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REGRESSION QUANTILES WITH ERRORS-IN-VARIABLES

D.A. IOANNIDES AND E. MATZNER-LOBER

ABSTRACT. In a lot of situations, variables are measured with errors. While this problem has been previously studied in the context of kernel regression, no work has been done in quantile regression. To estimate this function we use deconvoluting kernel estimators. The asymptotic behaviour of these estimators depends on the smoothness of the noise distribution.

1. INTRODUCTION

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$, be a random sample from the joint distribution of X and Y , where X is valued in R^d and is valued in R. For a fixed $0 < p < 1$, let $q(x)$ denote the p-th quantile of the Cumulative Conditional Distribution Function (CCDF) of Y given X. We postulate by this way the existence of the following model

$$Y_i = q(X_i) + \varepsilon_i \tag{1.1}$$

where $q(.)$ is the unknown function which we want to estimate and ε_i are the errors satisfying $F_{\varepsilon|X}(u|X = x) = F_{Y|X}(u + q(x)|X = x)$, where $F_{\varepsilon|X}(.|.)$ (resp. $F_{Y|X}(.|.)$) is the cumulative conditional distribution of ε given X (resp. of Y given X).
In nonparametric estimation of the function $q(.)$, most investigation were done on the estimation of the conditional mean of Y given a value x of the predictor X.
Estimation of conditional quantiles has gained attention in the last years because their useful application in various fields such as econometrics and finance. Since the pioneer work of Hogg (1975), much work have been done on quantiles regression and Koenker & Bassett (1978) provided motivations from econometrics. However, in a lot of situations and in particular in econometrics, variables of interest are either measured with an error or are not observed directly but through a proxy. So instead of measuring (X_i, Y_i) we have

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the following observations (X_i^e, Y_i) where

$$X_i^e = X_i + \eta_i. \tag{1.2}$$

η is a random noise variable with a known distribution. Literature on measurement errors models is extensive and Fuller (1987) is one of the most interesting book on this topic.
The goal of this article is to estimate nonparametric regression quantiles when the explanatory variables are measured with errors both theoritecally and also practically. To achieve our goal, we will obtain asymptotic results for an estimator of conditional distribution of Y knowing $X = x$ from the observations of (X_i^e, Y_i). These results are novel and are interesting by their own.

The plan of the article is a follows. In section 2, we introduce deconvoluting kernels estimators of conditional cumulative distribution functions (CCDF) of Y given $X = x$. Section 3 is devoted to the case where the error distribution is an ordinary smooth distribution (the characteristic function of the error decays algebrically to zero). All technical proofs are given in the Appendix. For expositional purposes, we consider only the special case where X is a scalar.
The case where the error variable has a super smooth distribution (the chartacteristic function of the error variable decays exponentially to zero) can be treated similarly, but the convergence rate is expected to be of logarithmic order.

2. Notations

2.1. **Nonparametric conditional quantile estimate.** In the last 30 years, several authors studied the asymptotic properties of nonparametric estimation of conditional quantile (see Cai (2002), for a recent survey). Thoses estimators were either computed directly by minimising the "check" function or indirectly through the CCDF. We focus our attention on indirect estimators of conditional quantile.
Roussas (1969) was the first one to consider the kernel estimators of conditional quantiles. He considered a real valued random variable $X_1, \cdots, X_{n+1}$ being the first random variable from a strictly stationary Markov process and proposed the following kernel estimator for the CCDF

$$G_n(y|x) = \frac{1}{h_n} \frac{\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right) \int_{-\infty}^{y} K\left(\frac{z - X_{i+1}}{h_n}\right) dz}{\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right)}.$$

He proved the uniform consistency and asymptotic normality of $G_n(y|x)$ and the consistency of the associated conditional quantiles

