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# EULER-MARUYAMA AND MILSTEIN APPROXIMATIONS FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED MEMORY TERM

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## Abstract

We consider the problem of strong approximations of the solution of stochastic functional differential equations of Itô form with a distributed delay term in the drift and diffusion coefficient. We provide necessary background material, and give convergence proofs for the Euler-Maruyama and the Milstein scheme. Numerical examples illustrate the theoretical results.

## 1 Introduction

Consider a scalar stochastic differential delay equation (SDDE) of the form

$$(1) \quad dX(t) = F(t, X(t), Y(t)) dt + G(t, X(t), Y(t)) dW(t), \quad t \in [0, T]$$

where  $Y(t)$  represents a memory term of the type

$$(2) \quad Y(t) = \int_{-\tau}^0 K(t, s, X(t+s)) ds = \int_{t-\tau}^t K(t, s-t, X(s)) ds.$$

The drift and diffusion coefficients  $F$  and  $G$  are given functions. The fixed value  $\tau > 0$  is called the lag or delay, and  $W(t) = W(t, \omega)$ ,  $t \in [0, T]$ ,  $\omega \in \Omega$ , is a scalar standard Brownian motion on the filtered probability space  $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in [0, T]}, \mathbb{P})$ . (As usual, the dependence of  $W$  on  $\omega$  will be suppressed.) It is necessary to specify the whole *initial path*  $\Psi$  on  $J := [-\tau, 0]$ , i.e., we require

$$(3) \quad X(s) = \Psi(s) \quad \text{for } s \in J.$$

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Equation (1) is an abbreviation of the integral equation

$$(4) \quad X(t) = X(0) + \int_0^t F(X(s), Y(s)) ds + \int_0^t G(X(s), Y(s)) dW(s),$$

for  $t \in [0, T]$  and with  $X(t) = \Psi(t)$ , for  $t \in J$ . The second integral in (4) is a stochastic integral which is to be interpreted in the Itô sense.

**Remark 1.1:** For ease of exposition we consider only one memory term. There is no essential difficulty involved in extending the calculations to equations containing several memory terms.

Deterministic models with distributed memory terms have been well studied. More realistic mathematical models can be investigated by allowing for random perturbations incorporated into the equations. The dynamical behaviour of such equations with examples in population dynamics has been discussed in (e.g.) Kuang [14]. More references to applications in economy or other areas can be found in Kolmanovskii and Myshkis [13]. In Guttorp and Kulperger [9] the following model is considered as a model for the average behaviour of a large single-species population, incorporating randomness, that is allowing for individual differences and interaction between the individuals:

$$(5) \quad dX(t) = X(t) \left( \mu - \lambda \int_0^\tau g(s) 1_{[0, \infty)}(t-s) X(t-s) ds \right) dt + \sigma X(t) dW(t),$$

where  $g(s)$  is a positive integrable function,  $\tau$  is the resource renewal time and  $1_A(s)$  the indicator function of the set  $A$ . A model like (1) but with an additional discrete delay term is considered in [15, 7, 22]

$$(6) \quad \begin{aligned} dX(t) &= F(t, X(t), Y(t), Z(t)) dt + G(t, X(t), Y(t), Z(t)) dW(t), \\ Y(t) &= \int_{-\tau}^0 e^{\lambda s} X(t+s) ds, \quad Z(t) = X(t-\tau). \end{aligned}$$

More models with a financial mathematics background based on similar equations appear in Appleby [1] and Hobson and Rogers [10].

In this article the theorem on strong convergence of explicit one-step schemes, stated and proved in [4] for SDDEs with discrete lags, is extended to cover the case of a distributed lag. The standard technique to develop approximation schemes for stochastic differential equations and establish their order of convergence is to set up an Itô-Taylor expansion by application of the Itô-formula to the drift and diffusion functions, see [12, 18]. In the case of a discrete lag the authors in [11] have developed an Itô-formula for functionals of the solution of the form  $\Phi(X(t), X(t-\tau))$ . As it turned out, anticipating stochastic integrals appear in the process of establishing the formula and Malliavin calculus becomes the appropriate tool. In [2], stochastic analysis for stochastic functional differential equations is developed, in particular an Itô-formula for so-called quasi-tame

functions is proved. A quasi-tame function  $\Phi : L^2([-\tau, 0]) \times \mathbb{R} \rightarrow \mathbb{R}$  is such that  $\Phi(v, \theta) = h(v, \int_{-\tau}^0 f(\theta(s))g(s)ds)$ , where  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^\infty$ -bounded, and  $g : [-\tau, 0] \rightarrow \mathbb{R}$  is  $C^1$ -bounded, with  $g(-\tau) = 0$ . (Compare also Chapter IV, §4. in [21].) In [15, 7, 22] an Itô-formula for functions of the form  $\Phi(X(t), Y(t))$  and  $Y(t) = \int_{-\tau}^0 e^{\lambda s} X(t+s) ds$  is established. As  $e^{-\lambda\tau} \neq 0$  this  $\Phi$  is not a quasi-tame function. In the next section we give an Itô-formula for functions of the form  $F$  and  $G$  as in (1). As it turns out, and already did in [15, 7, 22], a term incorporating  $X(t-\tau)$  appears. An Itô-formula for functions of the form  $\Phi(X(t), Y(t), Z(t))$ , with  $Y, Z$  as in (6), would be necessary to obtain higher order approximation methods, or treat the equations considered in [15, 7, 22]. This is a subject of further work. In Section 4 we apply the Itô-formula to establish consistency estimates for the Euler-Maruyama and the Milstein method. Numerical experiments, presented in Section 5, confirm the theoretical results.

## 2 Analytical background

In this section we state the necessary assumptions, provide estimates on the moments of the solution of (1), as well as an Itô-formula for functions of the form  $F$  and  $G$  in (1).

### 2.1 Assumptions

By  $C([-\tau, 0], \mathbb{R})$  we mean the Banach space of all continuous paths from  $[-\tau, 0] \rightarrow \mathbb{R}$  equipped with the supremum norm  $\|\eta\| := \sup_{s \in [-\tau, 0]} |\eta(s)|$ ,  $\eta \in C$ .  $W(t) = W(t, \omega)$ ;  $t \geq 0$ ,  $\omega \in \Omega$  is a one-dimensional standard Brownian motion on the filtered probability space  $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in [0, T]}, \mathbb{P})$ .

(A1) The functions  $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $G : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $K : [0, T] \times [-\tau, 0] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous.

(A2) (Uniform Lipschitz conditions) There exist positive constants  $L_{F,i}$  and  $L_{G,i}$ ,  $i = 1, 2$  and  $L_K$  such that for all  $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathbb{R}$  and  $t \in [0, T]$ :

$$(7) \quad |F(t, \phi_1, \phi_2) - F(t, \psi_1, \psi_2)| \leq L_{F,1}|\phi_1 - \psi_1| + L_{F,2}|\phi_2 - \psi_2|,$$

$$(8) \quad |G(t, \phi_1, \phi_2) - G(t, \psi_1, \psi_2)| \leq L_{G,1}|\phi_1 - \psi_1| + L_{G,2}|\phi_2 - \psi_2|,$$

and for  $t \in [0, T]$  and  $s \in [-\tau, 0]$ :

$$(9) \quad |K(t, s, \psi_1) - K(t, s, \psi_2)| \leq L_K|\psi_1 - \psi_2|.$$

(A3) (Linear growth conditions) There exist positive constants  $Q_F, Q_G$  and  $Q_K$  such that for all  $\phi, \phi_1, \psi, \psi_1 \in \mathbb{R}$  and  $t \in [0, T]$ :

$$(10) \quad |F(t, \phi, \phi_1)|^2 \leq Q_F(1 + |\phi|^2 + |\phi_1|^2),$$

$$(11) \quad |G(t, \psi, \psi_1)|^2 \leq Q_G(1 + |\psi|^2 + |\psi_1|^2),$$

and for  $t \in [0, T]$  and  $s \in [-\tau, 0]$ :

$$(12) \quad |K(t, s, \phi)|^2 \leq Q_K(1 + |\phi|^2).$$

(A4) The initial path  $\Psi(t)$  is an  $\mathcal{A}_0$ -measurable  $C([-\tau, 0], \mathbb{R})$  valued random variable such that  $\mathbb{E}\|\Psi\|^2 < \infty$ .  $\Psi$  is Hölder-continuous with exponent  $\gamma \in (0, 1]$ , i.e., there exists a positive constant  $L_\Psi$  such that for  $t, s \in [-\tau, 0]$

$$(13) \quad \mathbb{E}(|\Psi(t) - \Psi(s)|^p) \leq L_\Psi |t - s|^{p\gamma}, \quad p = 1, 2.$$

(A5) We assume that  $F$ ,  $G$  and  $K$  have sufficient differentiability with respect to their arguments. The partial derivatives are uniformly bounded in modulus on the domain of definition of  $F$ ,  $G$  and  $K$ , respectively. We denote these bounds by the same constant  $M_3 < \infty$ . Precise conditions on differentiability will be given later.

We assume that there exists a path-wise unique strong solution to equation (1) (defined as (4)). Proofs of this can be found in [17], [19] and [20, 21].

