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Working Paper

The stable marriage problem with ties and restricted edges

CERS-IE Working Papers, No. CERS-IE WP - 2020/7

Provided in Cooperation with:

Institute of Economics, Centre for Economic and Regional Studies, Hungarian Academy of Sciences

Suggested Citation: Cseh, Ágnes; Heeger, Klaus (2020) : The stable marriage problem with ties and restricted edges, CERS-IE Working Papers, No. CERS-IE WP - 2020/7, Hungarian Academy of Sciences, Institute of Economics, Centre for Economic and Regional Studies, Budapest

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January 2020

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ABSTRACT

In the stable marriage problem, a set of men and a set of women are given, each of whom has a strictly ordered preference list over the acceptable agents in the opposite class. A matching is called stable if it is not blocked by any pair of agents, who mutually prefer each other to their respective partner. Ties in the preferences allow for three different definitions for a stable matching: weak, strong and super-stability. Besides this, acceptable pairs in the instance can be restricted in their ability of blocking a matching or being part of it, which again generates three categories of restrictions on acceptable pairs. Forced pairs must be in a stable matching, forbidden pairs must not appear in it, and lastly, free pairs cannot block any matching.

Our computational complexity study targets the existence of a stable solution for each of the three stability definitions, in the presence of each of the three types of restricted pairs. We solve all cases that were still open. As a byproduct, we also derive that the maximum size weakly stable matching problem is hard even in very dense graphs, which may be of independent interest.

JEL codes: C63, C78

Keywords: stable matchings, restricted edges, complexity

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A stabil párosítás probléma gyengén rendezett listákkal és korlátozott élekkel

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ÖSSZEFOGLALÓ

A klasszikus stabil párosítás problémában nők és férfiak egy halmaza adott. Mindenki felállít egy szigorúan rendezett preferencialistát az ellenkező nem néhány tagjáról. Akkor nevezünk egy párosítást stabilnak, ha azt nem blokkolja egyetlen pár sem. Egy pár akkor blokkolja a párosítást, ha mindkét tagja magasabban rangsorolja egymást, mint a párosításban hozzájuk rendelt személyeket. Ha a preferencialisták nem szigorúan, hanem gyengén rendezettek, akkor háromféle stabilitási definícióval dolgozhatunk: gyenge, erős és szuper-stabilitással. Egyes párok lehetnek korlátozottak is: a kötelező párokat muszáj, a tiltott párokat pedig tilos tartalmaznia a keresett stabil párosításnak. A szabad élek lehetnek párosítás tagjai, de nem blokkolhatnak párosítást.

Cikkünkben bonyolultságelméleti szempontból tanulmányozzuk a stabil megoldás létezését mindhárom stabilitási definícióra, mindhárom féle korlátozott él jelenlétében. Minden eddig nyitott kérdést megválaszolunk. Az is kiderül, hogy a maximális méretű gyengén stabil párosítás problémája még nagyon sűrű gráfokban is NP-teljes.

JEL: C63, C78

Kulcsszavak: stabil párosítás, korlátozott élek, bonyolultság

The stable marriage problem with ties and restricted edges

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Abstract

In the stable marriage problem, a set of men and a set of women are given, each of whom has a strictly ordered preference list over the acceptable agents in the opposite class. A matching is called stable if it is not blocked by any pair of agents, who mutually prefer each other to their respective partner. Ties in the preferences allow for three different definitions for a stable matching: weak, strong and super-stability. Besides this, acceptable pairs in the instance can be restricted in their ability of blocking a matching or being part of it, which again generates three categories of restrictions on acceptable pairs. Forced pairs must be in a stable matching, forbidden pairs must not appear in it, and lastly, free pairs cannot block any matching.

Our computational complexity study targets the existence of a stable solution for each of the three stability definitions, in the presence of each of the three types of restricted pairs. We solve all cases that were still open. As a byproduct, we also derive that the maximum size weakly stable matching problem is hard even in very dense graphs, which may be of independent interest.

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[☆]The authors were supported by the Cooperation of Excellences Grant (KEP-6/2019), the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3/2019), OTKA grant K128611, the DFG Research Training Group 2434 “Facets of Complexity”, and COST Action CA16228 European Network for Game Theory.

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1. Introduction

In the classical *stable marriage problem* (SM) [14], a bipartite graph is given, where one side symbolizes a set of men U , while the other side symbolizes a set of women W . Man u and woman w are connected by the edge uw if they find one another mutually acceptable. In the most basic setting, each participant provides a strictly ordered preference list of the acceptable agents of the opposite gender. An edge uw *blocks* matching M if it is not in M , but each of u and w is either unmatched or prefers the other to their respective partner in M . A *stable matching* is a matching not blocked by any edge. From the seminal paper of Gale and Shapley [14], we know that the existence of such a stable solution is guaranteed and a stable matching can be found in linear time.

