of the generated semigroup. This criterion is in general unsharp. An example for $s(A) < c_0$ can be found in [2], p.470. However, in special cases the characterization of the asymptotic behaviour of T in terms of its infinitesimal generator is possible. One of the special cases is the following.

DEFINITION 5.13
Let $T = \{\, T(t) \,\}_{t\in[0,\infty)}$ be a strongly continuous semigroup.
T is called **eventually compact** if and only if $\exists\, t_0 \in [0,\infty)\ \forall\, t \in [t_0,\infty)$: $T(t)$ is compact.

For eventually compact strongly continuous semigroups the following is known.

Theorem 5.14
Let $T = \{\, T(t) \,\}_{t\in[0,\infty)}$ be a strongly continuous, eventually compact semigroup,
A be the infinitesimal generator of T.

Then it holds:
(*i*) $\sigma(A) = \sigma_p(A)$,
(*ii*) $s(A) = c_0$.

Proof:
See [2], p.97. □

EXAMPLE 5.15
Consider affine DDE.
Then $T = \{\, T(t) \,\}_{t\in[0,\infty)}$ defined by $T(t)\phi := x_t$, $t \in [0,\infty)$ is eventually compact ([5], p.194.

5.5 Proofs of selected theorems and lemmas

Proof: (lemma 2.4)
(*i*) and (*iii*) are already proven. So (*ii*) remains to be shown.
First we represent the difference quotient as a sum. It holds:

$$
\begin{aligned}
&\tfrac{1}{-h}(\,\mathbb{E}[\,X(t-h)^2 - X(t)^2\,)\\
&\quad= \tfrac{1}{h}\mathbb{E}[\,X(t)+X(t-h)\,)(\,X(t)-X(t-h)\,)\,]\\
&\quad= \tfrac{1}{h}\mathbb{E}[\,(\,2X(t-h) + \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds + \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s)\,)\\
&\qquad\qquad (\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds + \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s)\,)\,]\\
&\quad= \tfrac{1}{h}(\quad \mathbb{E}[\,X(t-h)\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds\,]\\
&\qquad +2\,\mathbb{E}[\,X(t-h)\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s)\,]\\
&\qquad +\ \mathbb{E}[\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds\,]\\
&\qquad +2\,\mathbb{E}[\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,da(u)\,ds \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s)\,]\\
&\qquad +\ \mathbb{E}[\int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s) \int_{t-h-\tau}^{t}\int_{}^{0} X(s+u)\,db(u)\,dW(s)\,]\,).
\end{aligned}
\tag{5.2}
$$

Now we determine the limits of each of the five summands in (5.2):

$$(i)\ \ \tfrac{1}{h}\mathbb{E}[\,X(t-h)\int_{t-h}^{t}\int_{-\tau}^{0}X(s+u)da(u)\,ds\,]$$

$$\begin{aligned}
&= \tfrac{1}{h}\mathbb{E}[\int_{t-h}^{t}\int_{-\tau}^{0}X(t-h)X(s+u)da(u)\,ds\,]\\
&\underset{Fubini}{=} \tfrac{1}{h}\int_{t-h}^{t}\int_{-\tau}^{0}\mathbb{E}[\,X(t-h)X(s+u)\,]\,da(u)\,ds
\end{aligned}$$

$$\lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}[\,X(t-h)\int_{t-h}^{t}\int_{-\tau}^{0}X(s+u)da(u)\,ds\,] = \int_{-\tau}^{0}\mathbb{E}[\,X(t)X(t+u)\,]\,da(u),$$

$$(ii)\ \tfrac{1}{h}\mathbb{E}[\,X(t-h)\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,db(u)\,dW(s)\,]$$

$$\begin{aligned}
&\underset{Fubini}{=} \tfrac{1}{h}\mathbb{E}[\,X(t-h)\int_{-\tau}^{0}\int_{t-h}^{t}X(s+u)\,dW(s)\,db(u)\,]\\
&= \tfrac{1}{h}\mathbb{E}[\int_{-\tau}^{0}X(t-h)\int_{t-h}^{t}X(s+u)\,dW(s)\,db(u)\,]\\
&\underset{Fubini}{=} \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\,X(t-h)\int_{t-h}^{t}X(s+u)\,dW(s)\,]\,db(u)\\
&= \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\,X(t-h)\,\mathbb{E}[\int_{t-h}^{t}X(s+u)\,dW(s)\ |\ \mathcal{F}_{t-h}\,]\,]\,db(u)\\
&\underset{Itô-integral}{=} \tfrac{1}{h}\int_{-\tau}^{0}\mathbb{E}[\,X(t-h)\,0\,]\,db(u)\\
&= 0
\end{aligned}$$

