Bethmann, Dirk

**Working Paper**

# The open-loop solution of the Uzawa-Lucas Model of Endogenous Growth with N agents

Papers, No. 2004,42

**Provided in Cooperation with:**
CASE - Center for Applied Statistics and Economics, Humboldt University Berlin

*Suggested Citation:* Bethmann, Dirk (2004) : The open-loop solution of the Uzawa-Lucas Model of Endogenous Growth with N agents, Papers, No. 2004,42, Humboldt-Universität zu Berlin, Center for Applied Statistics and Economics (CASE), Berlin

This Version is available at:
https://hdl.handle.net/10419/22215

# The open-loop solution of the Uzawa-Lucas Model of Endogenous Growth with $N$ agents

Dirk Bethmann*
Department of Economics
Humboldt University Berlin
Spandauer Straße 1
D-10178 Berlin
bethmann@wiwi.hu-berlin.de

October 25, 2004

**Abstract**

We solve an $N \in \mathbb{N}$ player general-sum differential game. The optimization problem considered here is based on the Uzawa Lucas model of endogenous growth. Agents have logarithmic preferences and own two capital stocks. Since the number of players is an arbitrary fixed number $N \in \mathbb{N}$, the model's solution is more general than the idealized concepts of the social planer's solution with one player or the competitive equilibrium with infinitely many players. We show that the symmetric Nash equilibrium is completely described by the solution to a single ordinary differential equation. The numerical results imply that the influence of the externality along the balanced growth path decreases rapidly as the number of players increases. Off the steady state, the externality is of great importance, even for a large number of players.

***Key words:*** Value Function Approach, Nash-Equilibrium, Open-loop Strategies, Ordinary Differential Equation.

***JEL Classifications:*** C72, C61, O41.

*The financial support of the German Science Foundation National Research Center 373 is gratefully acknowledged. I would like to thank Michael C. Burda, Harald Uhlig, and seminar participants at Humboldt University for helpful comments. Thanks also go to Henry Y. Wan Jr., Elias Dinopoulos, and participants of the International Conference on Dynamics, Economic Growth and International Trade (DEGIT IX) in Reykjavik. All remaining errors are my own.

# 1 Introduction

This paper studies an $N$ player differential game that is based on the Uzawa (1965) and Lucas (1988) model of endogenous growth. Each player owns a goods-producing firm and has access to educational services. The game's time horizon is infinite and the number of players is fixed. The goods sector technology is Cobb-Douglas in human and physical capital while the schooling technology is linear in human capital only. At the outset of the game, the players' initial endowments are identical. Their task is to simultaneously choose the optimal consumption as well as the optimal allocation of human capital between the two sectors. Since the goods sector productivity is influenced by the economy-wide average level of human capital, the player has to know the decisions of his co-players in order to optimally determine his own controls. Furthermore, the game is symmetric in the players' constraints and their objective functions. This allows us to limit the set of potential solutions to symmetric Nash equilibria.

By generalizing the solution method introduced in Bethmann and Reiß (2004), we derive the game's open-loop solution. The main idea is to exploit the model's inherent homogeneity (cf. Caballé and Santos, 1993) in order to reduce the complexity of the optimization problem. This is achieved by using the geometric mean when defining the economy-wide average level of human capital. This way we are able to subsume the information of the relevant state variables in a weighted product, with the model's inherent homogeneity determining the respective weights. Following Mulligan and Sala-i-Martin (1991), the result of this transformation is referred to as the state-like variable. The multiplicative structure of this variable together with the logarithmic utility function allows us to rewrite the value function of one player as the sum of the 'value-function-like function' and expressions representing the influence of the $N-1$ human capital stocks of the other players on his life-time utility. The introduction of the value-function-like function then allows us to reduce the complexity of the player's Hamilton-Jacobi-Bellman (HJB) equation to an implicit partial differential equation. Along the balanced growth path, we are able to state an explicit solution to this equation. However, it is the generalization of the unstable solutions to the resulting HJB equations in the social planer's and the infinitely many agents case (cf. Bethmann and Reiß, 2004). Nevertheless, because we know this function we are able to determine the model's steady state. Using a standard transformation (cf. Bronstein and Semendjajew, 1999), we get an equivalent explicit partial differential equation, which can be further reduced to an ordinary differential equation by inserting the condition for a symmetric Nash equilibrium. Finally, we use the steady state in order to determine an initial value problem for this ordinary differential equation.

The numerical results show that the number of players indeed influences the optimal human capital allocation of the agents. However, the influence of $N$ decreases very rapidly. Moreover, the strength of this phenomenon strongly depends on the degree of the external effect of human capital on the goods sector productivity. In fact, we show that both parameters work in opposite directions. An increase in the degree of the external effect is similar to the effect

of a lower number of players on the outcome of the differential game. Although the conjecture that both effects are reciprocal seems obvious, we show that this only holds for the steady state human capital allocation. Off the balanced growth path, the influence of the number of players diminishes rapidly while the degree of the external effect is still driving the agents' decisions via its influence on the shadow prices of the capital stocks.

The paper is organized as follows. Section 2 introduces the model. Section 3 states the differential game implied by the players' coupled optimization problems. Section 4 presents the mathematical solution strategy. Section 5 examines the numerical results showing the above mentioned antagonistic effects of the two parameters. Section 6 concludes. The Appendix contains statements that are omitted in the main paper for clarity of exposition.

# 2 The model

We assume a closed economy populated by $N \in \mathbb{N}$ identical and infinitely-lived self-sustaining players. They are producing a single good and have access to a schooling sector providing educational services. The player[1] $A^{(i)}$, $i \in \{1, .., N\}$, has logarithmic preferences over consumption streams

$$U^{(i)} = \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^{(i)} dt, \tag{1}$$

where $c_t^{(i)}$ is the player's level of consumption at time $t$. We denote player-specific variables and functions by upper indices. The parameter $\rho > 0$ is the subjective discount rate. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. Agents have a fixed endowment of time, which is normalized as a constant flow of one unit. The variable $u_t^{(i)}$ denotes the fraction of time allocated to goods production at time $t$. The fraction of time $1 - u_t^{(i)}$ is spent in the schooling sector. Because agents do not benefit from leisure, the whole time budget is allocated to the two sectors. Hence, in any solution the condition

$$u_t^{(i)} \in [0, 1] \tag{2}$$

has to be fulfilled. The variables $c_t^{(i)}$ and $u_t^{(i)}$ are the agent's control variables. Human capital production is determined by a linear technology in human capital

$$\dot{h}_t^{(i)} = B(1 - u_t^{(i)}) h_t^{(i)}, \tag{3}$$

where we assume that $B$ is positive. This technology combined with constraint (2) implies that human capital will never decrease, i.e. the growth rate $\dot{h}^{(i)}$ is non-negative. The schooling technology implies that the realized marginal and average product is equal to $B(1-u_t^{(i)})$. Note that we abstract from depreciation.

[1] Henceforth, we use player and agent interchangeably.

