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On integrals with respect to Lévy processes*

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Abstract

Assume L is a non-deterministic real valued Lévy process and f is a smooth function on $[0,t]$. If for some Borel function H P-almost sure the equality

$$H(\int\limits_{[0,t]} f(s)d\,L_s) = L_t$$

holds, then f is constant on $[0,t]$.

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1 Introduction

In modeling term structures of interest rates driven by Lévy processes $L = (L_s, s \geq 0))$ arises the question, if for a fixed positive t and for a given deterministic continuously differentiable function f on $[0,t]$ there may exist a (deterministic) Borel function H with

$$H\Big(\int\limits_{[0,t]} f(s)dL_s\Big) = L_t \qquad P-a.s. \tag{1}$$

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See Küchler, Naumann [3]. Obviously (1) holds with $H(x) = c^{-1}x$ if f equals a nonzero constant c. In this note we will show that for any non-deterministic Lévy process L the existence of a measurable function H with (1) implies f to be constant.

Note that under assumption (1) the support of the distribution of the random vector (I_t, L_t) with $I_t = \int\limits_{[0,t]} f(s)dL_s$ is a zero set with respect to the (two-dimensional) Lebesgue measure λ_2. Therefore the assertion immediately follows if one can ensure that (I_t, L_t) has a common density with respect to λ_2. This is the case if L is a Wiener process with positive diffusion coefficient and arbitrary drift. Indeed, if f is not a constant then (F_t, L_t) is two-dimensional Gaussian distributed with a regular covariance matrix. For other Lévy processes L Eberlein and Raible [1] proved that the following condition on the characteristic function of L_t is sufficient for (I_t, L_t) to have a common density: There exist real constants $C, \gamma, \eta > 0$ such that

$$|E[\exp(i\,u\,L_1)]| \leq C \cdot \exp(-\gamma|u|^\eta) \quad \forall\, u \in R \tag{2}$$

But not for every Lévy process L the vector (I_t, L_t) has a common density, consider for example Poisson processes. Analyzing the arguments above it turns out to be enough for our purpose to show that the common distribution of (I_t, L_t) has a nonzero absolutely continuous part with respect to λ_2 if f is not equal to a constant. The key point of the proof of the theorem below is to show that this property holds for every non-deterministic Lévy process L.

2 Lévy processes

Assume $L = (L_s, s \geq 0)$ to be a real valued Lévy process defined on some probability space $(\Omega, \mathfrak{A}, P)$. This means

(i) $P(L_0 = 0) = 1$.

(ii) L has independent and stationary increments.

(iii) All trajectories $(L_s(\omega), s \geq 0), \omega \in \Omega$, are cadlag, i.e. continuous from the right and having limits from the left.

Examples of Lévy process are

- the Wiener process W with drift μ and diffusion $\sigma^2 > 0$, in this case $W_t - W_s$ is $N(\mu(t-s), \sigma^2(t-s))$-distributed, and all trajectories are continuous,

- the Poisson process N with jump intensity $\lambda > 0$, in this case $N_t - N_s$ is Poisson-distributed with parameter $\lambda(t-s)$ and the trajectories are piecewise constant, non-decreasing, jumping at times $\tau_k, k \geq 1$, with jump size one where $(\tau_k - \tau_{k-1}), k \geq 1$, with $\tau_0 = 0$ are mutual independent exponential distributed with parameter λ random variables.

- the compound Poisson process $Y_t = \sum_{k=1}^{N_t} Z_k$, where $N = (N_s, s \geq 0)$ is a Poisson process with jump intensity $\lambda > 0$ and the $(Z_k, k \geq 1)$ are independent identically distributed random variables the distribution function F of which satisfies $F(0+) = F(0-)$. Moreover, the $(Z_k, k \geq 1)$ and N are independent. The Z_k form the jump sizes of $(Y_t, t \geq 0)$.

We summarize some well known facts on Lévy processes that will be used below. For proofs and further details see for example Sato [4]. In general, to every Lévy process L is assigned a uniquely determined *characteristic triple* $\mathfrak{L} := (\mu, \sigma^2, \nu)$ where $\mu \in R, \sigma^2 \geq 0$ and ν is a measure on the real axis R satisfying

$$\nu(\{0\}) = 0, \qquad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \nu(dx) < \infty \tag{3}$$

such that it holds

$$E \exp(iyL_t) = \exp[t \cdot \psi(y)], \quad t \geq 0, y \in R$$

with

$$\psi(y) = i\mu y - \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} (\mathrm{e}^{iyx} - 1 - \frac{iyx}{1+x^2}) \nu(dx), y \in R.$$

Conversely, to any triple $\mathfrak{L} = (\mu, \sigma^2, \nu)$ with (3) there is a uniquely in law determined Lévy process L having $\mathfrak{L}$ as its characteristic triple. If $\sigma^2 + \nu(R) = 0$ it holds $L_s = \mu s$, $s \geq 0$. We call this case the deterministic one.
If $\nu \equiv 0$ then we obtain a Wiener process with μ as drift and with σ^2 as diffusion coefficient.