## 2.2 Moment estimates

**Theorem 2.1** *Suppose that Assumption (A3) holds. Then the solution of equation (1) has the property*

$$(14a) \quad \mathbb{E} \left( \sup_{-\tau \leq t \leq T} |X(t)|^2 \right) \leq M_1(T),$$

with

$$(14b) \quad M_1(T) := (1 + 4 \mathbb{E}\|\Psi\|^2) \exp(3(T Q_F + 4 Q_G)(1 + \tau^2 Q_K) T).$$

Moreover, for any  $0 \leq s < t \leq T$  with  $t - s < 1$ ,

$$(15a) \quad \mathbb{E}|X(t) - X(s)|^2 \leq M_2(T)(t - s),$$

where

$$(15b) \quad M_2(T) = 2(Q_F + Q_G)(1 + \tau^2 Q_K)(1 + M_1(T)).$$

**Proof:** The proof follows the standard lines of, for example, [17, Section 5.4]. The inequality  $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ , Cauchy-Schwarz's and the Burkholder-Davis-Gundy inequality (e.g. [17, Theorem 1.7.3]), the (continuous) Gronwall inequality (e.g. [17, Theorem 1.8.1]) and the linear growth conditions are the main ingredients. We omit the details.

### 2.3 Itô-Formula and Itô-Taylor-Expansion

The derivation of the Itô-formula presented in this section follows the work [2] and also generalizes the formula given in [15, 7, 22], regarding the distributed lag part.

We will apply a multi-dimensional Itô-formula which we cite from [8, Theorem 4.5.3].

**Theorem 2.2** *Let  $d\xi_i(t) = a_i(t)dt + b_i(t)dW(t)$  ( $1 \leq i \leq m$ ) and let  $f(z_1, \dots, z_m, t)$  be a continuous function in  $(z, t)$  where  $z = (z_1, \dots, z_m) \in \mathbb{R}^m$ ,  $t \geq 0$ , together with its first  $t$  derivative and second  $x$ -derivatives. Then  $f(\xi_1(t), \dots, \xi_m(t), t)$  has a stochastic differential, given by*

$$\begin{aligned}
 df(Z(t), t) &= \left[ f_t(Z(t), t) + \sum_{i=1}^m f_{x_i}(Z(t), t)a_i(t) \right. \\
 &\quad \left. + \frac{1}{2} \sum_{i,j=1}^m f_{x_i x_j}(Z(t), t)b_i(t)b_j(t) \right] dt \\
 (16) \quad &+ \sum_{i=1}^m f_{x_i}(Z(t), t)b_i(t)dW(t),
 \end{aligned}$$

where  $Z(t) = (\xi_1(t), \dots, \xi_m(t))$ .

In order to apply Theorem 2.2, we set  $m = 2$ ,  $Z(t) = (\xi_1(t), \xi_2(t))$ ,  $\xi_1(t) = X(t)$  and  $\xi_2(t) = Y(t)$ . In what follows we denote by  $D_i h(t_1, \dots, t_m)$  and  $D_i^2 h(t_1, \dots, t_m)$  the first and second order derivative of a function  $h$  with respect to the  $i$ -th argument. We then rewrite Equation (1):

$$\begin{aligned}
 d\xi_1(t) &= a_1(t)dt + b_1(t)dW(t), \\
 d\xi_2(t) &= a_2(t)dt + b_2(t)dW(t), \\
 \text{with} \quad a_1(t) &= F(t, X(t), Y(t)), \\
 b_1(t) &= G(t, X(t), Y(t)), \\
 a_2(t) &= \frac{d}{dt} \int_{t-\tau}^t K(t, s-t, X(s))ds \\
 &= \int_{t-\tau}^t D_1 K(t, s-t, X(s)) - D_2 K(t, s-t, X(s))ds \\
 &\quad + K(t, 0, X(t)) - K(t, -\tau, X(t-\tau)), \\
 b_2(t) &= 0.
 \end{aligned}$$

In integral form and with our original notation we obtain for a function  $h$  subject to the conditions of Theorem 2.2, and  $t \in [0, T]$  :

$$(17) \quad h(t, X(t), Y(t)) - h(t, X(0), Y(0))$$

$$\begin{aligned}
&= \int_0^t \left[ D_1 h(s, X(s), Y(s)) + D_2 h(s, X(s), Y(s)) F(s, X(s), Y(s)) \right. \\
&+ D_3 h(s, X(s), Y(s)) \left\{ \int_{s-\tau}^s D_1 K(s, u-s, X(u)) - D_2 K(s, u-s, X(u)) du \right. \\
&\quad \left. + K(s, 0, X(s)) - K(s, -\tau, X(s-\tau)) \right\} \\
&\quad \left. + \frac{1}{2} D_2^2 h(s, X(s), Y(s)) G^2(s, X(s), Y(s)) \right] ds \\
&+ \int_0^t D_2 h(s, X(s), Y(s)) G(s, X(s), Y(s)) dW(s).
\end{aligned}$$

For some function  $f$  we abbreviate  $f_t := f(t, X(t), Y(t))$  and introduce operators  $\Lambda_1, \Lambda_2, \Lambda_3$  in the following way:

$$\begin{aligned}
(18) \quad \Lambda_1 h_t &= D_1 h_t + F_t \cdot D_2 h_t + \frac{1}{2} G_t^2 \cdot D_2^2 h_t, \quad \Lambda_2 h_t = G_t \cdot D_2 h_t, \\
\Lambda_3 h_t &= \left( \int_{t-\tau}^t D_1 K(t, u-t, X(u)) - D_2 K(t, u-t, X(u)) du \right. \\
&\quad \left. + K(t, 0, X(t)) - K(t, -\tau, X(t-\tau)) \right) \cdot D_3 h_t.
\end{aligned}$$

We expand the solution of Equation (1) in the usual way for Itô-Taylor-expansions, see [12, 18]. We start with

$$X(t+h) = X(t) + \int_t^{t+h} F(s, X(s), Y(s)) ds + \int_t^{t+h} G(s, X(s), Y(s)) dW(s),$$

and apply the Itô-formula to the functions  $F$  and  $G$ . We obtain

$$(19) \quad X(t+h) = X(t) + F_t \int_t^{t+h} ds + G_t \int_t^{t+h} dW(s) + R_1 + R_2 + R_M,$$

$$(20) \quad R_1 = \int_t^{t+h} \int_t^s \Lambda_1 F_u + \Lambda_3 F_u du ds$$

$$(21) \quad R_2 = \int_t^{t+h} \int_t^s \Lambda_2 F_u dW(u) ds + \int_t^{t+h} \int_t^s \Lambda_1 G_u + \Lambda_3 G_u du dW(s),$$

$$(22) \quad R_M = \int_t^{t+h} \int_t^s \Lambda_2 G_u dW(u) dW(s).$$

From (19) we will derive the Euler-Maruyama method, the term  $R_1 + R_2 + R_M$  constitutes the remainder to be estimated. If we expand the integrand in  $R_M$

we can obtain the Milstein method from:

$$(23) \quad \begin{aligned} X(t+h) &= X(t) + F_t \int_t^{t+h} ds + G_t \int_t^{t+h} dW(s) \\ &+ \Lambda_2 G_t \int_t^{t+h} \int_t^s dW(u) dW(s) + R_1 + R_2 + R_3, \end{aligned}$$

$$(24) \quad \begin{aligned} \text{where } R_3 &= \int_t^{t+h} \int_t^s \int_t^u \Lambda_1 \Lambda_2 G_z + \Lambda_3 \Lambda_2 G_z dz dW(u) dW(s) \\ &+ \int_t^{t+h} \int_t^s \int_t^u \Lambda_2 \Lambda_2 G_u dW(z) dW(u) dW(s), \end{aligned}$$

and  $R_1, R_2$  are given by (20) and (21), respectively.

### 3 Convergence

We define a family of meshes with a uniform step on the interval  $[0, T]$  of the form

$$(25) \quad \mathcal{T}_h^N := \{0 = t_0 < t_1 < t_2, \dots < t_N\} \subseteq [0, T],$$

with  $t_n = nh$ ,  $n = 0, \dots, N$ ,  $hN \leq T$ ,  $N \in \mathbb{N}$ . In addition, the choice of  $h$  is not arbitrary, and we require, with the lag  $\tau$  given, that

$$(26) \quad h = \tau/N_\tau, \quad N_\tau \in \mathbb{N}.$$

On  $\mathcal{T}_h^N$  we consider strong approximations  $\tilde{X}_n$  of the solution to (1), using stochastic explicit one-step methods of the form

$$(27a) \quad \tilde{X}_n = \tilde{X}_{n-1} + \phi(h, t_{n-1}, \tilde{X}_{n-1}, \tilde{Y}_{n-1}, I_\phi(t_{n-1})), \quad n \geq 1,$$

$$(27b) \quad \tilde{Y}_m = \sum_{\ell=m-N_\tau}^{m-1} \Upsilon(h, t_m, t_\ell - t_m, \tilde{X}_\ell, I_\Upsilon(t_\ell)), \quad m \geq 0,$$

where the initial values are given by  $\tilde{X}_i := \Psi(t_i)$  for  $i \leq 0$ . The expression  $\tilde{Y}_m$  provides an approximation to the integral  $Y(t_m)$ .