The agent's consumption good is produced by using a Cobb-Douglas technology in the two inputs physical and human capital. If not consumed, the good is invested, thereby increasing the physical capital stock. The level of human capital utilized in goods production equals the total level of the stock of human capital multiplied by the fraction of time spent in the goods sector at time $t$. Total factor productivity $A$ is enhanced by the external effect $\gamma$ of the economy's average stock of human capital, $h_{a,t}$. Hence, the player's output $y_t^{(i)}$ is determined by

$$y_t^{(i)} = A(k_t^{(i)})^{\alpha}(u_t^{(i)}h_t^{(i)})^{1-\alpha}h_{a,t}^{\gamma}.$$

The parameter $\alpha$ is the output elasticity of physical capital, and we assume $\alpha \in (0,1)$. The exponent $\gamma$ is assumed to be non-negative. We assume that the economy-wide average level of human capital is defined by the geometric mean of the individual human capital stocks

$$h_a = \Big(\prod_{n=1}^{N} h^{(n)}\Big)^{\frac{1}{N}}. \tag{4}$$

Since all agents are homogeneous, the economy's average level of human capital must be equal to the individual human capital levels at any point in time. Hence,

$$h_t^{(i)} = h_{a,t}, \quad \forall t \geq 0, \quad \forall i \in \{1,..,N\}, \tag{5}$$

holds. The player can either consume or invest his output $y_t^{(i)}$. Since we abstract from depreciation, his budget constraint reads as follows

$$y_t^{(i)} = c_t^{(i)} + \dot{k}_t^{(i)}, \qquad \forall t \geq 0.$$

The right-hand side describes the spending of the player's earnings, where $\dot{k}_t^{(i)}$ is the rate of change of his physical capital stock $k_t^{(i)}$, i.e. his net investment in physical capital. The left-hand side represents the income streams from the agent's physical capital stock and from his work effort $u_t^{(i)}h_t^{(i)}$. We assume that the initial values $k_0$ and $h_0$ are strictly positive. By consuming more than current production, it is possible to disinvest in physical capital, i.e. the growth rate of physical capital becomes negative.

**Informational assumptions**

Clemhout and Wan (1994, p.812) argue that 'which equilibrium concept to use should be decided on the relative realism, and nothing else'. In order to quantify the influence of the external effect and of the number of players on the game's outcome, we compute the model's open-loop equilibrium strategies. The open-loop equilibrium concept is characterized by the fact that agents commit themselves at the outset of the game to entire temporal paths of human capital allocation and consumption that maximize their discounted utility given the decision paths of all other agents. Although this is a simplification we think that it is justified by the long-term character of schooling decisions. The next section discusses the optimization problem at hand as a differential game.

# 3 The N-player game

We consider a non-cooperative differential game with $N \in \mathbb{N}$ players. The game extends over the unbounded time interval $[0, \infty)$. The state of the game at each instant $t \in [0, \infty)$ is described by a vector $s_t := \left\{k_t^{(1)}, h_t^{(1)}; ..; k_t^{(N)}, h_t^{(N)}\right\} \in \mathbb{R}_{++}^{2N}$, where $\mathbb{R}_{++}^{2N}$ is the state space of the game. The entries $k^{(i)}$ and $h^{(i)}$ denote the physical and human capital stocks of agent $A^{(i)}$, $i \in \{1, .., N\}$. Let us define the set $\bar{S}$ as the subset of $\mathbb{R}_{++}^{2N}$ with elements of the form $\{z, y; z, y; ..; z, y\}$, where $y, z \in \mathbb{R}_{++}$. Therefore, the symmetric initial state of the game is a fixed vector $\bar{s}_0 = \left\{k_0, h_0; ..; k_0, h_0\right\} \in \bar{S} \subset \mathbb{R}_{++}^{2N}$. At each point in time $t \in [0, \infty)$, each player $A^{(i)}$ chooses the control variables $c^{(i)}$ and $u^{(i)}$ from his set of feasible controls $\chi$, with

$$\chi = \left\{(f, g) : \mathbb{R}_{++}^{2N} \mapsto [0, \infty) \times [0, 1] \,|\text{locally bounded and measurable}\right\}.$$

The state of the game evolves according to the differential equations

$$\begin{aligned} \dot{k}_t^{(i)} &= A\big(k_t^{(i)}\big)^{\alpha}\big(u_t^{(i)}\big)^{1-\alpha}\big(h_t^{(i)}\big)^{1-\alpha}\big(h_{a,t}\big)^{\gamma} - c_t^{(i)}, \\ \dot{h}_t^{(i)} &= B\big(1 - u_t^{(i)}\big)h_t^{(i)}. \end{aligned}$$

These equations hold for all $t \in [0, \infty)$ and all $i \in \{1, .., N\}$. Note that the assumption (4) allows us to rewrite the term $h_a^{\gamma}$, which influences total factor productivity in the goods sector, as follows

$$h_a^{\gamma} = \left(h^{(i)}\right)^{\frac{\gamma}{N}} \left(\prod_{j \neq i} h^{(j)}\right)^{\frac{\gamma}{N}},$$

where $j \neq i$ is a shortcut and stands for $j \in \{1, .., N\} \setminus \{i\}$. We assume that agents make their choices simultaneously and solve the following dynamic optimization problems (DOPs). Agent $A^{(i)}$ seeks to maximize lifetime utility

$$U^{(i)} = \max_{c_t^{(i)}, u_t^{(i)}} \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^{(i)} dt,$$

subject to the state dynamics

$$\begin{aligned} &\dot{k}_t^{(i)} = A(k_t^{(i)})^{\alpha}\big(u_t^{(i)}\big)^{1-\alpha}(h_t^{(i)})^{1-\alpha+\frac{\gamma}{N}}\left(\prod_{j \neq 1} h_t^{(j)}\right)^{\frac{\gamma}{N}} - c_t^{(i)}, && \forall\, t \geq 0, \\ &\dot{h}_t^{(i)} = B\big(1 - u_t^{(i)}\big)h_t^{(i)}, && \forall\, t \geq 0, \\ &k_t^{(i)} \geq 0 \quad \text{and} \quad h_t^{(i)} \geq 0, && \forall\, t \geq 0. \end{aligned}$$

Since all these optimization problems are coupled via the values of the individual human capital stocks $h^{(j)}$ for positive $\gamma$, we interpret this optimization problem as a multiple players' non-cooperative game $\Gamma(t, s_t)$. Note that the above DOPs are homogeneous in the initial conditions (cf. Bethmann and Reiß, 2004) - a

fact that we will exploit extensively in the next section. The symmetry in the agents' initial endowments combined with the symmetry in preferences and technology encourages us to look for symmetric Nash equilibria. Applying the definition (e.g. Dockner et al. (2000), Chapter 4) to our game $\Gamma(0, \bar{s}_0)$, the policy functions $u^{(i*)}$ and $c^{(i*)}$ form a Nash equilibrium if

$$U^{(i)}\left((u^{(i*)}, c^{(i*)}); (u^{(j*)}, c^{(j*)}), j \neq i\right) \geq U^{(i)}\left((u^{(i)}, c^{(i)}); (u^{(j*)}, c^{(j*)}), j \neq i\right)$$

holds for all feasible controls $u^{(i)}$ and $c^{(i)}$ and for all agents $A^{(i)}$. In the following section, we assume that agents commit themselves to entire courses of action at the game's outset and are not allowed to revise them at any subsequent date.

# 4 The open-loop solution

In this section, we solve the non-cooperative game under the assumption that the agents are not allowed to revise their action paths once they have made their choices, i.e. we consider that agents play open-loop strategies. In Section 4.1 we exploit the game's homogeneity in the initial conditions in order to reduce the Hamilton-Jacobi-Bellman equation to an implicit partial differential equation. Along the balanced growth path, this equation is an implicit ordinary differential equation and we are able to give an explicit solution which is indeed the agent's value function. However, outside this path, the application of this function leads to non-feasible controls. In Section 4.2, we start with the implicit partial differential equation describing the solution to our problem. We define the symmetric Nash equilibrium and reduce the problem to an initial value problem for a single explicit ordinary differential equation.