$$\lim_{h\downarrow 0}\tfrac{1}{h}\mathbb{E}[\,X(t-h)\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,db(u)\,dW(s)\,] = 0$$

$$(iii)\ |\ \tfrac{1}{h}\mathbb{E}[\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,da(u)\,ds\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,da(u)\,ds\,]\ |$$

$$\begin{aligned}
&\underset{Fubini}{\leq} \tfrac{1}{h}\int_{t-h}^{t}\int_{t-h}^{t}\ |\ \mathbb{E}[\int_{-\tau}^{0}X(s+u)\,da(u)\int_{-\tau}^{0}X(r+u)\,da(u)\,]\ |\ ds\,dr\\
&\underset{Hölder}{\leq} \tfrac{1}{h}\int_{t-h}^{t}\int_{t-h}^{t}\mathbb{E}[\,|\int_{-\tau}^{0}X(s+u)da(u)\,|^2\,]^{1/2}\,\mathbb{E}[\,|\int_{-\tau}^{0}X(s+u)da(u)\,|^2\,]^{1/2}\,ds\,dr,
\end{aligned}$$

$$\begin{aligned}
&\lim_{h\downarrow 0}|\ \tfrac{1}{h}\mathbb{E}[\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)da(u)\,ds\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)da(u)\,ds\,]\ |\\
&\leq \lim_{h\downarrow 0}\tfrac{1}{h}\int_{t-h}^{t}\int_{t-h}^{t}\mathbb{E}[\,|\int_{-\tau}^{0}X(s+u)da(u)\,|^2\,]^{1/2}\,\mathbb{E}[\,|\int_{-\tau}^{0}X(s+u)da(u)\,|^2\,]^{1/2}\,ds\,dr,\\
&\leq 0,
\end{aligned}$$

$$(iv)\ |\ \tfrac{1}{h}\mathbb{E}[\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,da(u)\,ds\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,db(u)\,dW(s)\,]\ |$$

$$\begin{aligned}
&\underset{Fubini}{=} |\ \tfrac{1}{h}\mathbb{E}[\int_{t-h-\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}X(s+u)\int_{t-h}^{t}X(r+v)\,dW(r)\,db(v)\,da(u)\,ds\,]\ |\\
&\underset{Jensen}{\leq} \tfrac{1}{h}\int_{t-h-\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}\ |\ \mathbb{E}[\,X(s+u)\int_{t-h}^{t}X(r+v)\,dW(r)\,]\ |\ d|b|(v)\,d|a|(u)\ :ds\\
&\underset{Hölder}{\leq} \tfrac{1}{h}\int_{t-h-\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)^2\,]^{1/2}\,(\int_{t-h}^{t}\mathbb{E}[\,X(r+v)^2\,]\,dr)^{1/2}\,d|b|(v)\,d|a|(u)\,ds,
\end{aligned}$$

$$\begin{aligned}
&\lim_{h\downarrow 0}|\ \tfrac{1}{h}\mathbb{E}[\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,da(u)\,ds\int_{t-h-\tau}^{t}\int_{-\tau}^{0}X(s+u)\,db(u)\,dW(s)\,]\ |\\
&\leq \lim_{h\downarrow 0}\tfrac{1}{h}\int_{t-h-\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)^2\,]^{1/2}\,(\int_{t-h}^{t}\mathbb{E}[\,X(r+v)^2\,]\,dr)^{1/2}\,d|b|(v)\,d|a|(u)\,ds,\\
&\leq 0\ ,
\end{aligned}$$

$$(v)\ \mathbb{E}[\int_{t-h-\tau}^{t}\int^{0} X(s+u)\,db(u)\,dW(s)\int_{t-h-\tau}^{t}\int^{0} X(s+u)\,db(u)\,dW(s)\,]$$

$$\underset{Itô-integral}{=} \frac{1}{h}\int_{t-h}^{t}\mathbb{E}[\int_{-\tau}^{0} X(s+u)\,db(u)^2\,]\,ds$$

$$\underset{Fubini}{=} \frac{1}{h}\int_{t-h-\tau}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)X(s+v)\,]\,db(u)\,db(v)\,]\,ds$$

$$\lim_{h\to 0}\mathbb{E}[\int_{t-h-\tau}^{t}\int^{0} X(s+u)\,db(u)\,dW(s)\int_{t-h-\tau}^{t}\int^{0} X(s+u)\,db(u)\,dW(s)\,]$$

$$= \int_{-\tau}^{0}\int_{-\tau}^{0}\mathbb{E}[\,X(s+u)\,X(s+v)\,]\,db(u)\,db(v)\,]\,ds\quad .$$

Plugging (i)-(v) into (5.2) yields then the limit to be shown. □

Proof: (lemma 2.6)