Let L be a Lévy process. For any $\varepsilon > 0$ one can decompose L into two mutual independent Lévy processes $L^{\leq \varepsilon}$ and $L^{> \varepsilon}$ defined by

$$\begin{aligned} L_s^{>\varepsilon} &:= \sum_{u \leq s} 1\!\!1_{\{|\Delta L_u| > \varepsilon\}} \Delta L_u, s \geq 0, \text{ where} \\ \Delta L_u &:= L_u - L_{u-0}, \end{aligned}$$

and

$$L_s^{\leq \varepsilon} := L_s - L_s^{>\varepsilon}, \quad s \geq 0.$$

The process $L^{>\varepsilon}$ is a compound Poisson process with jump intensity $\lambda_\varepsilon := \nu(R\backslash[-\varepsilon,\varepsilon])$ and jump size distribution $F_\varepsilon(dz) := \lambda_\varepsilon^{-1}\nu(dz)\cdot 1\!\!1_{R\backslash[-\varepsilon,\varepsilon]}(z)$.
If f is a continuously differentiable function on $[0,t]$ we define the integral $\int\limits_{[0,t]} f(s)dL_s$ by

$$\int\limits_{[0,t]} f(s)dL_s := f(t)L_t - \int\limits_{[0,t]} L_s \frac{d}{ds} f(s)ds$$

and denote it shortly by I_t.
Note that L is a (possibly degenerated to the deterministic case if $\sigma^2 = 0$) Wiener process if and only if $L_t^{>\varepsilon} \equiv 0$ for all $\varepsilon > 0$.

3 Results

The result of this note is the following theorem.

Theorem 3.1. *Fix a positive number t, let f be a real valued continuously differentiable function on $[0,t]$ and assume $L = (L_s, s \geq 0)$ to be a non-deterministic Lévy process. If for some Borel function H it holds*

$$H\Big(\int\limits_{[0,t]} f(s)dL_s\Big) = L_t \qquad P-a.s. \tag{4}$$

then f is necessarily a constant.

Proof. Let us firstly consider the case that L is a Wiener process with parameters $\mu \in R$ and $\sigma^2 > 0$. Then $(I_t, L_t) = (\int\limits_0^t f(s)d\,L_s, L_t)$ is a Gaussian vector. Its distribution is degenerated by assumption (4). Thus its covariance matrix is singular, because of $\sigma^2 > 0$ this implies $(\int\limits_{[0,t]} f(s)d\,s)^2 = t\cdot\int\limits_{[0,t]} f^2(s)d\,s$. By the Cauchy-Schwarz inequality this is only possible if f is a constant.

As a second step we assume that L is a compound Poisson process with jump times $\tau_1, \tau_2, \cdots$ and jumps sizes Z_k at time $\tau_k, k \geq 1$. Denote the distribution function of Z_k by F. Let z be a point of increase of F. This means $P(z-\varepsilon < Z_1 < z+\varepsilon) > 0$ for all $\varepsilon > 0$.

We may suppose $z > 0$, otherwise consider $-L$ instead of L. If f is not a constant function, based on the assumptions on f we find a subinterval $U := (u_0, u_1)$ of $(0,t)$ with $u_0 < u_1$ and $f'(x) \neq 0$ at all points x from U. In particular f' is either strictly positive or strictly negative on U. Thus f is strictly monotone on U and consequently maps U one-to-one on an interval $V = (v_0, v_1)$ with $v_0 < v_1$.

Without restriction we can assume $v_0 > 0$, otherwise restrict U to a smaller open non-void interval and/or consider, if it is needed, $-f$ instead of f.

We continue the proof by choosing a positive ε such that

$$\varepsilon < z \cdot \frac{c-1}{c+1} \tag{5}$$

with $c := (v_1/v_0)^{\frac{1}{2}}$. For every $n \geq 1$ put

$$C_{\varepsilon,n} := \{\omega \in \Omega \,|\, Z_k(\omega) \in (z-\varepsilon, z+\varepsilon), k = 1, \ldots, n\}$$

and

$$D_n := \{\omega \in \Omega : u_0 < \tau_1(\omega) < \ldots < \tau_n(\omega) < u_1, \tau_{n+1}(\omega) < t\}.$$