## 4.1 The Hamilton Jacobi Bellman equation

Agent $A^{(i)}$ defines the value function as the solution to his dynamic optimization problem given the symmetric initial state $\bar{s}_0$ and given the optimal decisions of his co-players $A^{(j)}$, with $j \neq i$:

$$V^{(i)}(\bar{s}_0) := \max_{(c^{(i)}, u^{(i)}) \in \mathcal{X}} \begin{cases} \int_0^\infty e^{-\rho t} \ln c_t^{(i)} \, dt, & \tau = \infty \\ -\infty, & \tau < \infty, \end{cases}$$

where $\tau$ denotes the stopping time $\tau := \inf\{t \geq 0 \,|\, k_t = 0\}$. In order to determine the value function, we write down the Hamilton-Jacobi-Bellman (HJB) equation for the value function $V^{(i)} := V^{(i)}(s_t)$:

$$\rho V^{(i)} = \max_{(c^{(i)}, u^{(i)}) \in X} \left\{ \ln c^{(i)} + V_k^{(i)} \dot{k}_t^{(i)} + V_h^{(i)} \dot{h}_t^{(i)} + \sum_{j \neq i} V_{h^{(j)}}^{(i)} \dot{h}_t^{(j)} + V_t^{(i)} \right\} \quad (6)$$

Here, $V_k^{(i)}$, $V_h^{(i)}$, and, $V_{h^{(j)}}^{(i)}$ denote the partial derivatives with respect to the agent's own capital stocks $k^{(i)}$ and $h^{(i)}$ and to the other agents' human capital

stocks $h^{(j)}$. These derivatives can be interpreted as the shadow prices of relaxing the corresponding constraints. $V_t^{(i)}$ denotes the partial derivative with respect to time $t$. The first order necessary conditions are:

$$c^{(i*)} = \tfrac{1}{V_k^{(i)}}, \tag{7}$$

$$u^{(i*)} = \left(\frac{A(1-\alpha)V_k^{(i)}}{BV_h^{(i)}}\right)^{\frac{1}{\alpha}} \frac{k^{(i)}\big(\prod_{j\neq 1} h^{(j)}\big)^{\frac{\gamma}{\alpha N}}}{\big(h^{(i)}\big)^{1-\frac{\gamma}{\alpha N}}}. \tag{8}$$

The agent chooses the consumption stream such that the marginal utility is equal to the marginal change of wealth with respect to physical capital. The optimal allocation of human capital between the two sectors is achieved if the weighted marginal changes in the two sectors due to marginal shifting are equal. The weights are the shadow prices of the state variable, which is generated in the respective sector. Since it is obvious, that the value function $V^{(i)}$ is increasing in its arguments, expression (7) ensures that the consumption rate is positive. Equally, $u^{(i*)} \in (0,\infty)$ holds, but $u^{(i*)} > 1$ may well occur. For the moment, let us suppose that the controls $(c^{(i*)}, u^{(i*)})$ found above are feasible. We continue by inserting our findings into the HJB equation (6). We obtain:

$$\begin{aligned}&\rho V^{(i)} + 1 + \ln V_k^{(i)} - V_t^{(i)}\\ &= \alpha\big(AV_k^{(i)}\big)^{\frac{1}{\alpha}}\big(\tfrac{1-\alpha}{BV_h^{(i)}}\big)^{\frac{1-\alpha}{\alpha}} k^{(i)}\Big(\prod_{n=1}^{N} h^{(j)}\Big)^{\frac{\gamma}{\alpha N}} + BV_h^{(i)}h^{(i)} + \sum_{j\neq i} BV_{h^{(j)}}^{(i)}(1-u_t^{(j)})h^{(j)}.\end{aligned}$$

The homogeneity in the initial conditions allows us to generalize Mulligan and Sala-i-Martin's (1991) idea to introduce the agent's state-like variable $x_t^{(i)}$ and his control-like variable $q_t^{(i)}$. Here, we define them as follows:

$$x_t^{(i)} = \frac{k_t^{(i)}}{h_t^{(i)}\big(\prod_{n=1}^{N} h_t^{(n)}\big)^{\frac{\gamma/N}{1-\alpha}}} \quad \text{and} \quad q_t^{(i)} = \frac{c_t^{(i)}}{h_t^{(i)}\big(\prod_{n=1}^{N} h_t^{(n)}\big)^{\frac{\gamma/N}{1-\alpha}}}. \tag{9}$$

The evolution of the state-like variable $x^{(i)}$ can be determined by taking the derivative with respect to time and inserting the state dynamics for $k^{(i)}$ and $h^{(n)}$ respectively, $n \in \{1,..,N\}$:

$$\dot{x}_t^{(i)} = A\big(x_t^{(i)}\big)^{\alpha}\big(u_t^{(i)}\big)^{1-\alpha} - q_t^{(i)} - \frac{(1-\alpha+\gamma-\frac{\gamma}{N}\sum_{j\neq i} u_t^{(j)})Bx_t^{(i)}}{1-\alpha} + \frac{(1-\alpha+\frac{\gamma}{N})Bu_t^{(i)}x_t^{(i)}}{1-\alpha}. \tag{10}$$

We see that the evolution of $x_t^{(i)}$ is completely described by three groups of variables. First, by the state-like variable $x^{(i)}$ itself. Second, by $A^{(i)}$'s controls $u^{(i)}$ and $q^{(i)}$. Third, by all other agent's human capital allocation decision paths $u_t^{(j)}$ with $j \neq i$. The homogeneity in the initial conditions implies that we are led to apply the same controls $\tilde{u}_t^{(i)} = u_t^{(i)}$ and $\tilde{q}_t^{(i)} = q_t^{(i)}$ for any symmetric initial state $\tilde{s}_0$ with $\tilde{x}_0^{(i)} = x_0^{(i)}$. The only difference is that the consumption rate $\tilde{c}_t^{(i)}$ differs from $c_t^{(i)}$ by the factor

$$\frac{\tilde{h}_0^{(i)}}{h_0^{(i)}}\left(\prod_{n=1}^{N}\frac{\tilde{h}_0^{(n)}}{h_0^{(n)}}\right)^{\frac{\gamma/N}{1-\alpha}} = \left(\frac{\tilde{h}_0^{(i)}}{h_0^{(i)}}\right)^{\frac{1-\alpha+\gamma}{1-\alpha}}.$$

Following Bethmann and Reiß (2004), we derive the symmetric solution from the value function-like function $f^{(i)}$ with $V^{(i)}(\bar{s}_t) = f^{(i)}(x^{(i)}, t; u_t^{(j)}, j \neq i)$ via

$$V^{(i)}(\bar{s}_t, t; u_t^{(j)}, j \neq i) = f^{(i)}(x^{(i)}, t; u_t^{(j)}, j \neq i) + \frac{(1-\alpha+\frac{\gamma}{N})\ln h^{(i)}}{\rho(1-\alpha)} + \frac{\frac{\gamma}{N}\sum_{j\neq i}\ln h^{(j)}}{\rho(1-\alpha)},$$

where the semicolon indicates that we restrict the function to given time-dependent paths of the other agents' allocations of human capital. The derivatives of $V^{(i)}$ with respect to the agent's capital stocks $k^{(i)}$ and $h^{(i)}$, to the other agents' human capital stocks $h^{(j)}$ where $j \neq i$, and to time $t$ can be expressed in terms of the function $f^{(i)}$. Hence, we consider:

$$\begin{aligned}
V_{k^{(i)}}^{(i)} &= \tfrac{f_x^{(i)} x^{(i)}}{k^{(i)}}, \\
V_{h^{(i)}}^{(i)} &= \tfrac{1-\alpha+\gamma/N}{1-\alpha}\left(\tfrac{1}{\rho h^{(i)}} - \tfrac{f_x^{(i)} x^{(i)}}{h^{(i)}}\right), \\
V_{h^{(j)}}^{(i)} &= \tfrac{\gamma/N}{1-\alpha}\left(\tfrac{1}{\rho h^{(j)}} - \tfrac{f_x^{(i)} x^{(i)}}{h^{(j)}}\right), \\
V_t^{(i)} &= f_t^{(i)},
\end{aligned}$$

where $f_x^{(i)} := \frac{\partial f^{(i)}}{\partial x^{(i)}}$ and $f_t^{(i)} := \frac{\partial f^{(i)}}{\partial t}$. Thus, the function $f^{(i)}$ determines decisively the shadow prices of the two private production factors and of the other agents' stocks of human capital. The introduction of the state-like variable $x^{(i)}$, of the control-like variable $q^{(i)}$, and of the value function-like function $f^{(i)}$ allows us to rewrite the first order necessary conditions (7) and (8):

$$q^{(i*)}(x^{(i)}) = \left(\tfrac{\partial f^{(i)}}{\partial x^{(i)}}\right)^{-1} \tag{11}$$

$$u^{(i*)}(x^{(i)}) = \left(\frac{(1-\alpha)^2 A}{B\left(1-\alpha+\frac{\gamma}{N}\right)\left(\frac{1}{\rho f_x^{(i)}} - x^{(i)}\right)}\right)^{\frac{1}{\alpha}} x^{(i)}. \tag{12}$$

Let $u_{a,t}$ denote the arithmetic mean of all other agents' decisions of allocating their stocks of human capital ($A^{(j)}$, $j \neq i$), i.e. we define the co-players' optimal average human capital allocation decision at date $t$ as follows:

$$u_{a,t}^* := \tfrac{1}{N-1}\sum_{j\neq i} u_t^{(j*)}.$$

Using this definition combined with the first order necessary conditions and the above shadow prices, we rewrite the HJB equation as follows:

$$f^{(i)} - \tfrac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N})}{\rho^2(1-\alpha)} + \tfrac{1-f_t^{(i)}+\ln f_x^{(i)}}{\rho} = \tfrac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N})x^{(i)} f_x^{(i)}}{\rho(1-\alpha)}\left[\left(\tfrac{f_x^{(i)}\varphi_t}{\frac{1}{\rho}-x^{(i)} f_x^{(i)}}\right)^{\frac{1-\alpha}{\alpha}} - 1\right]$$

with

$$\varphi_t := \tfrac{1-\alpha}{1-\alpha+\frac{\gamma}{N}}\left(\frac{\alpha^\alpha(1-\alpha)A}{B\left(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N}\right)^\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Note that we have reduced the HJB equation to an implicit partial differential equation in the variables $x^{(i)}$ and $t$. The parameter $\varphi_t$ is strictly positive. In a

first step, we only look for the balanced growth path solution, where the state-like variable $x^{(i)}$ and the controls $q^{(i)}$ and $u^{(i)}$ remain constant over time. In this case, we set the exogenous value $u^*_{a,t}$ equal to a constant $u^*_a \in [0,1]$ such that the coefficients of the reduced HJB-equation are time-independent. We then look for time autonomous solutions $f^{(i)}$. In this case $f^{(i)}_t = 0$ holds and we obtain:

$$f^{(i)} - \tfrac{B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})}{\rho^2(1-\alpha)} + \tfrac{1+\ln f^{(i)}_x}{\rho} = \tfrac{B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})x^{(i)} f^{(i)}_x}{\rho(1-\alpha)} \left[ \left( \tfrac{f^{(i)}_x \varphi}{\frac{1}{\rho} - x^{(i)} f^{(i)}_x} \right)^{\frac{1-\alpha}{\alpha}} - 1 \right],$$

where the parameter $\varphi$ is given by:

$$\varphi := \tfrac{1-\alpha}{1-\alpha+\frac{\gamma}{N}} \left( \tfrac{\alpha^\alpha (1-\alpha) A}{B\left(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N}\right)^\alpha} \right)^{\frac{1}{1-\alpha}}.$$

Now we consider an implicit ordinary differential equation, where $\varphi$ is strictly positive. We claim that a solution to this equation is given by:

$$g^{(i)}(x^{(i)}) = \tfrac{B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})}{\rho^2(1-\alpha)} + \tfrac{\ln \rho - 1}{\rho} + \tfrac{1}{\rho} \ln(x^{(i)} + \varphi). \tag{13}$$

Indeed, we have $g^{(i)}_x = 1/(\rho x^{(i)} + \rho\varphi)$ and $1/\rho - x^{(i)} g^{(i)}_x = \varphi g^{(i)}_x$. Hence:

$$g^{(i)} - \tfrac{B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})}{\rho^2(1-\alpha)} + \tfrac{1+\ln g^{(i)}_x}{\rho} = 0 \qquad \text{and} \qquad \left( \tfrac{g^{(i)}_x \varphi}{\frac{1}{\rho} - x^{(i)} g^{(i)}_x} \right)^{\frac{1-\alpha}{\alpha}} = 1.$$

The controls derived from $g^{(i)}$ are given by:

$$u^{(i*)} = \left( \tfrac{(1-\alpha)^2 A}{B(1-\alpha+\frac{\gamma}{N})\varphi} \right)^{\frac{1}{\alpha}} x^{(i)} \quad \text{and} \quad q^{(i*)} = \rho \left( x^{(i)} + \varphi \right).$$

The insertion of these findings into the dynamics equation (10) for the state-like variable $x_t$ gives us the following quadratic equation:

$$\dot{x}^{(i)}_t = \mathrm{a}\big(x^{(i)}_t\big)^2 + \big(\mathrm{a}\varphi - \rho\big) x^{(i)}_t - \varphi\rho \quad \text{with} \quad \mathrm{a} := \tfrac{B(1-\alpha+\gamma-\gamma u_a \frac{N-1}{N})}{\alpha\varphi}. \tag{14}$$

A search for the steady states of $x^{(i)}$ shows that on the positive axis, $\dot{x}^{(i)}_t$ only vanishes for the value:

$$x^{(i)}_{ss} = \frac{\rho}{\mathrm{a}} = \tfrac{\rho(1-\alpha)}{B(1-\alpha+\frac{\gamma}{N})} \left( \tfrac{\alpha(1-\alpha)A}{B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})} \right)^{\frac{1}{1-\alpha}}.$$

Linearizing the right hand side of equation (10) at $x^{(i)} = x^{(i)}_{ss}$ shows that $x^{(i)}_{ss}$ is locally unstable:

$$\dot{x}^{(i)}_t \simeq (\rho + \mathrm{a}\varphi) \left( x^{(i)}_t - x^{(i)}_{ss} \right), \quad \text{with} \quad \rho + \mathrm{a}\varphi > 0.$$

Therefore, we infer that $g^{(i)}$ yields the unstable solution branch in the phase diagram. Unfortunately, an analytic expression for the stable solution branch is unknown. The steady state controls implied by $f^{(i)}$ are as follows:

$$u^{(i*)}_{ss} = \tfrac{\rho(1-\alpha)}{B(1-\alpha+\frac{\gamma}{N})} \quad \text{and} \quad q^{(i*)}_{ss} = \tfrac{\rho\varphi}{B} \tfrac{\alpha\rho + B(1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N})}{1-\alpha+\gamma-\gamma u_a^* \frac{N-1}{N}}. \tag{15}$$

We stress that $g^{(i)}$ determines $u^{(i*)}_{ss}$ independently of $u^*_a$, so that the steady state is unique. As a result, $u^*_a = u^{(i*)}_{ss}$ holds along the balanced growth path.