As in the previous lemma we use the integral representation of $X(t)-X(t-h)$ for $h\geq 0$ resulting from the SDDE to find a suitable representation of the difference quotient and take then the limit $h\downarrow 0$.

$$\frac{1}{-h}(\,\mathbb{E}[\,(X(t-h)-X(t))X(r)\,]$$

$$= \frac{1}{h}\mathbb{E}[\,(\int_{t-h-\tau}^{t}\int^{0} X(s+u)da(u)\,ds+\int_{t-h-\tau}^{t}\int^{0} X(s+u)db(u)\,dW(s)\,)\,X(r)\,]$$

$$\underset{Fubini}{=} \frac{1}{h}(\int_{t-h-\tau}^{t}\int^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)\int_{t-h}^{t} X(s+u)\,dW(s)\,]\,db(u)\,)$$

$$= \frac{1}{h}(\int_{t-h-\tau}^{t}\int^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)\mathbb{E}[\int_{t-h}^{t} X(s+u)\,dW(s)\ \mid\ \mathcal{F}_{t-h}\,]\,]\,db(u)\,)$$

$$\underset{Itô-integral}{=} \frac{1}{h}(\int_{t-h-\tau}^{t}\int^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds+\int_{-\tau}^{0}\mathbb{E}[\,X(r)0\,]\,db(u)\,)$$

$$= \frac{1}{h}(\int_{t-h-\tau}^{t}\int^{0}\mathbb{E}[\,X(s+u)X(r)\,]\,da(u)\,ds,$$

$$\lim_{h\downarrow 0}\tfrac{1}{-h}\,\mathbb{E}[\,(X(t-h)-X(t))X(r)\,]=\int_{-\tau}^{0}\mathbb{E}[\,X(t+u)X(r)\,]\,da(u).$$

□

Proof: (theorem 2.22)

(1) We investigate the differences of consecutive Picard-iterates. $\forall\, n \in \mathbb{N}$, $\forall\, t,s \in [-\tau,\infty)$ define:

$$\begin{aligned}\Delta_{n+1}K(t,s) &:= K_{n+1}(t,s) - K_n(t,s)\ ,\\ \delta_{n+1}(t) &:= \sup\{\,|\,\Delta_{n+1}K(s_1,s_2)\,|\ \mid\ t \geq s_1 \geq s_2 \geq -\tau\,\}\ .\end{aligned}$$

By (1) and the previous definition, for all $n \in \mathbb{N}$ and for all $0 \geq t \geq s \geq -\tau$ it holds:

$$\Delta_{n+1}K(t,s) = 0$$

and for all $n \in \mathbb{N}$, $t \in (0,\infty)$, $s \in [-\tau,t)$ it holds:

$$\begin{aligned}\Delta_{n+2}K(t,t) &= \Delta_{n+1}K(0,0) + 2\int_{0}^{t}\int_{-\tau}^{0}\Delta_{n+1}K(r+u,r)\,da(u)\,dr\\ &\qquad + \int_{0}^{t}\int_{-\tau}^{0}\int_{-\tau}^{0}\Delta_{n+1}K(r+u,r+v)\,db(u)\,db(v)\,dr\ ,\\ \Delta_{n+2}K(t,s) &= \Delta_{n+2}K(s,s) + \int_{s}^{t}\int_{-\tau}^{0}\Delta_{n+1}K(r+u,s)\,da(u)\,dr\ .\end{aligned}$$

(2) By the definition of the Picard iterates $\forall\, n \in \mathbb{N}$, $s \in [-\tau,\infty)$, $t \in [0 \vee s,\infty)$ it holds:

$$\begin{aligned}\Delta_{n+1}K(0,0) &:= 0\ ,\\ |\,\Delta_{n+2}K(t,t)\,| &\leq 2\int_{-\tau}^{0}d|a|(u)\int_{0}^{t}\delta_{n+1}(r)\,dr + \Big(\int_{-\tau}^{0}d|b|(u)\Big)^2\int_{0}^{t}\delta_{n+1}(r)\,dr \leq C(a,b,\tau)\int_{0}^{t}\delta_{n+1}(r)\,dr\ ,\\ |\,\Delta_{n+2}K(t,s)\,| &\leq |\,\Delta_{n+2}K(0\vee s,s)\,| + \Big|\int_{0\vee s}^{t}\int_{-\tau}^{0}\Delta_{n+1}K(r+u,s)\,da(u)\,dr\,\Big|\\ &\leq C(a,b,\tau)\int_{0}^{0\vee s}\delta_{n+1}(r)\,dr + \int_{-\tau}^{0}d|a|(u)\int_{0\vee s}^{t}\delta_{n+1}(r)\,dr \leq C(a,b,\tau)\int_{0}^{t}\delta_{n+1}(r)\,dr\ .\end{aligned}$$