Because of the special choice of the point z and the independence of the $Z_k, k \geq 1$ we have $P(C_{\varepsilon,n}) > 0$. The jump times $\tau_n, n \geq 1$ have the property that the differences $\tau_{k+1} - \tau_k, k \geq 1$, are independent and exponentially distributed. This implies in particular $P(D_n) > 0$. Now using the independence of the $(Z_k, k \geq 1)$ from the $(\tau_k, k \geq 1)$ we conclude that

$$P(D_n \cap C_{\varepsilon,n}) = P(D_n)P(C_{\varepsilon,n}) > 0. \tag{6}$$

We have P-almost surely for all $\omega \in D_n \cap C_{\varepsilon,n}$

$$I_t = \int\limits_{[0,t]} f(s)dL_s = \sum_{k=1}^{n} Z_k f(\tau_k) \in (n(z-\varepsilon)v_0, n(z+\varepsilon)v_1) \tag{7}$$

Consequently, the distribution of I_t given $D_n \cap C_{\varepsilon,n}$ can be expressed as

$$\begin{aligned} &P(I_t \in B \,|\, D_n \cap C_{\varepsilon,n}) = \\ &[P(C_{\varepsilon,n})]^{-1} \int\limits_{(z-\varepsilon,z+\varepsilon)^n} P\Big(\sum_{k=1}^{n} z_k f(\tau_k) \in B | D_n\Big) F(dz_1) \ldots F(dz_n) \end{aligned} \tag{8}$$

where B is any Borel set. Here we have used once again the independence of the $(Z_k, k \geq 1)$ from the $(\tau_k, k \geq 1)$.

At this point we formulate and prove an auxiliary result on the distribution of $\sum_{k=1}^{n} z_k f(\tau_k)$ as a lemma.

Lemma 3.1. *Assume $n \geq 1$, as well as $z_k > 0, k = 1, \ldots, n$, to be fixed. Then the distribution of $S_n := \sum_{k=1}^{n} z_k f(\tau_k)$ given D_n has a density being strictly positive on $\mathcal{J} := \Big(v_0 \sum_{k=1}^{n} z_k, v_1 \sum_{k=1}^{n} z_k\Big)$.*

Proof. Given $D_n^0 := \{\tau_n \le t < \tau_{n+1}\}$ the random vector $(\tau_1, \tau_2, \ldots, \tau_n)$ has a strictly positive density on $\Delta_n := \{(s_1, \ldots, s_n) : 0 \le s_1 \le s_2 \le \ldots \le s_n \le t\}$, indeed it is uniformly distributed thereon (see for example Sato [4], Chapter 1.3). This implies immediately that given D_n the vector $(\tau_1, \ldots, \tau_n)$ is uniformly distributed on $\Delta_n \cap (u_0, u_1)^n$.

Put $S_n(s_1, \ldots, s_n) := \sum_{k=1}^{n} z_k f(s_k)$. The function G defined by $G(s_1, \ldots, s_n) = (s_1, s_2, \ldots, s_{n-1}, S_n(s_1, \ldots, s_n))$ maps $\Delta_n \cap (u_0, u_1)^n$ continuously differentiable one-to-one on $(\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times \mathcal{J}$. This is a consequence of the supposed smoothness and the strict monotonicity of the mapping f from (u_0, u_1) onto (v_0, v_1). Thus, given D_n the vector $G(\tau_1, \ldots, \tau_n) = (\tau_1, \ldots, \tau_{n-1}, \sum_{k=1}^{n} z_k f(\tau_k))$ has a density being strictly positive on $(\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times \mathcal{J}$ which can be expressed as

$$\gamma \cdot 1\!\!1_{\Delta_n \cap (u_0, u_1)^n}(G^{-1}(v_1, v_2, \ldots, v_n))(z_n |f'(v_n)|\,)^{-1}$$

where γ is the normalizing constant. Consequently given D_n the random variable $S_n = S_n(\tau_1, \ldots, \tau_n)$ has a density $\psi_{S_n}(v)$ equal to

$$\gamma {z_n}^{-1} \cdot \int\limits_{\Delta_{n-1} \cap (u_0, u_1)^{n-1}} 1\!\!1_{\Delta_n \cap (u_0, u_1)^n}(G^{-1}(v_1, \ldots, v_{n-1}, v))\, dv_1, \ldots, dv_{n-1}$$

$$\cdot |f'(v)|^{-1} \cdot 1\!\!1_{\mathcal{J}}(v) \quad , \quad v \in R,$$

which is strictly positive on $\mathcal{J}$. Thus the lemma is proved. □

Corollary 3.1: *Given $D_n \cap C_{\varepsilon,n}$ the integral I_t has a density being strictly positive at least on the non-void interval $(nv_0(z+\varepsilon), nv_1(z-\varepsilon))$.*

Proof. We use the Lemma 3.1 to express the integrand in formula (8)

$$P\Big(\sum_{k=1}^{n} z_k f(\tau_k) \in B \,|\, D_n\Big) = \int\limits_B \psi_{S_n}(v; z_1, \ldots, z_n) dv \tag{9}$$

Inserting (9) into (8) and changing the order of integration we get

$$P(I_t \in B \,|\, D_n \cap C_{\varepsilon,n}) \cdot P(C_{\varepsilon,n}) =$$
$$\int\limits_B \Big(\int\limits_{(z-\varepsilon, z+\varepsilon)^n} \psi_{S_n}(v; z_1, \ldots, z_n) F(dz_1) \ldots F(dz_n) \Big) dv$$

Thus given $D_n \cap C_{\varepsilon,n}$ the integral I_t has a density which is strictly positive on $(nv_0(z+\varepsilon), nv_1(z-\varepsilon))$.