## 4.2 The symmetric Nash equilibrium

By symmetry, $u^{(i*)} = u^*$ and $c^{(i*)} = c^*$ do not depend on the agent $A^{(i)}$. In particular the average human capital allocation rule satisfies $u_a = u^{(i*)}$. Hence, the agent's lifetime utility $U^{(i)}$ only depends on his own controls, $u^{(i)}$ and $c^{(i)}$, and on the average decision rule $u_a^*$ concerning the optimal allocation of human capital $h_a$. Thus, $u^*$ and $c^*$ satisfy the Nash condition if

$$U\Big((u_t^{(*)})_{t\geq 0}, (c_t^{(*)})_{t\geq 0}, (u_{a,t}^*)_{t\geq 0}\Big) \geq U\Big((u_t)_{t\geq 0}, (c_t)_{t\geq 0}, (u_{a,t}^*)_{t\geq 0}\Big)$$

holds for all feasible controls $(u_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$. Note that from now on we drop the superscript $(i)$ in the notation. Furthermore, we restrict our attention of the HJB-equation to the homogeneous form $G^{(t)}(x, f_x(x,t), f_t(x,t))$ with:

$$G^{(t)}(x,p,d) := \tfrac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N})}{\rho^2(1-\alpha)} + \tfrac{d-\ln p}{\rho} + \tfrac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N})xp}{\rho(1-\alpha)}\left[\tfrac{\varphi_t^{\frac{1-\alpha}{\alpha}}}{(\frac{1}{\rho p}-x)^{\frac{1-\alpha}{\alpha}}} - 1\right],$$

where we have defined

$$\varphi_t := \tfrac{1-\alpha}{1-\alpha+\gamma/N}\left(\tfrac{\alpha^\alpha(1-\alpha)A}{B\left(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N}\right)^\alpha}\right)^{\frac{1}{1-\alpha}}.$$

The function $G^{(t)}$ equals up to an additive constant and to the factor $\rho$ the Hamiltonian of the transformed DOP. Consequently, $p(x,t) := f_x(x,t)$ solves the partial differential equation:

$$p = G_x^{(t)} + G_p^{(t)} p_x + G_d^{(t)} p_t,$$

where the respective derivatives of the homogeneous form $G^{(t)}$ are given by

$$\begin{aligned}
G_x^{(t)}(x,p,d) &= \tfrac{B\left(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N}\right)p}{\rho(1-\alpha)}\left[\tfrac{\varphi_t^{\frac{1-\alpha}{\alpha}}(\frac{1}{\rho p}+\frac{1-2\alpha}{\alpha}x)}{(\frac{1}{\rho p}-x)^{\frac{1}{\alpha}}} - 1\right],\\
G_p^{(t)}(x,p,d) &= \tfrac{B\left(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N-1}{N}\right)x}{\rho(1-\alpha)}\left[\tfrac{\varphi_t^{\frac{1-\alpha}{\alpha}}(\frac{1}{\alpha\rho p}-x)}{(\frac{1}{\rho p}-x)^{\frac{1}{\alpha}}} - 1\right] - \tfrac{1}{\rho p},\\
G_d^{(t)}(x,p,d) &= \tfrac{1}{\rho}.
\end{aligned}$$

Note that the dynamics equation (10) for the state-like variable $x_t$ combined with the restated first order necessary conditions in (11) and (12) imply that $\dot{x}_t = \rho G_p^{(t)}$ holds along the optimal control path. From economic theory, we immediately infer that $x_t$ converges monotonically to its steady state value $x^{ss}$ such that $\dot{x}_t = \rho G_p^{(t)} \neq 0$ holds off the balanced growth path. Denoting the inverse function of $t \mapsto x_t$ by $x \mapsto t(x)$ we put $\tilde{p}(x) = p(x, t(x))$ for $x > 0$ and $x \neq x^{ss}$. From $G_d^{(t)} = G_p^{(t)} \dot{x}_t^{-1}$ we thus infer

$$\tilde{p}(x) = G_x^{(t)} + G_p^{(t)}\left(\tilde{p}'(x) - \left(t'(x) - \dot{x}_t^{-1}\right)p_t\right) = G_x^{(t)} + G_p^{(t)}\tilde{p}'(x),$$

where $G^{(t)}$ means obviously $G^{(t(x))}$ along the solution path. We know from the Nash condition that $u_{a,t} = u^*(x_t)$ holds, which we can now insert to obtain the ordinary differential equation $\tilde{p} = G_x + G_p\tilde{p}'$ with

$$G_x(x,p) = \frac{B(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u(p,x))p}{\rho(1-\alpha)}\left[\big(\varphi(p,x)\big)^{\frac{1\text{-}\alpha}{\alpha}}\big(\tfrac{1}{\rho p}-x\big)^{\frac{\text{-}1}{\alpha}}\big(\tfrac{1}{\rho p}+\tfrac{1-2\alpha}{\alpha}x\big)-1\right],$$
$$G_p(x,p) = \frac{B(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u(p,x))x}{\rho(1-\alpha)}\left[\big(\varphi(p,x)\big)^{\frac{1\text{-}\alpha}{\alpha}}\big(\tfrac{1}{\rho p}-x\big)^{\frac{\text{-}1}{\alpha}}\big(\tfrac{1}{\alpha\rho p}-x\big)-1\right]-\tfrac{1}{\rho p}.$$

Here, the notation $\varphi(p,x)$ reflects that $\varphi_t$ depends on $u(p,x)$, which is itself derived from formula (12), where

$$u(p,x) = \left(\frac{(1-\alpha)^2 A}{B(1-\alpha+\frac{\gamma}{N})\big(\frac{1}{\rho p}-x\big)}\right)^{\frac{1}{\alpha}} x. \tag{16}$$

Hence, the co-players' optimal average decision $u^*_{a,t}$ is replaced by (16), which stresses the fact that the agents commit themselves to time dependent control paths $(u_t)_{t\geq 0}$ at the outset of the game. The differential equation is now explicit with

$$\tilde{p}' = \frac{\tilde{p} - G_x}{G_p}.$$

Because $q = \tilde{p}^{-1}$ and hence $q' = -\tilde{p}^{-2}\tilde{p}'$ holds, we can now rewrite this equation in terms of the optimal control-like variable such that the following explicit ordinary differential equation has to be solved:

$$q' = q\frac{1-\frac{B(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u(q,x))}{(1-\alpha)\rho}\left[\big(\varphi(q,x)\big)^{\frac{1\text{-}\alpha}{\alpha}}\big(\frac{q}{\rho}-x\big)^{\frac{\text{-}1}{\alpha}}\big(\frac{q}{\rho}+\frac{1-2\alpha}{\alpha}x\big)-1\right]}{\frac{q}{\rho}-\frac{B(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u(q,x))x}{(1-\alpha)\rho}\left[\big(\varphi(q,x)\big)^{\frac{1\text{-}\alpha}{\alpha}}\big(\frac{q}{\rho}-x\big)^{\frac{\text{-}1}{\alpha}}\big(\frac{q}{\alpha\rho}-x\big)-1\right]}. \tag{17}$$

This fraction is indeterminate at $x_{ss}$:

$$q'(x) = \frac{K\left(x,q(x)\right)}{L\left(x,q(x)\right)} \quad \text{and} \quad K\left(x_{ss},q(x_{ss})\right) = L\left(x_{ss},q(x_{ss})\right) = 0.$$