The increment functions  $\phi$  and  $\Upsilon$  incorporate a finite number of multiple Itô-integrals (see [12] or [18]) of the form

$$(28) \quad I_{(j_1, \dots, j_l), h, t} = \int_t^{t+h} \int_t^{s_l} \dots \int_t^{s_2} dW^{j_1}(s_1) \dots dW^{j_{l-1}}(s_{l-1}) dW^{j_l}(s_l),$$

where  $j_i \in \{0, 1\}$  and  $dW^0(t) = dt$  and with  $t = t_{n-1}$  in the case (27). We denote by  $I_\phi(t)$  and  $I_\Upsilon(t)$  the collection of Itô-integrals required to compute the increment functions  $\phi$  and  $\Upsilon$ . The increment function  $\phi$  is assumed to generate approximations  $\tilde{X}_n$  which are  $\mathcal{A}_{t_n}$ -measurable.



**Assumptions on the increment functions  $\phi$  and  $\Upsilon$ :**

We suppose there exist positive constants  $L_{\phi,1}, L_{\phi,2}, L_{\Upsilon,2}$ , such that for all random variables  $\xi(t), \xi'(t), \eta(t), \eta'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  the following Lipschitz-like conditions hold for all  $t \in [0, T]$ :

$$(29) \quad \begin{aligned} & \mathbb{E} \left( |\phi(h, t, \xi(t), \eta(t), I_{\phi}(t)) - \phi(h, t, \xi'(t), \eta'(t), I_{\phi}(t))| \middle| \mathcal{A}_t \right) \\ & \leq L_{\phi,1} h \mathbb{E}(|\xi(t) - \xi'(t)| + |\eta(t) - \eta'(t)| \middle| \mathcal{A}_t), \end{aligned}$$

$$(30) \quad \begin{aligned} & \mathbb{E} \left( |\phi(h, t, \xi(t), \eta(t), I_{\phi}(t)) - \phi(h, t, \xi'(t), \eta'(t), I_{\phi}(t))|^2 \middle| \mathcal{A}_t \right) \\ & \leq L_{\phi,2} h \mathbb{E}(|\xi(t) - \xi'(t)|^2 + |\eta(t) - \eta'(t)|^2 \middle| \mathcal{A}_t) \end{aligned}$$

and for all  $t \in [0, T], s \in [-\tau, 0]$ :

$$(31) \quad \begin{aligned} & \mathbb{E} \left( |\Upsilon(h, t, s, \xi(t), I_{\Upsilon}(t)) - \Upsilon(h, t, s, \xi'(t), I_{\Upsilon}(t))|^2 \middle| \mathcal{A}_t \right) \\ & \leq L_{\Upsilon,2} h^2 \mathbb{E}(|\xi(t) - \xi'(t)|^2 \middle| \mathcal{A}_t). \end{aligned}$$

We will consider convergence of our approximations in the following sense.

**Definition 3.1** *The approximations  $\{\tilde{X}_n\}$  for the solution  $X$  of equation (1), defined on  $\mathcal{T}_h^N$  with the step-size  $h$  constraint by (26), are said to be convergent in the mean-square-sense, with order  $p$ , on the mesh-points, when*

$$(32) \quad \max_{t_n \in \mathcal{T}_h^N} \left( \mathbb{E} |X(t_n) - \tilde{X}_n|^2 \right)^{\frac{1}{2}} \leq C h^p \quad \text{as } h \rightarrow 0,$$

where  $C < \infty$  is independent of  $h$ , but may depend on  $T$  and on the initial data.

We shall establish a relationship between convergence, that is, the behaviour of the global error of the approximation to consistency, that is the local truncation error, measured in an appropriate way.

**Definition 3.2** *The approximations  $\{\tilde{X}_n\}$  for the solution  $X$  of (1) are said to be consistent with order  $p_1$  in the mean and with order  $p_2$  in the mean-square sense if, with*

$$(33) \quad p_2 \geq \frac{1}{2} \quad \text{and} \quad p_1 \geq p_2 + \frac{1}{2},$$

and with  $\delta_n, n = 1, \dots, N$  the sequence of random variables

$$(34) \quad \delta_n := X(t_n) - X(t_{n-1}) - \phi(h, t_{n-1}, X(t_{n-1}), \hat{Y}(t_{n-1}), I_{\phi}(t_{n-1})),$$

$$(35) \quad \hat{Y}(t_m) := \sum_{\ell=m-N_{\tau}}^{m-1} \Upsilon(h, t_m, t_{\ell} - t_m, X(t_{\ell}), I_{\Upsilon}(t_{\ell})), \quad m \geq 0,$$

the following estimates hold:

$$(36) \quad \max_{1 \leq n \leq N} |\mathbb{E}(\delta_n)| \leq C_1 h^{p_1}, \quad \text{as } h \rightarrow 0$$

and

$$(37) \quad \max_{1 \leq n \leq N} (\mathbb{E}|\delta_n|^2)^{\frac{1}{2}} \leq C_2 h^{p_2}, \quad \text{as } h \rightarrow 0,$$

where the constants  $C_1$  and  $C_2$  do not depend on  $h$ , but may depend on  $T$  and the initial data.

Before proceeding with our main theorem, we state a discrete Gronwall inequality, familiar in the (numerical) analysis of Volterra integral equations (see, e.g. [3, 5, 6, 16]).

**Lemma 3.1** *Let the positive sequence  $Z_0, Z_1, \dots$  satisfy*

$$Z_n \leq h C \sum_{i=0}^{n-1} Z_i + D_n, \quad n \geq r, \quad r \geq 0,$$

where  $h C > 0$ ,  $\sum_{i=0}^{r-1} Z_i \leq \eta$ . If, for  $r \leq n \leq N$ ,  $|D_n| \leq D$ , then

$$Z_n \leq (D + hC\eta) \exp(NhC), \quad n = r, r+1, \dots, N.$$

We now state the main result of this article.

**Theorem 3.2** *We require that assumptions (A1) to (A4) hold. We suppose the method defined by (27) satisfies the estimates (36) and (37), the increment function  $\phi$  in equation (27) satisfies the estimates (29) and (30) and  $\Upsilon$  satisfies (31). Then the approximation (27) for the solution of equation (1) is convergent in the sense of Definition 3.1 with order  $p = p_2 - \frac{1}{2}$ .*

**Proof:** Define  $\mathbf{e}_n := X(t_n) - \tilde{X}_n$ ,  $n \geq 1$ . Note that  $\mathbf{e}_n$  is  $\mathcal{A}_{t_n}$ -measurable since both  $X(t_n)$  and  $\tilde{X}_n$  are  $\mathcal{A}_{t_n}$ -measurable random variables.

Using (34), adding and subtracting  $X(t_{n-1})$  and, with  $\hat{Y}(t_m)$  given as (35),  $\phi(h, t_{n-1}, X(t_{n-1}), \hat{Y}(t_{n-1}), I_\phi(t_{n-1}))$  and rearranging we obtain

$$\begin{aligned} \mathbf{e}_n &= X(t_n) - \tilde{X}_n \\ &= \mathbf{e}_{n-1} + \delta_n + u_{n-1}, \end{aligned}$$

where  $u_m$  is defined as

$$(38) \quad u_m := \phi(h, t_m, X(t_m), \hat{Y}(t_m), I_\phi) - \phi(h, t_m, \tilde{X}_m, \tilde{Y}_m, I_\phi).$$

Thus squaring, employing the conditional mean with respect to the  $\sigma$ -algebra  $\mathcal{A}_{t_0}$ , and taking the modulus, we obtain

$$\begin{aligned}
 & \mathbb{E}(|\mathbf{e}_n|^2 | \mathcal{A}_{t_0}) \\
 & \leq \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + \underbrace{\mathbb{E}(|\delta_n|^2 | \mathcal{A}_{t_0})}_{1)} + \underbrace{2 |\mathbb{E}(\delta_n \cdot \mathbf{e}_{n-1} | \mathcal{A}_{t_0})|}_{2)} \\
 (39) \quad & + \underbrace{2 |\mathbb{E}(\mathbf{e}_{n-1} \cdot u_{n-1} | \mathcal{A}_{t_0})|}_{3)} + \underbrace{2 |\mathbb{E}(\delta_n \cdot u_{n-1} | \mathcal{A}_{t_0})|}_{4)} + \underbrace{\mathbb{E}(|u_{n-1}|^2 | \mathcal{A}_{t_0})}_{5)},
 \end{aligned}$$

which holds almost surely.

We will now estimate the separate terms in (39) individually and in sequence; all the estimates hold almost surely. We will frequently use the Hölder inequality, the inequality  $2ab \leq a^2 + b^2$  and properties of conditional expectation. The latter immediately imply for the term labelled 4) that  $2 |\mathbb{E}(\delta_n \cdot u_{n-1} | \mathcal{A}_{t_0})| \leq \mathbb{E}(|\delta_n|^2 | \mathcal{A}_{t_0}) + \mathbb{E}(|u_{n-1}|^2 | \mathcal{A}_{t_0})$ , therefore we have twice the terms labelled 1) and 5) below. The structure of the proof is similar to the proof of Theorem 3 in [4], we thus omit the details of some of the computations and refer to [4].