Let (X, Y) be a pair of random variables valued in $\mathbb{R}^d \times \mathbb{R}$ with unknown distribution. Having at hand a realization of the process (X_i, Y_i), Roussas estimator is

$$F_{1,n}(y|x) = \frac{1}{h_n} \frac{\sum_{i=1}^{n} K_x\left(\frac{x - X_i}{h_{xn}}\right) \int_{-\infty}^{y} K_y\left(\frac{z - Y_i}{h_{yn}}\right) dz}{\sum_{i=1}^{n} K_x\left(\frac{x - X_i}{h_{xn}}\right)}, \tag{2.1}$$

where K_x (respectively K_y) is a measurable function on $\mathbb{R}^d$ (respectively on $\mathbb{R}$) integrating to one and the bandwidths are sequences tending to zero with n. From now on, we write the bandwidths without the subscript n for simplification. One of the interest of estimator $F_{1,n}(y|x)$ is that it gives a continuous estimator of CCDF. Thus, if the CCDF is continuous, it is of interest to use an estimator sharing the same properties.

Collomb (1980) proposed an empirical estimator of CCDF

$$F_{2,n}(y|x) = \frac{\sum_{i=1}^{n} K_x\left(\frac{x - X_i}{h_x}\right) \mathbf{1}_{\{Y_i \leq y\}}}{\sum_{i=1}^{n} K_1\left(\frac{x - X_i}{h_x}\right)}. \tag{2.2}$$

Since those precursors, a lot of authors studied the nonparametric estimation of conditional quantiles using $F_{1,n}(y|x)$, $F_{2,n}(y|x)$ or modified version of them.

All those works were done on data having a standard structure but in a lot of situations and in various fields such as economics, biological studies, applied physics, variables are not directly observable and are measured with error. With such data, parametric estimation of the function g are numerous. See for example the recent monograph Cheng & Van Ness (1999). In such a context, most investigations in nonparametric estimation are devoted to the estimation of the density of X from the observations of X^e. Nonparametric estimation of the mean regression function g were studied by Fan et al. (1991), see others references in chapter 12 of Carroll et al. (1995). Recently, that topic has gained new interest see for example the papers of Carroll et al. (1999), Berry et al. (2002).

2.2. **Deconvoluting kernels.** The deconvoluting kernel estimator was first considered by Stefanski & Carroll (1990) and Fan (1991b). For self-completeness of the paper we recall briefly their construction. Let $X^e = X + \eta$ and let $X_1^e, \cdots, X_n^e$ denote a training sample from the distribution of X^e. Denote by $\Phi_{X^e}(t)$ (resp. $\Phi_X(t)$ and $\Phi_\eta(t)$) the characteristic

function of X^e (resp. X and η), We have $\Phi_{X^e}(t) = \Phi_X(t)\Phi_\eta(t)$. If we assume that the error η has a non-vanishing characteristic function, then by Fourier inversion the density of X is given by

$$f_X(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itu)\frac{\Phi_{X^e}(t)}{\Phi_\eta(t)}dt. \tag{2.3}$$

The problem is now to estimate $\Phi_{X^e}(t) = \int_{-\infty}^{\infty} \exp(-itu) f_{X^e}(u)du$. First, using a classical kernel estimator of $f_{X^e}(.)$ with a kernel K and the bandwidth h_x, we easily get an estimator for $\Phi_{X^e}(t)$. Replacing this estimator in (2.3), we get

$$f_{nX}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itu)\frac{\hat{\Phi}_{X^e}(t)}{\Phi_\eta(t)}dt = \frac{1}{nh_x}\sum_{j=1}^{n} W_\eta\left(\frac{u - X_j^e}{h_x}\right)$$

where the deconvoluting kernel is given by

$$W_\eta(u) = \frac{1}{(2\pi)}\int_{\mathbb{R}} e^{-iut}\frac{\Phi_K(t)}{\Phi_\eta(\frac{t}{h})}dt. \tag{2.4}$$

The deconvoluting kernel takes into account the error distribution by constructing the correct neighbourhood around u where correct is in the sense of taking into account the error.