Several real-world applications [6] require a relaxation of the strict order to weak order, or, in other words, preference lists with ties, leading to the *stable marriage problem with ties* (SMT) [16, 18, 24]. When ties occur, the definition of a blocking edge needs to be revisited. In the literature, three intuitive definitions are used, namely weakly, strongly and super-stable matchings [16]. According to weak stability, a matching is *weakly blocked* by an edge uw if agents u and w both strictly prefer one another to their partners in the matching. A *strongly blocking* edge is preferred strictly by one end vertex, whereas it is not strictly worse than the matching edge at the other end vertex. A *super-blocking edge* is at least as good as the matching edge for both end vertices in the super-stable case. Super-stable matchings are strongly stable and strongly stable matchings are weakly stable by definition, because weakly blocking edges are strongly blocking, and strongly blocking edges are super-blocking at the same time.

Weak and strong stability serve as the goal to achieve in most applications, such as college admission programs. In most countries, colleges are not required to rank all applicants in a strict order of preference, hence large ties occur in their lists. According to the equal treatment policy used in Chile and Hungary for example, it may not occur that a student is rejected from a college preferred by her, even though other students with the same score are admitted [7, 29]. Other countries, such as Ireland [9], break ties with lottery, which gives way to a weakly stable solution according to the original, weak order. Super-stable matchings can represent safe solutions if agents provide

	uw must be in M	uw can be in M	uw must not be in M
uw can block M	forced	unrestricted	forbidden
uw cannot block M	forced	free	irrelevant

Table 1: The three types of restricted edges are marked with bold letters. The columns tells edge uw ’s role regarding being in a matching, while the rows split cases based on uw ’s ability to block a matching.

uncertain preferences that mask an underlying strict order [30, 5, 4]. If two edges are in the same tie because of incomplete information derived from the agent, then super-stable matchings form the set of matchings that guarantee stability for all possible true preferences.

Another classical direction of research is to distinguish some of the edges based on their ability to be part of or to block a matching. Table 1 provides a structured overview of the three sorts of restricted edges that have been defined in earlier papers [20, 11, 12, 3, 22, 10]. The mechanism designer can specify three sets of restricted edges: *forced* edges must be in the output matching, *forbidden* edges must not appear in it, and finally, *free* edges cannot block the matching, regardless of the preference ordering.

The market designer’s motivation behind forced and forbidden edges is clear. By adding these restricted edges to the instance, one can shrink the set of stable solutions to the matchings that contain a particularly important or avoid an unwelcome partnership between agents. Free edges model a less intuitive, yet ubiquitous scenario in applications [3]. Agents are often not aware of the preferences of others, not even once the matching has been specified. This typically occurs in very large markets, such as job markets [2], or if the preferences are calculated rather than just provided by the agents, such as in medical [8] and social markets [1]. Agents who cannot exchange their preferences are connected via a free edge. If a matching is only blocked by free edges, then no pair of agents can undermine the stability of it.

In this paper, we combine weakly ordered lists and restricted edges, and determine the computational complexity of finding a stable matching in all cases not solved yet.

1.1. Literature review

We first focus on the known results for the SMT problem without restricted edges, and then switch to the SM problem with edge restrictions. Finally, we list all progress up to our paper in SMT with restricted edges.

65 *Ties.* If all edges are unrestricted, a weakly stable matching always exists,
66 because generating any linear extension to each preference list results in a
67 classical SM instance, which admits a solution [14]. This solution remains
68 stable in the original instance as well. On the other hand, strong and super-
69 stable matchings are not guaranteed to exist. However, there are polynomial-
70 time algorithms to output a strongly/super-stable matching or a proof for
71 its nonexistence [16, 25].

72 *Restricted edges.* Dias et al. [11] showed that the problem of finding a stable
73 matching in a SM instance with forced and forbidden edges or reporting that
74 none exists is solvable in $O(m)$ time, where m is the number of edges in
75 the instance. Approximation algorithms for instances not admitting any
76 stable matching including all forced and avoiding all forbidden edges were
77 studied in [10]. The existence of free edges can only enlarge the set of stable
78 solutions, thus a stable matching with free edges always exists. However, in
79 the presence of free edges, a maximum-cardinality stable matching is **NP**-
80 hard to find [3]. Kwanashie [22, Sections 4 and 5] performed an exhaustive
81 study on various stable matching problems with free edges. The term “stable
82 with free edges” [8, 13] is equivalent to the adjective “socially stable” [3, 22]
83 for a matching.