- For (twice) the term labelled 1) in (39) we have, due to (37),

$$2 \mathbb{E}(|\delta_n|^2 | \mathcal{A}_{t_0}) \leq 2 C_2^2 h^{2p_2}.$$

- For the term labelled 2) we have, due to the consistency condition (36),

$$2 |\mathbb{E}(\delta_n \cdot \mathbf{e}_{n-1} | \mathcal{A}_{t_0})| \leq C_1^2 h^{2p_1-1} + h \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}).$$

- For the term labelled 3) in (39) we obtain by the assumptions (29) on  $\phi$  and

(31) on  $\Upsilon$  and that  $h = \tau/N_\tau$

$$\begin{aligned}
2 \mathbb{E}(\mathbf{e}_{n-1} \cdot u_{n-1} | \mathcal{A}_{t_0}) &\leq 2 \mathbb{E}(|\mathbf{e}_{n-1}| \cdot |\mathbb{E}(u_{n-1} | \mathcal{A}_{t_{n-1}})| | \mathcal{A}_{t_0}) \\
&\leq 2 L_{\phi,1} h \left\{ \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + \sum_{\ell=n-1-N_\tau}^{n-2} \mathbb{E}(|\mathbf{e}_{n-1}| \times \right. \\
&\quad \times |\Upsilon(h, t_{n-1}, t_{\ell-n-1}, X(t_\ell), I_\Upsilon(t_\ell)) - \Upsilon(h, t_{n-1}, t_{\ell-n-1}, \tilde{X}_\ell, I_\Upsilon(t_\ell))| | \mathcal{A}_{t_0}) \Big\} \\
&\leq 2 L_{\phi,1} h \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + L_{\phi,1} h \sum_{\ell=n-1-N_\tau}^{n-2} \left\{ 2 (\mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}))^{\frac{1}{2}} \times \right. \\
&\quad \times (\mathbb{E} \mathbb{E}(|\Upsilon(h, t_{n-1}, t_{\ell-n-1}, X(t_\ell), I_\Upsilon(t_\ell)) - \\
&\quad \left. - \Upsilon(h, t_{n-1}, t_{\ell-n-1}, \tilde{X}_\ell, I_\Upsilon(t_\ell))|^2 | \mathcal{A}_{t_\ell}) | \mathcal{A}_{t_0}))^{\frac{1}{2}} \Big\} \\
&\leq 2 L_{\phi,1} h \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) \\
&\quad + L_{\phi,1} L_{\Upsilon,2} h^2 \sum_{\ell=n-1-N_\tau}^{n-2} \left\{ 2 (\mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}))^{\frac{1}{2}} \times (\mathbb{E}|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}))^{\frac{1}{2}} \right\} \\
&\leq 2 L_{\phi,1} h \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) \\
&\quad + L_{\phi,1} L_{\Upsilon,2} h^2 \sum_{\ell=n-1-N_\tau}^{n-2} \left\{ \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + \mathbb{E}(|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}) \right\} \\
&\leq L_{\phi,1} h (2 + L_{\Upsilon,2} \tau) \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + L_{\phi,1} L_{\Upsilon,2} h^2 \sum_{\ell=n-1-N_\tau}^{n-2} \mathbb{E}(|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}).
\end{aligned}$$

• For (twice) the term labelled 5) in (39) we have from the assumptions (30) on  $\phi$  and (31) on  $\Upsilon$

$$\begin{aligned}
2 \mathbb{E}(|u_{n-1}|^2 | \mathcal{A}_{t_0}) &= 2 \mathbb{E}(\mathbb{E}(|u_{n-1}|^2 | \mathcal{A}_{t_{n-1}}) | \mathcal{A}_{t_0}) \\
&\leq 2 L_{\phi,2} h \left\{ \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + \mathbb{E} \left( \sum_{\ell=n-1-N_\tau}^{n-2} \right. \right. \quad (\text{cont.}) \\
&\quad \left. \left. \Upsilon(h, t_{n-1}, t_{\ell-n-1}, X(t_\ell), I_\Upsilon(t_\ell)) - \Upsilon(h, t_{n-1}, t_{\ell-n-1}, \tilde{X}_\ell, I_\Upsilon(t_\ell)) \right|^2 | \mathcal{A}_{t_0} \right) \Big\} \\
&\leq 2 L_{\phi,2} h \left\{ \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + N_\tau \sum_{\ell=n-1-N_\tau}^{n-2} \left\{ \right. \quad (\text{cont.}) \right. \\
&\quad \left. \mathbb{E} \left( \mathbb{E}(|\Upsilon(h, t_{n-1}, t_\ell - t_{n-1}, X(t_\ell), I_\Upsilon(t_\ell)) - \right. \right. \\
&\quad \left. \left. - \Upsilon(h, t_{n-1}, t_\ell - t_{n-1}, \tilde{X}_\ell, I_\Upsilon(t_\ell))|^2 | \mathcal{A}_{t_\ell}) | \mathcal{A}_{t_0} \right) \right\} \\
&\leq 2 L_{\phi,2} h \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) + 2 L_{\phi,2} L_{\Upsilon,2} h^2 \tau \sum_{\ell=n-1-N_\tau}^{n-2} \mathbb{E}(|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}).
\end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}
\mathbb{E}(|\mathbf{e}_n|^2 | \mathcal{A}_{t_0}) &\leq \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) \\
&+ h \underbrace{(1 + 2 L_{\phi,2} h + L_{\phi,1} (2 + L_{\Upsilon,2} \tau))}_{=: \Gamma_1} \mathbb{E}(|\mathbf{e}_{n-1}|^2 | \mathcal{A}_{t_0}) \\
&+ h^2 \underbrace{(2 L_{\Upsilon,2} L_{\phi,2} h \tau + L_{\phi,1} L_{\Upsilon,2})}_{=: \Gamma_2} \sum_{\ell=n-1-N_\tau}^{n-2} \mathbb{E}(|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}) \\
&+ \underbrace{2 C_2^2 h^{2p_2} + C_1^2 h^{2p_2-1}}_{=: \eta(h)}.
\end{aligned}$$

Iterating yields

$$\begin{aligned}
\mathbb{E}(|\mathbf{e}_n|^2 | \mathcal{A}_{t_0}) &\leq h \Gamma_1 \sum_{j=1}^n \mathbb{E}(|\mathbf{e}_{n-j}|^2 | \mathcal{A}_{t_0}) \\
&+ h^2 \Gamma_2 \sum_{j=1}^n \sum_{\ell=n-N_\tau}^{n-1} \mathbb{E}(|\mathbf{e}_{\ell-j}|^2 | \mathcal{A}_{t_0}) + \sum_{j=1}^n \eta(h) \\
(40) \quad &= h \sum_{j=0}^{n-1} \left\{ \Gamma_1 \mathbb{E}(|\mathbf{e}_j|^2 | \mathcal{A}_{t_0}) + h \Gamma_2 \sum_{\ell=j-N_\tau}^{j-1} \mathbb{E}(|\mathbf{e}_\ell|^2 | \mathcal{A}_{t_0}) \right\} + n \eta(h).
\end{aligned}$$

We write

$$(41) \quad R_0 = 0 \quad \text{and} \quad R_k = \max\{\mathbb{E}(\mathbf{e}_i^2 | \mathcal{A}_{t_0}) \mid i = k - N_\tau + 1, k - N_\tau + 2, \dots, k\}.$$

Note that the sequence  $\{R_k\}_{k \geq 1}$  is monotonically non-decreasing. Then (40) and (41) give

$$(42) \quad R_n \leq h \sum_{j=0}^{n-1} \{\Gamma_1 R_j + h N_\tau \Gamma_2 R_j\} + n \eta(h).$$

Now  $h N_\tau = \tau$  and  $n \eta(h) = n(2 C_2^2 h^{2p_2} + C_1^2 h^{2p_2-1}) \leq (2 C_2^2 + C_1^2) n h^{2p_2} \leq (2 C_2^2 + C_1^2) T h^{2p_2-1}$ , as  $2p_2 \leq 2p_1 - 1$  and  $nh \leq T$ . We deduce

$$(43) \quad R_n \leq h \Gamma_3 \sum_{j=0}^{n-1} R_j + \Delta_h,$$

with  $\Gamma_3 = \Gamma_1 + \tau \Gamma_2$  and  $\Delta_h = (2 C_2^2 + C_1^2) T h^{2p_2-1}$ . We now apply Lemma 3.1 with  $r = 0$  and thus  $\eta = 0$ . This establishes the theorem.

**Remark 3.1:** For simplicity we have assumed exact initial values, therefore there is no dependency on starting errors. Similarly, the dependence of the convergence error on the error of the quadrature method is implicit in the assumptions on the consistency error.