Fan et al. (1991) proposed the folowing CCDF estimator

$$F_{2,n}(y|x) = \frac{\sum_{i=1}^{n} W_\eta\left(\frac{x - X_i^e}{h_x}\right)\mathbf{1}_{\{Y_i \le y\}}}{\sum_{i=1}^{n} W_\eta\left(\frac{x - X_i^e}{h_x}\right)} = \frac{\Psi_{2,n}(x,y)}{f_{nX}(x)} \tag{2.5}$$

but without theoretical or practical results. The modified version of Roussas estimator is given by

$$F_{1,n}(y|x) = \frac{1}{h_y}\frac{\sum_{i=1}^{n} W_\eta\left(\frac{x - X_i^e}{h_x}\right)\int_{-\infty}^{y} K_y\left(\frac{z - Y_i}{h_y}\right)dz}{\sum_{i=1}^{n} W_\eta\left(\frac{x - X_i^e}{h_n}\right)} = \frac{\Psi_{1,n}(x,y)}{f_{nX}(x)}. \tag{2.6}$$

A natural estimator of the p-th conditional quantile, $p \in (0,1)$, is defined for $i = 1, 2$ by

$$q_{i,n}(x) = \inf_{y \in \mathbb{R}}\{y : F_{i,n}(y|x) \ge p\}. \tag{2.7}$$

3. Ordinary smooth case

The error is said ordinary smooth if its characteristic function $\Phi_\eta(t)$ satisfies

$$d_0|t|^{-\beta} < |\Phi_\eta(t)| \leq d_1|t|^{-\beta}, \quad \text{as} \quad |t| \to \infty,$$

where d_0, d_1, and β are positive constants. Examples of such distributions are Gamma and symmetric Gamma distributions.

3.1. **Assumptions.**

The assumptions are summarised here for easy reference.

Assumption (A)

(i) The error η is independent of X, ε and Y.

(ii) The marginal density $f_X(.)$ is lower bounded by $\delta > 0$ on a compact subset C of $\mathbb{R}$ on which the and its derivatives $f_X^{(l)}(x)$ exist, are bounded for $0 \leq l \leq 2$.

(iii) $\dfrac{\partial^{i+j} f_{X,Y}(x,y)}{\partial x^i \partial y^j} = f_{X,Y}^{(i,j)}(x,y)$ exist, are bounded and integrable for $0 \leq i+j \leq 2$.

Assumption (E.OS)

(i) $\Phi_\eta(t) \neq 0$ for all $t \in \mathbb{R}$,

(ii) $|t|^\beta \Phi_\eta(t) \to d_0$, and $|t|^{\beta+1}\Phi'_\eta(t) \to d_1$ for $\beta \geq d_1$, $d_0 \neq 0$ and $d_1 \neq 0$ as $|t| \to \infty$.

Assumption (K.OS)

(i) The kernel $K_x(.)$, is an even bounded probability density function on $\mathbb{R}$ with a finite second moment denoted $\mu_2(K_x)$ and its characteristic function satisfies

$$\begin{aligned} \int_{\mathbb{R}} |t|^{\beta_\eta} |\Phi_{K_x}(t)| dt &< \infty \\ \int_{\mathbb{R}} |t|^{\beta_\eta+1} |\Phi'_{K_x}(t)| dt &< \infty \\ \int_{\mathbb{R}} |t|^{2\beta_\eta} |\Phi_{K_x}(t)|^2 dt &< \infty \end{aligned}$$

(ii) The kernel $K_y(.)$, is an even bounded probability density function on $\mathbb{R}$ with a finite second moment $(\mu_2(K_y))$.

The derivation of an approximation of bias and variance of the $F_{i,n}(x,y)$ is classical and will involve the terms of $\mathbb{E}f_n(x)$, $\mathbb{E}\Psi_{i,n}(x,y)$, $\mathrm{Var}(f_n(x))$ and $\mathrm{Cov}(f_n(x) - \mathbb{E}f_n(x), \Psi_{i,n}(x,y) -$

$\mathbb{E}\Psi_{i,n}(x,y)$). Those two last terms are not error free. All those term are given in the next lemma.