(3) Fix $t \in (0,\infty)$. By (2) for all $\bar{s},\tilde{s}$ with $t > \bar{s} \geq \tilde{s} \geq -\tau$ it holds:

$$|\,\Delta_{n+2}K(\bar{s},\tilde{s})\,| \leq C(a,b,\tau)\int_{0}^{0\vee\bar{s}}\delta_{n+1}(r)\,dr \leq C(a,b,\tau)\int_{0}^{t}\delta_{n+1}(r)\,dr\ .$$

(4) Using (2),(3) $\forall\, n \in \mathbb{N}$ and $\forall\, t \in [0,\infty)$ it holds:

$$\delta_{n+2}(t) = \sup\{\,|\,\Delta_{n+2}K(\bar{s},\tilde{s})\,|\ \mid\ t \geq \bar{s} \geq \tilde{s} \geq -\tau\,\} \leq C(a,b,\tau)\int_{0}^{t}\delta_{n+1}(r)\,dr.$$

(5) As in (3) of the proof of lemma 2.14, we introduce two index sets and a constant:

$$\begin{aligned}A_\tau &:= \{\,(t,u) \in \mathbb{R}^2 \mid t \in [-\tau,\tau], u \in [-t\wedge-\tau,-t]\,\}\ ,\\ B_\tau &:= \{\,(t,u,v) \in \mathbb{R}^3 \mid t \in [-\tau,\tau], u,v \in [-t\wedge-\tau,-t]\,\}\ ,\\ S^0_\tau &:= \sup_{t,s\in[-\tau,0],\,t\vee s=0}\{\,|\,k(t,s)\,|\,\}\ ,\\ S^1_\tau &:= \sup_{s\in[-\tau,0]}\{\int_{0}^{\tau}\int_{-\tau}^{-r}|\,k(r+u,s)\,|\,d|a|(u)\,dr\,\}\ ,\\ S^2_\tau &:= \int_{0}^{\tau}\int_{-\tau}^{-r}\int_{-\tau}^{-r}|\,k(r+u,r+v)\,|\,d|b|(u)\,d|b|(v)\,dr\ .\end{aligned}$$

(6) Let $t \in [0,\infty)$. Then it holds:

$$\begin{aligned}
|\,\Delta_1 K(t,t)\,| \;&\leq\; 2\,|\int_0^t\int_{-\tau}^0 K_0(r+u,r)\,da(u)\,dr\,| + |\int_0^t\int_{-\tau}^0\int_{-\tau}^0 K_0(r+u,r+v)\,db(u)\,db(v)\,dr\,| \\
&\leq\; 2\int_0^t\int_{-\tau}^0 |\,K_0(r+u,r)|\,1_{A_\tau}((r,u))\,d|a|(u)\,dr \\
&\quad + 2\int_0^t\int_{-\tau}^0 |\,K_0(r+u,r,\ldots,r)|\,(1-1_{A_\tau}((r,u)))\,d|a|(u)\,dr \\
&\quad + \int_0^t\int_{-\tau}^0\int_{-\tau}^0 |\,K_0(r+u,r+v)|\,1_{B_\tau}((r,u,v))\,d|b|(u)\,d|b|(v)\,dr \\
&\quad + \int_0^t\int_{-\tau}^0\int_{-\tau}^0 |\,K_0(r+u,r+v)|\,(1-1_{B_\tau}((r,u,v)))\,d|b|(u)\,d|b|(v)\,dr \\
&\leq\; 2\,S_\tau^1 + 2\int_{-\tau}^0 d|a|(u)\;S_\tau^0\,t + S_\tau^2 + (\int_{-\tau}^0 d|b|(u)\,)^2\;S_\tau^0\,t \\
&\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})} + t\,C(a,b,\tau)\,S_\tau^0\;\;.
\end{aligned}$$

(7) For all $s \in [-\tau,\infty), t \in (0 \vee s,\infty)$ it holds:

$$\begin{aligned}
|\,\Delta_1 K(t,s)\,| \;&\leq\; |\int_{0\vee s}^t\int_{-\tau}^0 K_0(r+u,s)\,da(u)\,dr\,| \\
&\leq\; \int_0^t\int_{-\tau}^0 |\,K_0(r+u,s)\,|\,d|a|(u)\,dr\,1_{[-\tau,0]}(s) \\
&\quad + \int_s^t\int_{-\tau}^0 |\,K_0(r+u,s)\,|\,1_{A_\tau}((t,u))\,d|a|(u)\,dr\,1_{(0,\infty)}(s) \\
&\quad + \int_s^t\int_{-\tau}^0 |\,K_0(r+u,s)\,|\,(1-1_{A_\tau}((t,u)))\,d|a|(u)\,dr\,1_{(0,\infty)}(s) \\
&\leq\; S_\tau^1 + \int_{-\tau}^0 d|a|(u)\;S_\tau^0\,t \\
&\leq\; S_\tau^1 + t\,C(a,b,\tau)\,S_\tau^0\;\;.
\end{aligned}$$