That this interval is non void follows from assumption (5). Indeed, (5) implies $z+\varepsilon < c(z-\varepsilon)$ as well as $z > \varepsilon$ and thus we have

$$\frac{v_1(z-\varepsilon)}{v_0(z+\varepsilon)} = c^2\frac{(z-\varepsilon)^2}{(z+\varepsilon)^2}\cdot\frac{z+\varepsilon}{z-\varepsilon} > \frac{z+\varepsilon}{z-\varepsilon} > 1. \tag{10}$$

This completes the proof of the corollary. □

We continue the proof of the theorem. By assumption we have

$$H(I_t) = L_t \qquad P-a.s.$$

For any $n \geq 1$ this implies

$$H(I_t) = \sum_{k=1}^{n} Z_k \in (n(z-\varepsilon), n(z+\varepsilon)) \quad P(\cdot|D_n \cap C_{\varepsilon,n}) - a.s. \tag{11}$$

Now choose two positive integers l and m with

$$\frac{z+\varepsilon}{z-\varepsilon} < \frac{l}{m} < c^2\frac{z-\varepsilon}{z+\varepsilon} \tag{12}$$

which is possible because of (10).
Introduce for any $n \geq 1$ the two intervals

$$V_n := (n(z-\varepsilon), n(z+\varepsilon)) \text{ and } W_n := (nv_0(z+\varepsilon), nv_1(z-\varepsilon)).$$

By construction of l and m we get from (12) $l > m$ and

$$\underline{w} := lv_0(z+\varepsilon) < mv_1(z-\varepsilon) =: \overline{w}. \tag{13}$$

Thus it holds

$$W_l \cap W_m = (\underline{w}, \overline{w}) \neq \emptyset. \tag{14}$$

From Corollary 3.1. and (11) it follows for $n = l$

$$H(v) \in V_l \quad \text{Lebesgue - a.e. on} \, W_l$$

and for $n = m$

$$H(v) \in V_m \quad \text{Lebesgue - a.e. on} \, W_m.$$

Thus (14) implies

$$H(v) \in V_l \cap V_m \quad \text{for all } v \text{ from } (\underline{w}, \overline{w}). \tag{15}$$

But from the first inequality of (12) we have $l > m$ and $l(z-\varepsilon) > m(z+\varepsilon)$, this means $V_l \cap V_m = \emptyset$. This is a contradiction to (15). Consequently, the assumption, that f is not constant cannot be valid.

The proof of the theorem is finished for L being a compound Poisson process or a Wiener process, both non degenerated.

In the third part of the proof we assume that L is a general non-deterministic Lévy process. We choose an $\varepsilon > 0$ such that $L^{>\varepsilon}$ is non-trivial. (If no such ε exists then L necessarily is a Wiener process, and for this case the proof was given at the beginning.)

By assumption we have

$$H(I_t^{>\varepsilon} + I_t^{\leq\varepsilon}) = L_t^{>\varepsilon} + L_t^{\leq\varepsilon} \quad \text{P-a.s.} \tag{16}$$

with

$$I_t^{>\varepsilon} := \int\limits_{[0,t]} f(s) dL_s^{>\varepsilon}, \quad I_t^{\leq\varepsilon} := \int\limits_{[0,t]} f(s) dL_s^{\leq\varepsilon}.$$

We know that $(I_t^{>\varepsilon}, L_t^{>\varepsilon})$ is independent of $(I_t^{\leq\varepsilon}, L_t^{\leq\varepsilon})$.

Therefore from (16) we get

$$H(I_t^{>\varepsilon} + y) = L_t^{>\varepsilon} + z \qquad \text{P-a.s.} \tag{17}$$

for $P^{(I_t^{\leq\varepsilon}, L_t^{\leq\varepsilon})}$ - almost all (y, z). Thus there exists at least one pair (y, z) such that (17) holds. As a consequence from the second part of the proof we obtain

$$\tilde{H}(I_t^{>\varepsilon}) = L_t^{>\varepsilon} \qquad \text{P-a.s.} \tag{18}$$

for the Borel function $\tilde{H}(x) := H(x+y) - z$. Because $L_t^{>\varepsilon}$ is a nontrivial compound Poisson process, the function f has to be constant. Now the proof of the theorem is complete. □

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