84 *Ties and restricted edges.* Table 2 illustrates the known and our new re-
85 sults on problems that arise when ties and restricted edges are combined
86 in an instance. Weakly stable matchings in the presence of forbidden edges
87 were studied by Scott [32], where the author shows that deciding whether a
88 matching exists avoiding the set of forbidden edges is **NP**-complete. A simi-
89 lar hardness result was derived by Manlove et al. [26] for the case of forced
90 edges, even if the instance has a single forced edge. Forced and forbidden
91 edges in super-stable matchings were studied by Fleiner et al. [12], who gave a
92 polynomial-time algorithm to decide whether a stable solution exists. Strong
93 stability in the presence of forced and forbidden edges is covered by Kun-
94 ysz [21], who gave a polynomial-time algorithm for the weighted strongly
95 stable matching problem with non-negative edge weights. Since strongly sta-
96 ble matchings are always of the same cardinality [23, 18], a stable solution
97 or a proof for its nonexistence can be found via setting the edge weights to
98 0 for forbidden edges, 2 for forced edges, and 1 for unrestricted edges.

Existence	weak	strong	super
forbidden	NP-complete [32] even if $P = 1$	$O(nm)$ [21]	$O(m)$ [12]
forced	NP-complete even if $Q = 1$ [26]	$O(nm)$ [21]	$O(m)$ [12]
free	always exists	NP-complete	NP-complete

Table 2: Previous and our results summarized in a table. The contribution of this paper is marked by bold gray font. The instance has n vertices, m edges, $|P|$ forbidden edges, and $|Q|$ forced edges.

1.2. Our contributions

In Section 3 we prove a stronger result than the hardness proof in [32] delivers: we show that finding a weakly stable matching in the presence of forbidden edges is **NP-complete** even if the instance has a single forbidden edge.

As a byproduct, we gain insight into the well-known maximum size weakly stable matching problem (without any edge restriction). This problem is known to be **NP-complete** [19, 26], even if preference lists are of length at most three [17, 27]. On the other hand, if the graph is complete, a complete weakly stable matching is guaranteed to exist. It turns out that this completeness is absolutely crucial to keep the problem tractable: as we show here, if the graph is a complete bipartite graph missing exactly one edge, then deciding whether a perfect weakly stable matching exists is **NP-complete**.

We turn to the problem of free edges under strong and super-stability in Section 4. We show that deciding whether a strongly/super-stable matching exists when free edges occur in the instance is **NP-complete**. This hardness is in sharp contrast to the polynomial-time algorithms for the weighted strongly/ super-stable matching problems. Afterwards, we show that deciding the existence of a strongly or super-stable matching in an instance with free edges is fixed-parameter tractable parameterized by the number of free edges.

2. Preliminaries

The input of the stable marriage problem with ties consists of a bipartite graph $G = (U \cup W, E)$ and for each $v \in U \cup W$, a weakly ordered preference list O_v of the edges incident to v . We denote the number of vertices in G by n , while m stands for the number of edges. An edge connecting vertices u and w is denoted by uw . We say that the preference lists in an instance are

126 derived from a *master list* if there is a weak order O of $U \cup W$ so that each O_v
 127 where $v \in U \cup W$ can be obtained by deleting entries from O .

128 The set of restricted edges consists of the set of *forbidden edges* P , the set
 129 of *forced edges* Q , and the set of *free edges* F . These three sets are disjoint.

130 **Definition 1.** A matching M is weakly/strongly/super-stable with restricted
 131 edge sets P, Q , and F , if $M \cap P = \emptyset$, $Q \subseteq M$, and the set of edges blocking M
 132 in a weakly/strongly/super sense is a subset of F .

133 3. Weak stability

134 In Theorem 1 we present a hardness proof for the weakly stable matching
 135 problem with a single forbidden edge, even if this edge is ranked last by
 136 both end vertices. The hardness of the maximum-cardinality weakly stable
 137 matching problem in dense graphs (Theorem 2) follows easily from this result.

138 **Problem 1.** SMT-FORBIDDEN-1

139 *Input:* A complete bipartite graph $G = (U \cup W, E)$, a forbidden edge $P = \{uw\}$
 140 and preference lists with ties.

141 *Question:* Does there exist a weakly stable matching M so that $uw \notin M$?

142 **Theorem 1.** SMT-FORBIDDEN-1 is NP-complete, even if all ties are of length
 143 two, they appear only on one side of the bipartition and at the beginning of
 144 the complete preference lists, and the forbidden edge is ranked last by both its
 145 end vertices.