Lemma 3.1. *If assumptions* **(A)**, **(E.OS)** *and* **K.OS** *are satisfied, we have*

$$(i) \qquad \mathbb{E}(f_{nX}(x)) = f_X(x) + \frac{h_x^2}{2} f^{(2)}(x)\mu_2(K) + o(h_x^2)$$

$$(ii) \qquad Var(f_{nX}(x)) = \frac{1}{nh_x^{1+2\beta_\eta}} \frac{f_{X^e}(x)}{2\pi d_o^2} \int_{\mathbb{R}} |t|^{2\beta} |\Phi_{K_x}(t)|^2 dt(1+o(1))$$

$$(iii) \qquad \mathbb{E}(\Psi_{i,n}(x,y)) = \Psi(x,y) + \mu_2(K_x)\frac{h_x^2}{2}\int_{-\infty}^{y} f^{(2,0)}(x,z)dz + o(h_x^2)$$
$$+ \quad \delta_{i,1}\mu_2(K_y)\frac{h_y^2}{2}\int_{-\infty}^{y} f^{(0,2)}(x,z)dz + \delta_{i,1}o(h_y^2)$$

$$(iv) \qquad Var(\Psi_{i,n}(x,y)) = \frac{1}{nh_x^{1+2\beta}} \frac{\Psi(x,y)}{2\pi c^2} \int_{\mathbb{R}} |t|^{2\beta} |\Phi_K(t)| 2dt(1+o(1))$$

$$(v) \qquad Cov(f_{nX}(x), \Psi_{i,n}) = \frac{1}{nh_x^{1+2\beta}} . \Psi(x,y) \frac{1}{2\pi c^2} \int_{\mathbb{R}} |t|^{2\beta} |\Phi_K(t)| 2dt(1+o(1))$$

Using the Taylor expansion of the ratio of two random variables we get our first theorem

Theorem 3.2 (Asymptotic bias)**.** *If all assumptions are satisfied, we get*

$$\begin{aligned} B(F_{1,n}(y|x)) &= \frac{h_x^2}{2}\xi(x,y) + \frac{h_y^2}{2}\omega(x,y) + o(h_y^2) + o(h_x^2) + o(h_y h_x) + O\left(\frac{1}{nh_x^{1+2\beta_\eta}}\right), \\ B(F_{2,n}(y|x)) &= \frac{h_x^2}{2}\xi(x,y) + o(h_x^2) + O\left(\frac{1}{nh_x^{1+2\beta_\eta}}\right), \end{aligned}$$

where

$$\xi(x,y) = \left(F^{(2,0)}(y|x) + 2\frac{g^{(1)}(x)F^{(1,0)}(y|x)}{g(x)}\right)\mu_2(K_x) \quad \text{and} \quad \omega(x,y) = F^{(0,2)}(y|x)\mu_2(K_y).$$

Comments The estimator $F_{1,n}(y|x)$ is smoother than $F_{2,n}(y|x)$, and this difference appears in the bias where a term $\omega(x)$ is coming from the use of the kernel K_y. Comparing the bias, we conclude that the additional term $F^{(0,2)}(y|x)\mu_2(K_1)$ is positive on the convex part of the CCDF and negative on the concave part. However, nothing could be said on the bias because bias depends also on partial derivative of the marginal density and the CCDF. This additional term is zero if evaluated at the conditional median with a symmetric conditional density.

Our second result concerns the variance of our estimator $F_{i,n}(y|x)$, for $i = 1, 2$.

Theorem 3.3 (Asymptotic variance)**.** *If all assumptions are satisfied, we get*

$$Var(F_{i,n}(y|x)) = \frac{1}{nh_x^{1+2\beta_\eta}}\frac{F(y|x)(1-F(y|x))}{f_X(x)}\mu_0(K_x^2) + o\left(\frac{1}{nh_x^{1+2\beta_\eta}}\right).$$

Having the asymptotic bias and variance, we get the asymptotic MSE

Theorem 3.4 (Asymptotic MSE)**.** *If all assumptions are satisfied, we get*

$$\begin{aligned} MSE(F_{1n}(y|x)) &= \left(h_x^4\xi^2(x,y) + h_x^2h_y^2\xi(x,y)\omega(x,y) + h_y^4\omega^2(x,y)\right) \\ &\quad + \frac{1}{nh_x^{1+2\beta_\eta}}\frac{F(y|x)(1-F(y|x))}{f_X(x)}\mu_0(K_x^2) + o\left(h_x^4 + h_y^4 + h_x^2h_y^2 + \frac{1}{nh_x^{1+2\beta}}\right) \\ MSE(F_{2n}(y|x)) &= h_x^4\xi^2(x,y) + \frac{1}{nh_x^{1+2\beta}}\frac{F(y|x)(1-F(y|x)}{f_X(x)}\mu_0(K_x^2) + o\left(h_x^4 + \frac{1}{nh_x^{1+2\beta}}\right), \end{aligned}$$