(8) Due to (6)-(7) for all $s \in [-\tau,\infty)$, $t \in [0 \vee s,\infty)$ it holds:

$$|\,\Delta_1 K(t,s)\,| \;\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;(\,1 + t\,C(a,b,\tau)\,)\;\;.$$

(9) By definition of K_0 for all $s \in [-\tau,0]$, $t, \in [0 \vee s,\infty)$ it holds:

$$|\,K_0(t,s)\,| \;\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;\;.$$

(10) From (8) and (10) we can conclude via complete induction that for all $T \in [0,\infty)$:

$$\begin{aligned}
\delta_{n+1}(T) \;&\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;(\,\tfrac{1}{n!}(C(a,b,\tau)\,T)^n + \tfrac{1}{(n+1)!}(C(a,b,\tau)\,T)^{n+1}\,)\;\;, \\
\sum_{n=0}^\infty \delta_{n+1}(T) \;&\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;(\,2\,e^{C(a,b,\tau)\,T} - 1\,) < \infty\;\;.
\end{aligned}$$

With (9) it follows then that for all $n \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{0\}$, $T \geq 0$ and $(t,s) \in [-\tau,T]^2 \setminus [-\tau,0]^2$:

$$\begin{aligned}
|\,K_{n+1}(t,s)\,| \;&\leq\; |\,K_0(t,s)\,| + \sum_{k=1}^{n+1} |\,\Delta K_k(t,s)\,| \\
&\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;2\,e^{C(a,b,\tau)\,T}\;\;, \\
|\,K_{n+1}(t,s) - K_{n+m+1}(t,s)\,| \;&\leq\; \sum_{k=n+1}^{n+m+1} |\,\Delta K_k(t,s)\,| \\
&\leq\; \|k\|_{\mathcal{K}_{a,b}([-\tau,0]^2,\mathbb{R})}\;2\,(e^{C(a,b,\tau)\,T} - \sum_{k=0}^{n+1}\tfrac{(C(a,b,\tau)\,T)^k}{k!})\;\;.
\end{aligned}$$

Hence $\{K_n\}_{n\in\mathbb{N}}$ is uniformly bounded and a Cauchy-sequence on $[-\tau,T]^2 \setminus [-\tau,0)^2$. But then the pointwise limit $K := \lim_{n\to\infty} K_n$ does exist on $[-\tau,T]^2$.

(11) Now we show, that K found in (11) satisfies (2.4). Let n be in $\mathbb{N} \setminus \{0\}$. Then for all $t \in (0, \infty)$ it holds:

$$
\begin{aligned}
&| K(t,t) - K(0,0) - 2\int_0^t \int_{-\tau}^0 K(r+u,r)\, da(u)\, dr - \int_0^t \int_{-\tau}^0 \int_{-\tau}^0 K(r+u,r+v)\, db(u)\, db(v)\, dr| \\
&\quad \leq \quad | K(t,t) - K_{n+1}(t,t) | + | K(0,0) - K_n(0,0) | \\
&\qquad + | 2\int_0^t \int_{-\tau}^0 K(r+u,r) - K_n(r+u,r)\, da(u)\, dr | \\
&\qquad + | \int_0^t \int_{-\tau}^0 \int_{-\tau}^0 K(r+u,r+v) - K_n(r+u,r+v)\, db(u)\, db(v)\, dr | \\
&\quad \leq \quad \|K - K_{n+1}\|_{\mathcal{C}([0,T]^2,\mathbb{R})} + C(a,b,\tau)\ \|K - K_n\|_{\mathcal{C}([0,T]^2,\mathbb{R})} \ ,
\end{aligned}
$$

Furthermore, for any $s \geq -\tau$, $t \in (0 \vee s, \infty)$:

$$
\begin{aligned}
&| K(t,s) - K(0 \vee s, s) - \int_{0 \vee s - \tau}^t \int_{-\tau}^0 K(r+u,s)\, da(u)\, dr | \\
&\leq \ | K(t,s) - K_{n+1}(t,s) | + \ | K(0 \vee s) - K_n(0 \vee s) | + \ | \int_{0 \vee s - \tau}^t \int_{-\tau}^0 K(r+u,s) - K_n(r+u,s)\, da(u)\, dr | \\
&\leq \ \|K - K_{n+1}\|_{\mathcal{C}([0,T]^2,\mathbb{R})} + C(a,b,\tau)\, \|K - K_n\|_{\mathcal{C}([0,T]^2,\mathbb{R})} \ .
\end{aligned}
$$

As $\lim_{n \to \infty} \|K - K_n\|_{\mathcal{C}([0,T]^2,\mathbb{R})} = 0$ by (10) the theorem is proved.