146 *Proof.* SMT-FORBIDDEN-1 is clearly in NP, as any matching can be checked
 147 for weak stability in linear time.

148 We reduce from the PERFECT-SMTI problem defined below, which is
 149 known to be NP-complete even if all ties are of length two, and appear on
 150 one side of the bipartition and at the beginning of the preference lists, as
 151 shown by Manlove et al. [26].

152 **Problem 2.** PERFECT-SMTI

153 *Input:* An incomplete bipartite graph $G = (U \cup W, E)$, and preference lists
 154 with ties.

155 *Question:* Does there exist a perfect weakly stable matching M ?

156 *Construction.* To each instance $\mathcal{I}$ of PERFECT-SMTI, we construct an in-
 157 stance $\mathcal{I}'$ of SMT-FORBIDDEN-1.

158 Let $G = (U \cup W, E)$ be the underlying graph in instance $\mathcal{I}$. When con-
 159 structing G' for $\mathcal{I}'$, we add two men u_1 and u_2 to U , and two women w_1
 160 and w_2 to W . On vertex classes $U' = U \cup \{u_1, u_2\}$ and $W' = W \cup \{w_1, w_2\}$,
 161 G' will be a complete bipartite graph. As the list below shows, we start with
 162 the original edge set $E(G)$ in stage 0, and then add the remaining edges in
 163 four further stages. An example for the built graph is shown in Figure 1.

- 164 0. $E(G)$
 165 We keep the edges in $E(G)$ and also preserve the vertices' rankings on
 166 them. These edges are solid black in Figure 1.
- 167 1. $(U \times \{w_1\}) \cup (\{u_1\} \times W)$
 168 We first connect u_1 to all women in W , and w_1 to all men in U . Man u_1
 169 (woman w_1) ranks the women from W (men from U) in an arbitrary
 170 order. Each $u \in U$ ($w \in W$) ranks w_1 (u_1) after all their edges in $E(G)$.
 171 These edges are loosely dashed green in Figure 1.
- 172 2. $(U \times W) \setminus E(G)$
 173 Now we add for each pair $(u, w) \in U \times W$ with $uw \notin E(G)$ the edge uw ,
 174 where u (w) ranks w (u) even after w_1 (u_1). These edges are densely
 175 dashed blue in Figure 1.
- 176 3. $[(U \cup \{u_1\}) \times \{w_2\}] \cup [\{u_2\} \times (W \cup \{w_1\})]$
 177 Man u_2 is connected to all women from $W \cup \{w_1\}$, and ranks all these
 178 women in an arbitrary order. The women from $W \cup \{w_1\}$ rank u_2
 179 worse than any already added edge. Similarly, w_2 is connected to all
 180 men from $M \cup \{u_1\}$, and ranks all these men in an arbitrary order.
 181 The men from $M \cup \{u_1\}$ rank w_2 worse than any already added edge.
 182 These edges are dotted red in Figure 1.
- 183 4. u_1w_1 and u_2w_2
 184 Finally, we add the edges u_1w_1 and u_2w_2 , which are ranked last by both
 185 of their end vertices. Edge u_2w_2 is the only forbidden edge and it is
 186 the violet zigzag edge in Figure 1, while u_2w_2 is wavy gray.

187 **Claim:** $\mathcal{I}$ admits a perfect weakly stable matching if and only if $\mathcal{I}'$ admits
 188 a weakly stable matching not containing u_2w_2 .