Now, the MSE for the associated quantiles is obtained easily and is given by the following theorem

Theorem 3.5. *If all the assumptions are satisfied and if $f(q(x)|x) \neq 0$, we have*

$$\begin{aligned} MSE(q_{2,n}(x)) &= \frac{h_x^4}{4f^2(q(x)|x)}\xi^2(x,q(x)) + \frac{1}{nh_x^{1+2\beta}}\frac{p\,(1-p)}{f^2(q(x)|x)g(x)}\mu_0(K_x^2) + o(h_x^4) + o\left(\frac{1}{nh_x^{1+2\beta}}\right), \\ MSE(q_{1,n}(x)) &= MSE(q_{2n}(x) + \frac{h_x^2h_y^2\xi(x,q(x))\omega(x,q(x)) + h_y^4\omega^2(x,q(x))}{4f^2(q(x)|x)} + o(h_y^4) + o(h_x^2h_y^2). \end{aligned}$$

Comments : Finding the optimal bandwidth for $q_{2n}(x)$ is easy and in our context involves the error distribution through the value of β. It is much more complicated for $q_{1n}(x)$ because 2 bandwidths are involved. Hyndman et al. (1996) dealt with a similar problem when estimating the MSE of the conditional distribution in the error free case.
The next theorem deals with the uniform strong consisrency for the estimators of CCDF. Its proof is standard in the literature, see for example Roussas (1991) and the references cited therein.

Theorem 3.6. *If all the assumptions are satisfied, then*

$$\begin{aligned} \sup_{y\in\mathbb{R}}|F_{1,n}(y|x) - F(y|x)| &= O\left(\frac{\log n}{nh_x^{1+2\beta}}\right) + O(h_x^2) + O(h_y^2) + O(h_xh_y) \quad a.s. \\ \sup_{y\in\mathbb{R}}|F_{2,n}(y|x) - F(y|x)| &= O\left(\frac{\log n}{nh_x^{1+2\beta}}\right) + O(h_x^2) \quad a.s. \end{aligned}$$

The strong consistency for the conditional quantiles is obtained by the previous theorem taking into account the Taylor expansion of $F_{i,n}(q_{in}|x)$ at point $q(x)$.

Theorem 3.7. *If all the assumptions are satisfied and if the conditional quantile $q_p(x)$ of order $p \in (0,1)$ is unique and $f(q_p(x)|x) \neq 0$ then*

$$|q_{i,n}(x) - q(x)| = O\left(\frac{\log n}{n h_x^{1+2\beta}}\right) + O(h_x^2) + O(h_y^2) + O(h_x h_y) \quad a.s.$$

Theorem 3.8. *If all the assumptions are satisfied, and if the limits of $nh_x^{5+2\beta}$ and $nh_y^4 h_x^{1+\beta}$ are two positive constants then*

$$\sqrt{nh_x^{1+\beta}}\left(F_{i,n}(y|x) - F(y|x) - B(F_{i,n}(y|x)\right) \to \mathcal{N}\left(0, \sigma^2(y|x)\right)$$

where

$$\sigma^2(y|x) = \frac{F(y|x)(1-F(y|x))}{f_X(x)}\mu_0(K_x^2).$$

Proof:

A standard decomposition for $F_{1,n}(y/x)$ gives

$$F_{1,n}(y|x) - F(y|x) = \frac{1}{f_{nX}(x)}\left[\Psi_{1,n}(x,y) - \Psi(x,y) + F(y|x)(f_{nX}(x) - f_X(x))\right],$$

where $\Psi_{1,n}(x,y) = W_\eta\left(\frac{x-X_1^e}{h_x}\right)\int_{-\infty}^{y} K_1\left(\frac{u-z_1}{h_y}\right)dz$, and $f_{nX}(x)$ is the smooth estimator for the density function $f(x)$. Now, if we are working similar as in Lemma 3.1 all the conditions of the Liapunov's central limit theorem are satisfied for the terms involved in the quantity $\Psi_{1,n}(x,y)$ and our result is obvious.