□

Proof: (theorem 2.26)

(1) We want to partition the total difference into suitable summands. First we introduce some necessary variables. Let $t > s \geq 0$, $h = (h_1, h_2) \in \mathbb{R}^2$ such that $\|h\|_2 < \frac{1}{2}|t-s|$.
Define $M(a,b) = max\{a+b, a\}$, $m(a,b) = min\{a+b, a\}$ for all $a, b \in \mathbb{R}$.
Now we derive partition the total difference. It holds:

$$
\begin{aligned}
& K(t+h_1, s+h_2) - K(t,s) \\
= \;& \mathbb{E}[\,(X(t+h_1) - X(t))\,(X(s+h_2) - X(s))\,] + \mathbb{E}[\,(X(t+h_1) - X(t))\,X(s)\,] \\
& + \mathbb{E}[\,X(t)\,(X(s+h_2) - X(s))\,] \\
= \;& (-1)^{\delta_{[0,\infty)}(h_1)+\delta_{[0,\infty)}(h_2)}\,(\,\mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,da(v)\,dr\,] \\
& + \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,] \\
& + \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dW(r) \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,da(v)\,dr\,] \\
& + \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,db(v)\,dW(r) \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,] \\
& + (-1)^{\delta_{[0,\infty)}(h_1)}\,(\,\mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr\,X(s)\,] + \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,X(s)\,] \\
& + (-1)^{\delta_{[0,\infty)}(h_2)}\,(\,\mathbb{E}[\,X(t)\int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,da(v)\,dr\,] + \mathbb{E}[\,X(t)\int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,]\,)
\end{aligned}
$$

(2) We deal now with one particular summand of (1) and can show that it is of order $o(\|h\|_2)$.
It holds:

$$
\begin{aligned}
& |\,\mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,da(v)\,dr\,]\,| \\
\underset{Fubini}{=} & \int_{m(t,h_1)}^{M(t,h_1)}\int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{-\tau}^{0}\int_{-\tau}^{0} \mathbb{E}[\,X(r_1+v_1)X(r_2+v_2)\,]\,da(v_1)\,dr_1\,da(v_2)\,dr_2\,| \\
\underset{Jensen}{\leq} & |h_1 \cdot h_2|\,\tfrac{1}{|h_1|}\int_{m(t,h_1)}^{M(t,h_1)}\tfrac{1}{|h_2|}\int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{-\tau}^{0}\int_{-\tau}^{0} |\,\mathbb{E}[\,X(r_1+v_1)X(r_2+v_2)\,]\,|\,da(v_1)\,dr_1\,da(v_2)\,dr_2
\end{aligned}
$$

As $\lim\limits_{h\to 0}\frac{|h_1\cdot h_2|}{\|h\|_2} \leq \lim\limits_{h\to 0}\frac{1}{2}\|h\|_2 = 0$, the above term converges to 0 for $h \to 0$.

(3) As in (2) we deal with particular summands of (1) and show that they vanish. Due to the choice of h, $[m(t,h_1), M(t,h_1)] \cap [m(s,h_2), M(s,h_2)] = \emptyset$. But then, using the properties of the Ito-integral, it holds:

$$
\begin{aligned}
& \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dW(r) \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,da(v)\,dr\,] = 0, \\
& \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,db(v)\,dW(r) \int_{m(s,h_2)-\tau}^{M(s,h_2)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,] = 0, \\
& \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,db(v)\,dW(r)\,X(s)\,] = 0.
\end{aligned}
$$

(4) Now we deal with a particular summand from (1) and filter out its h-linear fraction.
It holds:

$$
\begin{aligned}
& \tfrac{1}{\|h\|_2}\,|\,(-1)^{\delta_{[0,\infty)}(h_1)}\,\mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr\,X(s)\,] - h_1\int_{-\tau}^{0}\mathbb{E}[\,X(t+v)X(s)\,]\,da(v)\,| \\
= \;& \tfrac{|h_1|}{\|h\|_2}\,|\,(-1)^{\delta_{[0,\infty)}(h_1)}\,\tfrac{1}{|h_1|}\,(\,\mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0} X(r+v)\,da(v)\,dr\,X(s)\,] - \int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0}\mathbb{E}[\,X(t+v)X(s)\,]\,da(v)\,dr\,)\,| \\
= \;& \tfrac{|h_1|}{\|h\|_2}\,|\,\tfrac{1}{|h_1|}\,(\int_{m(t,h_1)-\tau}^{M(t,h_1)}\int_{}^{0}\mathbb{E}[\,X(r+v)X(s)\,] - \mathbb{E}[\,X(t+v)X(s)\,]\,da(v)\,dr\,)\,|
\end{aligned}
$$