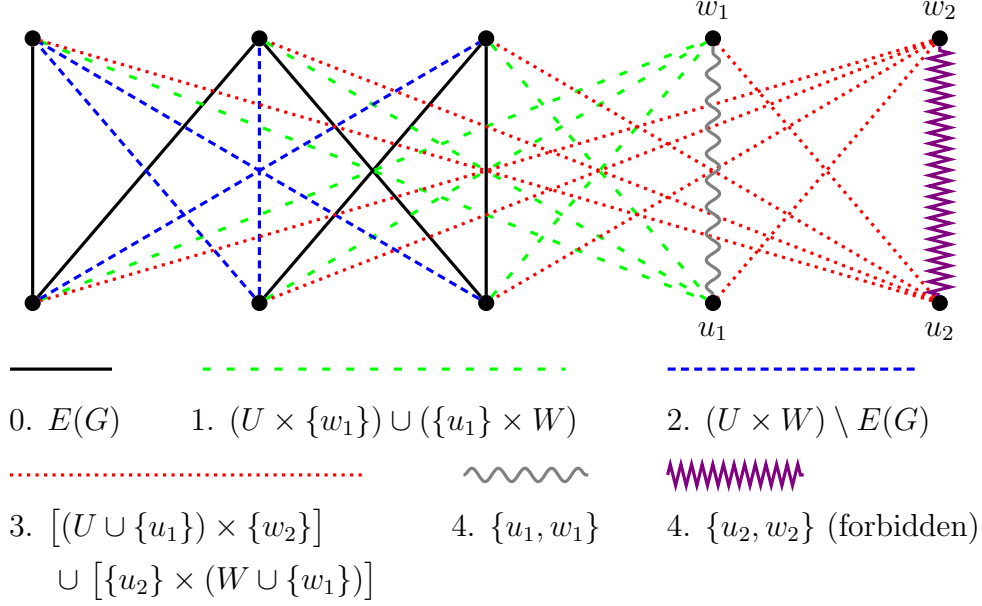


Figure 1: An example for the reduction. The legend below the graph lists the six groups of edges in the preference order at all vertices. The edges from the PERFECT-SMTI instance (drawn in solid black) keep their ranks. Every vertex ranks solid black edges best, then loosely dashed green edges, then densely dashed blue edges, then dotted red edges, then the wavy gray edge $\{u_1, w_1\}$ and the forbidden violet zigzag edge $\{u_w, w_2\}$.

189 $(\Rightarrow)$ Let M be a perfect weakly stable matching in $\mathcal{I}$. We construct M' as
 190 $M \cup \{u_1 w_2\} \cup \{u_2 w_1\}$. Clearly, M' is a matching not containing the forbidden
 191 edge $u_2 w_2$, so it only remains to show that M' is weakly stable. We do this
 192 by case distinction on a possible weakly blocking edge.

- 193 0. $E(G)$
 194 Since M does not admit a weakly blocking edge in $\mathcal{I}$, no edge from the
 195 original $E(G)$ can block M' weakly in $\mathcal{I}'$.
- 196 1. $(U \times \{w_1\}) \cup (\{u_1\} \times W)$
 197 All vertices in $U \cup W$ rank these edges lower than their edges in M' .
- 198 2. $(U \times W) \setminus E(G)$
 199 Edges in this set cannot block M' weakly because they are ranked worse
 200 than edges in M' by both of their end vertices.

- 201 3. $[(U \cup \{u_1\}) \times \{w_2\}] \cup [(W \cup \{w_1\}) \times \{u_2\}]$
 202 Vertices in $U \cup W$ prefer their edge in M' to all edges in this set. Since
 203 they are in M' , u_1w_2 and u_2w_1 also cannot block M' weakly.
- 204 4. u_1w_1 and u_2w_2
 205 These two edges are strictly worse than $u_1w_2 \in M'$ and $u_2w_1 \in M'$ at
 206 all four end vertices.

207 ($\Leftarrow$) Let M' be a weakly stable matching in $\mathcal{I}'$ and $u_2w_2 \notin M'$. Since G'
 208 is a complete bipartite graph with the same number of vertices on both
 209 sides, M' is a perfect matching. In particular, u_2 and w_2 are matched
 210 by M' , say to w and u , respectively. Since M' does not contain the for-
 211 bidden edge u_2w_2 , we have that $u \neq u_2$ and $w \neq w_2$. Then we have $w = w_1$
 212 and $u = u_1$, as uw blocks M' weakly otherwise.

213 If M' contains an edge $uw \notin E(G)$ with $u \in U$ and $w \in W$, then this
 214 implies that uw_1 is a weakly blocking edge. Thus, $M := M' \setminus \{u_1w_2, u_2w_1\} \subseteq$
 215 $E(G)$, i.e. it is a perfect matching in G . This M is also weakly stable, as any
 216 weakly blocking edge in G immediately implies a weakly blocking edge for M' ,
 217 which contradicts our assumption on M' being a weakly stable matching. $\square$

218 As a byproduct, we get that MAX-SMTI-DENSE, the problem of deciding
 219 whether an almost complete bipartite graph admits a perfect weakly stable
 220 matching, is also NP-complete.

221 **Problem 3.** MAX-SMTI-DENSE

222 *Input:* A bipartite graph $G = (U \cup W, E)$, where $E(G) = \{uw : u \in U, w \in$
 223 $W\} \setminus \{u^*w^*\}$ for some $u^* \in U$ and $w^* \in W$, and preference lists with ties.

224 *Question:* Does there exist a perfect weakly stable matching M ?

225 **Theorem 2.** MAX-SMTI-DENSE is NP-complete, even if all ties are of length
 226 two, are on one side of the bipartition, and appear at the beginning of the
 227 preference lists.