## 4 Two specific schemes

Recall definition (25) of the mesh  $\mathcal{T}_h^N$ . In this section we investigate two specific choices of the increment functions  $\phi$  and  $\Upsilon$ . First we consider the Euler-Maruyama scheme with the quadrature method for  $Y(t)$  chosen to be the composite Euler method. The second method is the Milstein scheme, together with an appropriate quadrature method for  $Y(t)$ . We assume that the initial values are given by  $\tilde{X}_r = \Psi(rh)$  for  $r \leq 0$ ,  $r \in \mathbb{Z}^-$ . We will use notation given in (18) and introduce corresponding operators  $\tilde{\Lambda}_1$  and  $\tilde{\Lambda}_2$  for  $K$  as:

$$\begin{aligned}\tilde{\Lambda}_1 K_{s,t} &= D_2 K(t, s-t, X(s)) + F_s \cdot D_3 K(t, s-t, X(s)) + \frac{1}{2} G_s^2 \cdot D_3^2 K(t, s-t, X(s)), \\ \tilde{\Lambda}_2 K_{s,t} &= G_s \cdot D_3 K(t, s-t, X(s)).\end{aligned}$$

### Method 1: Euler-Maruyama (EM)

For  $n \geq 0$ , the increment function  $\phi_{EM}$  of the EM-method is given by

$$(44) \quad \phi_{EM}(h, t_n, \tilde{X}_n, \tilde{Y}_n^E, I_{\phi_{EM}}(t_n)) = h F(t_n, \tilde{X}_n, \tilde{Y}_n^E) + G(t_n, \tilde{X}_n, \tilde{Y}_n^E) \Delta W_{n+1},$$

where  $\Delta W_{n+1}$  denotes  $\int_{t_n}^{t_{n+1}} dW(s)$ .

For  $m \geq 0$ ,  $m - N_\tau \leq \ell \leq m - 1$ , the increment function  $\Upsilon^E$  of the composite Euler method has the form

$$(45) \quad \Upsilon^E(h, t_m, t_\ell - t_m, \tilde{X}_\ell, I_{\Upsilon^E}(t_\ell)) = h K(t_m, t_\ell - t_m, \tilde{X}_\ell),$$

thus

$$\hat{Y}_m^E = \sum_{\ell=m-N_\tau}^{m-1} h K(t_m, t_\ell - t_m, \tilde{X}_\ell).$$

### Method 2: Milstein

For  $n \geq 0$ , the increment function  $\phi_M$  of the Milstein method is given as

$$(46) \quad \begin{aligned}\phi_M(h, t_n, \tilde{X}_n, \tilde{Y}_n^M, I_{\phi}(t_n)) &= h F(t_n, \tilde{X}_n, \tilde{Y}_n^M) + G(t_n, \tilde{X}_n, \tilde{Y}_n^M) \Delta W_{n+1} \\ &+ \Lambda_3 G(t_n, \tilde{X}_n, \tilde{Y}_n^M) \int_{t_n}^{t_{n+1}} \int_{t_n}^s dW(u) dW(s).\end{aligned}$$

For  $m \geq N_\tau$ ,  $m - N_\tau \leq \ell \leq m - 1$ , the increment function  $\Upsilon^M$  of the quadrature method has the form

$$(47) \quad \begin{aligned}\Upsilon^M(h, t_m, t_\ell - t_m, \tilde{X}_\ell, I_{\Upsilon^M}(t_\ell)) &= h K(t_m, t_\ell - t_m, \tilde{X}_\ell) \\ &+ D_3 K(t_m, t_\ell - t_m, \tilde{X}_\ell) \cdot G(t_\ell, \tilde{X}_\ell, \tilde{Y}_\ell^M) \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s dW(u) ds,\end{aligned}$$

for  $m < N_\tau$  we take  $\Upsilon^M$  to be the same as  $\Upsilon^E$ .

Thus for  $m \geq N_\tau$

$$\hat{Y}_m^M = \sum_{\ell=m-N_\tau}^{m-1} \left\{ h K(t_m, t_\ell - t_m, \tilde{X}_\ell) + \Lambda_2 K(t_m, t_\ell - t_m, \tilde{X}_\ell) \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s dW(u) ds \right\},$$

otherwise  $\hat{Y}_m^M$  is the same as  $\hat{Y}_m^E$ .

Our purpose in the subsequent exposition is to provide consistency estimates of the form (36) and (37) for the methods described by the increment functions (44) and (46) and verify that (44) and (46) satisfy conditions (29) and (30). We will also provide estimates of the consistency error of the quadrature schemes  $Y^E$  and  $Y^M$  and ascertain that  $\Upsilon^E$  and  $\Upsilon^M$  satisfy (31).

In the course of the corresponding proofs we will require estimates of multiple Itô-integrals and we begin with a lemma providing these. It is adapted to our notation, based on Lemmas 2.1 and 2.2 [18] and the discussion in that section, where its proof can also be found.

We denote multiple Wiener integrals with some function  $f(\cdot)$  as an integrand by

$$(48) \quad I_{(j_1, j_2, \dots, j_l), h, t}(f) = \int_t^{t+h} \int_t^{s_1} \dots \int_t^{s_l} f(s_1) dW^{j_1}(s_1) \dots dW^{j_l}(s_l),$$

where  $j_i \in \{0, 1\}$ ,  $i = 1, \dots, l$  and  $dW^0(s) = ds$ . If  $f \equiv 1$  we omit the argument, as in (28). The integrand  $f$  may be one of the coefficient functions  $F$  or  $G$ , the kernel  $K$  or their derivatives.

**Lemma 4.1** *Suppose that*

$$(49) \quad \mathbb{E}|f(\cdot)| \leq M_4.$$

*Then we have for the multiple integral (48) the following estimates:*

- a)  $\mathbb{E}(I_{(j_1, j_2, \dots, j_l), h, t}(f)) = 0$ , if at least one  $j_k \neq 0$ ,  $k = 1, \dots, l$ ;
- b)  $\mathbb{E}(I_{(j_1, j_2, \dots, j_l), h, t}(f)) \leq M_4 h^l$ , if all  $j_k = 0$ ,  $k = 1, \dots, l$ ;
- c)  $(\mathbb{E}|I_{(j_1, j_2, \dots, j_l), h, t}(f)|^2)^{\frac{1}{2}} \leq M_4 h^{l_1 + l_2/2}$ , where  $l_1$  is the number of zero indices  $j_k$  and  $l_2$  the number of non-zero indices  $j_k$  in (48).

### Method 1: Euler-Maruyama

We first treat the composite Euler scheme  $Y^E$ .

**Lemma 4.2** *Assume (9) and that  $\tilde{\Lambda}_1 K_{s,t}$  and  $\tilde{\Lambda}_2 K_{s,t}$  exist for  $s \in [-\tau, 0]$ ,  $t \in [0, T]$  and are uniformly bounded in modulus by some constant  $M_3$  (we specify condition (A5) concerning the function  $K$ ). Part 1.: The composite Euler quadrature  $Y^E$  satisfies for all  $m \geq 0$  the consistency estimates*

$$a) \quad |\mathbb{E}(Y(t_m) - \hat{Y}^E(t_m))| \leq M \tau h^{\min(\gamma, 1)},$$

$$b) \mathbb{E}|Y(t_m) - \hat{Y}^E(t_m)|^2 \leq M \tau^2 h^{\min(2\gamma, 1)},$$

where  $M$  is a constant, not depending on  $h$ . Part 2.: The increment function  $\Upsilon^E$  satisfies (31) for all random variables  $\xi(t), \xi'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T], s \in [-\tau, 0]$ .

**Proof:** For Part 1., we have to distinguish two cases, *i*)  $t_m - \tau \geq 0$  and *ii*)  $t_m \leq 0$  (if  $t_m - \tau < 0 < t_m$  one would need to separate the integral).

For part a), case *i*), we use the Itô-formula and Lemma 4.1 to obtain

$$\begin{aligned} |\mathbb{E}(Y(t_m) - \hat{Y}^E(t_m))| &= \left| \sum_{\ell=m-N_\tau}^{m-1} \mathbb{E} \int_{t_\ell}^{t_{\ell+1}} K(t_m, s-t_m, X(s)) - K(t_m, t_\ell-t_m, X(t_\ell)) ds \right| \\ &\leq \sum_{\ell=m-N_\tau}^{m-1} \mathbb{E} \left( \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s |\tilde{\Lambda}_1 K_{u,t_m}| du ds \right) \leq \sum_{\ell=m-N_\tau}^{m-1} M_3 h^2 = M_3 \tau h. \end{aligned}$$

For part a), case *ii*) we obtain, using conditions (9) and (13):

$$\begin{aligned} &\left| \sum_{\ell=m-N_\tau}^{m-1} \mathbb{E} \int_{t_\ell}^{t_{\ell+1}} K(t_m, s-t_m, \Psi(s)) - K(t_m, t_\ell-t_m, \Psi(t_\ell)) ds \right| \\ &\leq L_K L_\Psi \sum_{\ell=m-N_\tau}^{m-1} \int_{t_\ell}^{t_{\ell+1}} (s-t_\ell)^\gamma ds = M \tau h^\gamma. \end{aligned}$$