4. APPENDIX

Proof of lemma 3.1

i) The expectation of the deconvoluting kernel density estimator is the same as the expectation of the usual kernel density estimator. So under the assumption that $K_x(u)$ first moment is zero and second moment is finite and under the assumption that the derivative of $f_X(u)$ denoted by $f_X^{(l)}(u)$ exists and are bounded for $l \geq 2$, we have (see for example Fan (1991a) or Masry (1991))

$$\mathbb{E}f_{nX}(x) \quad = \quad f_X(x) + \frac{h_x^2}{2}f^{(2)}(x)\mu_2(K_x) + o(h_x^2).$$

ii) Under the assumption given in assumption K and E, we have (see lemma 3.1 in Fan (1991a))

$$h_x^{\beta_\eta} W_\eta(u) \in L_\infty(\mathbb{R}) \qquad h_x^{\beta_\eta} W_\eta(u) \in L_2(\mathbb{R}).$$

We can then obtain calculate the variance explicitly as done in Masry (1991) for example, so we have

$$\lim nh_x^{1+2\beta_\eta}\mathrm{Var}(f_{nX}(x)) \quad = \quad \frac{f_{X^e}(x)}{2\pi d_o^2}\int_{\mathbb{R}}|t|^{2\beta}|\Phi_{K_x}(t)|^2dt.$$

iii) Using Fubini theorem and assumptions (A), we obtain

$$\begin{aligned}
\mathbb{E}\Psi_{1,n}(x,y) &= \mathbb{E}\left(\frac{1}{nh_x^dh_y}W_1\left(\frac{x-X_1^e}{h_x}\right)\int_{-\infty}^{y}K_2\left(\frac{z-Y_1}{h_y}\right)dz\right)\\
&= \frac{1}{(2\pi)^d}\int_{-\infty}^{y}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}}e^{-it.x}e^{-it'z}\frac{\Phi_K(th_x)}{\Phi_\eta(t)}K_2(t'h_y)\phi_n(t,t')dtdt'\right]dz\\
&= \int_{-\infty}^{y}\frac{1}{h_x^dh_y}\mathbb{E}K\left(\frac{x-X_1}{h_x}\right)K_2\left(\frac{z-Y_1}{h_y}\right)dz,
\end{aligned}$$

where $\phi_n(t,t')$ is the empirical characteristic function of (X^e,Y). The expectation of $\Psi_{2,n}(x,y)$ is obtain in the same way and we get (iv).

The derivations of the variance of $\Psi_{i,n}(x,y)$ are similar, we derive it for $\Psi_{2,n}(x,y)$.

$$\begin{aligned}
\mathrm{Var}(\Psi_{2,n}(x,y)) &= \frac{1}{nh_x^2}\mathrm{Var}\left(W_\eta\left(\frac{x-X_1^e}{h_x}\right)1_{\{Y_1\le y\}}\right)\\
&= \frac{1}{nh_x^2}\left(\mathbb{E}\left[W_\eta\left(\frac{x-X_1^e}{h_x}\right)1_{\{Y_1\le y\}}\right]^2-\mathbb{E}^2\left[W_\eta\left(\frac{x-X_1^e}{h_x}\right)1_{\{Y_1\le y\}}\right]\right).
\end{aligned}$$

The leading term is the first one and we have

$$\begin{aligned}
\lim_{n\to\infty}nh_x^{1+2\beta_\eta}\mathrm{Var}(\Psi_{1,n}(x,y) &= \lim_{n\to\infty}nh_x^{2\beta_\eta}\int|W_\eta(u)|^21_{\{v\le y\}}f_{X^e,Y}(x-h_xu,v)dudv\\
&= \int\int_{-\infty}^{y}|h^{\beta_\eta}W_\eta(u)|^2f_{X^e,Y}(x,v)dudv\\
&= \frac{1}{2\pi d_o^2}\int_{-\infty}^{y}f_{X^e,Y}(x,v)dv\int|t|^2|\Phi_{K_x}(t)|^2dt.
\end{aligned}$$

The proof of the covariance term is of the same spirit.

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