228 *Proof.* MAX-SMTI-DENSE is in NP, as a matching can be checked for stability
 229 in linear time.

230 We reduce from SMT-FORBIDDEN-1. By Theorem 1, this problem is NP-
 231 complete even if the forbidden edge uw is at the end of the preference lists
 232 of u and w . For each such instance $\mathcal{I}$ of SMT-FORBIDDEN-1, we construct an
 233 instance $\mathcal{I}'$ of MAX-SMTI-DENSE by deleting the forbidden edge uw .

234 **Claim:** The instance $\mathcal{I}$ admits a weakly stable matching if and only if $\mathcal{I}'$
 235 admits a perfect weakly stable matching.

236 $(\Rightarrow)$ Let M be a weakly stable matching for $\mathcal{I}$. As SMT-FORBIDDEN-1 gets
 237 a complete bipartite graph as an input, M is a perfect matching. Since M
 238 does not contain the edge uw , it is also a matching in $\mathcal{I}'$. Moreover, M
 239 is weakly stable there, because the transformation only removed a possible
 240 blocking edge and added none of these.

241 $(\Leftarrow)$ Let M' be a perfect weakly stable matching in $\mathcal{I}'$. Since uw is at the
 242 end of the preference lists of u and w , and M' is perfect, uw cannot block M' .
 243 Thus, M' is weakly stable in $\mathcal{I}$. $\square$

244 Having shown a hardness result for the existence of a weakly stable match-
 245 ing even in very restricted instances with a single forbidden edge in Theo-
 246 rem 1, we now turn our attention to strongly and super-stable matchings.

247 4. Strong and super-stability

248 As already mentioned in Section 1.1, strongly and super-stable matchings
 249 can be found in polynomial time even if both forced and forbidden edges
 250 occur in the instance [12, 21]. Thus we consider the case of free edges, and
 251 in Theorem 3 and Proposition 4 we show hardness for the strong and super-
 252 stable matching problems in instances with free edges. The same construction
 253 suits both cases. Then, in Proposition 5 we remark that both problems are
 254 fixed-parameter tractable with the number of free edges $|F|$ as the parameter.

255 **Problem 4.** SSMTI-FREE

256 *Input:* A bipartite graph $G = (U \cup W, E)$, a set $F \subseteq E$ of free edges, and
 257 preference lists with ties.

258 *Question:* Does there exist a matching M so that $uw \in F$ for all $uw \in E$ that
 259 blocks M in the strongly/super-stable sense?

260 In SSMTI-FREE, we define two problem variants simultaneously, because
 261 all our upcoming proofs are identical for both of these problems. For the
 262 super-stable marriage problem with ties and free edges, all super-blocking
 263 edges must be in F , while for the strongly stable marriage problem with ties
 264 and free edges, it is sufficient if a subset of these, the strongly blocking edges,
 265 are in F .

266 **Theorem 3.** SSMTI-FREE is NP-complete even in graphs with maximum de-
 267 gree four, and if preference lists of women are derived from a master list.

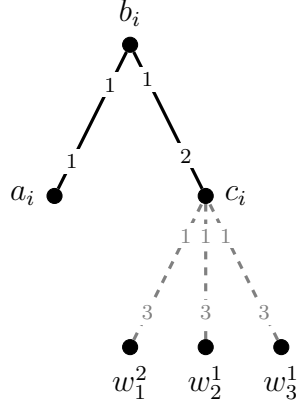


Figure 2: An example of a clause gadget for the clause C_i , containing the variables x_1 , x_4 , and x_5 . The interconnecting edges are dashed and gray.

268 *Proof.* SSMTI-FREE is clearly in **NP** because the set of edges blocking a match-
 269 ing can be determined in linear time.

270 We reduce from the 1-IN-3 POSITIVE 3-SAT problem, defined below,
 271 which is known to be **NP**-complete [31, 15, 28].

272 **Problem 5.** 1-IN-3 POSITIVE 3-SAT

273 *Input:* A 3-SAT formula, in which no literal is negated and every variable
 274 occurs in at most three clauses.

275 *Question:* Does there exist a satisfying truth assignment that sets exactly one
 276 literal in each clause to be true?

277 *Construction.* To each instance $\mathcal{I}$ of 1-IN-3 POSITIVE 3-SAT, we construct
 278 an instance $\mathcal{I}'$ of SSMTI-FREE.