For part b), case *i*), set  $\lfloor s \rfloor := t_k$  if  $t_k \leq s < t_{k+1}$  and  $s := t_N$  if  $s = t_N$ .

$$\begin{aligned} &\mathbb{E} \left| \int_{t_m-\tau}^{t_m} K(t_m, s-t_m, X(s)) ds - h \sum_{\ell=m-N_\tau}^{m-1} K(t_m, t_\ell-t_m, X(t_\ell)) \right|^2 \\ &\leq \tau \mathbb{E} \left( \int_{t_m-\tau}^{t_m} |K(t_m, s-t_m, X(s)) - K(t_m, \lfloor s \rfloor - t_m, X(\lfloor s \rfloor))|^2 ds \right) \\ &\leq \tau \sum_{\ell=m-N_\tau}^{m-1} \mathbb{E} \left( \int_{t_\ell}^{t_{\ell+1}} \left| \int_{t_\ell}^s \tilde{\Lambda}_1 K_{u,t_m} du + \int_{t_\ell}^s \tilde{\Lambda}_2 K_{u,t_m} dW(u) \right|^2 ds \right) \\ &\leq 2\tau \sum_{\ell=m-N_\tau}^{m-1} h \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s \mathbb{E} |\tilde{\Lambda}_1 K_{u,t_m}|^2 du ds + \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s \mathbb{E} |\tilde{\Lambda}_2 K_{u,t_m}|^2 du ds \\ &\leq 2\tau \sum_{\ell=m-N_\tau}^{m-1} M_3 (h^3 + h^2) \leq M_3 \tau^2 h. \end{aligned}$$



For part b), case *ii*) we obtain, using conditions (9) and (13):

$$\begin{aligned}
& \mathbb{E} \left| \int_{t_m - \tau}^{t_m} K(t_m, s - t_m, \Psi(s)) ds - h \sum_{\ell=m-N_\tau}^{m-1} K(t_m, t_\ell - t_m, \Psi(t_\ell)) \right|^2 \\
& \leq \tau \mathbb{E} \left( \int_{t_m - \tau}^{t_m} |K(t_m, s - t_m, \Psi(s)) - K(t_m, \lfloor s \rfloor - t_m, \Psi(\lfloor s \rfloor))|^2 ds \right) \\
& \leq \tau \sum_{\ell=m-N_\tau}^{m-1} \int_{t_\ell}^{t_{\ell+1}} L_K^2 \mathbb{E}(|\Psi(s) - \Psi(t_\ell)|^2) ds \\
& \leq \tau L_K^2 L_\Psi \sum_{\ell=m-N_\tau}^{m-1} \int_{t_\ell}^{t_{\ell+1}} (s - t_\ell)^{2\gamma} ds = \tau^2 L_K^2 L_\Psi \frac{1}{2\gamma + 1} h^{2\gamma}.
\end{aligned}$$

For Part 2. we have

$$\begin{aligned}
& \mathbb{E} (|\Upsilon^E(h, t, s, \xi(t), I_{\Upsilon^E}(t)) - \Upsilon^E(h, t, s, \xi'(t), I_{\Upsilon^E}(t))|^2 | \mathcal{A}_t) \\
& = h^2 \mathbb{E} (|K(t, s, \xi(t)) - K(t, s, \xi'(t))|^2 | \mathcal{A}_t) \\
& \leq h^2 L_K \mathbb{E} (|\xi(t) - \xi'(t)|^2 | \mathcal{A}_t).
\end{aligned}$$

We turn now our attention to the Euler-Maruyama scheme (44).

**Lemma 4.3** *Assume that  $\Lambda_1 F_t, \Lambda_2 F_t, \Lambda_3 F_t, \Lambda_1 G_t, \Lambda_2 G_t$  and  $\Lambda_3 G_t$  exist for  $t \in [0, T]$  and are uniformly bounded in modulus by a constant  $M_3$ . In addition the assumptions of Lemma 4.2 are supposed to hold. Part 1.: The Euler-Maruyama method (44) satisfies the consistency estimates a) (36) with  $p_1 = \min(1 + \gamma, 2)$  and b) (37) with  $2p_2 = \min(2\gamma + 1, 2)$ . Part 2.: The increment function  $\phi_{EM}$  satisfies the Lipschitz-conditions c) (29) and d) (30).*

**Proof:** Part 1.: We recall the definition of  $\delta_n$  (34) and using the Itô-expansion (19), we obtain:

$$\begin{aligned}
\delta_n &= h \left( F(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - F(t_{n-1}, X(t_{n-1}), \hat{Y}^E(t_{n-1})) \right) \\
&+ \left( G(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - G(t_{n-1}, X(t_{n-1}), \hat{Y}^E(t_{n-1})) \right) \int_{t_{n-1}}^{t_n} dW(s) \\
&+ R_1 + R_2 + R_M.
\end{aligned}$$

Part a) Only the first summand and the term  $R_1$  in the remainder contribute, when taking the expectation. Applying the mean value theorem and denoting  $\xi_{t_{n-1}} := Y(t_{n-1}) + \zeta(\hat{Y}^E(t_{n-1}) - Y(t_{n-1}))$ ,  $0 < \zeta < 1$ , we have using Lemmas

4.1 and 4.2

$$\begin{aligned}
|\mathbb{E}(\delta_n)| &\leq h \left| \mathbb{E} \left( F(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - F(t_{n-1}, X(t_{n-1}), \hat{Y}^E(t_{n-1})) \right) \right| \\
&\quad + \left| \mathbb{E} \left( \int_{t_{n-1}}^{t+h} \int_t^s \Lambda_1 F_u + \Lambda_3 F_u \, du \, ds \right) \right| \\
&\leq h \left| \mathbb{E} \left( D_3 F(t_{n-1}, X(t_{n-1}), \xi_{t_{n-1}}) (\hat{Y}^E(t_{n-1}) - Y(t_{n-1})) \right) \right| \\
&\quad + \left| \mathbb{E} \left( \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s |\Lambda_1 F_u| + |\Lambda_3 F_u| \, du \, ds \right) \right| \\
&\leq h M_3 \left| \mathbb{E} (Y(t_{n-1}) - \hat{Y}^E(t_{n-1})) \right| + M_3 h^2 \leq M h^{\min(1+\gamma, 2)}.
\end{aligned}$$

Part b)  $\mathbb{E}|\delta_n|^2$ 

$$\begin{aligned}
&\leq 5 h^2 \mathbb{E} \left| F(X(t_{n-1}), Y(t_{n-1})) - F(X(t_{n-1}), \hat{Y}^E(t_{n-1})) \right|^2 \\
&\quad + 5 \mathbb{E} \left| G(X(t_{n-1}), Y(t_{n-1})) - G(X(t_{n-1}), \hat{Y}^E(t_{n-1})) \int_{t_{n-1}}^{t_n} dW(s) \right|^2 \\
&\quad + 5 \mathbb{E}|R_1|^2 + 5 \mathbb{E}|R_2|^2 + 5 \mathbb{E}|R_M|^2 \\
&\leq 5 \{ h^2 L_{F,2}^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^E(t_{n-1})|^2 + h L_{G,2}^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^E(t_{n-1})|^2 \\
&\quad + \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \Lambda_1 F_u + \Lambda_3 F_u \, du \, ds \right|^2 + 2 \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \Lambda_2 F_u \, dW(u) \, ds \right|^2 \\
&\quad + 2 \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \Lambda_1 G_u + \Lambda_1 G_u + \Lambda_3 G_u \, du \, dW(s) \right|^2 \\
&\quad + \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \Lambda_2 G_u \, dW(u) \, dW(s) \right|^2 \} \\
&\leq 5 \{ h^2 L_{F,2}^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^E(t_{n-1})|^2 + h L_{G,2}^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^E(t_{n-1})|^2 \\
&\quad + M_3 h^4 + 2M_3 h^3 + 2M_3 h^3 + M_3 h^2 \} \leq M h^{\min(2\gamma+1, 2)}.
\end{aligned}$$

Part c), we have for all random variables  $\xi(t), \xi'(t), \eta(t), \eta'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T]$ :

$$\begin{aligned}
&\left| \mathbb{E} \left( \phi_{EM}(h, t, \xi(t), \eta(t), \int_t^{t+h} dW(s)) - \phi_{EM}(h, t, \xi'(t), \eta'(t), \int_t^{t+h} dW(s)) \middle| \mathcal{A}_t \right) \right| \\
&\leq h \left| \mathbb{E} (F(t, \xi(t), \eta(t)) - F(t, \xi'(t), \eta'(t)) \middle| \mathcal{A}_t) \right| \\
&\quad + \left| \mathbb{E} \left( (G(t, \xi(t), \eta(t)) - G(t, \xi'(t), \eta'(t))) \int_t^{t+h} dW(s) \middle| \mathcal{A}_t \right) \right|.
\end{aligned}$$

The result follows from the Lipschitz-continuity of  $F$  and Lemma 4.1, a).