In order to obtain determinacy at $x^{ss}$, we use L'Hôpital's rule, which gives

$$q'(x^{ss}) = \frac{K_x(x^{ss},q(x^{ss})) + K_q(x^{ss},q(x^{ss}))q'(x^{ss})}{L_x(x^{ss},q(x^{ss})) + L_q(x^{ss},q(x^{ss}))q'(x^{ss})}.$$

This leads us to a quadratic equation in $q'(x^{ss})$. One solution of this equation, we already know from $g$, namely $q'(x^{ss}) = \rho$. Therefore, there exists exactly one other possible solution of $q'(x^{ss})$ which is given by

$$q'(x^{ss}) = \tfrac{-K_x(x^{ss},q(x^{ss}))}{\rho L_q(x^{ss},q(x^{ss}))}.$$

As a result, the fraction is now determinate with

$$q'(x^{ss}) = \frac{\big(\alpha\rho+B(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u^{ss})\big)\big((1-\alpha+\gamma)u^{ss}+(1-\alpha)(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u^{ss})(1+\frac{1-\alpha+\gamma}{1-\alpha+\gamma/N})\big)}{\big(1-\alpha^2\big)\big(1-\alpha+\gamma-\gamma\frac{N\text{-}1}{N}u^{ss}\big)+\alpha\big(1-\alpha+\gamma\big)u^{ss}}. \tag{18}$$

The Appendix states some intermediate results that we obtained when determining this expression. Note that only a simple initial value problem remains to be solved, which is done in next section.

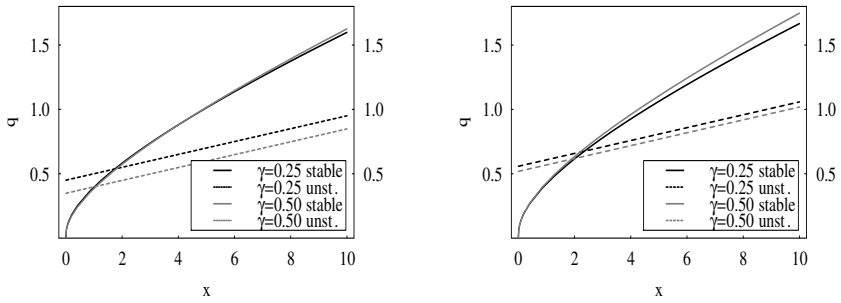


Figure 1: Phase diagrams for $q(x)$ with $\gamma = 0.25$ and $\gamma = 0.5$. Left: 2 players. Right: 1000 players.

# 5 Numerical results

In this section, we apply the findings of the preceding section and determine the model's solutions numerically. We examine the influence of the number of players, $N$, and the influence of the degree of the external effect, $\gamma$, on the outcome of the differential games. Our analysis focuses on the agents' two policy rules, and we consider the following typical calibration of the parameter values

$$A = 1, \quad B = \tfrac{1}{10}, \quad \rho = \tfrac{1}{20}, \quad \text{and} \quad \alpha = \tfrac{1}{3}\ . \tag{19}$$

Figure 1 contains the phase diagrams for the players' control-like variables $q(x)$ where we have set the degrees of the external effect $\gamma$ equal to 0.25 and 0.5, respectively. The left diagram displays the solutions for the two-player economy and the right diagram refers to the economy with 1000 players. The linear lines are derived from the unstable solutions (13) to the HJB-equations, with the parameters $u_a$ set equal to their steady state values (15). The concave lines, which start in the origin are the optimal controls $q$ derived from the true value functions $V$, that is the numerical solutions of the initial value problems given in equations (17) and (18). Both the stable and the unstable solution branches meet in the respective saddle points $(x^{ss}, q^{ss})$. In the $N = 1000$ case, the relatively low appreciation of human capital leads to higher steady-state values of $x$ compared to the corresponding values in the left diagram where $N = 2$. For increasing $\gamma$, however, we observe a shift of the steady states to the left, demonstrating that the $N$-effect and the $\gamma$-effect work in opposite directions.

The left diagram of Figure 2 shows the steady state ratios of consumption to physical capital for various economies[2]. These ratios depend on the external effect: The higher the degree of the external effect, the higher the ratio of consumption to physical capital. This relationship increases oppositely to the size of the economy, because the smaller the economy the more influence a single agent has on the evolution of the average level of human capital. The stronger

[2]These ratios are also represented in the phase diagrams (Figure 1) by the angles between the x-axis and the straight lines between the steady state $(x_{ss}, q_{ss})$ and the origin.

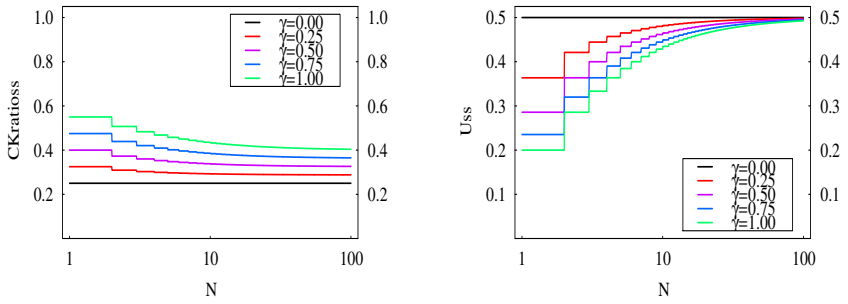


Figure 2: Steady state of $c/k = q/x$ and $u$ with respect to $\gamma$ and $N$. Left: Steady state ratio $\frac{q_{ss}}{x_{ss}}$. Right: Steady state allocation $u_{ss}$.

the influence on $h_a$, the less is the appreciation of physical capital, so that the steady state ratio of consumption to capital is higher. Since all costs and returns are internalized in the $N = 1$ player economies, the steady state consumption of physical capital is higher than in economies with more than one player. The right diagram of Figure 2 refers to the model's second control variable and displays the corresponding steady state allocations of human capital to the goods sector. Here, the internalization of all costs and returns in the $N = 1$ player economy leads to lower steady state values than in economies with more than one player. A similar reasoning holds for arbitrary economies with $N$ and $N'$ players, where $N < N'$. The higher marginal returns of the agents' human capital stocks in the $N$ player cases lead to lower steady state values of $u$ compared to the $N'$ player cases. While in the left diagram, we observe a remaining influence of the external effect $\gamma$ on $c/k$ even for large values of $N$, the right diagram indicates that $\gamma$ plays no role for the steady state human capital allocations for large vales of $N$. If we consider $N = \infty$, the intuition of this becomes clear. Along the balanced growth path, an agent benefits from a high degree of the external effect via a higher growth rate of total factor productivity. Thus he can afford sustainable higher consumption rates than in cases with a lower $\gamma$. For the optimal human capital allocation, however, the non-cooperative character of our games ensure that agents treat the evolution of $h_a$ as exogenously given. Thus, the allocation decisions are completely independent of $\gamma$.