$$\leq \frac{|h_1|}{\|h\|_2} \left(\frac{1}{|h_1|} \int_{m(t,h_1)-\tau}^{M(t,h_1)} \int_{-\tau}^{0} |K(r+v,s) - K(t+v,s)| \, d|a|(v) \, dr \right)$$

$$\leq \frac{|h_1|}{\|h\|_2} \sup_{r \in [m(t,h_1), M(t,h_1)]} \{ \int_{-\tau}^{0} |K(r+v,s) - K(t+v,s)| \, d|a|(v) \}.$$

It holds $0 \leq \frac{|h_1|}{\|h\|_2} \leq 1$. As K is continuous on $[0,\infty)^2$, it is uniformly continuous on compact sets and the above term tends to 0 for $h \to 0$.

(5) As in (4) we deal with a particular summand from (1) and filter out its h-linear fraction.
It holds:

$$\frac{1}{\|h\|_2} \left| (-1)^{\delta_{[0,\infty)}(h_2)} \, \mathbb{E}[X(t) \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} X(r+v) \, da(v) \, dr] - h_2 \int_{-\tau}^{0} \mathbb{E}[X(t) X(s+v)] \, da(v) \right|$$

$$= \frac{|h_2|}{\|h\|_2} \left| (-1)^{\delta_{[0,\infty)}(h_2)} \frac{1}{|h_2|} \left(\mathbb{E}[X(t) \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} X(r+v) \, da(v) \, dr] - \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} \mathbb{E}[X(t) X(s+v)] \, da(v) \, dr \right) \right|$$

$$= \frac{|h_2|}{\|h\|_2} \left| \frac{1}{|h_2|} \left(\int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} \mathbb{E}[X(t) X(r+v)] - \mathbb{E}[X(t) X(s+v)] \, da(v) \, dr \right) \right|$$

$$\leq \frac{|h_2|}{\|h\|_2} \left(\frac{1}{|h_2|} \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} |K(t,r+v) - K(t,s+v)| \, d|a|(v) \, dr \right)$$

$$\leq \frac{|h_2|}{\|h\|_2} \sup_{r \in [m(s,h_2), M(s,h_2)]} \{ \int_{-\tau}^{0} |K(t,r+v) - K(t,s+v)| \, d|a|(v) \}.$$

It holds $0 \leq \frac{|h_2|}{\|h\|_2} \leq 1$. As K is continuous on $[0,\infty)^2$, it is uniformly continuous on compact sets and the above term tends to 0 for $h \to 0$.

(6) Defining first for all $\hat{t}, \hat{s} \in [t_0, \infty)$, $\hat{h} \in \mathbb{R}_+$:

$$\overline{G}(\hat{t}; \hat{s}, \hat{h}) \quad = \quad \mathbb{E}[X(\hat{t}) \int_{\hat{s}}^{\hat{s}+\hat{h}} \int_{-\tau}^{0} X(\hat{r}+\hat{v}) \, db(\hat{v}) \, dW(\hat{r})] \ ,$$

we continue to evaluate the terms from (1).
Due to lemma (2.25) it holds:

$$\frac{1}{\|h\|_2} \left| \mathbb{E}[\int_{m(t,h_1)-\tau}^{M(t,h_1)} \int_{-\tau}^{0} X(r+v) \, da(v) \, dr \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} X(r+v) \, db(v) \, dW(r)] \right|$$

$$\underset{Fubini}{=} \frac{1}{\|h\|_2} \left| \int_{m(t,h_1)-\tau}^{M(t,h_1)} \int_{-\tau}^{0} \mathbb{E}[X(\bar{r}+\bar{u}) \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} X(r+v) \, db(v) \, dW(r)] \, da(\bar{u}) \, d\bar{r} \right|$$

$$\leq \frac{1}{\|h\|_2} \int_{m(t,h_1)-\tau}^{M(t,h_1)} \int_{-\tau}^{0} | \, \overline{G}(\bar{r}+\bar{u}; m(s,h_2), |h_2|, v) \, | \, d|a|(\bar{u}) \ d\bar{r}$$