279 Let $x_1, \dots, x_n$ be the variables and $C_1, \dots, C_m$ be the clauses of the 1-
 280 IN-3 POSITIVE 3-SAT instance $\mathcal{I}$. For each clause C_i , we add a clause gadget
 281 consisting of three vertices a_i , b_i , and c_i , where b_i is connected to a_i and c_i , as
 282 shown in Figure 2. While vertices a_i and b_i do not have any further edge, c_i
 283 will be incident to three *interconnecting* edges leading to variable gadgets.
 284 Vertex b_i is ranked first by a_i and last by c_i , and these two vertices are placed
 285 in a tie by b_i .

286 For each variable x_i , occurring in the three clauses C_{i_1} , C_{i_2} , and C_{i_3} , we
 287 add a variable gadget with nine vertices y_i^j , z_i^j , and w_i^j for $j \in [3]$, as indicated
 288 in Figure 3. Each vertex z_i^j is connected only to y_i^j by a free edge, and these

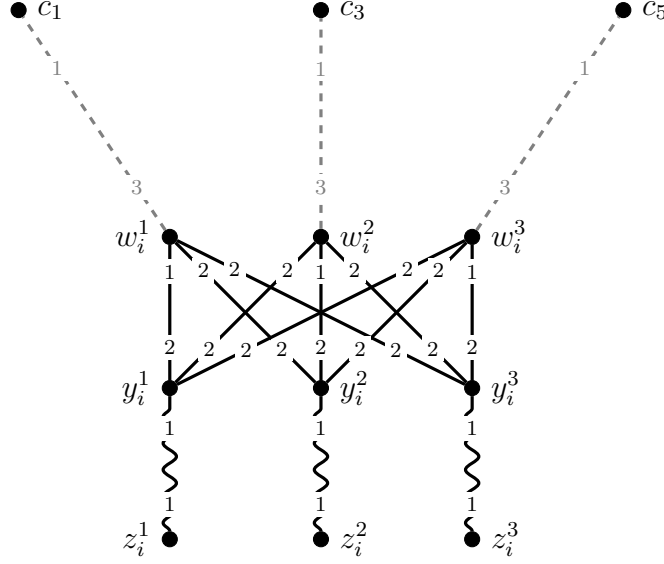


Figure 3: An example of a variable gadget for the variable x_i occurring, where x_i occurs exactly in the clauses C_1 , C_3 , and C_5 . Free edges are marked by wavy lines, while interconnecting edges are dashed and gray.

are the only free edges in our construction. For each $(\ell, j) \in [3]^2$, we add an edge $w_i^\ell y_i^j$, which is ranked second (after z_i^j) by y_i^j . The vertex w_i^ℓ ranks this edge at position one if $\ell = j$ and else at position two. Finally, we connect the vertex w_i^ℓ to the vertex c_{i_ℓ} by an interconnecting edge, ranked at position one by c_{i_ℓ} and position three by w_i^ℓ .

The resulting instance is bipartite: $U = \{z_i^j, w_i^j, b_i\}$ is the set of men and $W = \{y_i^j, c_i, a_i\}$ is the set of women. One easily sees that the maximum degree in our reduction is four.

Note that the preference lists of the women in the SSMTI-FREE instance are derived from a master list. The master list for the women $W = \{y_i^j, c_i, a_i\}$ is the following. At the top are all vertices of the form $\{z_i^j\}$ in a single tie, followed by all vertices of the form $\{w_i^j\}$ in a single tie, and finally, all other vertices $\{b_i\}$ at the bottom of the preference list, in a single tie.

Claim: $\mathcal{I}$ is a YES-instance if and only if $\mathcal{I}'$ admits a strongly/super-stable matching.

($\Rightarrow$) Let T be a satisfying truth assignment such that for each clause, exactly one literal is true. For each true variable x_i in this assignment, let M

306 contain the edges $w_i^\ell c_{i_\ell}$ and $y_i^\ell z_i^\ell$ for each $\ell \in [3]$. For all other variables,
 307 let M contain $w_i^\ell y_i^\ell$ for each $\ell \in [3]$. For each clause C_i , add the edge $a_i b_i$
 308 to M .

309 Following these rules, we have constructed a matching. It remains to
 310 check that M is super-stable (and thus also strongly stable). Since a_i is
 311 matched to its only neighbor, it cannot be part of a super-blocking edge.
 312 Since each c_i is matched along an interconnecting edge, which is better
 313 than b_i , no super-blocking edge involves b_i . A super-blocking interconnect-
 314 ing edge $c_i w_j^\ell$ implies that w_j^ℓ is not matched to any y_j^ℓ , however this is only
 315 true if $c_i w_j^\ell \in M$. A super-blocking edge $w_i^\ell y_i^j$ does not appear. Either w_i^ℓ
 316 is matched to its unique first choice y_i^ℓ and therefore not part of a super-
 317 blocking edge, or y_i^j is matched to its unique first choice z_i^j , and thus, y_i^j is
 318 not part of a super-blocking edge.