Figure 3 shows the optimal human capital allocation $u$ in the $(x, \gamma)$ space as a surface. As in the first figure, the left diagram represents the two-player economy and the right diagram the economy with $N = 1000$. The black lines correspond to the respective steady-state values $u_{ss}$. Since for $N = 2$, an agent has more influence on the evolution of the average human capital stock, his valuation of human capital is higher than that of an agent in the $N = 1000$ player games. This explains why for a positive $\gamma$ and $N = 2$, the allocation of human capital to the goods sector is always smaller than the corresponding values of $u$ in the right diagram where $N = 1000$ holds. For small values of $x$ and $\gamma$, agents in both economies would like to set $u$ larger than one. There we

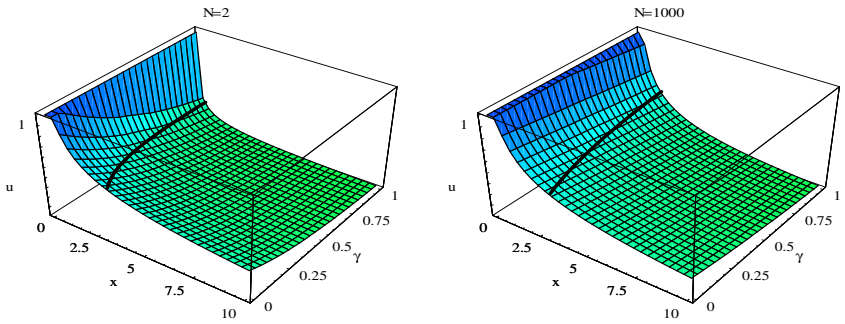


Figure 3: Optimal time share $u$ in $(x, \gamma)$ space.
Left: 2 players. Right: 1000 players.

have to set $u = 1$ and solve for the optimal consumption rates of the remaining one sector growth models. Note that for fixed values of $\gamma$, the fraction of time allocated to goods production decreases when $x$ increases. High values of $x$ indicate that an economy's endowment of human capital is low. This leads to high marginal returns of human capital in goods production and thus to high opportunity costs of schooling. Arbitrage reasoning implies that the marginal productivity of human capital in the schooling sector must also be high[3]. Hence a high fraction of human capital is attracted by the schooling sector and we observe a low optimal value of $u$. Arbitrage can also explain the negative slope of the surface with respect to $\gamma$. An increase in the number of players causes the influence of a single player on the average level of human capital to decline. Thus the optimal human capital allocation is decreasing in $N$.

At first glance, one may conjecture that considering different numbers of players is equivalent to a rescaling of the $\gamma$-axis, so that the surface is stretched like a rubber blanket (see also Figure 6 in the Appendix). In the following two figures, we study the interplay of $\gamma$ and $N$ and show that although they work in opposite directions their influence is not reciprocal. It is shown that the fraction $\gamma/N$ is not sufficient to determine the allocation of human capital off the balanced growth path. Thus, we need to know both parameters explicitly. This is in contrast to the steady state allocation, where $\gamma/N$ is a sufficient statistic.

Keeping the above conjecture in mind, we now consider optimal allocations along the transition paths towards the steady states. The diagrams of Figure 4 refer to the optimal human capital allocation if the state-like variable is fixed and equal to a constant. The two constants in the figure refer to the steady state of the state-like variable in the case where the external effect is zero[4]. For illustrational purposes, we consider half the value of the steady state ($\underline{a}$) in the left diagram and twice the value of the steady state ($\overline{a}$) in the right diagram. Consider two economies of size $N$ and $N'$ with $N < N'$. Suppose that $\gamma/N = \gamma'/N'$ holds. Then the degree of increasing returns is the same in

[3]The opportunity costs of schooling are also determined by the shadow prices of the agent's capital stocks, cf. first order necessary condition (8).

[4]The absence of the external effect entails this value to be the steady state for all $N$.

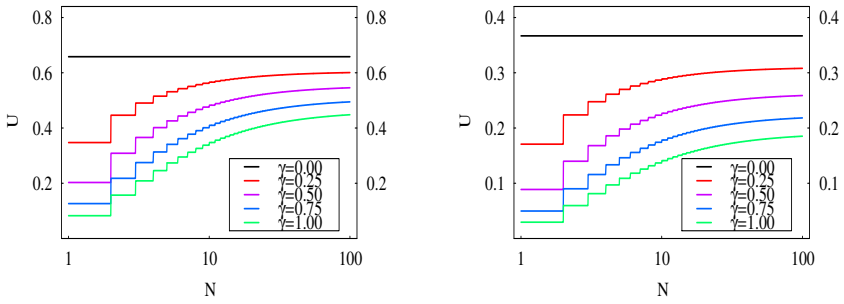


Figure 4: Optimal time share $u(x)$ with respect to $\gamma$ and $N$, for $x \in \{\underline{a}, \overline{a}\}$. Left: For $x = \underline{a}$. Right: For $x = \overline{a}$.

both economies. However, because of $1/N > 1/N'$, the agents' influence on the evolution of total factor productivity in the $N$ case is relatively larger than in the $N'$ case. Thus, the relationship between the shadow prices of the two capital stocks is shifted towards physical capital in the $N$-case. Hence, the opportunity cost of schooling must be higher and agents in the $N$ economy engage in more schooling activities than agents in the $N'$ economy. As a consequence, the optimal allocation of human capital in the two cases cannot be identical. Thus, we have disproved our initial idea that the influences of the degree of the external effect and the number of players are reciprocal when fixing the state-like variable.

Finally, we examine if the influences of the two parameters are reciprocal when considering relative deviations of the state-like variables from their respective steady state. Figure 5 shows the values of $u$ when considering the state-like variable to be far away from the steady state of the respective economy. In the left diagram, we consider $x = 1/2x_{ss}$. This corresponds to a relative scarcity of physical capital, thereby implying a relatively low marginal productivity of human capital in goods production. Interestingly, we find an increasing influence of the degree of the external effect on the optimal human capital allocation for economies with a larger number of players. In contrast, $u$ is decreasing in $\gamma$ for small economies. Let two games be given. In the first game, the parameter $\gamma$ is larger than in the second game. Furthermore, suppose that the number of players $N$ is relatively large. As a consequence, we can neglect the agents' influence on the economy-wide average human capital stock. If we assume firstly that the individual levels of the capital stocks and secondly that the growth rates of the average human capital stocks are the same in both games, the agents know that the total factor productivity of goods production will be higher in the next moment for the economy with the larger $\gamma$. Consequently, agents would like to consume more in the large $\gamma$ case. Thus, agents allocate more human capital to the goods sector in order to compensate the higher consumption rate. In other words, for large economies the relationship between the shadow prices of the two capital stocks is shifted towards human capital if the parameter $\gamma$ increases. Furthermore, the original assumption that the average stock of human capital grow in both cases at equal rates does not

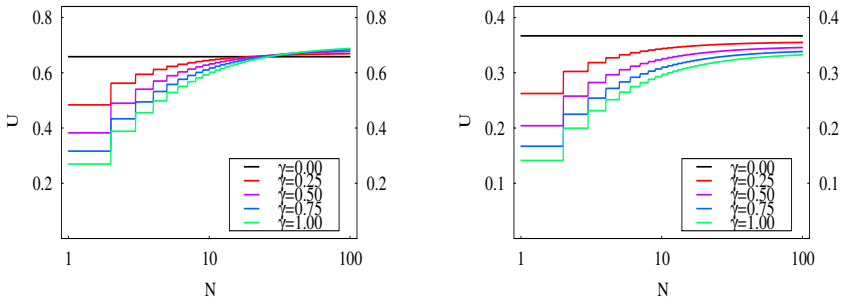


Figure 5: Optimal time share $u(x)$ with respect to $\gamma$ and $N$. Left: For $x = \frac{1}{2}x_{ss}$. Right: For $x = 2x_{ss}$.

hold. As a result, they cannot be symmetric Nash equilibria, at least in one of the two games considered. We conclude that the average stock of human capital grows faster in the high $\gamma$ economy if the agents' influence on $h_a$ is negligible. In the right diagram, $x = 2x_{ss}$ holds, i.e. we consider a relative scarcity of human capital. This causes the productivity of human capital in the goods sector to be very high. When disregarding the respective shadow prices, it follows that the opportunity cost of schooling is also very high. Therefore, the players optimal allocation of human capital to the goods sector has to be relatively small in order to balance the marginal productivity of human capital in the two sectors. Obviously, the larger the parameter $\gamma$ and the smaller the size of the economy, the more distinct this effect becomes.