$$\leq \frac{|h_1|}{\|h\|_2} \frac{1}{|h_1|} \int_{m(t,h_1)-\tau}^{M(t,h_1)} \int_{-\tau}^{0} \int_{-\tau}^{0} |h_2| \sup_{r \in [\bar{r}+\bar{u}-s-|h_2|, \bar{r}+\bar{u}]} \{ |x_0(r)| \}$$
$$\sup_{r \in [m(s,h_2), M(s,h_2)]} \{ | \int_{-\tau}^{0} K(r+u, r+v) db(u) | \} \, d|a|(\bar{u}) \, d|b|(v) \, d\bar{r}$$

$$\leq \frac{|h_1| \cdot |h_2|}{\|h\|_2} \int_{-\tau}^{0} \sup_{r \in [\bar{r}+\bar{u}-s-|h_2|, \bar{r}+\bar{u}]} \{ |x_0(r)| \} \, da(\bar{u})$$
$$\sup_{r \in [m(s,h_2), M(s,h_2)]} \{ | \int_{-\tau}^{0} \int_{-\tau}^{0} K(r+u, r+v) \, d|b|(u) \, d|b|(v) | \}$$

As $0 \leq \lim_{h \to \infty} \frac{|h_1| \cdot |h_2|}{\|h\|_2} \leq \lim_{h \to \infty} \frac{1}{2} \|h\|_2 = 0$ and the suprema are bounded, the above term converges to zero.

Furthermore, it holds:

$$\frac{1}{\|h\|_2} \left| \mathbb{E}[X(t) \int_{m(s,h_2)-\tau}^{M(s,h_2)} \int_{-\tau}^{0} X(r+v) \, db(v) \, dW(r)] - h_2 \, x_0(t-s) \int_{-\tau}^{0} \int_{-\tau}^{0} K(s+u, s+v) \, db(u) \, db(v) \right|$$

$$= \frac{|h_2|}{\|h\|_2}\frac{1}{|h_2|}\Big|\int_{m(s,h_2)}^{t\wedge M(s,h_2)} x_0(t-r)\int_{-\tau}^{0}\int_{-\tau}^{0} K(r+u,r+v)\,db(u)\,db(v)\,dr$$
$$- \int_{m(s,h_2)}^{M(s,h_2)} x_0(t-s)\int_{-\tau}^{0}\int_{-\tau}^{0} K(s+u,s+v)\,db(u)\,db(v)\,dr\Big|$$

$$\underset{choice\ of\ t,s,h}{=} \frac{|h_2|}{\|h\|_2}\frac{1}{|h_2|}\Big|\int_{m(s,h_2)}^{M(s,h_2)} x_0(t-r)\int_{-\tau}^{0}\int_{-\tau}^{0} K(r+u,r+v)\,db(u)\,db(v)\,dr$$
$$- \int_{m(s,h_2)}^{M(s,h_2)} x_0(t-s)\int_{-\tau}^{0}\int_{-\tau}^{0} K(s+u,s+v)\,db(u)\,db(v)\,dr\Big|$$

$$\leq \frac{|h_2|}{\|h\|_2}\Big(\,|x_0(t-s)|\sup_{r\in[m(s,h_2,M(s,h_2)]}\Big\{\Big|\int_{-\tau}^{0}\int_{-\tau}^{0} K(r+u,r+v)-K(s+u,s+v)\,db(u)\,db(v)\Big|\Big\}$$
$$+\sup_{r\in[m(s,h_2),M(s,h_2)]}\{\,|\,x_0(t-r)-x_0(t-s)\,|\,\}\sup_{r\in[m(s,h_2),M(s,h_2)]}\Big\{\Big|\int_{-\tau}^{0}\int_{-\tau}^{0} K(r+u,r+v)\,db(u)\,db(v)\Big|\Big\}\,\Big).$$

The last inequality holds as x_0, K are continuous.
As $\frac{|h_2|}{\|h\|_2} \in [0,1]$ and as x_0, K are continuous, the above term converges to 0 for $h \to 0$.

(7) Collecting the $\mathcal{O}(\|h\|_2)$-terms from (4), (5), (6), we get:

$$\lim_{h\to 0}\frac{1}{\|h\|_2}\Big|\,K(t+h_1,s+h_2)-K(t,s)-h_1\int_{-\tau}^{0}K(t+u,s)\,da(u)$$
$$-h_2\Big(\int_{-\tau}^{0}K(t,s+v)\,da(v)+x_0(t-s)\int_{-\tau}^{0}\int_{-\tau}^{0}K(s+u,s+v)\,db(u)\,db(v)\Big)\Big| \;=\; 0.$$

This proves the differentiability of K on $\mathbb{K}^2_{+,>}(\tau)$.
As K is continuous on $\mathbb{K}^2_{+}(0)$, symmetric on $\mathbb{K}^2_{+,\Delta}(-\tau)$, it has continuous derivatives on $\mathbb{K}^2_{+,\Delta}(\tau)$.
□

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