For Part d), we have for all random variables  $\xi(t), \xi'(t), \eta(t), \eta'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T]$ :

$$\begin{aligned} & \mathbb{E} \left( \left| \phi_{EM}(h, t, \xi(t), \eta(t), \int_t^{t+h} dW(s)) - \phi_{EM}(h, t, \xi'(t), \eta'(t), \int_t^{t+h} dW(s)) \right|^2 \middle| \mathcal{A}_t \right) \\ & \leq 2 h^2 \mathbb{E} \left( \left| F(t, \xi(t), \eta(t)) - F(t, \xi'(t), \eta'(t)) \right|^2 \middle| \mathcal{A}_t \right) \\ & \quad + 2 \mathbb{E} \left( \left| \int_t^{t+h} G(t, \xi(t), \eta(t)) - G(t, \xi'(t), \eta'(t)) dW(s) \right|^2 \middle| \mathcal{A}_t \right). \end{aligned}$$

Now the result follows from the Lipschitz-continuity of  $F$  and  $G$  and Lemma 4.1, c).

### Method 2: Milstein

The remainder of this section is devoted to the analysis of the Milstein method (46).

**Lemma 4.4** *Assume that  $\tilde{\Lambda}_1 K_{s,t}$  and  $\tilde{\Lambda}_2 K_{s,t}$  exist for  $s \in [-\tau, 0]$ ,  $t \in [0, T]$  and are uniformly bounded in modulus by  $M_3$  (we specify condition (A5) concerning the function  $K$ ). Part 1.: The composite quadrature  $Y^M$  with its increment function  $\Upsilon^M$  given by (47) satisfies for all  $m \geq 0$  the consistency estimates*

$$\begin{aligned} a) \quad & |\mathbb{E}(Y(t_m) - \hat{Y}^M(t_m))| \leq M \tau h^{\min(\gamma, h)}, \\ b) \quad & \mathbb{E}|Y(t_m) - \hat{Y}^M(t_m)|^2 \leq M \tau^2 h^{\min(2\gamma, h^2)}, \end{aligned}$$

where  $M$  is a constant, not depending on  $h$ . Part 2.: The increment function  $\Upsilon^M$  satisfies (31) for all random variables  $\xi(t), \xi'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T]$ ,  $s \in [-\tau, 0]$ .

**Proof:** Part 1: As in Lemma 4.2 we have to distinguish the two cases, i)  $t_m - \tau \geq 0$  and ii)  $t_m \leq 0$ .

For part a), case i), we use the Itô-formula and Lemma 4.1. We have

$$\begin{aligned} & |\mathbb{E}(Y(t_m) - \hat{Y}^M(t_m))| \\ & = |\mathbb{E} \left( \int_{t_m - \tau}^{t_m} K(t_m, s - t_m, X(s)) ds \right. \\ & \quad \left. - \sum_{\ell=m-N_\tau}^{m-1} (h K(t_m, t_\ell - t_m, X(t_\ell)) + \tilde{\Lambda}_2 K_{t_\ell, t_m} \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s dW(u) ds) \right)| \\ & = \left| \sum_{\ell=m-N_\tau}^{m-1} \mathbb{E} \int_{t_\ell}^{t_{\ell+1}} K(t_m, s - t_m, X(s)) - K(t_m, t_\ell - t_m, X(t_\ell)) ds \right|, \end{aligned}$$

then the proof proceeds as in the proof of Lemma 4.2, part a), case i), giving the desired result. In the same way we can prove part a), case ii).

For part b), case i), set  $\lfloor s \rfloor := t_k$  if  $t_k \leq s < t_{k+1}$  and  $s := t_N$  if  $s = t_N$ .

$$\begin{aligned}
\mathbb{E}|Y(t_m) - \hat{Y}^M(t_m)|^2 &\leq \tau \mathbb{E} \left( \int_{t_m-\tau}^{t_m} |K(t_m, s - t_m, X(s)) \right. \\
&\quad \left. - K(t_m, \lfloor s \rfloor - t_m, X(\lfloor s \rfloor)) - \tilde{\Lambda}_2 K_{\lfloor s \rfloor, t_m} \int_{\lfloor s \rfloor}^s dW(u)|^2 ds \right) \\
&\leq 2 \tau \sum_{\ell=m-N_\tau}^{m-1} \left\{ h \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s \mathbb{E} | \tilde{\Lambda}_1 K_{u, t_m} |^2 du ds \right. \\
&\quad \left. + \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s \mathbb{E} | \tilde{\Lambda}_2 K_{u, t_m} - \tilde{\Lambda}_2 K_{t_\ell, t_m} |^2 du ds \right\} \\
&\leq 2 \tau \sum_{\ell=m-N_\tau}^{m-1} M_3 h^3 + \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s L_{\tilde{\Lambda}_2 K}^2 \mathbb{E} |X(u) - X(t_\ell)|^2 du ds \\
&\leq 2 (M_3 + L_{\tilde{\Lambda}_2 K}^2 M_2(T)) \tau \sum_{\ell=m-N_\tau}^{m-1} h^3 \leq M_3 \tau^2 h^2.
\end{aligned}$$

Part b), case ii) is the same as in the case of the Euler-Maruyama method.

For Part 2. we have for all random variables  $\xi(t), \xi'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T], s \in [-\tau, 0]$ :

$$\begin{aligned}
&\mathbb{E} (| \Upsilon(h, t, s, \xi(t), I_\Upsilon(t)) - \Upsilon(h, t, s, \xi'(t), I_\Upsilon(t)) |^2 | \mathcal{A}_t ) \\
&\leq 2 h^2 \mathbb{E} (|K(t, s, \xi(t)) - K(t, s, \xi'(t))|^2 | \mathcal{A}_t ) \\
&\quad + 2 \mathbb{E} \left( |G_s D_3 K(t, s, \xi(t)) - G_s D_3 K(t, s, \xi'(t))|^2 \mid \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^s dW(u) ds \mid \mathcal{A}_t \right) \\
&\leq \underbrace{2 (L_K + L_{\tilde{\Lambda}_2 K})}_{:= L_{\Upsilon, 2}} h^2 \mathbb{E} (|\xi(t) - \xi'(t)|^2 | \mathcal{A}_t ).
\end{aligned}$$

**Lemma 4.5** Assume that  $\Lambda_1 F_t, \Lambda_2 F_t, \Lambda_3 F_t, \Lambda_1 G_t, \Lambda_2 G_t, \Lambda_3 G_t, \Lambda_1 \Lambda_3 G_t, \Lambda_2 \Lambda_3 G_t$  and  $\Lambda_3 \Lambda_3 G_t$  exist and are uniformly bounded in modulus by a constant  $M_3$  and  $\Lambda_2 G_t$  is Lipschitz-continuous with respect to its second and third argument. In addition the assumptions of Lemma 4.2 are supposed to hold. Part 1.: The Milstein method satisfies the consistency estimates a) (36) with  $p_1 = \min(1 + \gamma, 2)$  and b) (37) with  $2p_2 = \min(2\gamma + 1, 3)$ . Part 2.: The increment function  $\phi_M$  satisfies the Lipschitz-conditions c) (29) and d) (30).

**Proof:**

Using the Itô-expansion, we obtain for the consistency error  $\delta_n$  (34)

$$\begin{aligned} \delta_n = & h \left( F(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - F(t_{n-1}, X(t_{n-1}), \hat{Y}^M(t_{n-1})) \right) \\ & + \left( G(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - G(t_{n-1}, X(t_{n-1}), \hat{Y}^M(t_{n-1})) \right) \int_{t_{n-1}}^{t_n} dW(s) \\ & + \left( \Lambda_2 G_{t_{n-1}} G(t_{n-1}, X(t_{n-1}), \hat{Y}^M(t_{n-1})) \cdot D_2 G(t_{n-1}, X(t_{n-1}), \hat{Y}^M(t_{n-1})) \right) \\ & \quad \times \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s dW(u) dW(s) \\ & + R_1 + R_2 + R_3. \end{aligned}$$

Part 1., a) We obtain the same result as in the computations for the Euler-Maruyama method.

Part 1., b)

$$\begin{aligned} \mathbb{E} |\delta_n|^2 & \leq 6 L_{F,2} h^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^M(t_{n-1})|^2 + 6 L_{G,2} h \mathbb{E} |Y(t_{n-1}) - \hat{Y}^M(t_{n-1})|^2 \\ & + 6 L_{\Lambda_2 G} h^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^M(t_{n-1})|^2 + 6 M_3 h^4 + 12 M_3 h^3 + 12 M_3 h^3 \\ & + 12 \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \int_{t_{n-1}}^u \left( \Lambda_1 \Lambda_3 G(X(z), Y(z)) + \Lambda_2 \Lambda_3 G(X(z), Y(z)) \right. \right. \\ & \quad \left. \left. + L_4 \Lambda_3 G(X(z), Y(z)) \right) dz dW(u) dW(s) \right|^2 \\ & + 12 \mathbb{E} \left| \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^s \int_{t_{n-1}}^u \Lambda_3 \Lambda_3 G(X(z), Y(z)) dW(z) dW(u) dW(s) \right|^2 \\ & \leq 6(L_{F,2} + L_{\Lambda_2 G}) h^2 \mathbb{E} |Y(t_{n-1}) - \hat{Y}^M(t_{n-1})|^2 + 6 L_{G,2} h \mathbb{E} |Y(t_{n-1}) - \hat{Y}^M(t_{n-1})|^2 \\ & + 6 M_3 h^4 + 12 M_3 h^3 + 12 M_3 h^3 + 12 M_3 h^4 + 12 M_3 h^3. \end{aligned}$$

Part c), we have for all random variables  $\xi(t), \xi'(t), \eta(t), \eta'(t) : [0, T] \rightarrow \mathbb{R}$ , adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T]$ :

$$\begin{aligned} & |\mathbb{E}(\phi_M(h, t, \xi(t), \eta(t), \int_t^{t+h} dW(s), \int_t^{t+h} \int_t^s dW(u) dW(s)) \\ & \quad - \phi_M(h, t, \xi'(t), \eta'(t), \int_t^{t+h} dW(s), \int_t^{t+h} \int_t^s dW(u) dW(s)) | \mathcal{A}_t)| \\ & \leq h |\mathbb{E}(F(t, \xi(t), \eta(t)) - F(t, \xi'(t), \eta'(t)) | \mathcal{A}_t)|, \end{aligned}$$

using Lemma 4.1, a), the result follows from the Lipschitz-continuity of  $F$ .