319 ($\Leftarrow$) Let M be a strongly stable matching (note that any super-stable
 320 matching is also strongly-stable). Then M contains the edge $a_i b_i$, and c_i is
 321 matched to a vertex w_j^ℓ for all $i \in [m]$, as else $c_i b_i$ or $a_i b_i$ blocks M strongly.
 322 If $w_j^\ell c_i \in M$, then $y_j^a z_j^a \in M$ for all $a \in [3]$, as else $w_j^\ell y_j^a$ would be a strongly
 323 blocking edge. This, however, implies that $w_j^a c_{j_a} \in M$ for all $a \in [3]$, as
 324 else $w_j^a c_{j_a}$ would be a strongly blocking edge.

325 Thus, for each variable x_i , the matching M contains either all edges $w_i^\ell c_{i_\ell}$
 326 for $\ell \in [3]$ or none of these edges. Thus, the variables x_i such that M
 327 contains $w_i^\ell c_{i_\ell}$ for $\ell \in [3]$ induce a truth assignment such that for each clause,
 328 exactly one literal is true. $\square$

329 This proof aimed at the hardness of the restricted case, in which the
 330 underlying graph has a low maximum degree. For the sake of completeness,
 331 we add another variant, which is defined in a complete bipartite graph.

332 **Proposition 4.** *SSMTI-FREE is NP-complete, even in complete bipartite graphs,*
 333 *where each tie has length at most three.*

334 *Proof.* We reduce from SSMTI-FREE. Given a SSMTI-FREE instance in graph
 335 G , we add all non-present edges between men and women as free edges,
 336 ranked worse than any edge from $E(G)$. We call the resulting graph H .

337 Clearly, a strongly/super-stable matching in G is also strongly/super-
 338 stable in H , as we only added free edges.

339 Vice versa, let M be a strongly/super-stable matching in H . Let $M' :=$
 340 $M \cap E(G)$ arise from M by deleting all edges not in $E(G)$. Then M' clearly
 341 is a matching in G , so it remains to show that M' is strongly/super-stable.

342 Assume that there is a blocking edge uw in G , in the strongly/super-
343 stable sense. Since uw is not blocking in H , at least one of u and w has to
344 be matched in H , but not in G . However, this vertex prefers uw also to its
345 partner in H , and thus, uw is also blocking in H , which is a contradiction. $\square$

346 Note that SSMTI-FREE becomes polynomial-time solvable if only a con-
347 stant number of edges is free in the same way as MAX-SSMI, the problem
348 of finding a maximum-cardinality stable matching with strict lists and free
349 edges [3].

350 **Proposition 5.** *SSMTI-FREE can be solved in $\mathcal{O}(2^k nm)$ time in the strongly*
351 *stable case, and in $\mathcal{O}(2^k m)$ time in the super-stable case, where $k := |F|$ is the*
352 *number of free edges, $n := |V(G)|$ is the number of vertices, and $m := |E(G)|$*
353 *is the number of edges.*

354 *Proof.* For each subset $Q \subseteq F$ of free edges, we construct an instance of
355 SSMTI-FORCED as follows. Mark all edges in Q as forced, and delete all edges
356 in $F \setminus Q$.

357 If any of the SSMTI-FORCED instances admits a stable matching, then
358 this is clearly a stable matching in the SSMTI-FREE instance, as only free
359 edges were deleted. Vice versa, any solution M for the SSMTI-FREE instance
360 containing exactly the set of forced edges Q (i.e. $Q = M \cap F$) immediately
361 implies a solution for the SSMTI-FORCED instance with forced edges Q .

362 Clearly, there are 2^k subsets of F . Since any instance of SSMTI-FORCED
363 can be solved in $\mathcal{O}(nm)$ time in the strongly stable case [12] and in $\mathcal{O}(m)$
364 time in the super-stable case [21], the running time follows. $\square$

365 5. Conclusion

366 Studying the stable marriage problem with ties combined with restricted
367 edges, we have shown three NP-completeness results. Our computational
368 hardness results naturally lead to the question whether imposing master
369 lists on both sides makes the problems easier to solve. Moreover, it is open
370 whether SMT-FORBIDDEN-1 remains hard in bounded-degree graphs. In ad-
371 dition, one may try to identify relevant parameters for our problems and then
372 decide whether they are fixed-parameter tractable or admit a polynomial-
373 sized kernel with respect to these parameters.

374 *Acknowledgments.* The authors thank David Manlove and Rolf Niedermeier
375 for useful suggestions that improved the presentation of this paper.

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