# 6 Conclusion

In this paper, we have derived the open loop solutions of symmetric differential games with logarithmic objective functions. The focus on time dependent control paths and symmetric Nash equilibria has permitted to solve the differential games for an arbitrary number of players $N$. We have shown that the games' solutions are completely described by an initial value problem for an ordinary differential equation. Since the allocation of human capital between the two production sectors is crucial for our understanding of the transitional dynamics (cf. Mulligan and Sala-i-Martin, 1993), we have examined the influences of the number of players, $N$, and of the degree of the external effect, $\gamma$, on the agents' optimal choices of $u$. Although the fraction $\gamma/N$ is a sufficient statistic to determine the steady state value of the optimal human capital allocation, we have demonstrated that we need to know both parameters when searching for the optimal $u$ outside the steady state.

Finally, future research will have to investigate the game's Markovian outcome. The Markovian equilibrium concept requires that at each point in time players are able to observe the state of the game and commit to an optimal state-dependent decision rule. While this concept also captures the strategic

externality that arises from the other players' feedback on an agent's human capital allocation decision, the open-loop concept only captures the cost externality due to changes in the opportunity cost of schooling. The difference in the two equilibrium concepts, therefore, quantifies the strategic externality. This externality provides an upper bound of the inaccuracy introduced by the simplifying open-loop equilibrium concept.

# Appendix

## The initial value $q'(x^{ss})$

Note that the derivatives of $\varphi(q,x)$ at $q=q(x^{ss})$ and $x=x^{ss}$ are found to be

$$\varphi_x = \frac{\varphi_a^{ss}\alpha\gamma(N-1)u_x}{(1-\alpha)[N(1-\alpha+\gamma)-\gamma(N-1)u_{ss}]} \quad \text{and} \quad \varphi_q = \frac{\varphi_a^{ss}\alpha\gamma(N-1)u_q}{(1-\alpha)[N(1-\alpha+\gamma)-\gamma(N-1)u_{ss}]}.$$

The respective derivatives of (16) are given by

$$u_x = u_{ss}\frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u_{ss})+\rho}{\alpha\rho\varphi} \quad \text{and} \quad u_q = \frac{u_{ss}}{\rho\varphi_a^{ss}}.$$

Let $K(q(x^{ss}),x^{ss})$ and $L(q(x^{ss}),x^{ss})$ denote the numerator and denominator of the differential equation (17). Since $q'(x^{ss}) = \frac{-K_x(q(x^{ss}),x^{ss})}{\rho L_q(q(x^{ss}),x^{ss})}$ holds for the second root of the quadratic equation we nedd the following two derivatives:

$$L_q(x^{ss},q(x^{ss})) = \frac{1-\alpha^2}{\alpha} + \frac{(1-\alpha+\gamma)u_{ss}}{1-\alpha+\gamma-\gamma\frac{N-1}{N}u_{ss}}$$

$$K_x(x^{ss},q(x^{ss})) = \frac{-\rho q^{ss}}{\alpha\varphi}\left\{\frac{(1-\alpha+\gamma)(1-\alpha)}{1-\alpha+\gamma/N} + \frac{(1-\alpha)^2(1-\alpha+\gamma-\gamma\frac{N-1}{N}u^{ss})}{(1-\alpha+\gamma/N)u^{ss}}\left(1+\frac{1-\alpha+\gamma}{1-\alpha+\gamma/N}\right)\right\}.$$

This implies the expression given in equation (18).

## Human capital allocation

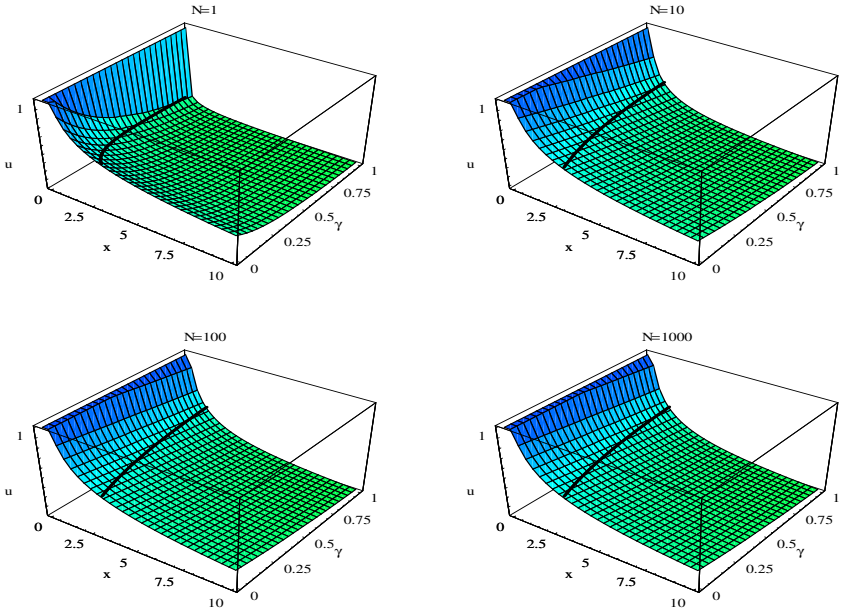


Figure 6: Optimal time share $u$ in $(x,\gamma)$ space, where $N \in \{1,10,100,1000\}$.

Figure 6 shows similar plots as Figure 2 above. In Figure 6, we have set the number of players, $N$, equal to 1, 10, 100, and 1000. Indeed $N$ causes the surface to rise and one might conjecture that it is stretched like a rubber blanket.

# References


[1] Bethmann, Dirk and Markus Reiß (2004): "A simple method to study the Transitional Dynamics in Endogenous Growth Models", preprint No 917, Weierstraß-Institut für angewandte Analysis und Stochastik.

[2] Bronshtein, Ilya N. and Konstantin A. Semendyayew (1997): "*Handbook of Mathematics*", third edition, Springer Verlag GmbH & Co, Berlin.

[3] Caballé, Jordi and Manuel S. Santos (1993): "On Endogenous Growth with Physical and Human Capital", *Journal of Political Economy* 101(6), 1042-1067.

[4] Clemhout, Simone and Henry Y. Wan Jr. (1994): "Differential Games - Economic Applications", in Aumann, Robert J. and Sergiu Hart (eds.), *Handbook of Game Theory with Economic Applications II*, Elsevier, Amsterdam, 801-825.

[5] Dockner, Engelbert; Jørgensen, Steffen; Long, Ngo Van and Gerhard Sorger (2000): "*Differential Games in economics and management science*", Cambridge University Press, Cambridge (UK).

[6] Lucas, Robert E. (1988): "On The Mechanics Of Economic Development", *Journal of Monetary Economics 22(1)*, 3-42.

[7] Mulligan, Casey B. and Xavier Sala-i-Martin (1991): "A Note On The Time-Elimination Method For Solving Recursive Dynamic Economic Models", *NBER Technical Working Paper Series*, TWP No. 116.

[8] Mulligan, Casey B. and Xavier Sala-i-Martin (1993): "Transitional Dynamics in Two-Sector Models of Endogenous Growth", *Quarterly Journal of Economics 108(3)*, 739-773.

[9] Uzawa, Hirofumi (1965): "Optimum Technical Change In An Aggregate Model Of Economic Growth", *International Economic Review 6(1)*, 18-31.