For Part d), we have for all random variables  $\xi(t), \xi'(t), \eta(t), \eta'(t) : [0, T] \rightarrow \mathbb{R}$ ,

adapted to the filtration  $\mathcal{A}_t$  and for all  $t \in [0, T]$ :

$$\begin{aligned}
& \mathbb{E}(|\phi_M(h, t, \xi(t), \eta(t), \int_t^{t+h} dW(s), \int_t^{t+h} \int_t^s dW(u) dW(s)) \\
& \quad - \phi_M(h, t, \xi'(t), \eta'(t), \int_t^{t+h} dW(s), \int_t^{t+h} \int_t^s dW(u) dW(s))|^2 | \mathcal{A}_t) \\
& \leq 3 h^2 \mathbb{E}(|F(t, \xi(t), \eta(t)) - F(t, \xi'(t), \eta'(t))|^2 | \mathcal{A}_t) \\
& \quad + 3 \mathbb{E}\left(|\int_t^{t+h} G(t, \xi(t), \eta(t)) - G(t, \xi'(t), \eta'(t)) dW(s)|^2 | \mathcal{A}_t\right) \\
& \quad + 3 \mathbb{E}(|\int_t^{t+h} \int_t^s G(t, \xi(t), \eta(t)) \cdot D_2 G(t, \xi(t), \eta(t)) \\
& \quad - G(t, \xi'(t), \eta'(t)) \cdot D_2 G(t, \xi'(t), \eta'(t)) dW(u) dW(s)|^2 | \mathcal{A}_t)
\end{aligned}$$

Now the result follows from the Lipschitz-continuity of  $F, G$  and  $G \cdot D_2 G$  and Lemma 4.1, c).

## 5 Numerical experiments

We present some results of numerical experiments corresponding to an example of (1). Our objective is to illustrate the convergence of the Euler-Maruyama and the Milstein method with respect to decreasing step-size.

**Example 5.1** Consider

$$(50) \quad dY(t) = \left[ -0.2 \int_{t-1}^t Y(s) ds - 0.3Y(t) \right] dt + 0.5Y(t)dW(t),$$

for  $t \in [0, 2]$  and  $Y(s) = 1 + t$  for  $-1 \leq s \leq 0$  and  $W(t)$  is a one dimensional Wiener process.

If we square both sides of (32) we obtain the mean-square error  $\mathcal{E}|X(T) - \tilde{X}_N|^2$  which should be bounded by  $C h^{2p}$ . An ‘explicit solution’ was computed on a very fine mesh (4096 steps). To illustrate the convergence of the method, we have simulated 2,000 sample trajectories with each of the step-sizes  $h = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$  and computed the error

$$(51) \quad \epsilon^h = \frac{1}{2000} \sum_{j=1}^{2000} |Y_N^h - Y(T)|^2$$

at the final time  $T = 2$ . In Table 1 the results of an experiment are presented, in Figure 1, we have plotted  $\log_2(\epsilon^h)$  versus  $\log_2(h)$  together with reference lines with the slope 1 and 2.

Table 1: Example (50)

Time step $h$	EM $\epsilon^h$	Mil $\epsilon^h$
0.25	0.0072224	0.002671
0.125	0.0028965	0.00057764
0.0625	0.0011966	0.00013745
0.03125	0.00051249	3.7755E-05

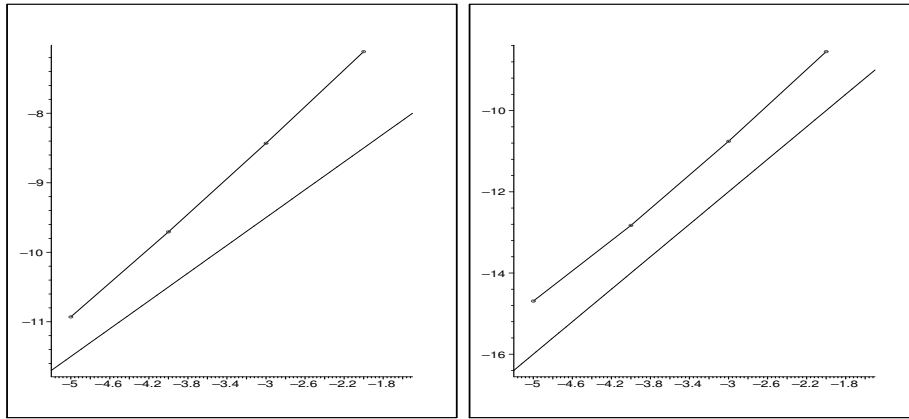


Figure 1:  $\log_2(\epsilon^h)$  versus  $\log_2(h)$  for (50) with left: Euler-Maruyama method ( $y = x - 6.5$ ), right: Milstein-method ( $y = 2x - 6$ ).

## References

- [1] J. Appleby, *A complete market model with feedback*, Tech. Report, MS-00-010, Dublin City University, Ireland, 2000.
- [2] M. Arriojas, *A stochastic calculus for functional differential equations*, PhD. thesis, Southern Illinois University at Carbondale, USA, 1997.
- [3] C. T. H. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.
- [4] C. T. H. Baker and E. Buckwar, *Numerical analysis of explicit one-step methods for stochastic delay differential equations*, LMS J. Comput. Math. 3, pp. 315–335, 2000.
- [5] C. T. H. Baker and N. Ford, *Convergence of linear multistep methods for a class of delay-integro-differential equations*, in International Series of Numerical Mathematics 86, pp. 47–59, Birkhäuser Verlag Basel, 1988.

- [6] H. Brunner and P. J. van der Houwen, *The Numerical Solution of Volterra Equations*, CWI Monographs, North-Holland, 1986.
- [7] I. Elsanosi, B. Øksendal and A. Sulem, *Some solvable stochastic control problems with delay*, Stoch. Stoch. Rep. 71 (1-2), pp. 69–89, 2000.
- [8] A. Friedman, *Stochastic Differential Equations and Applications*, Volume 1, Academic Press, 1975.
- [9] P. Guttorp and R. Kulperger, *Statistical inference for some Volterra population processes in a random environment*, The Canadian Journal of Statistics 12 (4), pp. 289–302, 1984.
- [10] D. G. Hobson and L. C. G. Rogers, *Complete models with stochastic volatility*, Math. Finance 8 (1), pp. 27–48, 1998.
- [11] Y. Hu, S. E. A. Mohammed and F. Yan, *Discrete-time approximations of stochastic delay equations: the Milstein scheme*, Tech. report, Southern Illinois University, 2001. (electronic, URL: <http://salah.math.siu.edu/>).
- [12] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.
- [13] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1999.
- [14] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego, 1993.
- [15] B. Larssen, *Dynamic programming in stochastic control of systems with delay*. PhD. thesis, University of Oslo, Norway, 2002.
- [16] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, 1985.
- [17] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing Limited, Chichester, 1997.
- [18] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Kluwer Academic Publishers Group, Dordrecht, 1995. Translated and revised from the 1988 Russian original.
- [19] V. J. Mizel and V. Trutzer, *Stochastic hereditary equations: existence and asymptotic stability*, J. Int. Eq. 7, pp. 1–72, 1984.
- [20] S. E. A. Mohammed, *Stochastic differential systems with memory. Theory, examples and applications*, in Stochastic Analysis and Related Topics VI. The Geilo Workshop, 1996, eds. L. Decreasefond, Jon Gjerde, B. Øksendal, A.S. Üstünel, Progress in Probability, Birkhauser (1998), pp. 1-77.



- [21] S. E. A. Mohammed, *Stochastic Functional Differential Equations*, Pitman (Advanced Publishing Program), Boston, Mass., 1984.
- [22] B. Øksendal and A. Sulem, *A maximum principle for optimal control of stochastic systems with delay, with applications to finance*, in Optimal Control and Partial Differential Equations- Innovations and Applications, (J. M. Menaldi, E. Rofman and A. Sulem, eds.), IOS Press, Amsterdam, 2000.